MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

PROBLEM SET 2

REFERENCES: Peskin and Schroeder, Chapter 2

Problem 1: Complex scalar fields

Peskin and Schroeder, Problem 2.2. The problem as stated asks you to find 4 conserved currents for the theory with two complex scalar fields. There are actually 6 conserved currents, as is indicated on the Peskin and Schroeder corrections web page,

http://www.slac.stanford.edu/~mpeskin/QFT.html

You will get full credit for finding the same four that Peskin and Schroeder found, and their generalization for n fields. If you can find all six currents for two fields and their generalization for n fields, you will get extra credit.

Problem 2: Lorentz transformations and Noether's theorem for scalar fields

Infinitesimal Lorentz transformations, including both rotations and boosts, can be described by

$$x'^{\lambda} = x^{\lambda} - \Sigma^{\lambda}{}_{\sigma}\, x^{\sigma} \ ,$$

where

$$\Sigma_{\lambda\sigma} = -\Sigma_{\sigma\lambda} \qquad \text{and} \qquad \Sigma^{\lambda}{}_{\sigma} \equiv \eta^{\lambda\kappa}\Sigma_{\kappa\sigma} \ .$$

(a) Show that to first order in Σ, $x'^2 \equiv x'^{\lambda}\, x'_{\lambda} = x^{\lambda} x_{\lambda}$. That is, show that the length of a vector is invariant under this transformation.

Now consider a scalar field $\phi(x)$, with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_{\mu}\phi\,\partial^{\mu}\phi - V(\phi) \ ,$$

which transforms as

$$\phi'(x') = \phi(x) \ .$$

Expanding to lowest order,

$$\phi'(x') = \phi(x'^{\lambda} + \Sigma^{\lambda}{}_{\sigma}\, x'^{\sigma}) = \phi(x'^{\lambda}) + \Sigma^{\lambda}{}_{\sigma} x'^{\sigma}\, \partial_{\lambda}\, \phi(x') \ .$$

Dropping the primes on the coordinates,

$$\phi'(x) = \phi(x) + \Sigma_{\lambda\sigma}\, x^\sigma \partial^\lambda \phi(x) \ .$$

(b) Show that the conserved Noether current for this transformation can be written as

$$j^{\mu\lambda\sigma} = x^\lambda\, T^{\mu\sigma} - x^\sigma\, T^{\mu\lambda} \ , \qquad \text{where} \qquad \partial_\mu\, j^{\mu\lambda\sigma} = 0 \ ,$$

where

$$T^{\mu\nu} = \partial^\mu \phi\, \partial^\nu \phi - \eta^{\mu\nu}\, \mathcal{L}$$

is the conserved energy-momentum tensor which we have already derived. You should apply Noether's theorem, and then verify from the equations of motion that your expression for the current is actually conserved.

(c) When λ and σ are spacelike, the conserved quantity is the angular momentum (e.g., $j^{012} = j_z$, the density of angular momentum in the z direction). What is the significance of the conserved quantity when $\lambda = 0$, $\sigma = i$?

(d) Express the conserved quantity in terms of creation and annihilation operators.

Problem 3: Lorentz transformations and Noether's theorem for the electromagetic potential $A_\mu(x)$

Consider again the electromagnetic potential $A_\mu(x)$, as discussed in problem 2.1 of Peskin and Schroeder. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}\, F^{\mu\nu} \ , \tag{1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{2}$$

In problem 2.1 we learned that translation invariance and Noether's theorem lead to a nonsymmetric energy-momentum tensor, which can be made symmetric by adding a piece that has the form of a total derivative that is automatically conserved regardless of the equations of motion. We discussed in lecture, however, how the conservation of angular momentum forces one to use a symmetric energy-momentum tensor, so that the cross product of $\vec{r}$ and the momentum density T^{0i} gives a conserved angular momentum density

$$\mathcal{J}_i = \epsilon_{ijk}\, x^j\, T^{0k} \ . \tag{3}$$

If $T^{\mu\nu}$ is both conserved and symmetric, then this angular momentum density can be written as the 0th component of the divergenceless current

$$K^{\mu\lambda\sigma} = x^\lambda\, T^{\mu\sigma} - x^\sigma\, T^{\mu\lambda} \ , \tag{4}$$

where

$$\partial_\mu K^{\mu\lambda\sigma} = 0 \qquad \text{and} \qquad \mathcal{J}_i = \epsilon_{ijk}\, K^{0jk} \ . \tag{5}$$

One might hope, therefore, that if one derived the conservation of angular momentum by using rotational symmetry and Noether's theorem, then one would be led directly to a symmetric energy-momentum tensor. This hope, however, is not realized, as will be shown in this problem.

We are interested mainly in rotations, but for the sake of generality we will consider arbitrary Lorentz transformations, which include rotations as a special case. Since $A_\mu(x)$ is a Lorentz vector, under a Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu\, x^\nu$ it transforms as

$$A'^\mu(x') = \Lambda^\mu{}_\nu\, A^\nu(x) \ . \tag{6}$$

For an infinitesimal Lorentz transformation $\Lambda^\mu{}_\nu = \delta^\mu_\nu - \Sigma^\mu{}_\nu$, where $\Sigma_{\mu\nu}$ is antisymmetric as discussed in the previous problem, the symmetry tranformation becomes

$$A'^\mu(x) = A^\mu(x) + \Sigma_{\lambda\sigma}\left\{x^\sigma\, \partial^\lambda A^\mu(x) - \eta^{\mu\lambda}\, A^\sigma(x)\right\} \ . \tag{7}$$

(a) Show that the above symmetry leads via Noether's theorem to the conserved current

$$j^{\mu\lambda\sigma} = x^\sigma \left\{F^{\mu\kappa}\, \partial^\lambda A_\kappa + \eta^{\mu\lambda}\, \mathcal{L}\right\} - F^{\mu\lambda}\, A^\sigma - (\lambda \leftrightarrow \sigma) \ . \tag{8}$$

Here "$-(\lambda \leftrightarrow \sigma)$" means to subtract an expression identical to everything previously written on the right-hand side, except that the subscripts λ and σ are interchanged.

(b) Show directly from the equations of motion that the above current is conserved ($\partial_\mu j^{\mu\lambda\sigma} = 0$). Thus Noether's theorem leads to a conserved current, as it must, but since Eq. (8) does not match the form of Eq. (4), the conserved angular momentum current can be constructed without using a symmetric energy-momentum tensor.

(c) As in Peskin and Schroeder's problem 2.1, we can construct a modified form of the conserved current by adding a derivative term:

$$\hat{\jmath}^{\mu\lambda\sigma} = j^{\mu\lambda\sigma} + \partial_\kappa N^{\kappa\mu\lambda\sigma} \ , \tag{9}$$

where $N^{\kappa\mu\lambda\sigma}$ is antisymmetric in its first two indices and in its last two indices. Show that if

$$N^{\kappa\mu\lambda\sigma} = x^\lambda\, F^{\mu\kappa}\, A^\sigma - x^\sigma\, F^{\mu\kappa}\, A^\lambda \ , \tag{10}$$

then $\hat{\jmath}^{\mu\lambda\sigma}$ can be written in the form of Eq. (4), with a symmetric energy-momentum tensor.

Problem 4: Uniqueness of the harmonic oscillator quantization

It is tempting to think that any classical quantity $x(t)$ that obeys the equation of motion

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega^2\, x \tag{1}$$

is a simple harmonic oscillator of angular frequency ω, to be quantized in the standard way. This, however, is not always true.

In this problem we will consider a harmonic oscillator described by the Lagrangian

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega_0^2 q^2 \ , \tag{2}$$

so $p = \partial L/\partial \dot{q} = \dot{q}$, and we as usual require that $[q,p] = i$. The creation and annihilation operators

$$\begin{aligned} a &= \sqrt{\frac{\omega_0}{2}}q + \frac{i}{\sqrt{2\omega_0}}\,p \\ a^\dagger &= \sqrt{\frac{\omega_0}{2}}q - \frac{i}{\sqrt{2\omega_0}}\,p \end{aligned} \tag{3}$$

can then be shown to satisfy the commutation relation $[a, a^\dagger] = 1$, and the Hamiltonian can be expanded as $H = \omega_0 \left(a^\dagger a + \frac{1}{2}\right)$. The coordinate q can then be written in the Heisenberg picture as

$$q(t) = e^{iHt} q e^{-iHt} = \frac{1}{\sqrt{2\omega_0}}\left(a\, e^{-i\omega_0 t} + a^\dagger\, e^{i\omega_0 t}\right) \ . \tag{4}$$

(a) Consider the "first harmonic" operator x_2, which we will define in the Schrödinger picture by

$$x_2 = \frac{1}{\omega_0}\left(a^2 + a^{\dagger 2}\right) \ . \tag{5}$$

(Note that we are still talking about the system defined by the Lagrangian of Eq. (2)—we are simply considering a new operator defined on the same quantum mechanical Hilbert space.) Find an expression for the Heisenberg operator $x_2(t)$ analogous to Eq. (4). Use this expression to show that

$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} = -\omega^2\, x_2, \tag{6}$$

where $\omega = 2\omega_0$. Thus x_2 has the equation of motion of a harmonic oscillator of angular frequence $2\omega_0$, yet it is defined on the Hilbert space of a harmonic oscillator of angular frequency ω.

(b) Show that x_2 can be expressed in terms of the original operators p and q by

$$x_2 = q^2 - \frac{p^2}{\omega_0^2}. \tag{7}$$

Now use the Heisenberg equations of motion for q and p, namely $\dot{q} = p$ and $\dot{p} = -\omega_0^2 q$, to confirm that $x_2(t)$ obeys Eq. (6).

(c) Construct a second harmonic operator x_3, which oscillates with angular frequency $3\omega_0$. Construct it first in terms of creation and annihilation operators, analogous to Eq. (5), and then re-express it in terms of q and p. Note that the overall normalization of x_3 is arbitrary, so don't worry about how to fix it.