MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

PROBLEM SET 11

(Corrected Version)

REFERENCE: Peskin and Schroeder, Chapter 4.

NOTE ABOUT CORRECTED VERSION:
A sign in Eq. (1.15) has been corrected.

Problem 1: The Linked Cluster Theorem

The cancellation of disconnected diagrams that we found when we calculated $\langle\Omega\,|T\{\phi(x_1)\phi(x_2)\ldots\phi(x_n)\}|\,\Omega\rangle$ is actually part of a more general "linked cluster" theorem that is applicable to any system for which the concept of an expectation value makes sense. If one wishes to calculate the expectation value of an exponential function, or of some quantity times an exponential function, then the linked cluster theorem is useful.

To express this theorem, let $A_1, A_2, \ldots$ denote "random quantities," i.e., quantities for which expectation values can be defined. They might be quantum-mechanical operators, like $\phi(x_1), \phi(x_2), \ldots$, or they might be classical random variables, such as the position or momentum of particles in a statistical mechanics ensemble. We assume that the operations of addition and multiplication are well-defined, and that the expectation value function $\langle\quad\rangle$ has the following two properties:

(1) $\langle\quad\rangle$ is linear: $\quad\langle A+\lambda B\rangle = \langle A\rangle + \lambda\langle B\rangle$ for any A, B .

(2) $\langle A_1 A_2 \ldots\rangle$ is independent of the order of the factors.

Note that (2) is not usually valid for quantum-mechanical operators, but does hold for time-ordered products of operators. For time-ordered products it is the time arguments that control the order of evaluation, so the order in which the operators are written has no effect.

Our first task is to define the *connected part,* or *cumulant,* of a product of random quantities. For the product of two quantities, the definition will be

$$\langle AB\rangle_c = \langle AB\rangle - \langle A\rangle\langle B\rangle \ , \tag{1.1}$$

which is also called the correlator of A and B. If A and B are uncorrelated, then

$$\langle AB\rangle_c = 0 \ . \tag{1.2}$$

A generalization of this concept can be defined by induction, but first we need to define what is meant by a partition of the integers $\{1, 2, \ldots, N\}$. A partition of $\{1, \ldots, N\}$ into subsets is defined as a specific way of dividing the integers 1–N into non-empty subsets. More formally, a partition of $\{1, \ldots, N\}$ is a list of sets $S_1, S_2, \ldots, S_k$, with the properties

i) Each S_α is non-empty.
ii) Each $S_\alpha \subset \{1, \ldots, N\}$.
iii) $S_\alpha \cap S_\beta = 0$ if $\alpha \neq \beta$.
iv) $\bigcup_{\alpha=1,\ldots,k} S_\alpha = \{1, \ldots, N\}$.

Two partitions which differ only by the ordering of the subscript labels on the S's are considered identical. Note that properties (iii) and (iv) above can be summarized by saying that each of the integers $\{1, \ldots, N\}$ belongs to one and only one set S_α. Assuming that the connected part has been defined for all products of size less than N, the connected part for $\langle A_1 \ldots A_N \rangle$ is defined by

$$\begin{aligned} \langle A_1 \ldots A_N \rangle_c &= \langle A_1 \ldots A_N \rangle \\ &- \sum_{\substack{\text{all proper} \\ \text{partitions of} \\ \{1,\ldots,N\}}} \left\langle \prod_{i \in S_1} A_i \right\rangle_c \cdots \left\langle \prod_{i \in S_k} A_i \right\rangle_c , \end{aligned} \tag{1.3}$$

where "proper partitions" excludes the case $k = 1$, $S_1 = \{1, \ldots, N\}$. The definition (1.3) applies even when some or all of the A_i's are equal; for example, Eq. (1.1) implies that $\left\langle A^2 \right\rangle_c = \left\langle A^2 \right\rangle - \langle A \rangle^2$.

(a) Use this definition to calculate $\langle ABC \rangle_c$ in terms of ordinary expectation values. Find also the expression for $\left\langle A^3 \right\rangle_c$ in terms of ordinary expectation values.

(b) Generalize Eq. (1.2) by showing that if two sets of random quantities $A_1, \ldots, A_M$ and $B_1, \ldots, B_N$ are independent of each other in the sense that

$$\langle A_{i_1} \ldots A_{i_m} B_{j_1} \ldots B_{j_n} \rangle = \langle A_{i_1} \ldots A_{i_m} \rangle \langle B_{j_1} \ldots B_{j_n} \rangle \tag{1.4}$$

for any subset of the A's and B's, then

$$\langle A_{i_1} \ldots A_{i_m} B_{j_1} \ldots B_{j_n} \rangle_c = 0 \tag{1.5}$$

whenever m and n are both greater than or equal to one. [*Hint:* Mathematical induction is likely to be useful here.]

We now consider expectation values of products of random quantities and exponentials of random quantities. The quantities in the product can be labeled $A_1, A_2, \ldots, A_N$, and the argument of the exponential can be written as

$$B \equiv \sum_{i=1}^{M} \lambda_i A_i , \tag{1.6}$$

where $M \geq N$ and M could be infinite. (Since the A_i's had no particular ordering before writing this expression, there is no loss in generality in labeling a particular subset of the A_i as $A_i, \ldots, A_N$.) Our goal is to prove the identity

$$\langle A_1 \ldots A_N e^B \rangle = \exp\left\{ \left\langle e^B \right\rangle_c - 1 \right\} \times \sum_{\substack{\text{all partitions} \\ \text{of } \{1,\ldots,N\} \\ \text{into subsets}}} \left\langle \left(\prod_{i \in S_1} A_i \right) e^B \right\rangle_c \cdots \left\langle \left(\prod_{i \in S_k} A_i \right) e^B \right\rangle_c . \tag{1.7}$$

We will prove this by a method that is somewhat indirect.

Instead of trying to directly prove Eq. (1.7) from the definition (1.3), we instead introduce a new definition of connected part to help us complete the proof. We use $\bar{c}$ to denote the new definition of connected part, which is defined by a special case of Eq. (1.7):

$$\left\langle e^{\sum_i \lambda_i A_i} \right\rangle \equiv \exp\left\{ \left\langle e^{\sum_i \lambda_i A_i} \right\rangle_{\bar{c}} - 1 \right\} , \tag{1.8}$$

where

$$\left\langle e^{\sum_i \lambda_i A_i} \right\rangle_{\bar{c}} \tag{1.9}$$

is to be viewed as a shorthand for

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1} \cdots \sum_{i_n} \lambda_{i_1} \ldots \lambda_{i_n} \langle A_{i_1} \ldots A_{i_n} \rangle_{\bar{c}} . \tag{1.10}$$

We define $\langle A_{i_1} \ldots A_{i_n} \rangle_{\bar{c}}$ to be independent of the ordering of the factors, so any terms in the above sum that differ only by ordering are to be considered equivalent.

To use Eq. (1.8) as a definition, consider expanding both sides to some fixed order in the λ's. To zero order in λ,

$$\begin{aligned} 1 &= \exp\{\langle 1 \rangle_{\bar{c}} - 1\} \\ \Longrightarrow \quad &\langle 1 \rangle_{\bar{c}} = 1 . \end{aligned} \tag{1.11}$$

To first order,

$$\begin{aligned} \sum_i \lambda_i \langle A_i \rangle &= \sum_i \lambda_i \langle A_i \rangle_{\bar{c}} \\ \Longrightarrow \langle A_i \rangle_{\bar{c}} &= \langle A_i \rangle . \end{aligned} \tag{1.12}$$

(c) Continue expanding Eq. (1.8) through third order in the λ_i's, verifying that $\langle AB \rangle_{\bar{c}} = \langle AB \rangle_c$ and that $\langle ABC \rangle_{\bar{c}} = \langle ABC \rangle_c$.

(d) Now use a proof by induction to show that

$$\langle A_1 \dots A_N e^B \rangle = \exp\left\{\left\langle e^B \right\rangle_{\bar{c}} - 1\right\} \times \sum_{\substack{\text{all partitions}\\ \text{of } \{1,\dots,N\}\\ \text{into subsets}}} \left\langle \left(\prod_{i\in S_1} A_i\right) e^B \right\rangle_{\bar{c}} \cdots \left\langle \left(\prod_{i\in S_k} A_i\right) e^B \right\rangle_{\bar{c}} \quad . \tag{1.13}$$

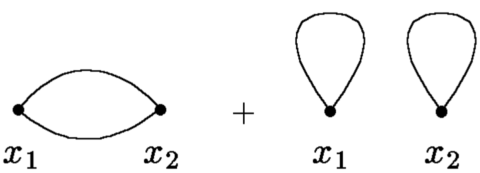
Note that this formula differs from Eq. (1.7) only by using $\bar{c}$ rather than c to define the connected part. The definition (1.8) guarantees that Eq. (1.13) holds for $N = 0$. You can therefore assume that Eq. (1.13) holds for some N, and show that it must hold for $N + 1$.

(e) Having derived Eq. (1.13), now use the special case $\lambda_i = 0$ for all i to show that

$$\langle A_1 \dots A_N \rangle_{\bar{c}} = \langle A_1 \dots A_N \rangle_c \tag{1.14}$$

for all N.

(f) To see how this general result applies to a free scalar quantum field theory, define $\langle X \rangle \equiv \langle 0 \,|T(X)|\, 0\rangle$, where T is the time-ordered product. Suppose that $A = \phi^2(x_1)$ and $B = \phi^2(x_2)$. Show that

$\langle AB \rangle =$ 

(1.15)

$$= 2\Delta_F(x_1 - x_2)^2 + \Delta_F(0)^2 \quad ,$$

and that $\langle AB \rangle_c$ is given by the first graph alone.

(g) Generalize the result in (f) to show that if each A_i is a normal-ordered product of free fields at the point x_i, then $\langle A_1 \dots A_N \rangle_c$ is given by the sum of all Wick contractions that correspond to connected diagrams. [*Hint:* Again, a proof by induction is appropriate. Assume that the statement holds for all values of N less than some N_0, and then use Eq. (1.3) to show that the only diagrams that make a net contribution to $\langle A_1 \dots A_N \rangle_c$ are the connected ones.

Once we identify the connected part $\langle A_1 \dots A_N \rangle_c$ with connected graphs, as you have done in part (g), note that one can take $B = -i \int \mathrm{d}^4 z \mathcal{H}_I(z)$, and then Eq. (1.7) becomes a generalization of Peskin and Schroeder's Eq. (4.53):

$$\left\langle 0 \middle| T\left\{ \phi_I(x)\phi_I(y) e^{-i\int \mathrm{d}^4 z \mathcal{H}_I(z)} \right\} \middle| 0 \right\rangle$$

is given by the sum of all connected graphs times the exponential of all disconnected graphs.

Problem 2: Decay of a Scalar Particle

Peskin and Schroeder, Problem 4.2.

Problem 3: Linear Sigma Model

Peskin and Schroeder, Problem 4.3.