MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

PROBLEM SET 10

(Corrected Version)

REFERENCE: Steven Weinberg, *The Quantum Theory of Fields, Volume 1: Foundations* (Cambridge University Press, Cambridge, 1995), Section 2.2 and Appendix A of Chapter 2.

Problem 1: Wigner's Symmetry Representation Theorem

This problem will be a guided exercise in which a proof of Wigner's theorem* will be constructed. The proof that you will construct is a modified version of the proof given in Weinberg's textbook. This version is in my opinion simpler, and it also avoids what I believe is a minor flaw† in Weinberg's argument. Weinberg in turn claims to have remedied a flaw in Wigner's original proof, so historical precedent seems to suggest that any proof of Wigner's theorem is flawed. If you find any flaws in this one, you will get extra credit.

First, we need some definitions that will be used in the statement of the theorem. Consider a quantum theory formulated on a Hilbert space $\mathcal{H}$. A physical state corresponds to a *ray* $\mathcal{R}$ in the Hilbert space, where a ray is defined as a set of normalized vectors ($\langle\Psi\,|\Psi\,\rangle = 1$), where $|\Psi\rangle$ and $|\Psi'\rangle$ belong to the same ray if they are equal up to a phase (i.e., if $|\Psi'\rangle = e^{i\theta}\,|\Psi\rangle$ for some real θ). I will use the notation $|\Psi\rangle \in \mathcal{R}$ or $\mathcal{R} \ni |\Psi\rangle$ to indicate that $|\Psi\rangle$ belongs to the ray $\mathcal{R}$, and I will define $\mathcal{R}(\Psi)$ to denote the ray that contains the vector $|\Psi\rangle$. We will consider a transformation T defined on physical states, so T maps one ray onto another. I will sometimes use the abbreviation $T(\Psi)$ to denote $T\big(\mathcal{R}(\Psi)\big)$, the image under T of the ray that contains the vector $|\Psi\rangle$. T will be said to be probability-preserving if

$$|\langle\psi_2'\,|\psi_1'\,\rangle| = |\langle\psi_2\,|\psi_1\,\rangle| \tag{1}$$

whenever

$$|\psi_1'\rangle \in T(\psi_1) \quad\text{and}\quad |\psi_2'\rangle \in T(\psi_2)\ . \tag{2}$$

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* The theorem was originally proven in *Gruppentheorie und ihre Anwendung auf die Quanten-mechanik der Atomspektren* (Braunschweig, 1931), pp. 251–3, by Eugene P. Wigner. An English translation was published by Academic Press in 1959.

† On pp. 92 and 93 of Weinberg's text, he uses a number of equations in which C_1 or C_1' appears in the denominator, where C_1 and C_1' are expansion coefficients of an arbitrary state in a particular basis. The argument is therefore inapplicable to states for which these particular coefficients vanish. The gap can be filled, but doing so makes the proof more cumbersome.

If U is an operator on the Hilbert space $\mathcal{H}$, then T is said to be represented by U if

$$|\Psi\rangle \in \mathcal{R} \;\text{ implies }\; U\,|\Psi\rangle \in T(\mathcal{R}) \; . \tag{3}$$

An operator U on $\mathcal{H}$ is said to be *linear* if

$$U\bigl(\alpha\,|\psi_1\rangle + \beta\,|\psi_2\rangle\bigr) = \alpha\,U\,|\psi_1\rangle + \beta\,U\,|\psi_2\rangle \;\; , \tag{4}$$

and it is said to be *antilinear* if

$$U\bigl(\alpha\,|\psi_1\rangle + \beta\,|\psi_2\rangle\bigr) = \alpha^*\,U\,|\psi_1\rangle + \beta^*\,U\,|\psi_2\rangle \;\; . \tag{5}$$

An operator is said to be *unitary* if

$$\langle U\psi_2\,|U\psi_1\,\rangle = \langle\psi_2\,|\psi_1\,\rangle \;\; , \tag{6}$$

and it is said to be *antiunitary* if

$$\langle U\psi_2\,|U\psi_1\,\rangle = \langle\psi_2\,|\psi_1\,\rangle^* \;\; . \tag{7}$$

Now Wigner's theorem can be stated:

> Given any probability-preserving invertible transformation T on the rays of a Hilbert space $\mathcal{H}$, then one and only one of the following two statements is true:
>
> (a) We can construct an operator U on the Hilbert space $\mathcal{H}$ which represents T and which is linear and unitary.
>
> (b) We can construct an operator U on the Hilbert space $\mathcal{H}$ which represents T and which is antilinear and antiunitary.
>
> In either case, the operator U is uniquely defined, up to an overall phase.

To prove the theorem, we begin by proving some properties that T must have if it is probability-preserving and invertible. Let $|\psi_1\rangle$, $|\psi_2\rangle$, $\ldots$ be a complete orthonormal set of vectors in $\mathcal{H}$. For each $k = 1, 2, \ldots$, choose some particular vector

$$|\tilde{\psi}_k\rangle \in T(\psi_k) \; . \tag{8}$$

(a) Show that the vectors $|\tilde{\psi}_1\rangle$, $|\tilde{\psi}_2\rangle$, $\ldots$ also form a complete orthonormal set of vectors in $\mathcal{H}$.

(b) Now consider the vectors

$$|\phi_k\rangle \equiv \frac{1}{\sqrt{2}}\bigl(\,|\psi_1\rangle + |\psi_k\rangle\bigr) \;\; , \tag{9}$$

for $k = 2, 3, \ldots$. Show that for each k,

$$T(\phi_k) \ni \frac{1}{\sqrt{2}}\big(|\tilde{\psi}_1\rangle + e^{i\theta_k} |\tilde{\psi}_k\rangle \big) \tag{10}$$

for some real θ_k.

Now define

$$\begin{aligned} |\psi_1'\rangle &= |\tilde{\psi}_1\rangle \\ |\psi_k'\rangle &= e^{i\theta_k} |\tilde{\psi}_k\rangle \text{ for } k = 2, 3, \ldots , \end{aligned} \tag{11}$$

so

$$T(\phi_k) \ni |\phi_k'\rangle \, , \quad \text{where} \quad |\phi_k'\rangle = \frac{1}{\sqrt{2}}\big(|\psi_1'\rangle + |\psi_k'\rangle \big) \; . \tag{12}$$

(c) Now consider the vectors

$$|\Phi(\theta)\rangle \equiv \frac{1}{\sqrt{2}}\big(|\psi_1\rangle + e^{i\theta} |\psi_2\rangle \big) \; , \tag{13}$$

where θ is a real number. By considering the inner product of these vectors with the $|\psi_k\rangle$ and with $|\phi_2\rangle$, show that either

$$T\big(\Phi(\theta)\big) \ni \big|\Phi_+'(\theta)\big\rangle \, , \quad \text{where} \quad \big|\Phi_+'(\theta)\big\rangle = \frac{1}{\sqrt{2}}\big(|\psi_1'\rangle + e^{i\theta} |\psi_2'\rangle \big) \qquad \text{(case A)} \tag{14a}$$

or

$$T\big(\Phi(\theta)\big) \ni \big|\Phi_-'(\theta)\big\rangle \, , \quad \text{where} \quad \big|\Phi_-'(\theta)\big\rangle = \frac{1}{\sqrt{2}}\big(|\psi_1'\rangle + e^{-i\theta} |\psi_2'\rangle \big) \qquad \text{(case B)} \; . \tag{14b}$$

If $\theta = n\pi$, where n is an integer, then these two cases are identical. Otherwise $\big|\Phi_+'(\theta)\big\rangle$ and $\big|\Phi_-'(\theta)\big\rangle$ belong to different rays, so only one of the two cases can apply. The choice between case A and case B is not our choice, but is determined by the properties of T, which defines the ray $T\big(\Phi(\theta)\big)$.

(d) Show that for a given transformation T, the same case in Eqs. (14a) and (14b) applies to all values of θ. (*Hint:* Suppose that case A applies for $\theta = \theta_A$ and case B applies for $\theta = \theta_B$, where $\theta_A \neq n\pi$ and $\theta_B \neq n\pi$. Consider the inner product $\langle \Phi(\theta_B) \, | \Phi(\theta_A) \, \rangle$.)

(e) Now consider the vectors

$$|\Psi_N(\alpha_2, \alpha_3, \ldots, \alpha_N)\rangle = \frac{1}{\sqrt{N}}\big(|\psi_1\rangle + e^{i\alpha_2} |\psi_2\rangle + e^{i\alpha_3} |\psi_3\rangle + \ldots + e^{i\alpha_N} |\psi_N\rangle \big) , \tag{15}$$

where α_2, α_3, ... , α_N are real numbers. For case A, show that

$$T\big(\Psi_N(\alpha_2,\ldots,\alpha_N)\big) \ni \big|\Psi'_{N,+}(\alpha_2,\ldots,\alpha_N)\big\rangle \ , \quad \text{where}$$

$$\big|\Psi'_{N,+}(\alpha_2,\ldots,\alpha_N)\big\rangle = \frac{1}{\sqrt{N}}\big(\,|\psi'_1\rangle + e^{i\alpha_2}\,|\psi'_2\rangle + e^{i\alpha_3}\,|\psi'_3\rangle + \ldots + e^{i\alpha_N}\,|\psi'_N\rangle\big) \ , \tag{16a}$$

and for case B, show that

$$T\big(\Psi_N(\alpha_2,\ldots,\alpha_N)\big) \ni \big|\Psi'_{N,-}(\alpha_2,\ldots,\alpha_N)\big\rangle \ , \quad \text{where}$$

$$\big|\Psi'_{N,-}(\alpha_2,\ldots,\alpha_N)\big\rangle = \frac{1}{\sqrt{N}}\big(\,|\psi'_1\rangle + e^{-i\alpha_2}\,|\psi'_2\rangle + e^{-i\alpha_3}\,|\psi'_3\rangle + \ldots + e^{-i\alpha_N}\,|\psi'_N\rangle\big) \ . \tag{16b}$$

(*Hint:* Note that for $N = 1$ and $N = 2$, this statement has already been proven. See if you can construct an argument using induction on N which demonstrates the result for all N.)

(f) Now we are ready to consider an arbitrary vector, which can be expanded in the complete orthonormal basis as

$$|\Psi\rangle = \sum_{k=1}^{\infty} C_k\,|\psi_k\rangle \quad . \tag{17}$$

Show that for case A,

$$T(\Psi) \ni \big|\Psi'_+\big\rangle \quad \text{where} \quad \big|\Psi'_+\big\rangle = \sum_{k=1}^{\infty} C_k\,|\psi'_k\rangle \tag{18a}$$

and that for case B,

$$T(\Psi) \ni \big|\Psi'_-\big\rangle \quad \text{where} \quad \big|\Psi'_-\big\rangle = \sum_{k=1}^{\infty} C_k^*\,|\psi'_k\rangle \ . \tag{18b}$$

(g) For case A, define

$$U\,|\Psi\rangle = \big|\Psi'_+\big\rangle = \sum_{k=1}^{\infty} C_k\,|\psi'_k\rangle \ , \tag{19a}$$

and for case B define

$$U\,|\Psi\rangle = \big|\Psi'_-\big\rangle = \sum_{k=1}^{\infty} C_k^*\,|\psi'_k\rangle \ , \tag{19b}$$

where $|\Psi\rangle$ is the state defined in Eq. (17). From part (f), U is clearly a representation of T, as defined by Eq. (3). Show for case A that U is linear and unitary, and for case B that it is antilinear and antiunitary.

(h) Finally, prove that U is unique up to an overall phase. (*Hint:* Assume that U_1 and U_2 both satisfy all the properties described in the theorem. Consider the product $U_2^{-1}U_1$, which in either case A or B is a linear transformation which maps each ray onto itself. Show that such a map is necessarily an overall phase times the identity operator.)