MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

PROBLEM SET 5

(Corrected Version)

REFERENCES: Peskin and Schroeder, Sections 3.1 and 3.2. Optional: *Quantum Field Theory*, by Lowell Brown, section 1.7 (coherent states).

Problem 1: Coherent states

In lecture we solved the problem of a quantized scalar field $\phi(x)$ interacting with a fixed classical source $j(x)$,

$$(\Box + m^2)\phi(x) = j(x) \ .$$

We found that the in and out operators are related by

$$\phi_{\text{out}}(x) = S^{-1}\,\phi_{\text{in}}(x)\,S$$

$$a_{\text{out}}(\vec{p}) = S^{-1}\,a_{\text{in}}(\vec{p})\,S \ ,$$

where S can be written

$$S = e^{-\frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}|\tilde{\jmath}(p)|^2}\,e^F\,e^G \ ,$$

where

$$F = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\tilde{\jmath}(p)\,a^\dagger_{\text{in}}(\vec{p})$$

$$G = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\tilde{\jmath}(-p)\,a_{\text{in}}(\vec{p}) \ ,$$

and

$$\tilde{\jmath}(p) \equiv \int \mathrm{d}^4y\,e^{ip\cdot y}j(y) \ .$$

(a) Show that S can also be written as

$$S = e^{-\frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}|\tilde{\jmath}(p)|^2}\,e^{F'}\,e^{G'} \ ,$$

where

$$F' = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\tilde{\jmath}(p)\,a^\dagger_{\text{out}}(\vec{p})$$

$$G' = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\tilde{\jmath}(-p)\,a_{\text{out}}(\vec{p}) \ ,$$

and hence that

$$|0_{\text{in}}\rangle = e^{-\frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}|\tilde{\jmath}(p)|^2}\, e^{F'}\, |0_{\text{out}}\rangle \ .$$

Note that the right-hand-side of the above equation gives a useful description of the final state, since the out operators have a straightforward interpretation at late times. States of this form — exponentials of creation operators acting on the vacuum — are called coherent states.

(b) To study further the properties of coherent states, it is useful to consider a single harmonic oscillator,

$$H = \frac{p^2}{2m} + \frac{\omega^2 m}{2} q^2 \ ,$$

which is simplified by the canonical transformation

$$p = \sqrt{m\omega}\ , \bar{p}\ , \quad q = \frac{1}{\sqrt{m\omega}}\,\bar{q}\ .$$

Dropping the overbars, the creation and annihilation operators are then given by

$$a^\dagger = \frac{1}{\sqrt{2}}(q - ip)$$

$$a_= \frac{1}{\sqrt{2}}(q + ip)\ .$$

A coherent state $|z\rangle$ can be defined by

$$|z\rangle \equiv e^{za^\dagger}\,|0\rangle\ .$$

Show that $|z\rangle$ is an eigenstate of the annihilation operator, and find its eigenvalue.

(c) Show that $\langle z_2\,|z_1\,\rangle = e^{z_2^* z_1}$. (If you look at Lowell Brown's book, note that I am defining $\langle z|$ to be the bra vector that corresponds to the ket $|z\rangle$, so my $\langle z|$ is equal to Brown's $\langle z^*|$.)

(d) Find

$$\langle q\rangle_z \equiv \frac{\langle z\,|q|\,z\rangle}{\langle z\,|z\,\rangle}$$

and

$$\langle p\rangle_z \equiv \frac{\langle z\,|p|\,z\rangle}{\langle z\,|z\,\rangle}\ .$$

(e) Compute the standard deviations of q and p,

$$\Delta q^2 = \left\langle (q - \langle q\rangle)^2\right\rangle$$

$$\Delta p^2 = \left\langle (p - \langle p\rangle)^2\right\rangle$$

and show that $|z\rangle$ is a minimal-uncertainty state, in the sense that

$$\Delta q\,\Delta p = \frac{1}{2}\ .$$

Problem 2: Commutation relations for the Lorentz group

Infinitesimal Lorentz transformations can be described by

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - iG^{\mu}{}_{\nu} \ ,$$

where

$$G^{\mu\nu} = -G^{\nu\mu} \ .$$

There are therefore 6 generators, since there are 6 linearly independent antisymmetric 4×4 matrices. One convenient way to choose a basis of 6 independent generators is to label them by two antisymmetric spacetime indices, $J^{\mu\nu} \equiv -J^{\nu\mu}$, with the explicit matrix definition

$$J^{\mu\nu}_{\alpha\beta} \equiv i\left(\delta^{\mu}_{\alpha}\,\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\,\delta^{\nu}_{\alpha}\right) \ .$$

Here μ and ν label the generator, and for each μ and ν (with $\mu \neq \nu$) the formula above describes a matrix with indices α and β. For the usual rules of matrix multiplication to apply, the index α should be raised, which is done with the Minkowski metric $g^{\mu\nu}$:

$$J^{\mu\nu\alpha}{}_{\beta} = i\left(g^{\mu\alpha}\,\delta^{\nu}_{\beta} - g^{\nu\alpha}\,\delta^{\mu}_{\beta}\right) \ .$$

(a) Show that the commutator is given by

$$[J^{\mu\nu}\,,\,J^{\rho\sigma}] = i\left(g^{\nu\rho}\,J^{\mu\sigma} - g^{\mu\rho}\,J^{\nu\sigma} - g^{\nu\sigma}\,J^{\mu\rho} + g^{\mu\sigma}\,J^{\nu\rho}\right) \ .$$

To minimize the number of terms that you have to write, I recommend adopting the convention that $\{\}_{\mu\nu}$ denotes antisymmetrization, so

$$\{\qquad\}_{\mu\nu} \equiv \frac{1}{2}\Big[\{\qquad\} - \{\ \mu \leftrightarrow \nu\ \}\Big] \ .$$

With this notation, the commutator can be written

$$[J^{\mu\nu}\,,\,J^{\rho\sigma}] = 4i\left\{\,\{\,g^{\nu\rho}\,J^{\mu\sigma}\,\}_{\mu\nu}\right\}_{\lambda\sigma} \ .$$

You might even want to adopt a more abbreviated notation, writing

$$[J^{\mu\nu}\,,\,J^{\rho\sigma}] = 4i\,\{\,g^{\nu\rho}\,J^{\mu\sigma}\,\}_{\substack{\mu\nu\\ \lambda\sigma}} \ .$$

(b) Construct a Lorentz transformation matrix $\Lambda^{\alpha}{}_{\beta}$ corresponding to an infinitesimal boost in the positive z-direction, and use this to show that the generator of such a boost is given by $K^3 \equiv J^{03}$. Signs are important here.

Problem 3: Representations of the Lorentz group

Peskin and Schroeder, Problem 3.1.