MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

PROBLEM SET 6

REFERENCES: Peskin and Schroeder, Sections 3.1 – 3.3.

Problem 1: The Dirac representation of the Lorentz group

Show that the defining property of the Dirac matrices,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \ ,$$

is sufficient to show that the matrices

$$S^{\mu\nu} = \frac{i}{4}\,[\gamma^\mu \, , \, \gamma^\nu]$$

have the commutation relations of the Lorentz group, as specified by Eq. (3.17) of Peskin and Shroeder. The notation for antisymmetrization introduced in Problem 2 of Problem Set 5 may prove useful.

Show also that

$$[\gamma^\mu \, , \, S^{\rho\sigma}] = (\mathcal{J}^{\rho\sigma})^\mu{}_\nu \, \gamma^\nu \ ,$$

where $(\mathcal{J}^{\mu\nu})_{\alpha\beta}$ is defined by Eq. (3.18) of Peskin and Schroeder,

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} \equiv i\left(\delta^\mu_\alpha \, \delta^\nu_\beta - \delta^\mu_\beta \, \delta^\nu_\alpha\right) \ .$$

Problem 2: Explicit tranformation matrices

Evaluate explicitly the 4×4 matrix used to represent a boost along the positive z-axis,

$$B_3(\eta) \equiv e^{-i\eta K^3} = e^{-i\eta S^{03}} \ .$$

Use Peskin and Schroeder's conventions for the Dirac matrices. How is η related to the velocity of the boost?

Similarly evaluate the 4×4 matrix used to represent a counterclockwise rotation about the positive z-axis,

$$R_3(\theta) \equiv e^{-i\theta J^3} = e^{-i\theta S^{12}} \ .$$

Problem 3: Wigner rotations and the transformation of helicity

The Lorentz transformation properties of spin-$\frac{1}{2}$ particles are actually completely dictated by the properties of the Lorentz group, even if we don't know anything about the Dirac equation.

Consider for example an electron in an eigenstate of momentum $\vec{p}$ with eigenvalue $\vec{p} = 0$; i.e., an electron at rest. We know from nonrelativistic quantum mechanics that the electron will have two possible spin states, which we can label as spin-up and spin-down along the z-axis. If we denote these states by $|\vec{p} = 0, \pm\rangle$, then

$$J^z |\vec{p} = 0, \pm\rangle = \pm\frac{1}{2} |\vec{p} = 0, \pm\rangle \ .$$

If we were to perform a rotation on such a state, the momentum would remain zero, and so the two-state system would transform under the spin-$\frac{1}{2}$ representation of the rotation group, as in nonrelativistic quantum theory. The nonrelativistic theory must apply, because the transformation properties in the nonrelativistic theory were dictated completely by properties of the rotation group, and the rotation group is a subgroup of the Lorentz group.

In the relativisitic theory, there must be a unitary operator $U(\Lambda)$ corresponding the each Λ in the Lorentz group. We can use the operators representing boosts to construct a state of nonzero momentum along the z-axis with a definite helicity h:

$$\left| p\hat{z}, h = \pm\frac{1}{2} \right\rangle = U\big(B_z(\eta(p))\big) \ |\vec{p} = 0, \pm\rangle \ ,$$

where $\eta(p)$ is the boost parameter (rapidity) that brings a rest vector to $p\hat{z}$. Note that $\eta(p)$ will depend on the mass m of the electron, so we assume that it has been specified. Note also that J^z commutes with K^z, so the state described by the equation above is still an eigenstate of J^z.

We can also define states of definite helicity in any other direction. Let

$$\left| \vec{p}, h = \pm\frac{1}{2} \right\rangle \equiv U(B_{\vec{p}}) \, |\vec{p} = 0, \pm\rangle \ ,$$

where

$$B_{\vec{p}} = R(\hat{p}) B_z\big(\eta(|\vec{p}|)\big) \ ,$$

where $R(\hat{p})$ is the rotation that rotates the positive z axis into the direction of $\vec{p}$. These states give a complete basis for the Hilbert space of free one-particle electron states.

(a) Consider the state

$$\left| p\hat{z}, h = \pm\frac{1}{2} \right\rangle \ ,$$

and imagine boosting it in the positive x-direction, by a velocity β

$$|\psi\rangle = U\big(B_x(\eta(\beta))\big)\left|p\hat{z}, h = \pm\frac{1}{2}\right\rangle$$
$$= U\big(B_x(\eta(\beta))B_z(\eta(p))\big)\,|\vec{p} = 0, \pm\rangle \ .$$

Compute the Lorentz transformation

$$B_x(\eta(\beta))B_z(\eta(p)) \ ,$$

expressing your answer in the form of a 4×4 Lorentz matrix $\Lambda^\mu{}_\nu$. What is the momentum $\vec{p}\,'$ of the state $|\psi\rangle$?

(b) To express $|\psi\rangle$ in terms of the original basis vectors, we need the inner products

$$\langle \vec{p}\,', h'\, |\psi\rangle = \left\langle \vec{p} = 0, h' \left| U^\dagger(B_{\vec{p}\,'})\, U\big(B_x(\eta(\beta))B_z(\eta(p))\big)\right| \vec{p} = 0, \pm \right\rangle \ .$$

Since $U(\Lambda)$ is a unitary representation of the group,

$$U^\dagger(B_{\vec{p}\,'})\, U\big(B_x(\eta(\beta))B_z(\eta(p))\big) = U\big(B_{\vec{p}\,'}^{-1}\, B_x(\eta(\beta))B_z(\eta(p))\big) \ .$$

Note, however, that

$$R_W \equiv B_{\vec{p}\,'}^{-1}\, B_x(\eta(\beta))B_z(\eta(p))$$

brings a momentum vector at rest back to a momentum vector at rest, and hence it is a pure rotation. It is called the Wigner rotation. Since the matrix elements of U for rotations are already known, the matrix element needed here is known. Compute the Wigner rotation for this case, describing it first as a Lorentz matrix Λ. What is the axis of rotation? What is the angle of the rotation?

(c) Now consider the $m \to 0$ limit, keeping p and η fixed. This would be the appropriate limit to describe a massless particle with momentum of magnitude p. Show that the Wigner rotation angle approaches zero in this limit, and hence that the helicity of a massless particle is Lorentz invariant.