MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

INFORMAL NOTES

DISTRIBUTIONS AND THE FOURIER TRANSFORM

Basic idea:

In QFT it is common to encounter integrals that are not well-defined. Last week we talked about the two point function $\langle 0\,|\phi(x)\phi(y)|\,0\rangle$ for spacelike separations $(x-y)^2=-r^2$, which is given formally by

$$D(r)=\frac{-i}{2(2\pi)^2 r}\int_{-\infty}^{\infty}\mathrm{d}p\,\frac{pe^{ipr}}{\sqrt{p^2+m^2}}\ .$$

If this integral is defined in the usual way as

$$\lim_{\Lambda\to\infty}\int_{-\Lambda}^{\Lambda}\mathrm{d}p\,\frac{pe^{ipr}}{\sqrt{p^2+m^2}}\ ,$$

then it does not exist. The integral can be defined by putting in a convergence factor $e^{-\epsilon|p|}$:

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\mathrm{d}p\,\frac{pe^{ipr}e^{-\epsilon|p|}}{\sqrt{p^2+m^2}}\ .$$

But how does one know whether a different convergence factor would get the same result? One way to resolve these issues is to treat the ambiguous quantity as a distribution, rather than a function. All tempered distributions (to be defined below) have Fourier transforms, which are also tempered distributions. Furthermore, we can show that the ϵ-prescription used above is equivalent to the tempered-distribution definition of the Fourier transform.

Distribution:

A distribution is a linear mapping from a space of test functions to real or complex numbers. (An operator-valued distribution maps test functions into operators.)

Test Functions:

The space of test functions $\{\varphi(t)\}$ determines what type of distribution one is discussing. The test functions for tempered distributions belong to "Schwartz space," the space of functions which are infinitely differentiable, and the function and each of its derivatives fall off faster than any power for large t. The Gaussian is a good example of a Schwartz function. Any function in Schwartz space has a Fourier transform in Schwartz space. (The Fourier transform of a Gaussian is a Gaussian.)

Functions as Distributions:

Given any function $f(t)$ which is piecewise continuous and bounded by some power of t for large t, one can define a distribution T_f by

$$T_f[\varphi] \equiv \int_{-\infty}^{\infty} \mathrm{d}t\, f(t)\varphi(t) \ .$$

Since $\varphi(t)$ falls off faster than any power, this integral will converge. Note that because the class of $\varphi(t)$'s is very restricted, the class of possible $f(t)$'s is very large.

Fourier Transform:

For any function $f(t)$ which is integrable, meaning that

$$\int_{-\infty}^{\infty} \mathrm{d}t\, |f(t)|$$

converges, define

$$\tilde{f}(\omega) \equiv \int_{-\infty}^{\infty} \mathrm{d}t\, e^{-i\omega t} f(t) \ .$$

Fourier Transform of a Distribution:

To motivate the definition, suppose $f(t)$ is integrable, and consider

$$\begin{aligned} T_{\tilde{f}}[\varphi] &= \int_{-\infty}^{\infty} \tilde{f}(\omega)\varphi(\omega)\, d\omega \\ &= \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} e^{-i\omega t}\, f(t)\, \varphi(\omega)\, d\omega \\ &= \int_{-\infty}^{\infty} \mathrm{d}t\, f(t)\, \tilde{\varphi}(t) \\ &= T_f[\tilde{\varphi}] \ . \end{aligned}$$

Note that these integrals are absolutely convergent, so there is no problem about interchanging the order of integration. So, for any distribution T, define its Fourier transform by

$$\boxed{\tilde{T}[\varphi] \equiv T[\tilde{\varphi}] \ .}$$

Note that any function $f(t)$ which is piecewise continuous and bounded by some power of t for large t can define a distribution, and can therefore be Fourier transformed as a distribution.

Relation to ϵ convergence factor:

Suppose $f(t)$ is not integrable, and so does not have a Fourier transform. Suppose, however, that there exists a continuous sequence of "regulated functions" $f_\epsilon(t)$ which are integrable for $\epsilon > 0$, which satisfy

$$|f_\epsilon(t)| < |f(t)| \ ,$$

and which for each t satisfy

$$\lim_{\epsilon\to 0} f_\epsilon(t) = f(t) \ .$$

Example: $f_\epsilon(t) = f(t)e^{-\epsilon|t|}$. Note that the regulator that we used for the two-point function at spacelike separations has this property. To show: if we Fourier transform $f_\epsilon(t)$ and take the limit $\epsilon \to 0$ at the end, it is the same as the distribution-theory definition of the Fourier transform.

Proof:

The distribution-theory definition of the Fourier transform is

$$\begin{aligned}\tilde{T}_f[\varphi] &\equiv T_f[\tilde{\varphi}] \\ &= \boxed{\int_{-\infty}^{\infty} \mathrm{d}t\, f(t)\, \tilde{\varphi}(t) \ .}\end{aligned}$$

The ϵ prescription is to use

$$T^*_f[\varphi] \equiv \lim_{\epsilon\to 0} T_{\tilde{f}_\epsilon}[\varphi] \ .$$

We need to show these are equivalent. Use

$$\begin{aligned}T^*_f[\varphi] &= \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \mathrm{d}\omega\, \tilde{f}_\epsilon(\omega)\varphi(\omega) \\ &= \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \mathrm{d}\omega \int_{-\infty}^{\infty} \mathrm{d}t\, e^{-i\omega t}\, f_\epsilon(t)\varphi(\omega) \\ &= \boxed{\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \mathrm{d}t\, f_\epsilon(t)\tilde{\varphi}(t) \ .}\end{aligned}$$

If we can take the limit inside the integral, we are done!

Last step is proven with Lebesgue's Dominated Convergence Theorem: If $h_\epsilon(t)$ is a sequence of functions for which

$$\lim_{\epsilon \to 0} h_\epsilon(t) = h(t) \qquad \text{for all } t,$$

and if there exists a function $g(t)$ for which

$$\int \mathrm{d}t\, g(t)$$

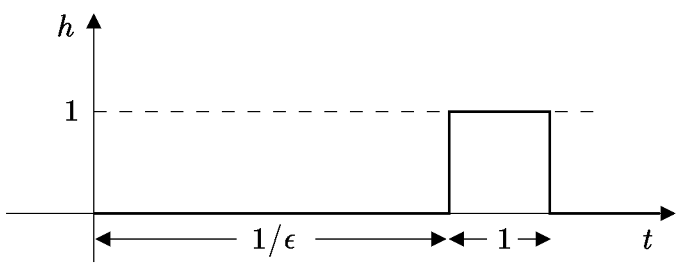
converges, and for which

$$g(t) \geq |h_\epsilon(t)| \qquad \text{for all } t \text{ and all } \epsilon,$$

then

$$\lim_{\epsilon \to 0} \int \mathrm{d}t\, h_\epsilon(t) = \int \mathrm{d}t\, h(t) \ .$$

Note, by the way, that the existence of the integrable bounding function $g(x)$ is absolutely necessary. A simple example of a function $h_\epsilon(t)$ for which one CANNOT bring the limit through the integral sign would be a function that looks something like:



Analytically, this function can be written as

$$h_\epsilon(t) = \begin{cases} 1 & \text{if } \frac{1}{\epsilon} < t < \frac{1}{\epsilon} + 1 \\ 0 & \text{otherwise} \ . \end{cases}$$

Note that the square well moves infinitely far to the right as $\epsilon \to 0$, so $h_\epsilon(t) \to 0$ for any t. But the integral of the curve is 1 for any ϵ, and hence it is 1 in the limit. The Lebesgue Dominated Convergence theorem excludes functions like this, because any bounding function $g(t)$ must be ≥ 1 for all t, so $g(t)$ cannot be integrable.

The theorem does apply, however, to

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \mathrm{d}t\, f_\epsilon(t) \tilde{\varphi}(t) \ .$$

Take

$$h_\epsilon(t) = f_\epsilon(t)\,\tilde{\varphi}(t) \ ,$$
$$h(t) = f(t)\,\tilde{\varphi}(t) \ ,$$

and

$$g(t) = |f(t)\,\tilde{\varphi}(t)| \ .$$

Bottom Line:

The ϵ prescription used by physicists is equivalent to the unambiguous definition of the Fourier transform in tempered-distribution theory. That is, if the function to be Fourier-transformed $f(t)$ is not integrable, one can proceed as long as one can find an integrable regulator $f_\epsilon(t)$ such that

$$|f_\epsilon(t)| < |f(t)| \ ,$$

and for each t,

$$\lim_{\epsilon\to 0} f_\epsilon(t) = f(t) \ .$$

One can then Fourier transform $f_\epsilon(t)$ instead. In the general case one cannot take the limit $\epsilon \to 0$ immediately, but one must leave ϵ in the expression for the distribution. Only after the distribution is evaluated for a particular test function can the limit $\epsilon \to 0$ be taken. Remember, for example, that we wrote the Fourier transform of the Feynman propagator as

$$\frac{i}{p^2 - m^2 + i\epsilon} \ .$$

With the ϵ in place one can carry out integrals involving the propagator, and then one can take the limit $\epsilon \to 0$ at the end. If one tried to set the ϵ term to zero immediately, then the poles in the propagator would lead to ill-defined integrations.

The two-point function at spacelike separation:

When we applied the regulator $e^{-\epsilon|p|}$ to the integral for the two-point function at spacelike separation, we found that we could obtain a definite function,

$$D(r) = \lim_{\epsilon\to 0} \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^{\infty} \frac{p e^{ipr} e^{-\epsilon|p|}}{\sqrt{p^2+m^2}} = \frac{m}{4\pi^2 r} K_1(mr) \ .$$

The distribution theory analysis, however, did not justify the last step of taking the limit $\epsilon \to 0$, but instead indicated that we should keep ϵ in place until after the distribution has been applied to a test function. In the previous lecture I showed that the integral shown above can be evaluated exactly with finite nonzero ϵ, and the result involved $K_1 m(r + i\epsilon)$, and also the imaginary part of a Struve function $\mathbf{H}_1(im(r + i\epsilon))$ and the real part of a Bessel function $J_1(im(r + i\epsilon))$. If one defines

$$\frac{-i}{2(2\pi)^2 r} \int_{-\infty}^{\infty} \frac{p e^{ipr} e^{-\epsilon|p|}}{\sqrt{p^2 + m^2}} \equiv \tilde{D}(r, \epsilon) \ ,$$

then the distribution theory approach implies that we should evaluate this distribution on a test function $\varphi(r)$ by computing

$$\lim_{\epsilon \to 0} \int_0^{\infty} \mathrm{d}r\, \tilde{D}(r, \epsilon)\, \varphi(r) \ .$$

We, however, would like to simplify this further by taking the limit through the integral sign and using

$$\lim_{\epsilon \to 0} \tilde{D}(r, \epsilon) = \frac{m}{4\pi^2 r} K_1(mr) \ .$$

Since $\varphi(r)$ is required to be very well behaved, one needs only a moderate amount of uniformity in the limit above to justify bringing the limit through the integral sign, using again the Lebesgue dominated convergence theorem. A uniform limit would mean that for every $\delta > 0$ there exists an ϵ such that

$$\left| \tilde{D}(r, \epsilon) - \frac{m}{4\pi^2 r} K_1(mr) \right| < \delta \ .$$

To bring the limit through the integral sign, it would be enough to show that for every $\delta > 0$ there exists an ϵ such that

$$\left| \tilde{D}(r, \epsilon) - \frac{m}{4\pi^2 r} K_1(mr) \right| < g(r)\, \delta \ ,$$

where $g(r)$ is some fixed function which blows up no faster than a power as $r \to \infty$, so that

$$\int_0^{\infty} \mathrm{d}r\, g(r)\, \varphi(r)$$

would be guaranteed to converge. I assume that this can be made to work, but I will not pursue it further.

There is, however, an interesting point to look at concerning the use of different regulators. The distribution analysis shows that any regulator should be equivalent to any other, provided only that

$$|f_\epsilon(t)| < |f(t)| \ ,$$

and for each t,

$$\lim_{\epsilon \to 0} f_\epsilon(t) = f(t) \ .$$

This includes the possibility of a sharp cutoff at $\Lambda = 1/\epsilon$. So

$$\int_0^{1/\epsilon} \mathrm{d}p \, \frac{pe^{ipr}}{\sqrt{p^2+m^2}}$$

should be an acceptable regulator. But earlier we rejected this definition, because it looked like the limit did not exist. To understand what is happening, it is easiest to integrate by parts:

$$\int_0^{1/\epsilon} \mathrm{d}p \, \frac{pe^{ipr}}{\sqrt{p^2+m^2}} = \frac{1}{ir} e^{ir/\epsilon} \frac{1}{\sqrt{1+m^2\epsilon^2}} - \frac{m^2}{ir} \int_0^{1/\epsilon} \mathrm{d}p \, \frac{e^{ipr}}{(p^2+m^2)^{3/2}} \ .$$

The second term is well-behaved, since it falls off at large p as $1/p^3$. This integration by parts is just the recipe that I recommended for a purely numerical evaluation of the two-point function. For the exponential regulator the surface term vanished, but this time we have the problematic term proportional to $e^{ir/\epsilon}$. This factor has no limit as $\epsilon \to 0$ if one thinks of it as a function, but as a distribution it vanishes:

$$\lim_{\epsilon \to 0} \int_0^\infty \mathrm{d}r \, e^{ir/\epsilon} \varphi(r)$$

is defined by first Fourier transforming $\varphi(r)$ at $p = 1/\epsilon$, and then taking the limit $\epsilon \to 0$, or equivalently $p \to \infty$. But the Fourier transform of a Schwartz function is also a function of Schwartz type, and therefore it falls off faster than any power as $p \to \infty$.

In the above argument I ignored the factor of $1/r$ that arose in the Fourier transform. For the spacelike separation problem r must be positive, so we restrict the $\varphi(r)$ to vanish if $r \leq 0$. But we have not yet specified how the test functions $\varphi(r)$ should behave as $r \to 0$. Following the general philosophy of choosing test functions to be extremely well-behaved, we can require each test functions to vanish in an open neighborhood about $r = 0$. We do not fix the size of these open neighborhoods, but just insist that for each acceptable $\varphi(r)$, there is some $\delta > 0$ such that $\varphi(r) = 0$ if $r < \delta$. Then multiplication by $1/r$ is perfectly well-defined operation on test functions, and we are justified in setting

$$\frac{1}{ir} e^{ir/\epsilon} \frac{1}{\sqrt{1+m^2\epsilon^2}}$$

to zero.