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COMPUTATIONALLY EFFICIENT MODELLING FOR LONG TERM PREDICTION OF GLOBAL POSITIONING SYSTEM ORBITS

by

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ABSTRACT

This thesis concentrates on computationally efficient modelling for the long-term prediction of Global Positioning System (GPS) orbits. Reduced force models for the more rapid computation of the averaged VOP equations are developed. This is accomplished by considering the 2:1 resonant condition of the GPS orbit as well as the very low nominal eccentricity. Explicit analytically averaged expressions, in non-singular equinoctial variables, are constructed for the potential and element rates of the primary GPS resonant tesseral harmonics [(2,2), (3,2), (4,2), (4,4)] . Numerical rates returned by these equations are in good agreement with those computed employing time-consuming numerical quadrature. Analysis of the explicit formulae suggests that a passive stationkeeping mechanism may be developed for the GPS constellation by selecting an inclination to zero the semi-major axis rate due to the dominant harmonic (3,2). The inclination is found to be $i \approx 70.53°$ and results in a dramatically stabilized groundtrack.

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Introduction

The Global Positioning System (GPS) is a navigational system consisting of a constellation of satellites that provide continuous radio frequency coverage of the Earth. This system is designed to fulfill a need for accurate position and velocity fixes not only for land based users, but also for users in the near-Earth environment. The transmitted satellite message contains information necessary for a user to determine his position and velocity given that he can acquire any four of the space vehicles in the constellation. This information includes accurate representations of the vehicle ephemerides as well as time and clock corrections. The actual fix is accomplished by measuring the range to several of the GPS satellites from which a user can reconstruct his position in three dimensions. This is accomplished by generating a replica of the satellite signal which is shifted in time until correlation with the transmitted signal is achieved. The time delay is then divided by the speed of light to produce the required range data. Likewise, range rate is measured to allow the computation of user velocity[1]. The selected orbit for GPS implementation is a 12 hour, low eccentricity trajectory with a repeating groundtrack which will be very closely monitored and controlled to insure maximum mission lifetime and to facilitate user acquisition.

The requirement for stationkeeping stems from the realization that the satellite motion is not Keplerian (not movement under a strictly inverse square gravitational field). If the motion were strictly Keplerian, then an initial set of orbital elements (see Appendix A for an overview of orbital elements and variation of parameters) could be chosen such that the subsequent orbit would satisfy the mission constraints in perpetuity. However, natural perturbations resulting from the non-sphericity of the Earth, third body effects, and solar radiation pressure cause deviations from the Keplerian orbit for which corrections must periodically be applied. Specifically, the GPS mission constraints are:

(1) ±2 second deviation in the period, P[2]. Since the semi-major axis, a, is connected to the mean motion, n, and the orbital period, τ, through the relations

$$n^2 a^3 = \mu$$

$$n \tau = 2\pi$$

where, μ = gravitational constant,
an equivalent bound can be given in terms of the semi-major axis which can be computed from $\frac{\delta P}{\left(3\pi\sqrt{\frac{a}{\mu}}\right)}$ for small δP. For a 12 hour orbit, the corresponding bound on a is approximately ±822 meters. This will be used for the sake of convenience in subsequent analysis.

(2) Bound on the eccentricity of .015, with a nominal value less than .005[3]. This will allow full coverage to be provided with the minimum number of satellites.

(3) Bound on the groundtrack that requires that the geographic node crossing stay within ±2° of the nominal value[2].

(4) Initial inclination of 63.44°. This is the so-called critical inclination for which the eccentricity growth due to J_3 is zero.

At this altitude, the inclination is expected to remain quite stable and will accordingly be allowed to drift without stationkeeping[2].

This thesis will be primarily concerned with the construction of computationally efficient, compact, reduced force models for the long-term prediction and orbital stability studies of the GPS trajectory. A corollary matter will be to estimate the time between stationkeeping maneuvers required to maintain the mission constraints. A technical approach with several components will guide the following presentation. A variation of parameters (VOP) formulation of the orbit prediction problem will be substituted for a brute-force Cowell integration of the equations of motion. This is desirable for a variety of reasons. First, VOP usually allows for the more rapid and efficient computation of the orbit when the perturbing accelerations are much smaller than the central force term as is the case here. Second, this formulation has the advantage of being amenable to averaging of the orbital dynamics to produce equations that are computationally more efficient. This point will now be discussed.

The unaveraged time history of the osculating orbital elements can be broken into several temporal categories, those being,

(A) short period - oscillations whose periods are less than the orbital period.

(B) medium period.

(C) long period.

(D) secular - unbounded, non-periodic drifts in the elements.

The short periodic variations in the elements are bounded and generally of low amplitude. In long-term orbit prediction and especially in studies which serve to establish preliminary stationkeeping guidelines, knowledge of these effects is rendered superfluous by the much more substantial contributions in the next three categories. Their presence then serves only to increase integration time needlessly since the required step-size is dictated by the highest frequency components. In this thesis, the short periodic effects will be eliminated by averaging (to first order, over two orbits) with respect to the phase angle variable to produce a set of VOP equations in mean elements.

The last component of the technical approach has to do with the judicious selection of orbital elements in which to express the VOP equations. The classical elements are poorly defined for inclinations of 0° and 180° and for eccentricities near 0. The variation of parameters equations for these elements become correspondingly intractable, both numerically and analytically in regions about these singularities. This requires inefficiently small step-sizes for their computation in numerical programs and induces non-physical oscillations in the elements. The GPS orbit does not have an inclination singularity problem, but does have a low e singularity making the pericenter very poorly defined. A change to another set of orbital elements that are well conditioned everywhere will serve to remove this singularity from the subsequently constructed analytical models. A set of non-singular equinoctial orbital elements will be employed to circumvent any numerical ill-conditioning. These are described in Appendix A.

The thesis will be divided into three main sections. Section I consists of a numerical study to determine reduced force models for the integration of the GPS variation of parameters equations. Analytical justification for these reductions will then be presented. Of major interest here will be the low orbital eccentricity which will be shown to cause only a mathematically prescribed subset of the gravitational field to have a significant influence on a given orbital element.

The GPS orbit has a repeating groundtrack, in part by virtue of its 2:1 commensurability with the Earth's rotation. As a result, this is a resonant orbit as well. Generally, the longitude dependent tesseral harmonics contribute rather low amplitude, short periodic oscillations which are overwhelmed by the effects of the first few zonal harmonics in the Earth's potential. In the computation of the mean element rates these effects are largely averaged out. However, commensurability of the satellite's period with the rotation of the Earth causes some of these terms to be amplified, the result of which is large, long period oscillations in the motion of the satellite in a way that is partially analogous to resonance in a linear mechanical system. These resonant tesserals can no longer be ignored since they now contribute a significant portion of the vehicle's long term motion. In fact, in some ways the tesserals are the only terms that affect the relative geometry of the satellites in the constellation. Until recently, there had been no analytical representation of the tesseral potential in non-singular elements as has existed classically for some years[4]. This deficiency has required a Gaussian form of the VOP equations in which the contributions of the tesserals is known only as a function position and velocity[5]. Numerical averaging of the tesserals must be performed at great computational expense.

Section II will present an analytically averaged potential for the GPS resonant tesserals. On the basis of this, it will be demonstrated that highly accurate computation of the mean element rates will be possible without the need of time consuming numerical quadrature. Appendix C presents a description of the computerized symbolic algebra involved in the construction of the analytically averaged potential.

The development of simple, explicit analytical models for the averaged element rates in Section II greatly assists in the search for passive stationkeeping mechanisms since the orbital physics are now more apparent. Section III will use these expressions to construct a modified 12 hour circular orbit which meets the orbital bounds almost entirely through a passive stationkeeping mechanism. An estimate of the required time between maneuvers necessary to maintain the absolute and relative positions of the satellites for this scheme is compared with that required given the nominal mission profile.

Section I: Reduced Force Models for GPS Orbits

In long-term prediction and stability analysis of orbits it is common practice to average the variation of parameters equations to remove short periodic components from the orbital dynamics. This means that the non-resonant tesseral harmonics, which contribute only short period effects, can be neglected. This allows for a significant reduction in the force models required to integrate the VOP equations. Also, since the satellite orbit for GPS will be well outside the earth's atmosphere, drag is negligible. However, integrating the averaged VOP equations forced by the remaining perturbations is still needlessly inefficient. As it will soon by shown numerically, with analytic justification to follow, the low GPS eccentricity strongly decouples the perturbations in their effects on the various element rates. It does so in such a way that only a subset of those remaining will have an appreciable effect on a given element. The same property will also allow for a dramatic truncation of the lunar potential.

The numerical study must begin with a set of orbital elements at epoch. All runs in the section were made with*:

$$e = 0 \rightarrow h = k = 0$$

$$M_o = \omega = \Omega = 0 \; ; \rightarrow \lambda_o = 0$$

$$i = 63.44^\circ \rightarrow p = 0, \; q = .618$$

In appendix B, an expression is derived whose solution yields the semi-major axis required for a repeating groundtrack, given a commensurability condition. For a 2:1 commensurability, the corresponding a = 26559.9 km. Section III, which deals with stationkeeping, will consider the effect on the long-term orbital evolution of adjusting these epoch values.

*All ESMAP simulations performed with an epoch of January 1, 1980 (0 hours, 0 minutes, 0 seconds).

Numerical Approach

The numerical approach to the problem was conducted using the Earth Satellite Mission Analysis Program (ESMAP)[6]. The modified version used[7] is presently resident on the Amdahl 470 at the Charles Stark Draper Laboratory. The program computes the mean element rates in the presence of a user defined subset of a 4 x 4 geopotential field, luni-solar effects, and atmospheric drag. The numerical attack was to select a certain subset of the full perturbation model deemed dominant in the long-term evolution of an element. This reduced force model was used to integrate the equations of motion for a 200 day arc, the result of which was compared with a similar run using all perturbations. Those models that compared most favorably (i.e., were closest for the longest time) were selected. A matrix of the runs is presented in Table 1.

Aiding in the selection of the models was the Harmonic Analysis Program[8], [9]. This is an implementation of first order solutions to the variation of parameters equations in classical elements. These solutions are based on Kaula's formulation of the gravitational potential in classical orbital coordinates,[4].

$$V_{\ell m} = \frac{\mu R_e^{\ell}}{a^{\ell+1}} \sum_{p=0}^{\ell} F_{\ell mp}(i) \sum_{q=-\infty}^{\infty} G_{\ell pq}(e) S_{\ell mpq}(\omega, M, \Omega, \theta) \tag{1}$$

where

$$S_{\ell mpq} = \begin{bmatrix} C_{\ell m} \\ -S_{\ell m} \end{bmatrix}_{\ell - m \text{ odd}}^{\ell - m \text{ even}} \cos \left[(\ell-2p)\omega + (\ell-2p+q)M + m(\Omega-\theta)\right]$$

$$+ \begin{bmatrix} S_{\ell m} \\ C_{\ell m} \end{bmatrix}_{\ell - m \text{ odd}}^{\ell - m \text{ even}} \sin \left[(\ell-2p)\omega + (\ell-2p+q)M + m(\Omega - \theta)\right]$$

	Run #	Zonals	Tesserals	Luni-Solar Effects
REFERENCE	1	J_2	NONE	NONE
	2	J_2, J_3, J_4, J_2^2	4 X 4 FIELD	1 SOLAR, LUNAR TERMS THROUGH $(\frac{a}{R_3})^6$
	3	J_2, J_3, J_4, J_2^2	4 X 4 FIELD, ODD ORDER HARMONICS EXCLUDED	1 SOLAR, LUNAR TERMS THROUGH $(\frac{a}{R_3})^6$
$\dot{a}$ MODEL	4	NONE	(2,2) (3,2) (4,4)	NONE
	5	NONE	(3,2) (4,4)	NONE
	6	NONE	(3,2)	NONE
	7	J_2	(3,2) (4,4)	NONE
$\dot{\Omega}$ MODEL	8	J_2	(3,2)	NONE
	9	J_2	(3,2)	1 SOLAR TERM, LUNAR TERMS THROUGH $(\frac{a}{R_3})^6$
	10	J_2	(3,2) (4,4)	1 SOLAR TERM, LUNAR TERMS THROUGH $(\frac{a}{R_3})^6$
	11	J_2	(3,2) (4,4)	1 SOLAR TERM, 1 LUNAR TERM α $(\frac{a}{R_3})^2$
$\dot{e}$ MODEL	12	J_3	(2,2) (4,2)	NONE
	13	J_3	(2,2) (4,2)	1 SOLAR TERM, 1 LUNAR TERM α $(\frac{a}{R_3})^2$
	14	J_3	(2,2) (4,2)	1 SOLAR TERM, 2 LUNAR TERMS α $(\frac{a}{R_3})^2$, $(\frac{a}{R_3})^3$
	15	$J_2^{\wedge}$, J_3	(2,2) (4,2) (3,2)	1 SOLAR TERM, 2 LUNAR TERMS $\alpha(\frac{a}{R_3})^2$, $(\frac{a}{R_3})^3$

Table 1. List of ESMAP Runs

θ = Greenwich hour angle

R_e = equatorial radius of the earth

$G_{\ell pq}(e)$ = polynomials in the eccentricity

$F_{\ell mp}(i)$ = polynomials in the cosine of the orbital inclination

μ = gravitational constant

ℓ = degree of harmonic

m = order of harmonic

The pertinent drift rate solutions are given by[4]

$$\Delta a_{\ell mpq} = \mu R_e^{\ell} \frac{2F_{\ell mp} G_{\ell pq} (\ell - 2p + q) S_{\ell mpq}}{na^{\ell+2} \left[(\ell-2p)\dot{\omega} + (\ell-2p+q)\dot{M} + m(\dot{\Omega} - \dot{\theta})\right]}$$

$$\Delta e_{\ell mpq} = \mu R_e^{\ell} \frac{F_{\ell mp} G_{\ell pq} (1-e^2)^{1/2} \left[(1-e^2)^{1/2} (\ell-2p+q) - (\ell-2p)\right] S_{\ell mpq}}{na^{\ell+3} \, e\left[(\ell-2p)\dot{\omega} + (\ell-2p+q)\dot{M} + m(\dot{\Omega} - \dot{\theta})\right]}$$

$$\Delta \Omega_{\ell mpq} = \mu R_e^{\ell} \frac{\left(\partial F_{\ell mp}/\partial i\right) G_{\ell pq} \overline{S}_{\ell mpq}}{na^{\ell+3} (1-e^2)^{1/2} \sin i \left[(\ell-2p)\dot{\omega} + (\ell-2p+q)\dot{M} + m(\dot{\Omega}-\dot{\theta})\right]}$$

$\overline{S}_{\ell mpq}$ is defined as the integral of $S_{\ell mpq}$ with respect to its argument. HAP seeks to identify the resonant tesserals and computes their contributions to the evolution of the orbital elements given epoch values of a, e, and i. The program suffers however from a zero eccentricity singularity in the solution for the eccentricity drift. Thus, the nominal epoch value of 0 for the GPS eccentricity was not used. The actual epoch values input to the program were

$$a = 4.164 \text{ Earth radii} = 26558.57 \text{ km}$$

$$e = .001$$

$$i = 63.44°$$

The resonant tesserals selected by the program were taken as good initial guesses for the construction of the reduced models. One interesting point is that no odd order (m = odd) tesserals are among those listed. This indicates that the odd order harmonics contribute only short period effects which would be filtered out of the long-term dynamics by the numerical quadrature. On this basis, the odd order harmonics were deleted from the models in a first step at reduction.

Inspection of the HAP output in Figure 1 gives an idea of what harmonics will dominate by degree L and order M. The node crossing rate (D [NODE]) is determined primarily by (3, 2) and (4, 4) with all other contributions at least an order of magnitude less. This is also true of the semi-major axis rate (D [A] column). The eccentricity growth, on the other hand, is dominated by (2, 2) and (4, 2) (D [E] column). This served as a basis for constructing the tesseral portion of the reduced models. The major zonals were then included and their effects weighed. The lunar potential has factors in the expansion which are of the form $\left(\frac{a}{R_3}\right)^n$ where R_3 is the Earth-moon distance. This ratio is small, so that an attempt was made to truncate the potential to at most two terms.

SATELLITE 12HUR -- RESONANT PERTURBATIONS

BEAT PERIOD =6632.5 DAYS PRIMARY RESONANCE WITH TERMS OF ORDER 2

A= 4.164 E.R. M DOT = 722.086 DEG/DAY
~~E=0.0010~~ ~~W DOT = -0.000 DEG/DAY~~ ~~PERIGEE HEIGHT = 0.2015D 05 KM~~
I= 63.44 DEG. NODE DOT= -0.030 DEG/DAY APOGEE HEIGHT = 0.2021D 05 KM
PERIOD = 11.965 HOURS ORBITAL FREQ. = 2.006 C/DAY

L	M	P	Q	BP (DAYS)	D(A) (M)	D(E)	D(I) (DEG)	D(W) (DEG)	D(M) (DEG)	D(NODE) (DEG)	POSITION COMPONENTS (M) RADIAL	TRANS.	NORMAL	VELOCITY COMP. (CM/SEC) RADIAL	TRANS.	NORMAL	
~~2~~	~~2~~	~~0~~		~~-16633.929~~	~~-7.23D 01~~	~~1.36D-03~~	~~9.64D-05~~	~~-7.80D 01~~	~~-8.11D 01~~	~~-2.63D-04~~	~~3.7D 04~~	~~1.6D 03~~	~~5.5D 01~~	~~5.4D 02~~	~~5.7D-01~~	~~4.6D-01~~	~~2~~
2	2	1		16631.015	1.66D 02	3.12D-03	-4.00D-04	1.79D 02	-1.86D 02	5.31D-05	8.5D 04	-3.4D 03	1.5D 02	1.2D 03	1.3D 00	1.9D 00	2
3	2	1		06632.471	-4.31D 03	4.06D-08	8.07D-03	-5.72D-02	1.85D 02	3.23D-02	6.1D 04	9.1D 04	1.7D 04	8.9D 02	9.4D-01	3.9D 01	2
~~4~~	~~2~~	~~1~~		~~-16633.929~~	~~6.60D-01~~	~~-1.24D-05~~	~~-8.80D-07~~	~~7.12D-01~~	~~-7.40D-01~~	~~1.68D-06~~	~~3.4D 02~~	~~-1.3D 01~~	~~7.6D-01~~	~~4.9D 00~~	~~5.2D-03~~	~~4.2D-03~~	~~2~~
4	2	2		16631.015	-1.03D 00	-1.95D-05	2.49D-06	-1.11D 00	1.16D 00	1.71D-05	5.3D 02	2.3D 01	6.8D 00	7.7D 00	8.1D-03	1.2D-02	2
4	4	1		03316.236	1.91D 03	-1.80D-08	-3.58D-03	6.64D-03	-4.11D 01	1.75D-03	1.4D 04	-4.0D 04	1.2D 03	2.0D 02	4.2D-01	1.7D 01	3
5	2	2		06632.471	-1.74D 02	1.64D-09	3.25D-04	-3.47D-03	7.48D 00	1.37D-04	2.5D 03	3.7D 03	2.4D 02	3.6D 01	3.8D-02	1.6D 00	2
~~5~~	~~4~~	~~2~~		~~13315.871~~	~~-7.44D-01~~	~~-7.00D-06~~	~~1.59D-06~~	~~-4.01D-01~~	~~4.17D-01~~	~~9.99D-07~~	~~1.9D 02~~	~~1.6D 01~~	~~7.3D-01~~	~~2.8D 00~~	~~5.8D-03~~	~~7.6D-03~~	~~2~~
6	2	3		16631.015	-9.64D-02	-1.81D-06	2.32D-07	-1.04D-01	1.08D-01	3.18D-07	4.9D 01	2.1D 00	1.2D-01	7.2D-01	7.6D-04	1.1D-03	2
6	4	2		03316.236	2.76D 01	-2.60D-10	-5.18D-05	2.38D-04	-5.95D-01	7.54D-05	2.0D 02	-5.8D 02	2.6D 01	2.8D 00	6.0D-03	2.5D-01	2
6	6	2		12210.662	2.21D-01	1.38D-06	-4.53D-07	7.93D-02	-8.25D-02	1.50D-07	3.8D 01	-4.5D 00	1.5D-01	[illegible]	1.7D-03	2.2D-03	2
7	2	3		06632.471	-4.15D 00	3.90D-11	7.77D-06	-1.28D-04	1.79D-01	-2.63D-05	5.9D 01	8.7D 01	7.7D 00	8.5D-01	9.0D-04	3.7D-02	2
7	6	2		02210.824	-5.76D-01	5.42D-12	1.08D-06	-8.00D-06	8.27D-03	1.08D-05	2.8D 00	1.2D 01	4.7D 00	4.0D-02	1.3D-04	5.2D-03	2
8	4	3		03316.236	2.39D-01	-2.25D-12	-4.48D-07	2.53D-06	-5.15D-03	3.92D-06	1.7D 00	-5.0D 00	1.5D 00	2.5D-02	5.2D-05	2.1D-03	2
~~8~~	~~8~~	~~2~~		~~01658.118~~	~~1.36D 00~~	~~1.28D-11~~	~~-2.54D-06~~	~~3.13D-06~~	~~-1.46D-02~~	~~5.26D-07~~	~~5.9D 00~~	~~-2.9D 01~~	~~8.4D-01~~	~~7.6D-02~~	~~3.8D-04~~	~~1.2D-02~~	~~3~~
9	6	3		02210.824	-1.05D-01	9.91D-13	1.97D-07	-1.36D-06	1.51D-03	3.12D-07	5.1D-01	2.2D 00	1.7D-01	7.2D-03	2.3D-05	9.4D-04	2

RMS BEAT PERIOD FOR L)16= 0.5082459D 04
RMS AMPLITUDE 0.2486511D 05 RSS AMPLITUDE 0.9946442D 05
RMS AMPLITUDE FOR Q=0 0.3313167D 05 RSS AMPLITUDE 0.9939501D 05
RMS AMPLITUDE FOR 0) Q 0.1404271D 04 RSS AMPLITUDE 0.3715351D 04
~~RATIO OF RMS AMPLITUDES Q=0 TO 0) Q - 23.594~~
RSS AMPLITUDES Q=0 TO 0) Q - 26.753

Figure 1. Output of Harmonic Analysis Program.

The selected set of reduced models that resulted is:

Element rate	Model	Illustration
$\dot{a}$	(3,2) , (4,4) , J_2	Figure 2
$\dot{\Omega}$	(3,2) , (4,4) , J_2, solar point mass term, 1 lunar term $\alpha \left(\frac{a}{R_3}\right)^2$	Figure 3
$\dot{e}$	(2,2), (3,2), (4,2) J_3, J_2, 2 lunar terms α $\left(\frac{a}{R_3}\right)^2$, $\left(\frac{a}{R_3}\right)^3$, 1 solar term	Figure 4

Figure 2 clearly shows that by far the most important contribution to the semi-major axis rate is the resonant tesseral (3,2). This observation is used in Section III to develop a passive stationkeeping procedure that can be implemented choosing epoch elements based on nulling just $\dot{a}_{(3,2)}$. The addition of (4,4) produces a model that is much closer to the all perturbations run, however there remains a discrepancy that results in a divergence of 50 meters after

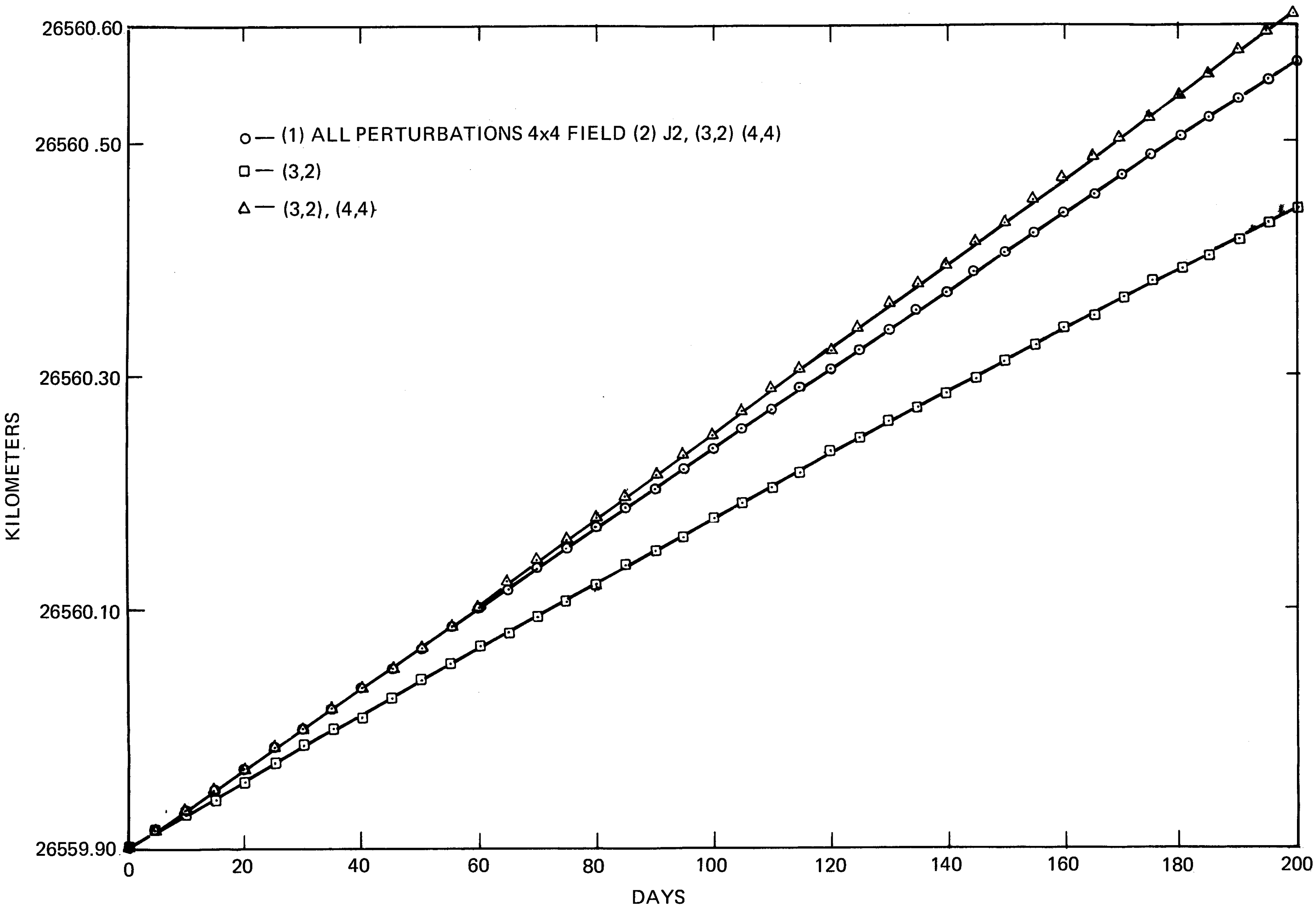

Figure 2. Semi-major Axis (km) vs. Time for the Nominal GPS Orbit

200 days. The inclusion of the J_2 zonal eliminates the discrepancy and completes a model which is virtually indistinguishable from the full force representation. The presence of J_2 is surprising since mathematically, the semi-major axis rate due to this harmonic is 0. The observed effect is a resonance phenomenon and is caused by the influence of J_2 perturbation on the mean longitude coupling into the a rate.

The geocentric longitude of the ascending node is controlled by the same geopotential harmonics as the semi-major axis. As previously stated, there is in fact considerable coupling between the two. In Figure 3 it is seen that a J_2, (3,2) model constitutes a significant portion of the perturbed motion. The addition of the full lunar potential (through $(^a/R_3)^6$) produces much closer agreement, but exposes the need to introduce (4,4). A truncation of the lunar potential in the final model was possible by including only one term, proportional to $(^a/R_3)^2$, with nearly no degradation. Evidently contributions from higher order terms are essentially negligible.

Figure 4 shows the k component of the eccentricity since some of the more interesting variations were found here. A model containing J_3, (2,2), and (4,2) produced the major portion of the growth. Inclusion of a lunar potential term proportional to $\left(\frac{a}{R_3}\right)^2$ in an attempt to model the medium period oscillation apparent in the all perturbations run, had virtually no effect, a rather intuitively surprising result. Adding the next term $(\alpha(^a/R_3)^3)$, however, yielded the desired evolution with all other terms contributing much less. The divergence between this model and the full field model is due to the same resonance phenomena observed in the semi-major axis rate. Since the mean longitude is coupled into the eccentricity it must be properly modelled as well. Therefore the addition of J_2 and (3,2) to the final model to correct for errors in the

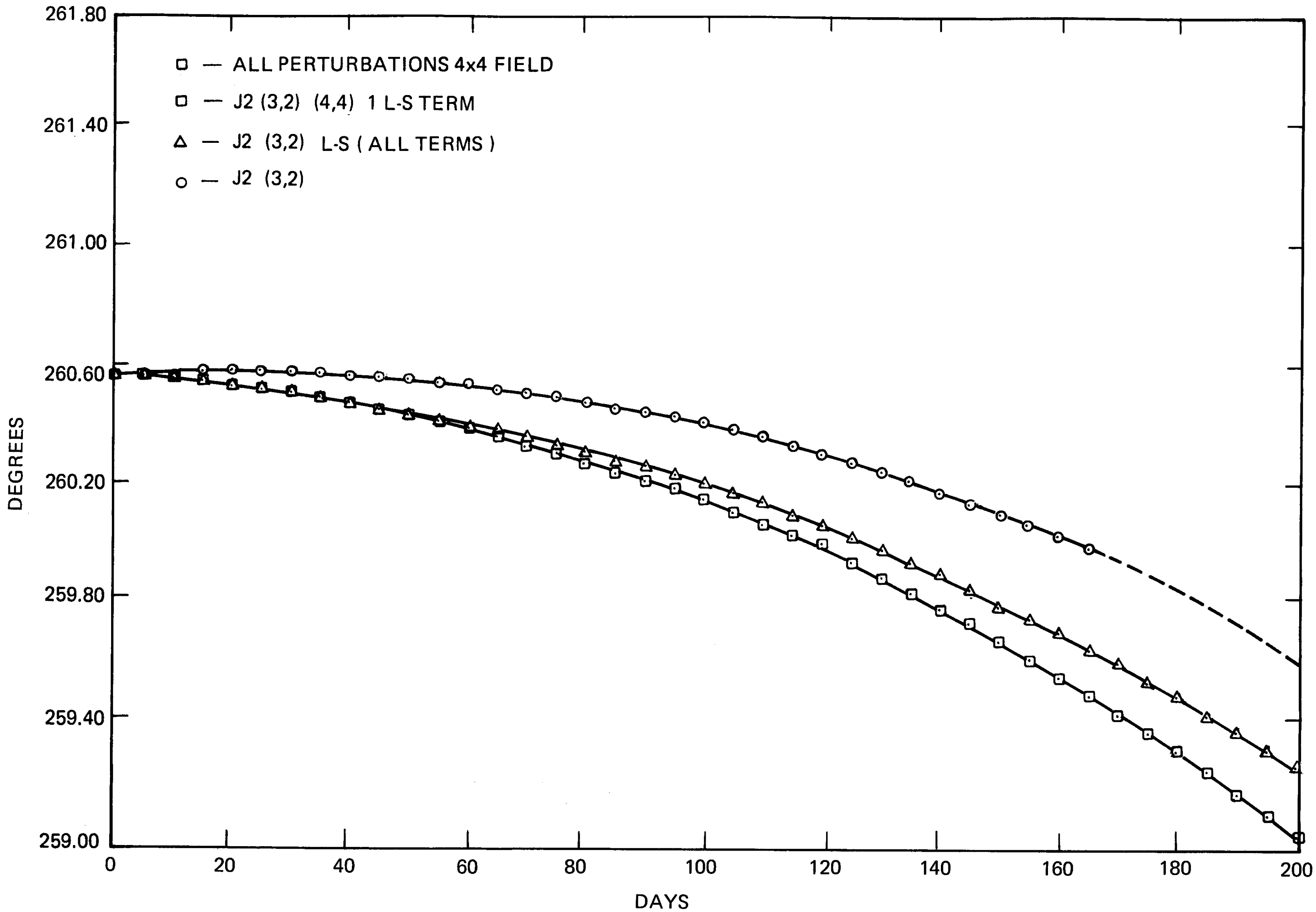

Figure 3. Geographic Longitude of Ascending Node (Deg) vs. TIME for the Nominal GPS Orbit

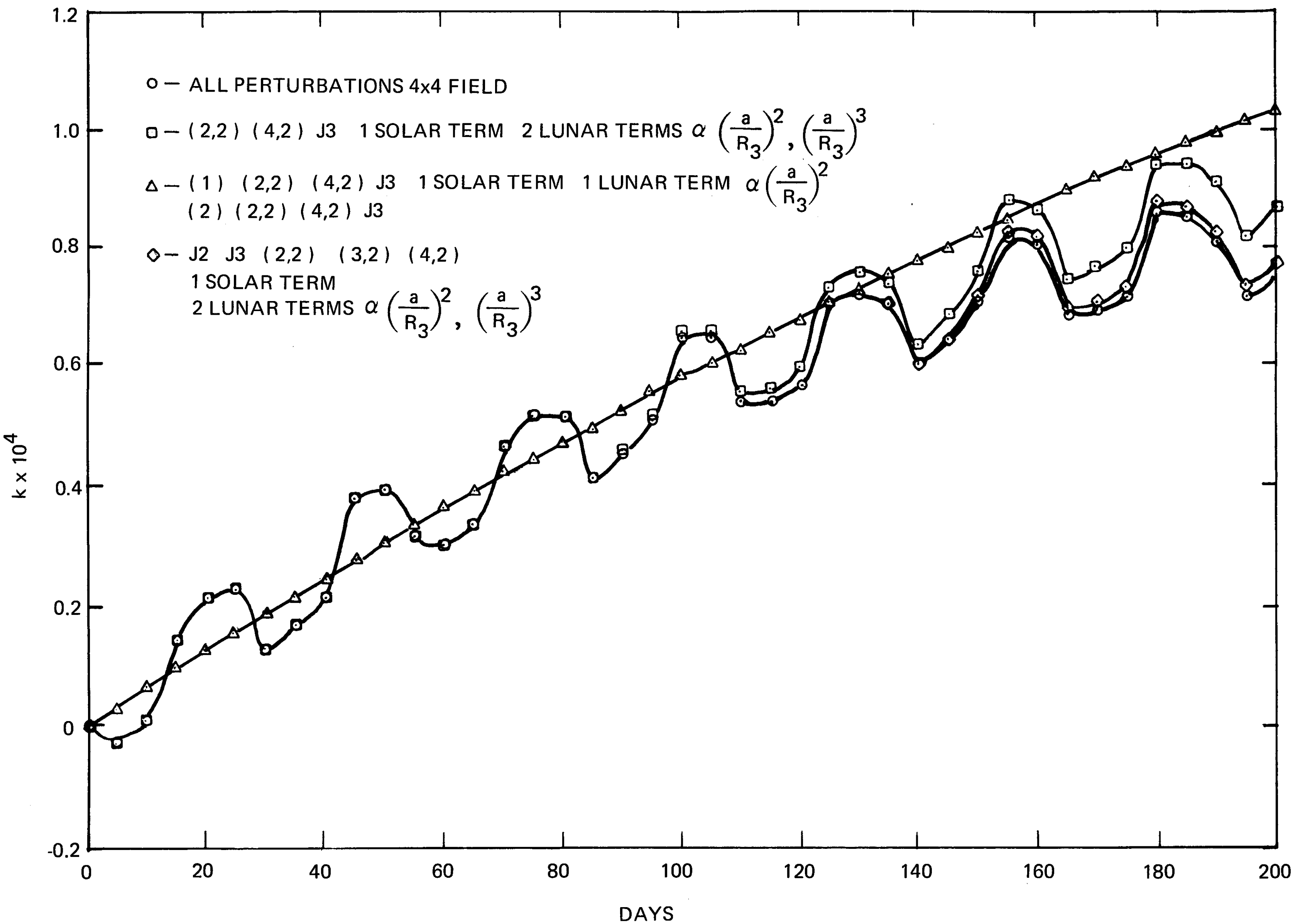

Figure 4. $k = e \cos(\omega + \Omega)$ Growth vs. Time (Days) for the Nominal GPS Orbit

mean longitude, results in excellent agreement with "ALL PERTS."*

All of the reduced models are suitable for judging the long-term stability of the GPS orbit. They are also found to be excellent replacements for full force representations over long time spans and may be used to integrate the averaged VOP equations with good results.

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*One is reminded that the "ALL PERTS" run is in all cases the full field of ESMAP perturbations, exclusive of the odd order tesseral harmonics.

Analytical Approach

The observed decoupling between the geopotential harmonics in their effects on the mean element rates can be justified on analytical grounds. Likewise, the neglect of terms in the lunar potential is mathematically justifiable. Each of these will now be considered in detail.

The basic expansion for the geopotential comes from a solution of Laplace's equation, $\underline{\nabla}^2 U = 0$, in spherical coordinates and is given by[4]

$$U = \frac{\mu}{r} \sum_{n=2}^{N} \sum_{m=0}^{n} \left(\frac{R_e}{r}\right)^n P_{nm}(\sin \phi) \left[C_{nm} \cos m\lambda + S_{nm} \sin m\lambda\right] \quad (2)$$

where, μ = gravitational constant = $G(m_e + m_s)$

m_s = mass of satellite

m_e = mass of Earth

r = distance from Earth to satellite

R_e = equatorial radius of the Earth

ϕ = geocentric latitude

λ = geographic longitude of the satellite

P_{nm} $(\sin \phi)$ = associated Legendre function of degree, n, and order, m

C_{nm}, S_{nm} = empirically determined gravity harmonic coefficients

The terms in this expansion for which $m = 0$ are called zonal harmonics and arise due to the nonsphericity of the Earth along a meridian. Accordingly, they possess symmetry about the Earth's rotation axis. The terms for which $m \neq 0$ are called tesseral harmonics and represent the longitudinally dependent deviations from sphericity.

As will be demonstrated more thoroughly in Section II, it is desirable to have the potential expressed directly in orbital elements. In implementing the Variation of Parameters equations this circumvents the computationally costly coordinate transformations otherwise required and allows for analytical averaging to remove short period components. An algebraic conversion of the spherical harmonic disturbing potential, (2), to classical elements has been available due to Kaula for several years, (1). A similar expression in the non-singular elements needed for GPS analysis has only recently been developed and for a general harmonic is the real part of [10]

$$U^*_{nm} = \frac{\mu}{a}\left(\frac{R_e}{a}\right)^n C^*_{nm} \sum_{s=-n}^{n} V^m_{n,s} \; S_{2n}^{(m,s)}(p,q) \tag{3}$$

$$X \sum_{t=-\infty}^{+\infty} Y_t^{-n-1,s}(k,h) \; \exp\left[j(t\,\lambda - m\theta)\right]$$

where, a = satellite orbit semi-major axis

$C_{nm*} = C_{nm} - jS_{nm}$; $j = \sqrt{-1}$

λ = mean longitude $\equiv M + \omega + \Omega$

Θ = Greenwich hour angle

$S_{2n}^{(m,s)}(p,q)$ is a special function, related to the Jacobi polynomials $P_n^{(\alpha,\beta)}$ that contains the equinoctial elements p and q which orient the orbital frame with respect to inertial space. The S function arises from the rotation of the spherical harmonics in (2) into the orbital reference frame. They are computed according to the following rules

$$
S_{2n}^{(m,r)} = \begin{cases} (-1)^{m-r}(1+p^2+q^2)^{-r}(p+jq)^{r-m} P_{n-r}^{(r-m,r+m)}(\gamma) & r \geq m \\ \\ \dfrac{(n+m)!(n-m)!}{(n+r)!(n-r)!}(1+p^2+q^2)^{-m}(p-jq)^{m-r} P_{n-m}^{(m-r,r+m)}(\gamma) & \\ & -m \leq r \leq +m \\ \\ (1+p^2+q^2)^{r}\ (p-jq)^{m-r} P_{n+r}^{(m-r,-m-r)}(\gamma) & r \leq -m \end{cases} \tag{4}
$$

where,

$$
\gamma = \frac{(1-p^2-q^2)}{(1+p^2+q^2)} = \cos i
$$

$$
P_n^{(\alpha,\beta)}(x) = \frac{1}{(-1)^n 2^n n!(1-x)^{\alpha}(1+x)^{\beta}} \frac{d^n}{dx^n}\left\{(1-x)^{\alpha}(1+x)^{\beta}(1-x^2)^n\right\}
$$

The expression of the form $Y_{\alpha}^{\beta,\gamma}(k,h)$ is another special function in the eccentricity analogs h and k, closely related to the standard Hansen coefficients, $X_{\alpha}^{\beta,\gamma}(e)$.

$$
Y_t^{n,m} = (k+jh)^{m-t}\sum_{\sigma=0}^{\infty} X_{\sigma+m-t,\sigma}^{n,-m}(h^2+k^2)^{\sigma} \qquad t \leq m
$$

$$
Y_t^{n,m} = (k-jh)^{t-m}\sum_{\sigma=0}^{\infty} X_{\sigma+t-m,\sigma}^{n,m}(h^2+k^2)^{\sigma} \qquad t \geq m \tag{5}
$$

where,

$$X_{\alpha,\beta}^{\delta,\gamma} \equiv \text{constant Newcomb operators}$$

(see Appendix C)

Last of all,

$$V_{n,s}^{m} \equiv \frac{(n-s)!}{(n-m)!} \quad P_{n,s}(0) \qquad (6)$$

where, $P_{n,s}(0) = \frac{d^s}{d\nu^s} \quad P_n(\nu) \Big|_{\nu = 0}$

$$P_{n,-s}(0) = (-1)^s \frac{(n-s)!}{(n+s)!} \quad P_{n,s}(0)$$

$P_n(\nu)$ = Legendre polynomial of order, n

The preliminary simplification of (3) is begun by recalling that the GPS orbit is resonant due to the 2:1 commensurability with the Earth's rotation. In general, the averaged elements of the orbit will be influenced by terms in the geopotential for which the trigonometric argument of (3) is slowly varying, or mathematically started, terms for which

$$t\,\dot{\lambda} - m\,\dot{\theta} \approx 0 \qquad (7)$$

or

$$t = m\,\frac{\dot{\theta}}{\dot{\lambda}} \qquad (8)$$

But $\dot{\theta}/\dot{\lambda}$ is equal to 1/2 so that equation (8) becomes

$$t = \frac{m}{2} \tag{9}$$

Now if m/2 is not an integer, then a harmonic of order m cannot contribute to the resonant potential . One immediately notes that all odd order harmonics are non-resonant. Consequently, they contribute short periodic oscillations which can be neglected for long-term averaged orbit studies. This result is consistent with the numerical section. (9) allows the resonant potential to be expressed as

$$U^*_{nm} = \frac{\mu}{a}\left(\frac{R_e}{a}\right)^n C^*_{nm} \exp\left[j\,\frac{m}{2}\,(\lambda - 2\theta)\right] \tag{10}$$

$$\times \sum_{s=-n}^{n} V^m_{n,s}\; S^{(m,s)}_{2n}(p,q)\; Y^{-n-1,s}_{\frac{m}{2}}(k,h)$$

Since $(\frac{R_e}{a})$ is approximately .24, further truncation of (10) is effected by eliminating terms associated with powers of this ratio greater than four. Thus, the full resonant potential can now be expressed as the real part of

$$U^* = \frac{\mu}{a}\sum_{n=2}^{4}\left(\frac{R_e}{a}\right)^n \sum_{m=0}^{n} C^*_{nm} \exp\left[j\frac{m}{2}(\lambda-2\theta)\right] \tag{11}$$

$$\times \sum_{s=-n}^{n} V^m_{n,s}\; S^{(m,s)}_{2n}\;(p,q)\; Y^{-n-1,s}_{\frac{m}{2}}\;(k,h)$$

$$m = 0,\ 2,\ 4$$

A final simplification is achieved by noting that the index s is constrained in three ways. First, since the eccentricity is not to exceed .015 between corrections, truncation of the Hansen coefficients (5) to the third power of h and k is allowable. This can be obtained by introducing a constraint on s of

$$\left| s - \frac{m}{2} \right| \leq 3 \qquad m = 0, 2, 4 \tag{12}$$

Second, the rotational transformation places further limits on s such that

$$\begin{aligned} n &= 2 \qquad & -2 \leq s \leq 2 \\ n &= 3 \qquad & -3 \leq s \leq 3 \\ n &= 4 \qquad & -4 \leq s \leq 4 \end{aligned} \tag{13}$$

Lastly, the coefficients (6), behave as

$$V^m_{n,s} = 0 \quad ; \quad |n - s| = \text{odd} \tag{14}$$

Intersecting these three constraints allows s to take on only the values displayed in Table 2.

Table 2 - Allowed Values of s for GPS Resonant Potential

n	m	s
2	0	-2, 0, 2
	2	-2, 0, 2
3	0	-3, -1, 1,3
	1	-1, 1, 3
4	0	-2, 0, 2
	2	-2, 0, 2,4
	4	0, 2, 4

An array of harmonics with their associated Hansen coefficients can now be constructed and is given in Table 3.

Table 3 - Required Hansen Coefficients for GPS Resonant Potential

n \ m	0	2	4
2	$Y_0^{-3,-2}$ $Y_0^{-3,0}$ $Y_0^{-3,2}$	$Y_1^{-3,-2}$ $Y_1^{-3,0}$ $Y_1^{-3,2}$	
3	$Y_0^{-4,-3}$ $Y_0^{-4,-1}$ $Y_0^{-4,1}$ $Y_0^{-4,3}$	$Y_1^{-4,-1}$ $Y_1^{-4,1}$ $Y_1^{-4,3}$	
4	$Y_0^{-5,-2}$ $Y_0^{-5,0}$ $Y_0^{-5,2}$	$Y_1^{-5,-2}$ $Y_1^{-5,0}$ $Y_1^{-5,2}, Y_1^{-5,4}$	$Y_2^{-5,0}$ $Y_2^{-5,2}$ $Y_2^{-5,4}$

From this array and the definition of the modified Hansen coefficients, (5), the resonant potential can be written term by term:

$$U_{20}^{*} = \frac{\mu}{a}\left(\frac{R_e}{a}\right)^2 C_{20}^{*} \Big[V_{2,-2}^{0} S_4^{(0,-2)} Y_0^{-3,-2} + V_{2,0}^{0} S_4^{(0,0)} Y_0^{-3,0} + V_{2,2}^{0} S_4^{(0,2)} Y_0^{-3,2}\Big] \quad (15)$$

where

$$Y_0^{-3,-2} = (k-jh)^2 \; X_{2,0}^{-3,-2}$$

$$Y_0^{-3,0} = X_{0,0}^{-3,0} + X_{1,1}^{-3,0} \; (h^2 + k^2)$$

$$Y_0^{-3,2} = (k + jh)^2 \; X_{2,0}^{-3,-2}$$

$$U_{22}^* = \frac{\mu}{a}\left(\frac{R_e}{a}\right)^2 C_{22}^* \; e^{j\left[\lambda-2\theta\right]} \left[V_{2,-2}^{2} \; S_4^{(2,-2)} \; Y_1^{-3,-2} \right. \tag{16}$$

$$\left. + V_{2,0}^{2} \; S_4^{(2,0)} \; Y_1^{-3,0} + V_{2,2}^{2} \; S_4^{(2,2)} \; Y_1^{-3,2}\right]$$

where

$$Y_1^{-3,-2} = (k-jh)^3 \; X_{3,0}^{-3,-2}$$

$$Y_1^{-3,0} = (k-jh) \left[X_{1,0}^{-3,0} + X_{2,1}^{-3,0} \; (h^2 + k^2)\right]$$

$$Y_1^{-3,2} = (k+jh) \left[X_{1,0}^{-3,-2} + X_{2,1}^{-3,-2} \; (h^2 + k^2)\right]$$

$$U_{30}^* = \frac{\mu}{a}\left(\frac{R_e}{a}\right)^3 C_{30}^* \left[V_{3,-3}^{0} S_6^{(0,-3)} Y_0^{-4,-3} + V_{3,-1}^{0} \; S_6^{(0,-1)} Y_0^{-4,-1} \right. \tag{17}$$

$$\left. + V_{3,1}^{0} \; S_6^{(0,1)} \; Y_0^{-4,1} + V_{3,3}^{0} \; S_6^{(0,3)} \; Y_0^{-4,3}\right]$$

where

$$Y_0^{-4,-3} = (k - jh)^3 \; X_{3,0}^{-4,-3}$$

$$Y_0^{-4,-1} = (k - jh) \left[X_{1,0}^{-4,-1} + X_{2,1}^{-4,-1} (h^2 + k^2)\right]$$

$$Y_0^{-4,1} = (k + jh) \left[X_{1,0}^{-4,-1} + X_{2,1}^{-4,-1} (h^2 + k^2)\right]$$

$$Y_0^{-4,3} = (k + jh) \; X_{3,0}^{-4,-3}$$

$$U_{32}^* = \frac{\mu}{a}\left(\frac{R_e}{a}\right)^3 C_{32}^* \; e^{j\left[\lambda - 2\theta\right]} \left[V_{3,-1}^{\;2} \; S_6^{(2,-1)} Y_1^{-4,-1} + V_{3,1}^{\;2} S_6^{(2,1)} Y_1^{-4,1} + V_{3,3}^{\;2} \; S_6^{(2,3)} \; Y_1^{-4,3}\right] \qquad (18)$$

where

$$Y_1^{-4,-1} = (k - jh)^2 \; X_{2,0}^{-4,-1}$$

$$Y_1^{-4,1} = X_{0,0}^{-4,1} + X_{1,1}^{-4,1} (h^2 + k^2)$$

$$Y_1^{-4,3} = (k + jh)^2 \; X_{2,0}^{-4,-3}$$

$$U_{40}^* = \frac{\mu}{a}\left(\frac{R_e}{a}\right)^4 C_{40}^* \left[V_{4,-2}^{\;0}\; S_8^{(0,-2)} Y_0^{-5,-2} + V_{4,0}^{\;0}\; S_8^{(0,0)} Y_0^{-5,0} \right. \quad (19)$$

$$\left. + V_{4,2}^{\;0}\; S_8^{(0,2)}\; Y_0^{-5,2}\right]$$

where

$$Y_0^{-5,-2} = (k - jh)^2\; X_{2,0}^{-5,-2}$$

$$Y_0^{-5,0} = X_{0,0}^{-5,0} + X_{1,1}^{-5,0}\;(h^2 + k^2)$$

$$Y_0^{-5,2} = (k + jh)^2\; X_{2,0}^{-5,-2}$$

$$U_{42}^* = \frac{\mu}{a}\left(\frac{R_e}{a}\right)^4 C_{42}^*\; e^{j\left[\lambda - 2\theta\right]}\left[V_{4,-2}^{\;2}\; S_8^{(2,-2)}\; Y_1^{-5,-2} \right. \quad (20)$$

$$\left. + V_{4,0}^{\;2}\; S_8^{(2,0)} Y_1^{-5,0} + V_{4,2}^{\;2}\; S_8^{(2,2)} Y_1^{-5,2} + V_{4,4}^{\;2} S_8^{(2,4)} Y_1^{-5,4}\right]$$

where

$$Y_1^{-5,-2} = (k - jh)^3\; X_{3,0}^{-5,-2}$$

$$Y_1^{-5,0} = (k - jh)\;\left[X_{1,0}^{-5,0} + X_{2,1}^{-5,0}\;(h^2 + k^2)\right]$$

$$Y_1^{-5,2} = (k + jh)\;\left[X_{1,0}^{-5,-2} + X_{2,1}^{-5,-2}\;(h^2 + k^2)\right]$$

$$Y_1^{-5,4} = (k + jh)^3\; X_{3,0}^{-5,-4}$$

$$U_{44}^{*} = \frac{\mu}{a}\left(\frac{R_e}{a}\right)^4 C_{44}^{*}\, e^{2j\left[\lambda-2\theta\right]}\left[V_{4,0}^{4}\, S_8^{(4,0)} Y_2^{-5,0}\right. \tag{21}$$

$$\left. + V_{4,2}^{4}\, S_8^{(4,2)}\, Y_2^{-5,2} + V_{4,4}^{4}\, S_8^{(4,4)} Y_2^{-5,4}\right]$$

where

$$Y_2^{-5,0} = (k - jh)^2\, X_{2,0}^{-5,0}$$

$$Y_2^{-5,2} = X_{0,0}^{-5,-2} + X_{1,1}^{-5,-2}\,(h^2 + k^2)$$

$$Y_2^{-5,4} = (k + jh)^2\, X_{2,0}^{-5,-4}$$

It is notable that some harmonics have associated Hansen coefficients which contain only even powers of the eccentricity, while others contain only odd powers. In general one finds, for the GPS orbit, that harmonics for which $n - m/2$ is even, are dependent only on the even powers of h and k. Similarly for $n - m/2$ odd, only odd powers appear.

Now, it is possible to tell how the resonant harmonics decouple in the element rates by inspection. To do this, a set of low eccentricity equinoctial VOP equations is presented for which powers of h and k greater than 1 have been neglected

$$\frac{da}{dt} = \frac{2}{na}\,\frac{\partial U}{\partial \lambda_o} \tag{22}$$

$$\frac{dh}{dt} = \frac{1}{na^2}\left(\frac{\partial U}{\partial k} - \frac{h}{2}\,\frac{\partial U}{\partial \lambda_o}\right) + \frac{k(1+p^2+q^2)}{2na^2}\left(p\,\frac{\partial U}{\partial p} + q\,\frac{\partial U}{\partial q}\right) \tag{23}$$

$$\frac{dk}{dt} = - \frac{1}{na^2}\left(\frac{\partial U}{\partial h} + \frac{k}{2}\frac{\partial U}{\partial \lambda_o}\right) - \frac{h(1+p^2+q^2)}{2na^2}\left(p\frac{\partial U}{\partial p} + q\frac{\partial U}{\partial q}\right) \qquad (24)$$

$$\frac{d\lambda}{dt} = - \frac{2}{na}\frac{\partial U}{\partial a} + \frac{1}{2na^2}\left(h\frac{\partial U}{\partial h} + k\frac{\partial U}{\partial k}\right) + \frac{(1+p^2+q^2)}{2na^2}\left(p\frac{\partial U}{\partial p} + q\frac{\partial U}{\partial q}\right) \qquad (25)$$

$$\frac{dp}{dt} = \frac{-p(1+p^2+q^2)}{2na^2}\left(k\frac{\partial U}{\partial h} - h\frac{\partial U}{\partial k} + \frac{\partial U}{\partial \lambda_o}\right) + \frac{(1+p^2+q^2)^2}{4na^2}\frac{\partial U}{\partial q} \qquad (26)$$

$$\frac{dq}{dt} = \frac{-q(1+p^2+q^2)}{2na^2}\left(k\frac{\partial U}{\partial h} - h\frac{\partial U}{\partial k} + \frac{\partial U}{\partial \lambda_o}\right) - \frac{(1+p^2+q^2)^2}{4na^2}\frac{\partial U}{\partial p} \qquad (27)$$

The procedure is a simple matter of selecting as dominant, in a particular element rate, those harmonics that contribute terms in the 0^{th} power of h and k. To facilitate the analysis all terms in (22) - (27) that have factors of h and k, will be neglected since they contain powers of the eccentricity of no less than one. The VOP equations can now be written as

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial U}{\partial \lambda_o} \qquad (22a)$$

$$\frac{dh}{dt} = \frac{1}{na^2}\frac{\partial U}{\partial k} \qquad (23a)$$

$$\frac{dk}{dt} = - \frac{1}{na^2}\frac{\partial U}{\partial h} \qquad (24a)$$

$$\frac{d\lambda_o}{dt} = \frac{-2}{na} \frac{\partial U}{\partial a} + \frac{(1+p^2+q^2)}{2na^2} \left(p \frac{\partial U}{\partial p} + q \frac{\partial U}{\partial q} \right) \quad (25a)$$

$$\frac{dp}{dt} = \frac{-p(1+p^2+q^2)}{2na^2} \frac{\partial U}{\partial \lambda_o} + \frac{(1+p^2+q^2)^2}{4na^2} \frac{\partial U}{\partial q} \quad (26a)$$

$$\frac{dq}{dt} = \frac{-q(1+p^2+q^2)}{2na^2} \frac{\partial U}{\partial \lambda_o} - \frac{(1+p^2+q^2)^2}{4na^2} \frac{\partial U}{\partial p} \quad (27a)$$

Now, in those rates whose remaining terms have no derivatives with respect to h and k, potential harmonics that contain 0^{th} powers of h and k will dominate (n-m/2 even). In cases where derivatives of h and k appear, harmonics that contain 1^{st} powers of h and k will dominate (n-m/2 odd). A quick check yields Table 4.

Table 4 - Analytically Predicted Models

Element rate	Dominant Harmonics
$\dot{a}$	J_2, J_4, (3,2), (4,4)
$\dot{h}$	J_3, (2,2), (4,2)
$\dot{k}$	J_3, (2,2), (4,2)
$\dot{\lambda}_o$	J_2, J_4, (3,2), (4,4)
$\dot{p}$	J_2, J_4, (3,2), (4,4)
$\dot{q}$	J_2, J_4, (3,2), (4,4)

The table is readily extended using the parity of n-m/2 as a guide. This decoupling is exactly that demonstrated in the numerical study performed by ESMAP. It is necessary to realize that the observed harmonic decoupling is not generally valid. However, in the case of the GPS orbit, the near zero eccentricity has caused this to happen. Support for this contention comes from a consideration of the Soviet Molniya communications satellites which move in highly eccentric (~.74), 12 hour orbits at the critical inclination of nearly 65°. These satellites have partially stabilized groundtracks which give rise to resonance as for GPS. However, it is found that the dominant harmonic affecting period change and nodal drift is (2,2) rather than (3,2). In fact the transition between (3,2) dominance and (2,2) control of the semi-major axis appears to be an eccentricity of approximately .05(11).

In the numerical section, terms of the lunar potential were required to adequately model the nodal drift and eccentricity growth. This was especially evident in the case of the eccentricity where the exclusion of the $\left(\frac{a}{R_3}\right)^3$ term would have resulted in the failure to model a significant medium period oscillation. The result was surprising since we might have expected the $\left(\frac{a}{R_3}\right)^2$ term to exert the dominant influence. An analytical approach is possible to explain the phenomenon as well as show which terms, if any, of the lunar potential will dominate in their effect on the other element rates.

To illustrate, the lunar potential is given by(12)

$$F^{(L)} = \frac{\mu_L}{R_L} \sum_{n=2}^{\infty} \left(\frac{r}{R_L}\right)^n P_n(\cos\psi) \tag{28}$$

where,

μ_L = lunar gravitational constant

R_L = earth-moon distance

r = earth-satellite distance

ψ = angle between $\underline{r}$ and $\underline{R}_L$

$P_n(\cos\psi)$ = n^{th} Legendre polynomial

The corresponding averaged potential of interest here can be written, after multiplying by $(a/a)^n$, as[12]

$$\overline{F}^{(L)} = \frac{\mu_L}{R_L} \sum_{n=2}^{\infty} \left(\frac{a}{R_L}\right)^n \frac{1}{2\pi} \int_o^{2\pi} \left(\frac{r}{a}\right)^n P_n(\cos\psi)\, d\lambda \tag{29}$$

where, λ = mean longitude $\equiv M + \omega + \Omega$

Now $(\cos\psi)$ can be rewritten in terms of the true longitude, $L(=f + \omega + \Omega$, f = true anomaly), as

$$\cos\psi = \alpha_1 \cos L + \beta_1 \sin L \tag{30}$$

where, α_1, β_1 $\equiv$ direction cosines of the moon relative to the equinoctial orbital frame.

It is well known that powers and products of trigonmetric functions in an argument can be rewritten as the same functions containing multiples of that argument. Thus the Legendre polynomials can be expressed in terms of even multiples of L if n is even and odd multiples if n is odd. The first two are[12]:

$$P_2(\cos\psi) = \frac{1}{2}\left(\frac{3}{2} S_2 + \frac{3}{2} S_3 \cos 2L + 3S_1 \sin 2L - 1\right) \tag{31}$$

$$P_3(\cos\Psi) = \frac{1}{2}\left(\frac{5}{4}\alpha_1 S_4 \cos 3L - \frac{5}{4}\beta_1 S_5 \sin 3L + 3\alpha_1\left(\frac{5}{4} S_2 - 1\right)\cos L + 3\beta_1\left(\frac{5}{4} S_2 - 1\right)\sin L\right) \tag{32}$$

where

S_Λ $\equiv$ functions of α_1 and β_1

In general an even order Legendre function will contain the arguments nL, (n-2)L ... 0L while an odd order function will contain nL, (n-2)L ... 1L. This transformation allows (29) to be rewritten in terms of sin NL and cos NL, which is especially convenient since the averaging integrals can now be solved via the zeroth order, modified Hansen coefficient[10]

$$Y_0^{n,m} = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{r}{a}\right)^n \exp(jmL)\, d\lambda \qquad (33)$$

A list of the first few integrals is[12]

$$P_2(\cos\psi): \quad \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{r}{a}\right)^2 d\lambda = 1 + \frac{3}{2}(h^2 + k^2) \qquad (34)$$

$$\frac{1}{2\pi}\int_0^{2\pi} \left(\frac{r}{a}\right)^2 \cos 2L d\lambda = \frac{5}{2}(k^2 - h^2)$$

$$\frac{1}{2\pi}\int_0^{2\pi} \left(\frac{r}{a}\right)^2 \sin 2L d\lambda = 5hk$$

$$P_3(\cos\psi): \quad \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{r}{a}\right)^3 \cos 3L d\lambda = \frac{35}{8} k(3h^2 - k^2) \qquad (35)$$

$$\frac{1}{2\pi}\int_0^{2\pi} \left(\frac{r}{a}\right)^3 \sin 3L d\lambda = \frac{35}{8} h(h^2 - 3k^2)$$

$$\frac{1}{2\pi}\int_0^{2\pi} \left(\frac{r}{a}\right)^3 \cos L d\lambda = -\frac{5}{2}k\left[\frac{3}{4}(h^2 + k^2) + 1\right]$$

$$\frac{1}{2\pi}\int_0^{2\pi} \left(\frac{r}{a}\right)^3 \sin L d\lambda = -\frac{5}{2}h\left[\frac{3}{4}(h^2 + k^2) + 1\right]$$

Generalizing one finds that for n even in eq. (29) even powers of the eccentricity result and for n odd, odd powers result.

Continuing, one notices that for $h = k \approx 0$, as in the GPS case, the dominant contributions to the VOP equations for h and k are

$$\frac{dh}{dt} \approx \frac{1}{na^2} \frac{\partial F}{\partial k} \tag{36}$$

$$\frac{dk}{dt} \approx - \frac{1}{na^2} \frac{\partial F}{\partial h} \tag{37}$$

After differentiation, terms in the lunar potential for which powers of h and k are even will yield at least first powers of the eccentricity. On the other hand, terms in the potential which contained odd powers may yield 0^{th} power contributions to the h and k element rates. Following this reasoning for very low eccentricity

$$\left.\frac{dk}{dt}\right|_{P_2} = \left.\frac{dh}{dt}\right|_{P_2} \approx 0 \tag{38}$$

$$\left.\frac{dh}{dt}\right|_{P_3} \approx \frac{\mu_L}{R_L} \left(\frac{a}{R_L}\right)^3 \left(-\frac{5}{2}\right)\left(\frac{1}{na^2}\right) \tag{39}$$

$$\left.\frac{dk}{dt}\right|_{P_3} \approx \frac{\mu_L}{R_L} \left(\frac{a}{R_L}\right)^3 \left(-\frac{5}{2}\right)\left(-\frac{1}{na^2}\right) \tag{40}$$

The result is consistent with the inability to model a medium period oscillation in h and k by the first term in the lunar potential. Thus it has been demonstrated that the second term $\left(\frac{a}{\bar{R}_L}\right)^3$ dominates due to the low GPS eccentricity. All higher order terms contribute negligible effects since they contain correspondingly higher powers of the ratio $\left(\frac{a}{\bar{R}_L}\right)$.

This type of analysis can be extended to the other elements treated in the numerical study. Eliminating all terms for which first or higher powers of h and k will be present, the rate $d\lambda_o/dt$ can be stated as

$$\frac{d\,\lambda_o}{dt} \approx -\frac{2}{na}\frac{\partial F}{\partial a} \tag{41}$$

No differentiation with respect to h and k is indicated so that only terms in the lunar potential for which there are 0^{th} powers of the eccentricity will exert any appreciable influence. Here only $\left(\frac{a}{\bar{R}_L}\right)^2$ will have a significant effect on the mean rate for λ_o as shown in the numerical study.

The model for the semi-major axis shows that no lunar terms are required to achieve good agreement with an all perturbations run. The correct expression for this rate is

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial F}{\partial \lambda_o} \tag{42}$$

However, since there is no explicit dependence of the averaged lunar potential on λ_o for any power of $\left(\frac{a}{\bar{R}_L}\right)$ the rate is zero, as expected.

Thus analytical justification can be found for the selection of the reduced force models for GPS, largely on the basis of the extremely low nominal eccentricity.

Section II A New Method for Treating Resonant Tesserals

The inclusion of resonant tesserals in long-term orbit prediction has, until recently, presented a real problem. The efficient computation of tesseral resonance effects has been hindered by the absence of an analytical expression for the disturbing potential in non-singular elements. This necessitated the use of the Gaussian formulation of the VOP equations in conjunction with a numerical quadrature process. In the Gaussian formulation, the element rates are expressed in terms of the distrubing acceleration via[12]

$$\dot{a}_i = \frac{\partial a_i}{\partial \underline{\dot{x}}} \cdot \underline{Q} \qquad i = 1,..6 \tag{43}$$

where,

$\underline{Q}$ = tesseral disturbing acceleration

$\frac{\partial a_i}{\partial \underline{\dot{x}}}$ = partial of the i^{th} element with respect to velocity.

The disturbing acceleration is given by[12]

$$\underline{Q}(t) = \frac{\partial U}{\partial \underline{r}} \left(\frac{\partial r}{\partial \underline{r}}\right)^T + \frac{\partial U}{\partial \phi} \left(\frac{\partial \phi}{\partial \underline{r}}\right)^T + \frac{\partial U}{\partial \lambda} \left(\frac{\partial \lambda}{\partial \underline{r}}\right)^T \tag{44}$$

where, $\underline{r}$ = position vector of the satellite in the coordinate frame of the acceleration vector

μ = potential function described in equation (2)

The indicated partials are computed from[12]

$$\frac{\partial U}{\partial r} = -\frac{\mu}{r^2}\sum_{n=2}^{\infty} (n+1)\left(\frac{a}{r}\right)^n \sum_{m=0}^{\infty}\left[S_{nm}\sin m\lambda + C_{nm}\cos m\lambda\right] P_{nm}(\sin\phi) \tag{45}$$

$$\frac{\partial U}{\partial \lambda} = \frac{\mu}{r}\sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^n \sum_{m=0}^{\infty} mP_{nm}(\sin\phi)\left[S_{nm}\cos m\lambda - C_{nm}\sin m\lambda\right] \tag{46}$$

$$\frac{\partial U}{\partial \phi} = \frac{\mu}{r}\sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^n \sum_{m=0}^{\infty}\left[S_{nm}\sin m\lambda + C_{nm}\cos m\lambda\right]\frac{\partial P_{nm}(\sin\phi)}{\partial\phi} \tag{47}$$

There is considerable overhead entailed in the numerical implementation of this method. As coded in ESMAP, each evaluation of the Gaussian VOP equations [Eq. (43)] requires the following manipulations:

(a) The computation of the partials of the equinoctial elements with respect to velocity, $\frac{\partial a_i}{\partial \underline{\dot{x}}}$ [6].

(b) A transformation of spacecraft coordinates from the mean of 1950 to body (Earth) fixed coordinates[6].

(c) The computation of spherical coordinates, (r,λ,ϕ), from the rectangular, (x,y,z), coordinates of the body fixed system.[6]

(d) Calculation of the partials of the potential with respect to the coordinates (r,λ,ϕ) from (45) - (47)[6].

(e) A transformation of these partials in spherical coordinates back into rectangular body-fixed coordinates according to[6]

$$\frac{\partial U}{\partial x} = \frac{x}{r}\frac{\partial U}{\partial r} - \frac{xz}{r^2\sqrt{x^2+y^2}}\frac{\partial U}{\partial \phi} - \frac{y}{x^2+y^2}\frac{\partial U}{\partial \lambda} \quad (48)$$

$$\frac{\partial U}{\partial y} = \frac{y}{r}\frac{\partial U}{\partial r} - \frac{yz}{r^2\sqrt{x^2+y^2}}\frac{\partial U}{\partial \phi} + \frac{x}{x^2+y^2}\frac{\partial U}{\partial \lambda} \quad (49)$$

$$\frac{\partial U}{\partial z} = \frac{z}{r}\frac{\partial U}{\partial r} + \frac{\sqrt{x^2+y^2}}{r^2}\frac{\partial U}{\partial \phi} \quad (50)$$

(f) A transformation of acceleration vector components to the mean of 1950 coordinate system after which the element rates are computed from equation (43)[6].

This overhead is now multiplied since a numerical process is used to obtain the averaged element rates. Mathematically, this numerical process is specified by[6]

$$\dot{a}_\alpha = \frac{1}{2\pi N}\sum_{i=1}^{n}\int_{F_o-N\pi+\frac{(i-1)2\pi N}{n}}^{F_o-N\pi+\frac{2\pi i N}{n}} \dot{a}_\alpha(F)\, dF \quad (51)$$

where N = number of orbits to be averaged over

F = eccentric longitude $\equiv E + \Omega + \omega$

E = eccentric anomaly

$\dot{a}_\alpha(F)$ = high precision element rate computed using Eq. (43)

n = number of quadratures used

The procedure used to evaluate this integral is to fit the integrand in Eq. (51) to an orthogonal polynomial in the eccentric longitude over N/n orbits. The integral of the orthogonal polynomial can be evaluated analytically. This method of computing integrals is known as numerical quadrature and is required when the integrand does not exist as a tractable analytic function of the integration variable. Considerable overhead is incurred, the extent of which is determined by the highest specified power of the interpolating polynomial. As shown in Table 5, the computation time entailed in computing the averaged orbital element rates is greatly increased through the inclusion of the numerical quadrature for resonant tesseral harmonics.

Table 5 - Computational Cost of Numerical Averaging

Test Case	Time, CPU Centi-seconds	( ) - (1)
(1) Reference All zonals, All luni-solar terms, no tesserals	314	0
(2) 12^{th} order quadrature, 2 orbits tesserals included	2020	1706
(3) 24^{th} order quadrature, 2 orbits tesserals included	3735	3421

The computation time is greatly increased when the tesseral harmonics are added to the perturbation field and goes up in direct relation to the quadrature order. If one subtracts the reference run from each of the other two assuming that what is left can be attributed to computation of the tesserals, it is seen that the 24^{th} order quadrature case is almost exactly twice as costly to run as the 12^{th} order case. Clearly, elimination of the averaging quadrature would greatly facilitate the rapid calculation of the tesseral resonance contributions to the element histories.

As mentioned in Section I, a way to circumvent this problem classically has been available due to Kaula for several years (1). This contribution has seen widespread use, but is of limited utility in the study of low eccentricity GPS type orbits.

However, recent results also stated in the first section, now allow for the construction of analytically averaged, explicit VOP equations in non-singular elements. The low eccentricity of the GPS orbit facilitates the rather radical truncation of the final expressions with respect to h and k to yield Variation of Parameters equations with terms containing powers of the eccentricity no greater than one. This form of analytical averaging removes short periodic terms and resonant terms proportional to high powers of e. The rates for (2,2), (3,2), (4,2) and (4,4) can be found in Appendix C along with the computerized algebra involved in this derivation.

Using the debug option of ESMAP[6] the element rates generated by the numerical averaged orbit prediction were compared with those produced by the new explicit formulae. The test cases run were

Case 1	Case 2
$\left.\begin{array}{l} h = 0 \\ k = 0 \end{array}\right\} e = 0$	$\left.\begin{array}{l} h = 0 \\ k = .01 \end{array}\right\} e = .01$
$p = 0$	$p = 0$
$q = .618095$	$q = .618095$
$\lambda = 0$	$\lambda = 0$
$a = 26559.9$ km	$a = 26559.9$ km
$\theta = 1.73553625$ radians	$\theta = 1.73553625$ radians

The Greenwich hour angle is based on an epoch date of January 1, 1980. A matrix of the results is seen in Table 6. The mean longitude rates are not included in this table. The difference of several orders of magnitude between the mean motion and the contributions to the mean longitude rate due to perturbations limited the utility of this comparison. Agreement is generally quite close,with the rates due to (3,2) dominating where expected. It will be noticed in the zero eccentricity case, that the ESMAP runs produce small non-zero rates when zero is predicted by the explicit formulation. Inspection of the expressions in Section 1 and Appendix C will verify that this discrepancy is not due to truncation on the eccentricity and that the prediction of zero is indeed correct. Rather,the difference is taken to be due largely to quadrature noise involved in the ESMAP numerical average. When the rates are extremely small as in the case of (4,2) and (4,4), it is expected that the apparent deviations can be attributed in greatest part to errors in the quadrature. Remaining discrepancies are probably due to uncertainty in the calculation of the correct Greenwich hour angle at epoch.

The quite close agreement of the explicit formulation of the averaged orbital element rates with the ESMAP numerical

A = ESMAP NUMERICAL AVERAGING*

B = EXPLICIT ANALYTICALLY AVERAGED THEORY

		(2,2)	(3,2)	(4,2)	(4,4)	
$\dot{a}$	A	$-.52665524\times10^{-10}$	$.32655463\times10^{-7}$	$.67463011\times10^{-11}$	$.64613481\times10^{-8}$	e=0
	B	0.00	$.32663116\times10^{-7}$	0.00	$.64369884\times10^{-8}$	
$\dot{h}$	A	$-.15124436\times10^{-10}$	$-.18350001\times10^{-15}$	$.77901248\times10^{-14}$	$.19364701\times10^{-14}$	
	B	$-.15105023\times10^{-10}$	0.00	$.78079905\times10^{-14}$	0.00	
$\dot{k}$	A	$.72077044\times10^{-11}$	$.16154185\times10^{-14}$	$.30750092\times10^{-12}$	$.66049711\times10^{-15}$	
	B	$.72294856\times10^{-11}$	0.00	$.30180676\times10^{-12}$	0.00	
$\dot{p}$	A	$-.36835041\times10^{-14}$	$-.86728999\times10^{-12}$	$-.29027992\times10^{-16}$	$.24183411\times10^{-13}$	
	B	0.00	$-.867415332\times10^{-12}$	0.00	$.235731083\times10^{-13}$	
$\dot{q}$	A	$-.19563118\times10^{-15}$	$-.74039139\times10^{-12}$	$.6533585\times10^{-16}$	$-.14570422\times10^{-12}$	
	B	0.00	$-.73767031\times10^{-12}$	0.00	$-.14537423\times10^{-12}$	
$\dot{a}$	A	$.13955590\times10^{-8}$	$.32674893\times10^{-7}$	$.55937390\times10^{-10}$	$.64651533\times10^{-8}$	e= .01
	B	$.15073953\times10^{-8}$	$.32663116\times10^{-7}$	$.35297236\times10^{-10}$	$.64369884\times10^{-8}$	
$\dot{h}$	A	$-.15135507\times10^{-10}$	$.746150032\times10^{-14}$	$.73980128\times10^{-14}$	$-.96582411\times10^{-14}$	
	B	$-.15105023\times10^{-10}$	$.82641452\times10^{-14}$	$.953004264\times10^{-14}$	$-.118882411\times10^{-13}$	
$\dot{k}$	A	$.72077418\times10^{-11}$	$.50586221\times10^{-13}$	$.31031528\times10^{-12}$	$.32055724\times10^{-14}$	
	B	$.72294856\times10^{-11}$	$.47746979\times10^{-13}$	$.300775915\times10^{-12}$	$.24697801\times10^{-15}$	
$\dot{p}$	A	$-.28841727\times10^{-12}$	$-.86746704\times10^{-12}$	$-.21144907\times10^{-14}$	$.24177542\times10^{-13}$	
	B	$-.28491258\times10^{-12}$	$-.867415332\times10^{-12}$	$-.21239864\times10^{-14}$	$.235731083\times10^{-13}$	
$\dot{q}$	A	$-.58693955\times10^{-13}$	$-.74096397\times10^{-12}$	$-.16580821\times10^{-14}$	$-.14576559\times10^{-12}$	
	B	$-.59016852\times10^{-13}$	$-.73745325\times10^{-12}$	$-.18361513\times10^{-14}$	$-.145361094\times10^{-12}$	

Table 6. Comparison of Analytical with Numerical Computation of Mean Element Rates.

*ESMAP Quadrature: 24th order Gaussian quadrature
2 orbital periods in averaging interval
1 quadrature partition per averaging interval

method, as seen in Table 6 , is very interesting. It represents the first numerical verification of the assumptions and consequent algebra involved in the construction of an analytically averaged potential for the tesseral harmonics in non-singular elements, and, as such, is quite valuable.

Several major advantages come from the existence of explicit, analytically averaged element rates. First, the computational overhead incurred by the use of the Gaussian formulation of VOP is completely eliminated. Since numerical quadrature was seen to be the primary determinant of CPU time it is reasonable to expect that the new expressions will run in a fraction of the time. Second, as will be shown in Section III, analytical expressions are more physically revealing and tend to suggest stationkeeping mechanisms that are not otherwise apparent.

Section III: Stationkeeping

The mission lifetime of the Global Positioning System will be determined by several factors, among them the mean time between failure of key satellite components and the onboard capability (reserve fuel supply) to maintain the mission constraints on the presence of natural perturbations. Ideally, the spacecraft reliability should be the primary determinant of the useful lifetime. Stationkeeping maneuvers should be minimized to circumvent the limited onboard fuel capacity of the satellite.

The results of Sections I and II will now be used as a basis to estimate the required time between stationkeeping maneuvers. Figure 2 shows that, for the nominal mission profile, the semi-major axis grows by approximately 670 meters in two hundred days. It will be recalled that the ±2 second bound on the orbital period equates to a ±822 meter change in the semi-major axis. Accordingly, it is seen, assuming containing linearity, that an orbital adjustment will be necessary in 245 days. Figure 3 shows a 1.6 degree regression in the geographic node crossing over two hundred days. Since the constraint on this parameter is ±2 degrees, stationkeeping would be required (based on the best case of linearity at the tail of the curve) every 250 days. The eccentricity grows from a nominal of zero to .000286 in the same span which is well below the upper bound of .015. Thus corrections will be necessary about every 8 months if no attempt is made to adjust the epoch orbital elements to provide better passive control.

One suggested solution to this problem is to target the orbital period for -1 second off nominal (-411 meters in a)[13]. The period would then be allowed to increase to its upper bound of +2 seconds (+822 meters). The need for stationkeeping the period would be reduced to every 368 days. Offsetting the semi-major axis to 26559.5 km to accomplish this also serves to stabilize the groundtrack as is evident in Figure 5. The eccentricity again presents no previous problem, so that this method will extend the interval between orbital adjustments to, approximately, once per year.

This scheme is dependent on the semi-major axis always increasing. The expressions for the averaged VOP equations, discussed in Section II and presented in Appendix C, provide a basis for testing the validity of this scheme. It was demonstrated earlier that the (3,2) harmonic is the dominant perturbation on the semi-major axis. Thus $(da/dt)_{3,2}$ from Appendix C will be taken to be a realistic analytical model and is given by

$$\left(\frac{da}{dt}\right)_{3,2} = -30((S_{3,2}q - C_{3,2}p)\ \sin(2\theta - \lambda)$$

$$+ (-C_{3,2}q - S_{3,2}p)\ \cos(2\theta - \lambda))\ \mu^{1/2}\ R_e^{\ 3}$$

$$X(2q^2 + 2p^2-1)\ /\ ((1 + p^2 + q^2)^3\ a^{7/2}) \qquad (52)$$

Eqn. (52) suggests that certain values of the trigonometric argument $(2\theta - \lambda)$ could actually cause the semi-major axis to decrease. Given that this argument would be expected to stay nearly constant for a resonant tesseral harmonic, the decrease in a would be due to the selection of a particular epoch value, λ_o, for the mean longitude. Figure 6 shows that for various epoch node placements, as well as different mean anomalies at epoch, the semi-major axis can actually decrease, rather than increase as previously predicted. Therefore, the magnitude and sign of the offset in semi-major axis will depend on individual consdieration of the motion of each satellite in the constellation. Even satellites moving within the same orbital plane would have to be controlled individually since differences in their mean anomalies

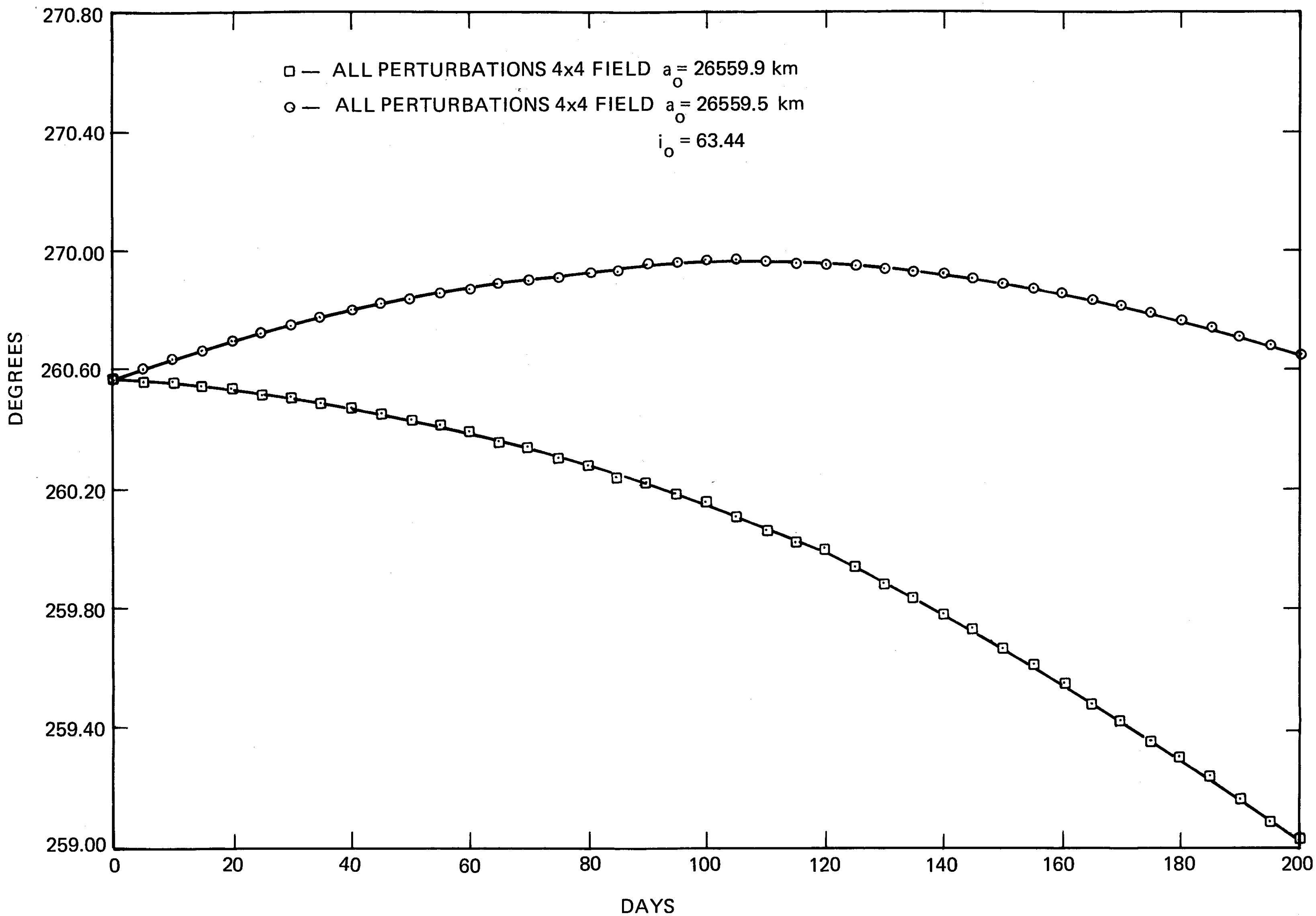


Figure 5. Geographic Longitude of Ascending Node (Deg) vs. Time with Semi-major Axis Bias

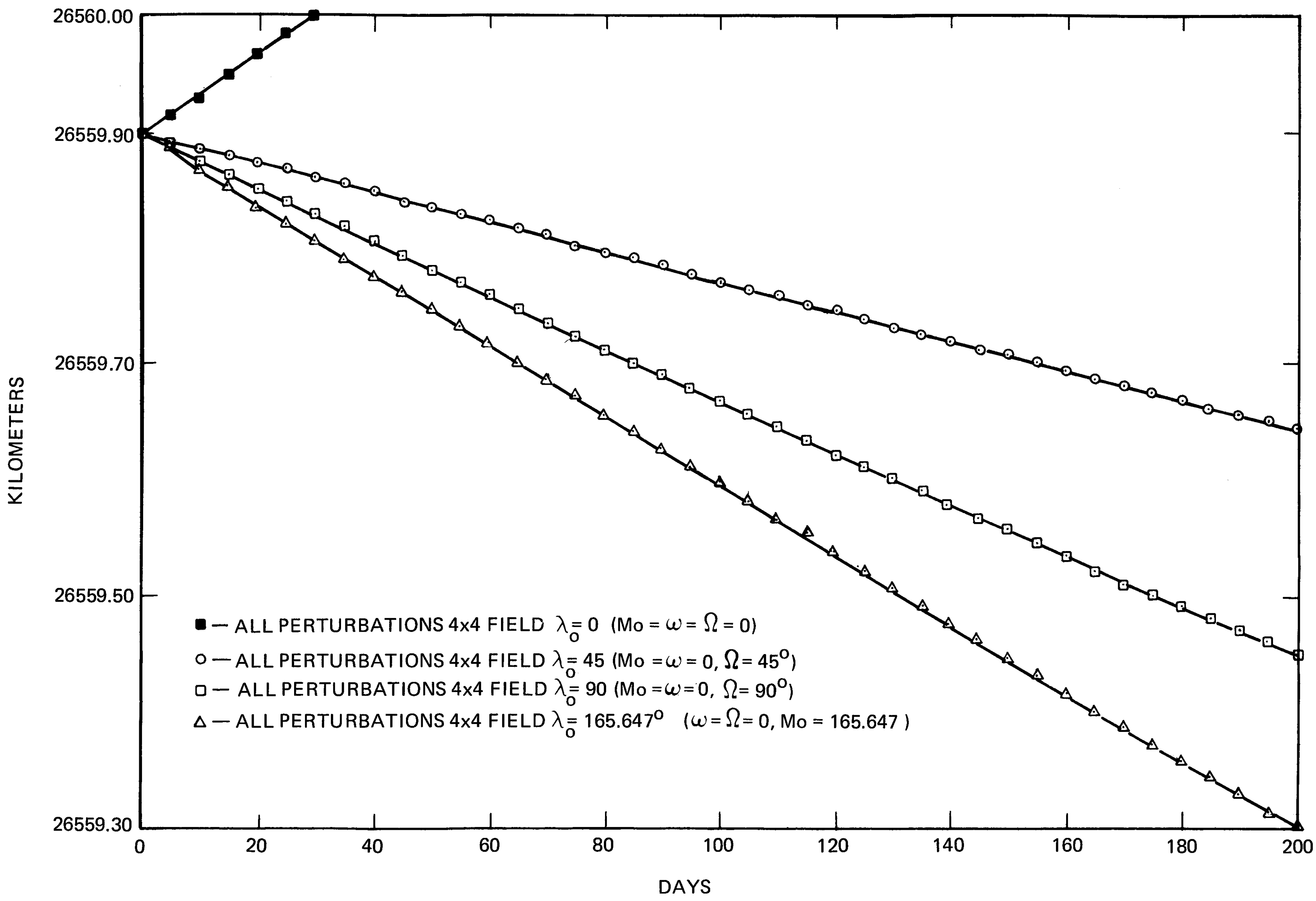


Figure 6. Semi-major Axis (km) vs. Time for Different Values of the Mean Longitude

at epoch might dictate that the period be targeted on the high side of nominal to achieve the desired interval between stationkeeping maneuvers.

Eqn. (52), however, presents a much more interesting physical result, one that promises to make stationkeeping almost entirely passive. It will be noticed that $\dot{a}_{(3,2)}$ contains a factor of the form $(2q^2 + 2p^2 - 1)$. Setting this to zero would yield an inclination for which the semi-major axis rate due to the dominant (3,2) harmonic would be zero.

$$2q^2 + 2p^2 - 1 = 0 \tag{53}$$

or

$$p^2 + q^2 = 1/2 \tag{54}$$

$$\tan^2\left(\frac{i}{2}\right) = 1/2 \quad , \tag{55}$$

which implies an inclination of $i = 70.52878°$. Figures 7 and 8 tell the story. The semi-major axis growth is greatly reduced, increasing only 100 meters in 200 days from a nominal value of 26559.9 km. The groundtrack also appears to stabilize, the geographic node regressing by 1.2 degree in the same span. The linear drift in the groundtrack is expected , since a repeating groundtrack semi-major axis has not been computed for the new inclination. Doing so, with the aid of Appendix B yields a = 26559.6465 km. Now the semi-major axis is seen to grow as before, but a really dramatic reduction in the node crossing drift has been achieved, amounting to only .16 degree in 200 days, a factor of ten improvement (Figures 7 and 8).

It is now proper to return to the stationkeeping scheme in which the semi-major axis is biased to improve orbital stability, keeping in mind that the magnitude and sign of the

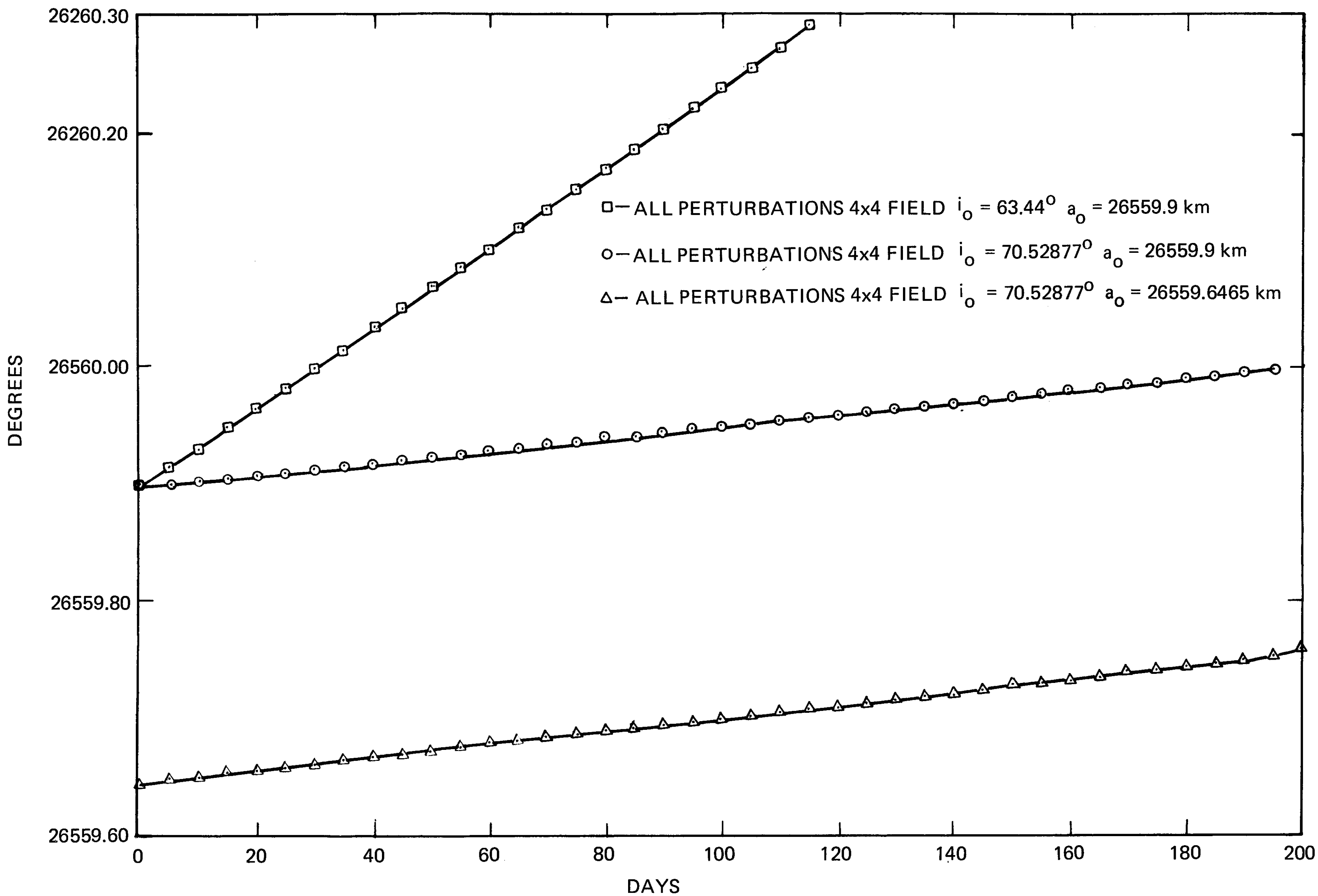


Figure 7. Semi-major Axis (km) vs. Time for Modified Mission Orbits

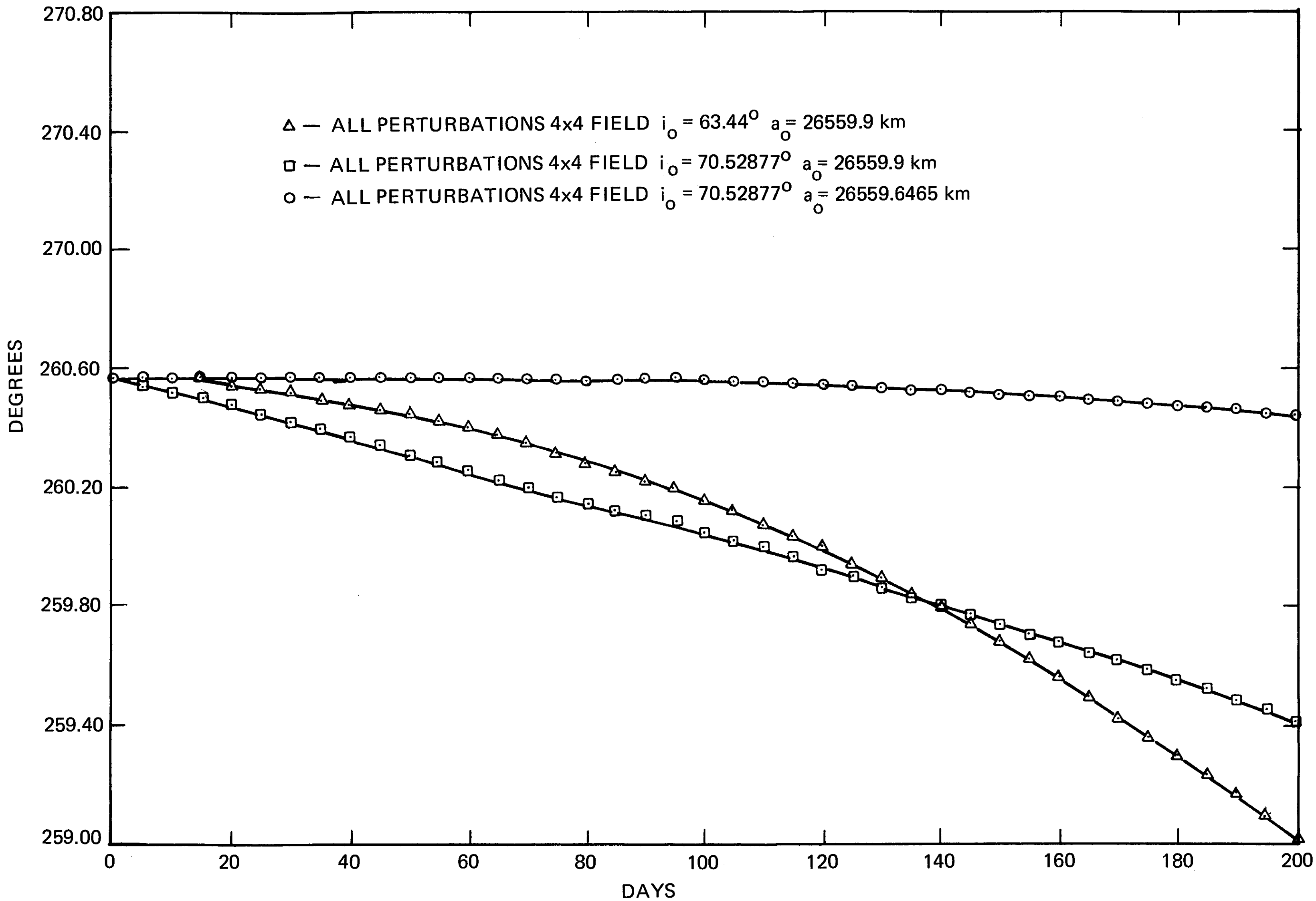


Figure 8. Geographic Longitude of Ascending Node (Deg) vs. Time for Modified Mission Orbits

bias required will be dependent on the initial selection of the mean longitude. Long arcs were run (1000 days) with all perturbations (odd order tesserals removed) for the following cases

a_o = 26559.5 km a_o = 26559.2465 km

i_o = 63.44° i_o = 70.52878°

λ_o = 0 λ_o = 0

Both semi-major axes represent a bias of -400 meters from nominal (-1 sec, period). Figure 9 shows that the semi-major axis, when $\mathring{a}_{(3,2)}$ has been zeroed, grows 853 meters in 1000 days, as opposed to 2.8 km in the i = 63.44° case. Likewise, Figure 10 demonstrates that the geographic node crossing remains extremely stable for the new inclination over this same 1000 day span, while in the other case, the drift is approximately 25°. Therefore it seems that the new inclination of 70.52878° and the corresponding nominal semi-major axis of 26559.6465 result in greatly reduced stationkeeping requirements for the GPS mission. An argument might be introduced that since the new inclination is not that required to zero the eccentricity growth due to J_3, the eccentricity will grow unacceptably fast, degrading the desirability of the new orbit. Figure 11 shows that this is not the case. The eccentricity at the end of 1000 days is .00175 for i = 70.52878°, well within acceptable limits. Thus, on the basis of an analytical model, great physical insight has been provided into the orbital dynamics of GPS and has suggested a new orbit, very close to the original, but more desirable since stationkeeping is required at much less frequent intervals. The result has arisen from the removal of the dominant (3,2) harmonic in its effect on the semi-major axis.

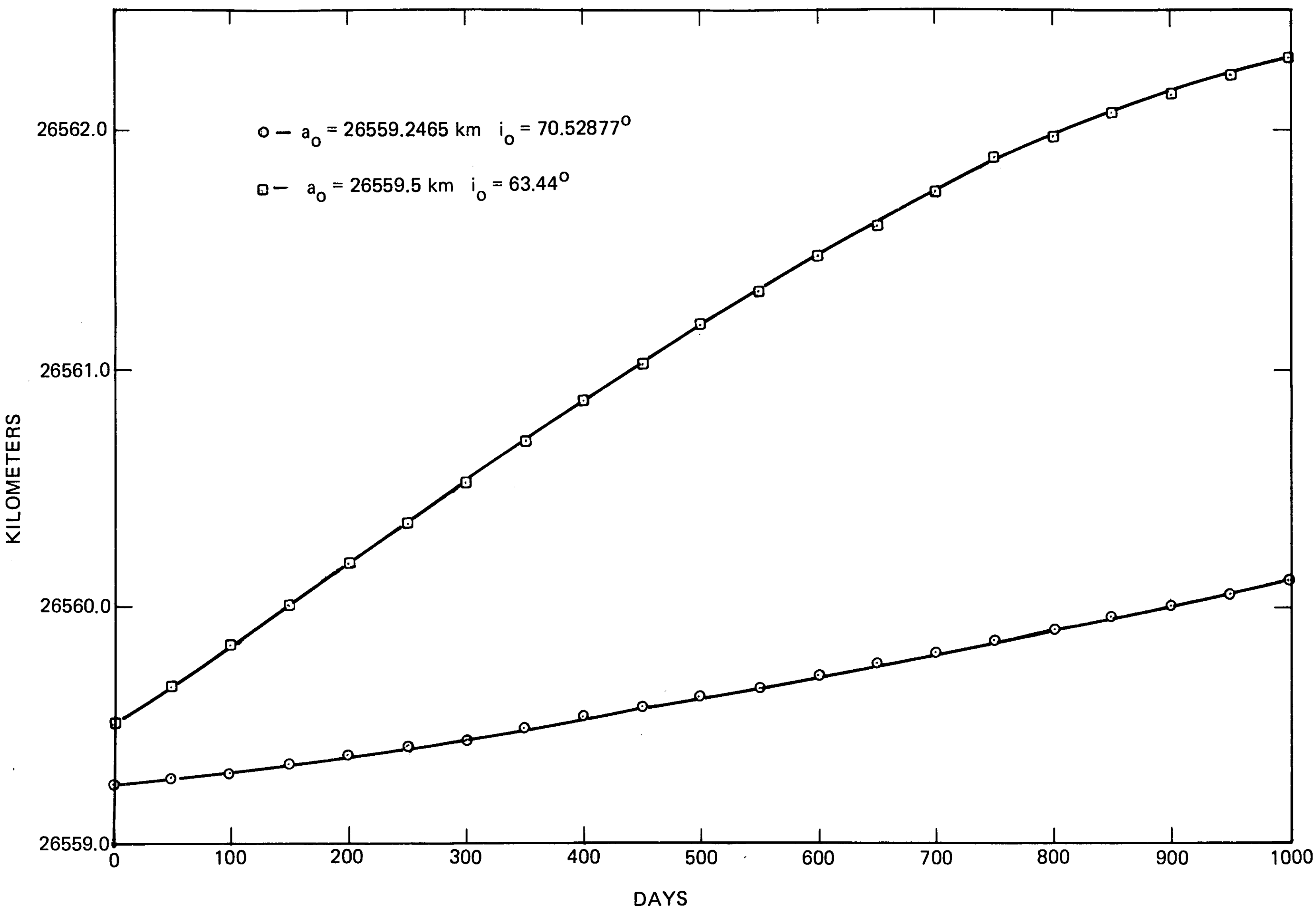


Figure 9. Long Term Semi-major Axis Growth (km) vs. Time (Days) for Modified Mission Orbits with Semi-major Axis Bias

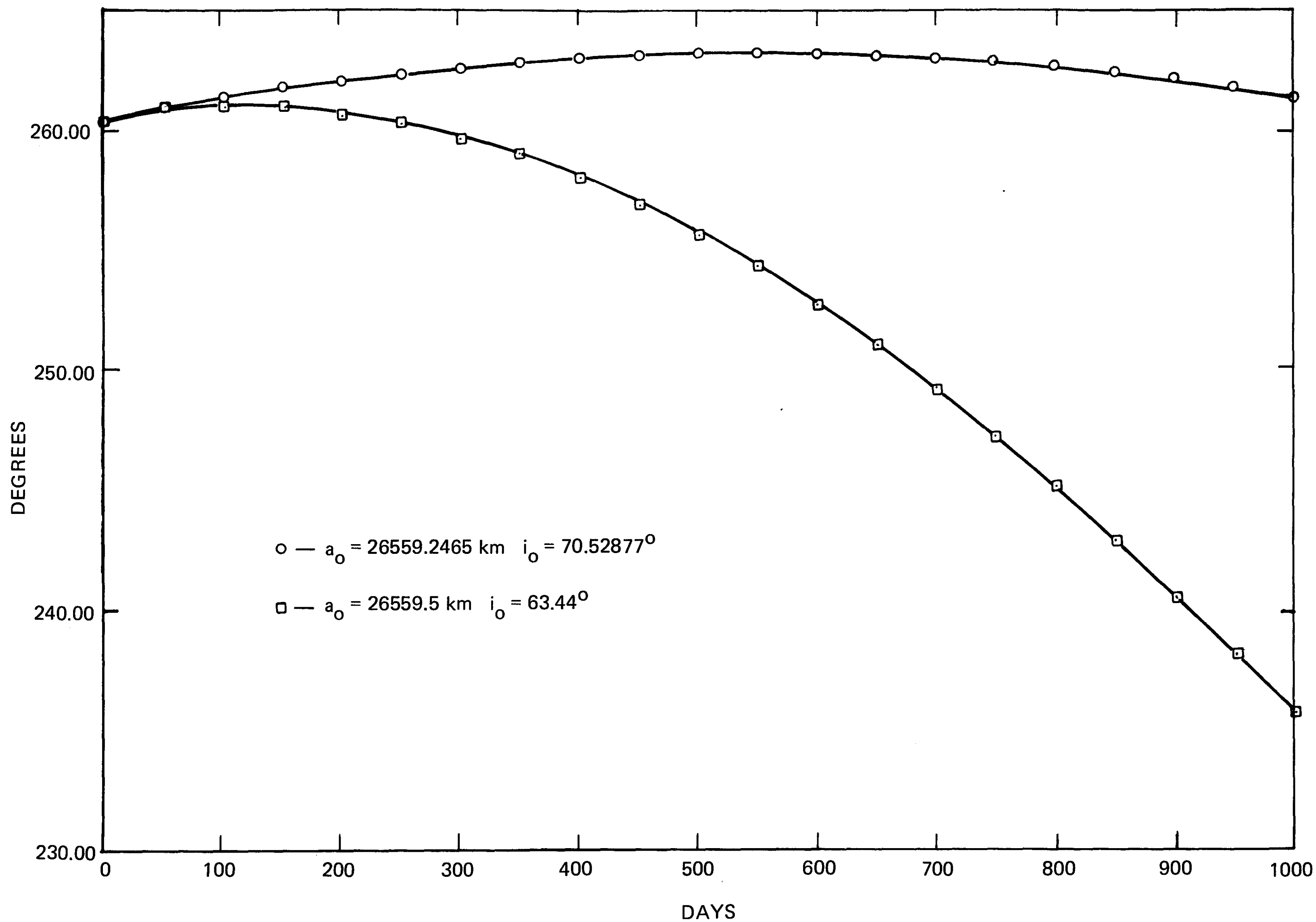


Figure 10. Long Term Evolution of Geographic Ascending Node Crossing (Deg) vs. Time (Days) for Modified Mission Orbits with Semi-major Axis Bias

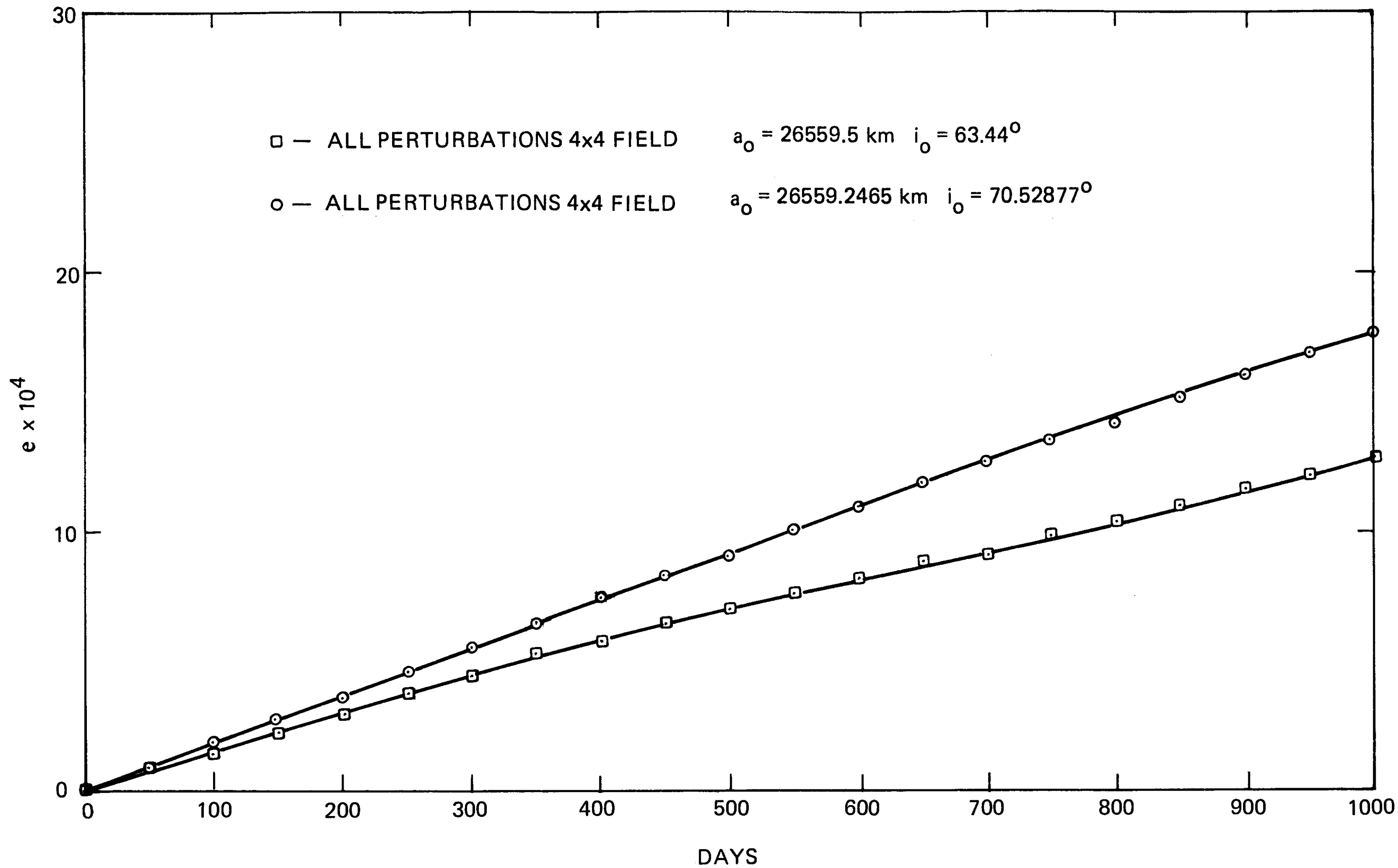


Figure 11. Long Term Eccentricity Growth vs. Time (Days) for Modified Mission Orbits with Semi-major Axis Bias

The preceding scheme is effective since the next contribution to the semi-major axis rate (and consequent groundtrack drift), the (4,4) harmonic term, is much smaller.

The foregoing analysis leads one to ask if the idea of an inclination that induces stability in the groundtrack for very low eccentricity, resonant orbits could be extended generally for any integer number of revolutions per day. The stability of the groundtrack is highly coupled to the constancy of the semi-major axis. Thus one would like to develop a procedure to yield that inclination, if it exists, for which the semi-major axis rate due to the dominant tesseral harmonic is zero.

For convenience the semi-major axis rate is restated as

$$\frac{da}{dt} = \frac{2}{na} \frac{\partial U}{\partial \lambda_o} \tag{56}$$

The dominant resonant tesseral harmonic, for a given commensurability, is the first (i.e., lowest degree and order) harmonic which contains terms with the zeroth powers of the eccentricity. For a commensurability of N revolutions per day, the lowest permissible order will be m = N. The degree is now prescribed. For N even the degree of the dominant harmonic must be n = N + 1 while for N odd, n = N. The two cases yield different results and will be treated separately.*

*The results are similar to those of R. R. Allan based on zeroing the inclination function $F_{\ell mp}(i)$ (14), (15). However, Allan does not exploit the result as a means of constructing passive stationkeeping orbits.

N even

In the a rate, the dominant resonant harmonic is (N + 1, N). The potential for this harmonic is expressed as the real part of

$$U^*_{N+1,N} = \frac{\mu}{a} \left(\frac{R_e}{a}\right)^{N+1} C^*_{N+1,N} \exp \left[j(\lambda - N\theta)\right]$$

$$x \sum_{s=-(N+1)}^{N+1} V^N_{N+1,s} \quad S^{(N,s)}_{2N+2} (p,q) \quad Y_1^{-N-2,s} (k,h) \tag{57}$$

Referring to the definition of the function $Y_1^{-N-2,s}(k,h)$ [eq. (5)] one sees that zeroth powers of h and k can appear in the semi-major axis rate for s - 1 = 0 or s = 1. The corresponding function $S^{N,1}_{2(N+1)}$ can now be written according to eqn. (4) as

$$S^{N,1}_{2(N+1)} (p,q) = \frac{(2N+1)!}{(N+2)!(N)!} (1+p^2+q^2)^{-N} (p-jq)^{N-1} P_1^{(N-1,N+1)} (\gamma)$$

where $\gamma = \cos i$

It is clear that the only way for (da/dt) to become zero aside from the trivial case of p = q = 0, is if the Jacobi polynomial $P_1^{(N-1,N+1)}$ (cos i) can be made to vanish. There is a convenient expression for the computation of the first degree Jacobi polynomial, of the form[16]

$$P_1^{(\alpha,\beta)}(x) = \frac{1}{2}\left[\alpha - \beta + (\alpha + \beta + 2)\ x\right] \qquad (58)$$

According to this formula,

$$P_1^{(N-1,N+1)}(\cos i) = -1 + (N+1)\cos i \qquad (59)$$

Now if this polynomial is set to zero one finds that

$$\cos i = \frac{1}{N+1} \qquad (60)$$

Solution of this equation for i, will yield an inclination for which $\mathring{a}_{(N+1,N)}$ is zero. As seen in the case of the GPS orbit this has the effect of greatly reducing the drift of the orbital period and the groundtrack. Table 7 shows the first few solutions to eqn. (60).

Table 7 - Stable Inclinations

Number of revolutions per day, N	Inclination, i (deg)
2	70.52878
4	78.46304
6	81.78679
8	83.62063
10	84.78409

N odd

In the semi-major axis rate, the dominant resonant harmonic, by virtue of containing 0^{th} power eccentricity terms, is (N, N). The S function corresponding to the zeroth power terms can be expressed as,

$$S_{2N}^{N,1}(p,q) = \frac{(2N)!}{(N+1)!(N-1)!}(1+p^2+q^2)^{-N}(p-jq)^{N-1}P_0^{(N-1,N+1)}(\cos i) \tag{61}$$

But the zeroth degree Jacobi polynomial is always equal to 1. Thus it may be seen that a locking inclination, for which the semi-major axis rate due to the dominant resonant harmonic is zero, does not exist for the odd number of revolutions per day case.

Section IV: Conclusions and Recommendations for Future Work

Reduced perturbation models have been constructed for the long-term analysis of GPS orbits. The compactness of these models, as displayed in the table on page 12, was a result of several factors. First, since the GPS orbit has a repeating groundtrack, it was found to be resonant, the consequence of which (for the 2:1 commensurability) was a justification for eliminating the odd order tesseral harmonics. Second, the very low nominal eccentricity of the GPS orbit was discovered to strongly decouple the resonant harmonics in such a way that not all of them had an appreciable effect on every mean element rate. A similar analysis, also based on the low eccentricity, led to a substantial truncation of the lunar potential. For the sake of computational efficiency, these reduced perturbation models could be used in lieu of full force representations since they produce accurate results over very long areas.

Explicit, analytically averaged VOP equations, truncated for low eccentricity, have been developed for the GPS resonant harmonics (Appendix C) employing the tesseral disturbing potential in non-singular equinoctial elements. Table 6 showed that for the test cases selected, these equations reproduced well the rates returned by ESMAP's numerical averaging. It is expected that despite the considerable number of terms to be evaluated in some of the explicit element rates, GPS long term orbit prediction may be performed in a fraction of the time required by programs employing numerical quadrature.

The avilability of explicit formulae for the VOP equations led to some rather interesting insights into the GPS staion-keeping problem. An inclination was found that zeroed the semi-major axis growth due to the dominant resonant harmonic (3,2). This had the effect of stabilizing the groundtrack dramatically. The inclination was $i \approx 70.53°$ in contrast to the nominal value of 63° cited in the literature. On the basis of the Section III results, it is recommended that the GPS orbits be retargeted to the new inclination. Doing so would significantly extend the arc over which the mission constraints could be maintained passively.

In the area of future work, several directions are foreseen. First, there has been as yet no good check on the computational savings gained by implementation of the reduced force models. In ESMAP, the undesired tesseral harmonic coefficients were set equal to zero to produce the required model. However, with the current program logic, the setup associated with the numerical average was still performed even though the ultimate contribution to the mean element rates was zero. As a consequence, the CPU time required to run the reduced models vs. that required for the full field was not appreciably different. It is therefore suggested that ESMAP be modified to accomodate the reduced models directly so that the extent of the expected advantages may be assessed.

It is also proposed to implement a software package in ESMAP which accesses the explicity analytically averaged VOP equations in Appendix C to compute the element rates due to GPS resonant tesserals. The package would be key in evaluating the accuracy and numerical efficiency of these new expressions. This would be similar to a module already incorporated for the long-term prediction of orbits perturbed by a third body.

For orbit prediction programs designed to model a wider range of orbital conditions, different explicit averaging theories would be required for each commensurability. This presents a non-trivial problem in analysis and software development. More desirably, one would like to construct a recursive theory which allows for the prediction of a general orbit without the need for extensive reprogramming. Accordingly, the explicit theory, developed for the GPS orbit, has very restricted applications in a general program. However, the results of Section II, notably Table 6, provide several excellent test cases for verifying a recursive scheme.

Additional work is also possible in the area of passive stationkeeping for the GPS constellation. As demonstrated in Section III, the drift in the geographic node crossing is dependent on the constancy of the semi-major axis. After having nulled the semi-major axis rate due to the dominant harmonic (3,2), the controlling harmonic becomes (4,4). Study of $(da/dt)_{4,4}$ in Appendix C suggests that added stability may be induced, by selecting mean longitudes at epoch for which the semi-major axis rate is zero. Investigaton of this method of extending passive stationkeeping is warranted. It may be possible to design other satellite constellation orbits in which the choice of inclination and longitude combine to null the orbital drift induced by tesseral resonance. Extension of the ESMAP gravity potential from 4X4 to 8X8 would facilitate such studies.

The results of this thesis also have several potential applications in the data support for GPS. One application might be a back-up cabability in which polynomial approximation of mean elements and analytical formulations for the short periodic corrections (which can be developed using the methods of this thesis) provide a nearly precise navigation ephemeris without frequent communication from the Master Control Station.

Appendix A Variation of Parameters: Orbital Elements

Classical Formulation

Variation of Parameters is a well known technique for solving differential equations of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y = f(x) \tag{A-1}$$

It can be implemented if a set of linearly independent solutions is known to the homogeneous case such that

$$y = Ay_1(x) + By_2(x) \tag{A-2}$$

where A and B are integration constants that uniquely determine the particular solution.

Variation of parameters contends that a particular solution to the inhomogeneous case can be found by assuming that A and B are actually functions of the independent variable. A knowledge of the exact functional dependence of A and B on x then uniquely defines y for all x when the system is forced by f(x).

As an example consider the forced spring mass problem with no damping

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = F \sin \omega t \tag{A-3}$$

The homogeneous solution is given by

$$x = A \sin \omega_n t \tag{A-4}$$

where

$$\omega_n = \sqrt{\frac{k}{m}}$$

Now assuming that A and B are actually functions of time and differentiating

$$\dot{x} = \overbrace{\dot{A}(t)\sin\omega_n t + \dot{B}\cos\omega_n t}^{(a)}$$

$$+ \underbrace{A\omega_n \cos\omega_n t - B\omega_n \sin\omega_n t}_{(b)} \qquad \text{(A-5)}$$

Part (a) is set to zero so that the expression for the actual velocity matches that of the unforced case.

Differentiating again and substituting into the original equation one gets

$$\dot{A}\,\omega_n \cos\omega_n t - \dot{B}\,\omega_n \sin\omega_n t \qquad \text{(A-6)}$$

$$\left[-A\omega_n^{\,2}\sin\omega_n t - B\omega_n^{\,2}\cos\omega_n t + A\omega_n \sin\omega_n t + B\omega_n \cos\omega_n t\right]$$

$$= F\sin\omega t$$

The part in brackets solves the homogeneous case and is therefore zero. Thus one is left with the two relations

$$\dot{A}\sin\omega_n t + \dot{B}\cos\omega_n t = 0$$

$$\text{(A-7)}$$

$$\dot{A}\,\omega_n\cos\omega_n t - \dot{B}\,\omega_n\sin\omega_n t = F\sin\omega t$$

Simultaneous solution for $\dot{A}$ and $\dot{B}$ yields

$$\dot{A} = -\frac{F}{2\omega_n}\sin\omega t\cos\omega_n t$$

$$\text{(A-8)}$$

$$\dot{B} = \frac{F}{2\omega_n}\sin\omega t\sin\omega_n t$$

Integration of these "constants" now serves to define the forced solution uniquely for all time. Notice that this method has obviated the need to actually integrate the forced differential equation itself. The position and velocity of the mass are immediately derivable from the instantaneous knowledge of A and B and a knowledge of how x and $\dot{x}$ depend on the integration constants in the homogeneous case.

Variation of parameters offers an extremely powerful means of predicting the position and velocity of an orbiting satellite. The general vector equation of motion is

$$\frac{d^2\underline{r}}{dt^2} + \frac{\mu}{r^3}\underline{r} = \underline{Q} \qquad \text{(A-9)}$$

where,

$\underline{r}$ = position vector of satellite in some coordinate system

μ = Gravitational constant = $G(m_s + m_e)$

m_s = mass of satellite

m_e = mass of earth

G = Universal gravitation constant

$\underline{Q}$ = an acceleration vector consisting of all perturbing effects exclusive of the inverse square gravitational acceleration. This includes solar radiation pressure, zonal and tesseral harmonics in the earth's geopotential, third body forces and atmospheric drag

Direct integration of the equation (called a Cowell procedure) is in general difficult and time consuming. However, if the perturbing acceleration is much smaller than the central force term, then variation of parameters can be utilized with great computational advantage.

Proceding as before we notice that the homogeneous two body problem

$$\frac{d^2\underline{r}}{dt^2} + \frac{\mu}{r^3}\underline{r} = 0 \qquad \text{(A-10)}$$

has six integration constants which serve to define a particular solution (orbit). One set of these could be the six components of initial position and velocity. A more commonly used set are the so-called classical orbital elements:

a = semi-major axis of orbit

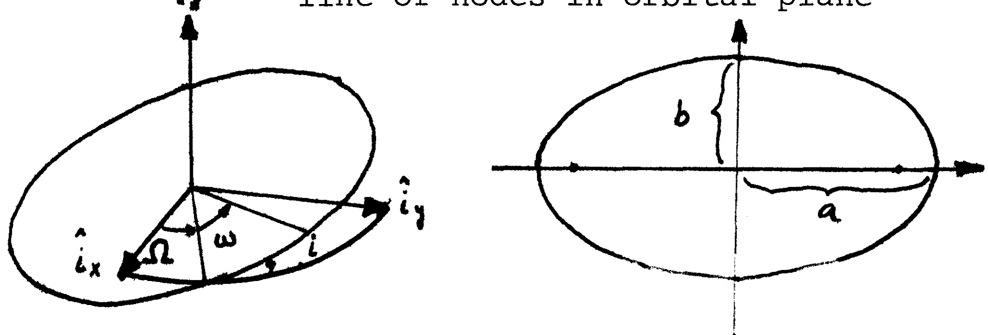
e = eccentricity of orbit

M_o = mean anomaly at epoch

i = inclination of the orbit with respect to the reference plane (ecliptic or equatorial)

Ω = longitude of the ascending node measured from the vernal equinox

ω = argument of pericenter measured from the line of nodes in orbital plane



$$e = \sqrt{\frac{a^2+b^2}{a^2}}$$

Figure A-1 - Definition of Classical Orbital Elements

These constants of two body motion define the orientation of the plane and the shape of the conic orbit.

In the spirit of variation of parameters the perturbed orbit equation can be solved by assuming the integration constants to be functions of time for which first order differential equations can be formulated. Given an epoch state, these equations can be integrated to yield parameters for any time which can then be converted, using the conic relations, to perturbed position and velocity. Essentially this is stating that a set of osculating orbital elements can be determined for a conic orbit that passes through a given point on the perturbed trajectory. Accordingly the position and velocity determined for the osculating (conic) orbit at the point are identically equal to the same quantities on the perturbed path.

If the disturbing acceleration is small relative to the central force term then the variation of parameters equations can be integrated with tremendous computational savings over Cowell routines. This is because the osculating element rates will be slow (with the exception of $\dot{M}$), allowing larger time steps in predicting the orbit. The general variation of parameters (VOP) equations are given by[12]

$$\dot{a}_i = - \sum_{j=1}^{6} (a_i, a_j) \frac{\partial R}{\partial a_j} + \frac{\partial a_i}{\partial \underline{\dot{x}}} \underline{Q} \qquad \text{(A-11)}$$

where,

a_i = osculating elements $i = 1, \ldots 6$

(a_i, a_j) = Poisson brackets of the elements

R = all perturbations for which a potential can be written (conservative forces)

$\underline{Q}$ = perturbing acceleration due to non-conservative forces (solar radiation pressure, drag)

The VOP equations in classical elements written for concervative forces only are:[4]

$$\frac{da}{dt} = \frac{2}{na} \frac{\partial R}{\partial M} \tag{A-12}$$

$$\frac{de}{dt} = \frac{1-e^2}{na^2 e} \frac{\partial R}{\partial M} - \frac{(1-e^2)^{1/2}}{na^2 e} \frac{\partial R}{\partial \omega} \tag{A-13}$$

$$\frac{d\omega}{dt} = - \frac{\cos i}{na^2 (1-e^2)^{1/2} \sin i} \frac{\partial R}{\partial i} + \frac{(1-e^2)^{1/2}}{na^2 e} \frac{\partial R}{\partial e} \tag{A-14}$$

$$\frac{di}{dt} = \frac{\cos i}{na^2 (1-e^2)^{1/2} \sin i} \frac{\partial R}{\partial \omega} - \frac{1}{na^2 (1-e^2)^{1/2} \sin i} \frac{\partial R}{\partial \Omega} \tag{A-15}$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2 (1-e^2)^{1/2} \sin i} \frac{\partial R}{\partial i} \tag{A-16}$$

$$\frac{dM}{dt} = n - \frac{1-e^2}{na^2 e} \frac{\partial R}{\partial e} - \frac{2}{na} \frac{\partial R}{\partial a} \tag{A-17}$$

Equinoctial Formulation

Inspection of the classical VOP equations reveals that they are singular for $i = 0$ or $180°$ and may become so for $e = 0$. The result is that the equations are numerically and analytically intractable in a region about these singularities causing rather wild behavior in the elements. Since this condition is not really a physical manifestation of the orbit it can be eliminated by a more judicious choice of orbital elements. One such set is the equinoctial elements defined as[12]

$$
\begin{aligned}
\lambda_0 &= \text{mean longitude} = M_0 + \omega + \Omega \\
a &= a \\
h &= e \sin(\omega + \Omega) \\
k &= e \cos(\omega + \Omega) \\
p &= \tan(i/2) \sin \Omega \\
q &= \tan(i/2) \cos \Omega
\end{aligned}
\qquad \text{(A-18)}
$$

In the orbital frame h and k are the components of the eccentricity vector. Similarly p and q represent the components of a vector pointing in the direction of the ascending node crossing having a magnitude of $\tan(i/2)$. VOP equations can be constructed from these elements which are defined everywhere except $i = 180°$*. Even when the classical elements exhibit non-physical oscillations the non-singular elements will be well behaved. Because of the low eccentricity of the proposed GPS orbit, a change to this new set is in order.

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* A retrograde set of elements can be used for $i = 180°$ cases

Appendix B Derivation of Repeating Groundtrack Equation

The following is the derivation of a repeating groundtrack equation. The solution of this equation yields a semi-major axis which has been corrected for nodal drift due to J_2. The expression will be presented in general form with the Global Positioning System orbit treated as a specific example.

From Gedeon[17] the defining equation for a repeating groundtrack is

$$s(\omega_e - \dot{\Omega}) = \dot{M} + \dot{\omega} \tag{B-1}$$

where, ω_e = Earth's rotational rate

s = an integer ratio which specifies the satellite-Earth commensurability

Thus

$$\dot{M} + \dot{\omega} + s\dot{\Omega} - s\,\omega_e = 0 \tag{B-2}$$

must be satisfied.

For J_2 perturbation $\dot{a} = \dot{e} = \dot{i} = 0$, so that the classical VOP equations reduce to the following set:

$$\frac{d\Omega}{dt} = \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i} \tag{B-3}$$

$$\frac{dM}{dt} = n - \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a} \tag{B-4}$$

$$\frac{d\omega}{dt} = -\frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i} + \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e} \tag{B-5}$$

The J_2 disturbing potential is[4]

$$R = \frac{3}{2} \mu J_2 \frac{R_e^2}{a^3} (1 - e^2)^{-3/2} \left\{ \frac{1}{3} - \frac{1}{2} \sin^2 i \right\} \qquad \text{(B-6)}$$

Substituting (B-6) successively into (B-3) (B-4) and (B-5)

$$\dot{\Omega} = - \frac{3}{2} \frac{J_2 n R_e^2 \cos i}{a^2 (1 - e^2)^2} \qquad \text{(B-7)}$$

$$\dot{M} = n \left[1 + \frac{9}{2} J_2 \frac{R_e^2}{a^2} (1 - e^2)^{-3/2} \left(\frac{1}{3} - \frac{1}{2} \sin^2 i \right) \right] \qquad \text{(B-8)}$$

$$\dot{\omega} = - \frac{3}{2} n \frac{R_e^2}{a^2} \frac{J_2}{(1-e^2)^2} \left[\frac{1}{2} - \frac{5}{2} \cos^2 i \right] \qquad \text{(B-9)}$$

Now substituting (B-7), (B-8), and (B-9) into (B-2) and simplifying

$$n \left[1 + \frac{3}{2} J_2 \frac{R_e^2}{a^2} \left\{ 3 (1 - e^2)^{-3/2} \left(\frac{1}{3} - \frac{1}{2} \sin^2 i \right) - \frac{1}{(1-e^2)^2} \left(\frac{1}{2} - \frac{5}{2} \cos^2 i \right) - \frac{s}{(1-e^2)^2} \cos i \right\} \right] - s\, \omega_e = 0 \qquad \text{(B-10)}$$

Define

$$\psi(s) \equiv 3 (1 - e^2)^{-3/2} \left(\frac{1}{3} - \frac{1}{2} \sin^2 i \right) - \frac{1}{(1-e^2)^2} \left(\frac{1}{2} - \frac{5}{2} \cos^2 i \right) - \frac{s}{(1-e^2)^2} \cos i \qquad \text{(B-11)}$$

so that (B-10) becomes

$$n \left[1 + \frac{3}{2} J_2 \frac{R_e^2}{a^2} \psi(s)\right] - s\,\omega_e = 0 \qquad \text{(B-12)}$$

Further defining

$$Q(s) \equiv \frac{3}{2} \frac{J_2}{\mu^{2/3}} R_e^2 \psi(s) \qquad \text{(B-13)}$$

and making the substitutions

$$a^2 = \left(\frac{\mu}{n^2}\right)^{2/3} \qquad \text{(B-14)}$$

$$\chi = n^{1/3} \qquad \text{(B-15)}$$

the final compact form can be expressed as,

$$\chi^3 + Q(s)\,\chi^7 - s\,\omega_e \qquad \text{(B-16)}.$$

The appropriate root may be extracted by Newton Raphson iteration with a bit of physical intuition.

Numerical Example: The GPS Orbit

The GPS orbit has the following properties:

$e \approx 0$

$i = 63.44°$

$s = 2$ (2:1 commensurability)

Also

$\omega_e = .729211585 \times 10^{-4}$ rad/sec

$\mu = 398600.8 \text{ km}^3/\text{sec}^2$

$R_e = 6378.145$ km

$J_2 = 1082.6517 \times 10^{-6}$

Using these values, the constants in (B-16) can be computed.

$$\psi(2) = -1.094375268 \quad \text{(B-17)}$$

$$Q(2) = -13.34870836 \quad \text{(B-18)}$$

$$2\,\omega_e = 1.458431170 \times 10^{-4} \quad \text{(B-19)}$$

Substitution of (B-18) and (B-19) into (B-16) yields, after simplification

$$\chi^7 - .0749129746\,\chi^3 + 1.09254818 \times 10^{-5} = 0 \quad \text{(B-20)}$$

The correct root is $\chi = .0526392092$

Combining Eqs. (B-14) and (B-15) gives,

$$a = \frac{\mu^{1/3}}{\chi^2} \quad \text{(B-21)}$$

from which the corresponding semi-major axis is found to be

$$a = 26559.9 \text{ km} \tag{B-22}$$

with an associated period of 11.966 hours. This value is in very close agreement with the actual GPS semi-major axis of 26560.123 km(P = 11.9661 hours) presented in the literature[3]. The computed value will be used for all GPS analysis in this thesis.

Appendix C: Explicit, Analytically Averaged Expressions for the VOP Equations in Non-Singular Elements.*

The following describes the algebra involved in the construction of the explicit, analytically averaged equations of motion for the GPS orbit as discussed in Section II. Final expressions for the potential and averaged element rates will be presented at the end of the Appendix. Parts of Section I will now be restated for convenience.

For the 2:1 commensurable GPS orbit, the resonant potential can be expressed as the real part of Equation (10)

$$U^*_{nm} = \frac{\mu}{a}\left(\frac{R_e}{a}\right)^n C^*_{nm} \exp\left[j\,\frac{m}{2}\,(\lambda - 2\theta)\right]$$

$$\times \sum_{s=-n}^{n} V^m_{n,s}\; S^{(m,s)}_{2n}\,(p,q)\; Y^{-n-1,s}_{\frac{m}{2}}\,(k,h)$$

where

$$V^m_{n,s} = \frac{(n-s)!}{(n-m)!}\; P_{n,s}(0)$$

$$V^m_{n,-s} = \frac{(n+s)!}{(n-m)!}\; P_{n,-s}(0)$$

*A directory for the algebraic results of this appendix is given on page 99 .

$$P_{n,s}(0) = \frac{d^s}{d\nu^s} P_n(\nu)\Big|_{\nu = 0}$$

$$P_{n,-s}(0) = (-1)^s \frac{(n - s)!}{(n + s)!} P_{n,s}(0)$$

$$S_{2n}^{(m,s)}(p,q) = \begin{cases} (1+p^2+q^2)^s (p-jq)^{m-s} P_{n+s}^{(m-s,-m-s)}(\gamma) & s \leq -m \\[2em] \dfrac{(n+m)!(n-m)!}{(n+s)!(n-s)!} (1+p^2+q^2)^{-m} (p-jq)^{m-s} P_{n-m}^{(m-s,s+m)}(\gamma) & -m \leq s \leq +m \\[2em] (-1)^{m-s} (1+p^2+q^2)^{-s} (p+jq)^{s-m} P_{n-s}^{(s-m,s+m)}(\gamma) & s \geq m \end{cases}$$

$$\gamma = \frac{1-p^2-q^2}{1+p^2+q^2} = \cos i$$

$$Y^{-n-1,s}_{\frac{m}{2}}(k,h) = (k+jh)^{s-\frac{m}{2}} \sum_{\sigma=0}^{\infty} X^{-n-1,-s}_{\sigma+s-\frac{m}{2},\sigma} (h^2+k^2)^{\sigma} \qquad \frac{m}{2} \leq s$$

$$Y^{-n-1,s}_{\frac{m}{2}}(k,h) = (k-jh)^{\frac{m}{2}-s} \sum_{\sigma=0}^{\infty} X^{-n-1,s}_{\sigma+\frac{m}{2}-s,\sigma} (h^2+k^2)^{\sigma} \qquad \frac{m}{2} \geq s$$

The functions $X^{n,-m}_{\sigma+m-t,\sigma}$ are constants called Newcomb operators and are computed according to the following rules(12)

$$X^{n,m}_{0,0} = 1 \qquad \text{(C-1)}$$

$$X^{n,m}_{1,0} = m - \frac{n}{2} \qquad \text{(C-2)}$$

$$4\rho \; X^{n,m}_{\rho,0} = 2(2m - n) \; X^{n,m+1}_{\rho-1,0} + (m - n) \; X^{n,m+2}_{\rho-2,0} \qquad \text{(C-3)}$$

$$\begin{aligned} 4\sigma \; X^{n,m}_{\rho,\sigma} &= -2(2m + n) \; X^{n,m-1}_{\rho,\sigma-1} - (m + n) \; X^{n,m-2}_{\rho,\sigma-2} \\ &- (\rho - 5\sigma + 4 + 4m + n) \; X^{n,m}_{\rho-1,\sigma-1} \\ &+2(\rho - \sigma + m) \sum_{\tau \geq 2} (-1)^{\tau} \binom{3/2}{\tau} X^{n,m}_{\rho-\tau,\sigma-\tau} \end{aligned} \qquad \text{(C-4)}$$

As an example of how the potential needed for computing the averaged element rates was constructed, the algebra for the harmonic (4,4) will be presented. The complex potential for (4,4) is given by

$$U^*_{44} = \frac{\mu}{a}\left(\frac{R_e}{a}\right)^4 C^*_{44} \exp\left[j2(\lambda - 2\theta)\right] \tag{C-5}$$

$$\times \sum_{s=0,2,4} V^4_{4,s}\; S_8^{(4,s)}(p,q)\; Y_2^{-5,s}(k,h)$$

Computation of the potential requires knowledge of the following functions as can be verified by reference to Section I

$$V^4_{4,0}\;,\; S_8^{(4,0)}\;,\; Y_2^{-5,0} \tag{C-6}$$

$$V^4_{4,2}\;,\; S_8^{(4,2)}\;,\; Y_2^{-5,2} \tag{C-7}$$

$$V^4_{4,4}\;,\; S_8^{(4,4)}\;,\; Y_2^{-5,4} \tag{C-8}$$

Using the definition of the V functions and recognizing that

$P_4(\nu) = \frac{1}{8}(35\nu^4 - 30\nu^2 + 3)$ one gets

$$V^4_{4,0} = \frac{4!}{0!}\frac{d^0}{d\nu^0} P_4(\nu)\Big|_0 = 9$$

$$V^4_{4,2} = \frac{2!}{0!}\frac{d^2}{d\nu^2} P_4(\nu)\Big|_0 = -15$$

$$V^4_{4,4} = \frac{0!}{0!}\frac{d^4}{d\nu^4} P_4(\nu)\Big|_0 = 105$$

The S functions are

$$S_8^{4,0} = \frac{8!0!}{4!4!} \frac{(p - jq)^4}{(1+p^2 + q^2)^4} \quad P_0^{(4,4)}(\gamma)$$

But since all 0^{th} degree Jacobi polynomials are equal to 1 this becomes

$$S_8^{4,0} = \frac{70(p - jq)^4}{(1 + p^2 + q^2)^4}$$

Continuing,

$$S_8^{(4,2)} = \frac{8!0!}{6!2!} \frac{(p-jq)^2}{(1 + p^2 + q^2)^4} \quad P_0^{(2,6)}(\gamma) = \frac{28(p - jq)^2}{(1 + p^2 + q^2)^4}$$

$$S_8^{(4,4)} = \frac{P_0^{(0,8)}(\gamma)}{(1 + p^2 + q^2)^4} = \frac{1}{(1 + p^2 + q^2)^4}$$

Last of all, the Y functions are expressible in terms of the Newcomb operators. For (4,4) one has

$$Y_2^{-5,0} = X_{2,0}^{-5,0} \quad (k - jh)^2$$

$$Y_2^{-5,2} = X_{0,0}^{-5,-2} + X_{1,1}^{-5,-2}(h^2 + k^2)$$

$$Y_2^{-5,4} = X_{2,0}^{-5,-4} \quad (k + jh)^2$$

The Newcomb operators are computed according to eqns. (C-1) - (C-4).

$$X_{0,0}^{-5,-2} = 1$$

$$8X_{2,0}^{-5,0} = 2(5)\ X_{1,0}^{-5,1} + 5\ X_{0,0}^{-5,2} = 10(1 + 5/2) + 5 = 40$$

$$\therefore\ X_{2,0}^{-5,0} = 5$$

$$8X_{2,0}^{-5,-4} = 2(-8+5)\ X_{1,0}^{-5,-3} + (-4+5)\ X_{0,0}^{-5,-2} = -6(-3+5/2) + 1 = 4$$

$$\therefore\ X_{2,0}^{-5,-4} = \frac{1}{2}$$

$$4X_{1,1}^{-5,-2} = -2(-4-5)\ X_{1,0}^{-5,-3} - (1-5+4-8-5)\ X_{0,0}^{-5,-2}$$

$$= 18(-3 + \frac{5}{2}) + 13 = 4$$

$$\therefore\ X_{1,1}^{-5,-2} = 1$$

Thus the final form of the Y functions for (4,4) is

$$Y_2^{-5,0} = 5(k - jh)^2$$

$$Y_2^{-5,2} = 1 + h^2 + k^2$$

$$Y_2^{-5,4} = \frac{1}{2}\ (k + jh)^2$$

The construction of the (4,4) potential from these blocks is not an algebraicly trivial matter. The terms must be multiplied out and the real part extracted. Computing by hand is extremely tedious and invites the near certainty of error. To avoid this, the algebra was performed using MACSYMA[13]. This is a symbolic manipulation program currently resident on several computers at MIT's Laboratory for the Computer Sciences. Its purpose is to manipulate strings of symbols, not necessarily numeric, according to the rules of algebra. What follows is a description of the MACSYMA steps required to produce the (4,4) potential and the related VOP equations, where MACSYMA uses the nomenclature

%I $\equiv$ j $\equiv$ $\sqrt{-1}$

%E $\equiv$ exp

L $\equiv$ λ $\equiv$ mean longitude

T $\equiv$ θ $\equiv$ Greenwich hour angle

^ $\equiv$ exponential

* $\equiv$ multiplication

The lines preceded by identifiers of the form (Cα), α an integer, are the command lines, typed in by the user. If a command line is ended with a semi-colon (;) the line immediately following has an identifier of the form (Dα). This is the display line returned by MACSYMA after having performed the indicated operation. If the C line has been terminated with a dollar sign ($), then the command will be executed and the result stored internally, the printout having been suppressed. This avoids the print of intermediate results, if desired. In this case, the next line following will be just a consecutively numbered C line.

In the example, (C1) is the complex potential U_{44}^{*}. FUNC represents the as yet unspecified summation

$$\sum_{s=0,2,4} V_{4,s}^{4} \; S_{8}^{(4,s)} \; (p,q) \; Y_{2}^{-5,s} \; (k,\ h) \qquad \text{(C-9)}$$

The result is seen in (D1).

```
(C1) MU/A*(RE/A)^4*(C[4,4]-%I*S[4,4])*EXP(2*%I*(L-2*T))*FUNC;

                                                4   2 %I (L - 2 T)
                 (C     - %I S    ) FUNC MU RE   %E
                   4, 4       4, 4
(D1)             ------------------------------------------------
                                        5
                                       A
```

Lines (C2), (C3) and (C4) along with their corresponding display lines represent the three terms of (C-9) constructed by multiplying together the individually computed V, S, and Y functions in (C-6), (C-7) and (C-8).

```
(C2) 9*(70*(P-%I*Q)^4/(1+P^2+Q^2)^4)*5*(K-%I*H)^2;

                                          2            4
                          3150 (K - %I H)   (P - %I Q)
(D2)                      ---------------------------
                                    2     2     4
                                  (Q  + P  + 1)

(C3) -15*(28*(P-%I*Q)^2/(1+P^2+Q^2)^4)*(1+H^2+K^2);

                                  2     2                   2
                          420 (K  + H  + 1) (P - %I Q)
(D3)                    - ----------------------------
                                    2     2     4
                                  (Q  + P  + 1)
```

```
(C4) 105*(1/(1+P^2+Q^2)^4)*(1/2)*(K+%I*H)^2;
```

$$\text{(D4)} \qquad \frac{105\ (K + \%I\ H)^2}{2\ (Q^2 + P^2 + 1)^4}$$

The three terms are then added to form FUNC in (C-5).

```
(C5) FUNC = D2+D3+D4;
```

$$\text{(D5) FUNC} = \frac{3150\ (K - \%I\ H)^2\ (P - \%I\ Q)^4}{(Q^2 + P^2 + 1)^4} - \frac{420\ (K^2 + H^2 + 1)\ (P - \%I\ Q)^2}{(Q^2 + P^2 + 1)^4} + \frac{105\ (K + \%I\ H)^2}{2\ (Q^2 + P^2 + 1)^4}$$

FUNC is then rationally substituted (all products expanded) into (D1) (in line (C6)) to yield the result, (D6).

```
(C6) RATSUBST(RHS(D5),FUNC,D1);
```

$$\text{(D6)} \ - \text{MU}\ (S_{4,\ 4}\ (\%I\ (6300\ K^2\ Q^4 + 50400\ H\ K\ P\ Q^3$$

$$+ 37800\ H^2\ P^2\ Q^2) + 12600\ H\ K\ Q^4 - 6300\ \%I\ H^2\ Q^4 - 25200\ K^2\ P\ Q^3$$

$$+ 25200\ H^2\ P\ Q^3 - \%I\ ((37800\ K^2\ P^2 - 840\ K^2 - 840\ H^2 - 840)\ Q^2$$

$$+ 50400\ H\ K\ P^3\ Q + 6300\ H^2\ P^4 + 105\ H^2) - 75600\ H\ K\ P^2\ Q^2$$

$$- ((1680 K^{2} + 1680 H^{2} + 1680) P - 25200 K^{2} P^{3}) Q - 25200 H^{2} P^{3} Q$$

$$+ \%I (6300 K^{2} P^{4} + (- 840 K^{2} - 840 H^{2} - 840) P^{2} + 105 K^{2})$$

$$+ 12600 H K P^{4} - 210 H K) + C_{4, 4}$$

$$(\%I (12600 H K Q^{4} + 25200 H^{2} P Q^{3}) - 6300 K^{2} Q^{4} + 6300 H^{2} Q^{4}$$

$$- \%I (25200 K^{2} P Q^{3} + 75600 H K P^{2} Q^{2} + 25200 H^{2} P Q^{3})$$

$$- 50400 H K P Q^{3} - (- 37800 K^{2} P^{2} + 840 K^{2} + 840 H^{2} + 840) Q^{2}$$

$$- 37800 H^{2} P^{2} Q^{2} + \%I ((25200 K^{2} P^{3}$$

$$+ (- 1680 K^{2} - 1680 H^{2} - 1680) P) Q + 12600 H K P^{4} - 210 H K)$$

$$+ 50400 H K P^{3} Q - 6300 K^{2} P^{4} + 6300 H^{2} P^{4}$$

$$+ (840 K^{2} + 840 H^{2} + 840) P^{2} - 105 K^{2} + 105 H^{2})) RE^{4}$$

$$\%E^{2 (\%I L - 2 \%I T)} / (A^{5} (2 Q^{8} + (8 P^{2} + 8) Q^{6}$$

$$+ (12 P^{4} + 24 P^{2} + 12) Q^{4} + (8 P^{6} + 24 P^{4} + 24 P^{2} + 8) Q^{2} + 2 P^{8}$$

$$+ 8 P^{6} + 12 P^{4} + 8 P^{2} + 2))$$

To get the (4,4) potential, the real part must be extracted. Invoking the REALPART command in MACSYMA, (C7), produces the desired expression, (D7).

```
(C7) REALPART(D6);

                4                    2  4           2  4                      3
(D7) - MU RE  ((C       (- 6300 K  Q  + 6300 H  Q  - 50400 H K P Q
                 4, 4

            2  2          2          2          2                2  2  2
 - (- 37800 K  P  + 840 K  + 840 H  + 840) Q  - 37800 H  P  Q

                3            2  4          2  4
 + 50400 H K P  Q - 6300 K  P  + 6300 H  P

       2          2           2          2          2
 + (840 K  + 840 H  + 840) P  - 105 K  + 105 H )

                          4           2    3           2    3
 + S      (12600 H K Q  - 25200 K  P Q  + 25200 H  P Q
    4, 4

                2  2            2           2                        2  3
 - 75600 H K P  Q  - ((1680 K  + 1680 H  + 1680) P - 25200 K  P )

          2  3                    4
 Q - 25200 H  P  Q + 12600 H K P  - 210 H K)) COS(2 (L - 2 T))

                  2  4            2  4                 3
 - (S      (6300 K  Q  - 6300 H  Q  + 50400 H K P Q
    4, 4

          2  2          2          2          2                2  2  2
 - (37800 K  P  - 840 K  - 840 H  - 840) Q  + 37800 H  P  Q

                3            2  4           2  4
 - 50400 H K P  Q + 6300 K  P  - 6300 H  P

          2          2           2          2          2
 + (- 840 K  - 840 H  - 840) P  + 105 K  - 105 H )

                          4           2    3           2    3
 + C      (12600 H K Q  - 25200 K  P Q  + 25200 H  P Q
    4, 4

                2  2              2  3
 - 75600 H K P  Q  + (25200 K  P

           2           2                         2  3
 + (- 1680 K  - 1680 H  - 1680) P) Q - 25200 H  P  Q
```

$$+ 12600\, H\, K\, P^{4} - 210\, H\, K))\, \mathrm{SIN}(2\, (L - 2\, T)))$$

$$/(A^{5}\, (2\, Q^{8} + (8\, P^{2} + 8)\, Q^{6} + (12\, P^{4} + 24\, P^{2} + 12)\, Q^{4}$$

$$+ (8\, P^{6} + 24\, P^{4} + 24\, P^{2} + 8)\, Q^{2} + 2\, P^{8} + 8\, P^{6} + 12\, P^{4} + 8\, P^{2}$$

$$+ 2))$$

To aid in visualization and in order to produce a more compact model, a factorization of (D7) is called for. The command FACTORSUM (line(C8)) had the effect of pulling out some common factors and factoring the denominator, while leaving each term multiplying a trigonometric function

(C8) FACTORSUM(D7);

$$\text{(D8)} \quad - 105\, MU\, RE^{4}\, (60\, S_{4,\,4}\, K^{2}\, Q^{4}\, \mathrm{SIN}(2\, (2\, T - L))$$

$$+ 120\, C_{4,\,4}\, H\, K\, Q^{4}\, \mathrm{SIN}(2\, (2\, T - L))$$

(this space purposely left blank. Expression continued on next page)

$$- 60 S_{4,4} H^2 Q^4 \sin(2(2T - L))$$

$$- 240 C_{4,4} K^2 P Q^3 \sin(2(2T - L))$$

$$+ 480 S_{4,4} H K P Q^3 \sin(2(2T - L))$$

$$+ 240 C_{4,4} H^2 P Q^3 \sin(2(2T - L))$$

$$- 360 S_{4,4} K^2 P^2 Q^2 \sin(2(2T - L))$$

$$- 720 C_{4,4} H K P^2 Q^2 \sin(2(2T - L))$$

$$+ 360 S_{4,4} H^2 P^2 Q^2 \sin(2(2T - L))$$

$$+ 8 S_{4,4} K^2 Q^2 \sin(2(2T - L))$$

$$+ 8 S_{4,4} H^2 Q^2 \sin(2(2T - L)) + 8 S_{4,4} Q^2 \sin(2(2T - L))$$

$$+ 240 C_{4,4} K^2 P^3 Q \sin(2(2T - L))$$

$$- 480 S_{4,4} H K P^3 Q \sin(2(2T - L))$$

$$- 240 C_{4,4} H^2 P^3 Q \sin(2(2T - L))$$

$$- 16 C_{4,4} K^2 P Q \sin(2(2T - L))$$

$$- 16 C_{4, 4} H^2 P Q \sin(2 (2 T - L))$$

$$- 16 C_{4, 4} P Q \sin(2 (2 T - L))$$

$$+ 60 S_{4, 4} K^2 P^4 \sin(2 (2 T - L))$$

$$+ 120 C_{4, 4} H K P^4 \sin(2 (2 T - L))$$

$$- 60 S_{4, 4} H^2 P^4 \sin(2 (2 T - L))$$

$$- 8 S_{4, 4} K^2 P^2 \sin(2 (2 T - L))$$

$$- 8 S_{4, 4} H^2 P^2 \sin(2 (2 T - L)) - 8 S_{4, 4} P^2 \sin(2 (2 T - L))$$

$$+ S_{4, 4} K^2 \sin(2 (2 T - L)) - 2 C_{4, 4} H K \sin(2 (2 T - L))$$

$$- S_{4, 4} H^2 \sin(2 (2 T - L)) - 60 C_{4, 4} K^2 Q^4 \cos(2 (2 T - L))$$

$$+ 120 S_{4, 4} H K Q^4 \cos(2 (2 T - L))$$

$$+ 60 C_{4, 4} H^2 Q^4 \cos(2 (2 T - L))$$

$$- 240 S_{4, 4} K^2 P Q^3 \cos(2 (2 T - L))$$

$$- 480 C_{4, 4} H K P Q^3 \cos(2 (2 T - L))$$

$$+ 240\, S_{4,4}\, H^2\, P\, Q^3 \cos(2\,(2\,T - L))$$

$$+ 360\, C_{4,4}\, K^2\, P^2\, Q^2 \cos(2\,(2\,T - L))$$

$$- 720\, S_{4,4}\, H\, K\, P^2\, Q^2 \cos(2\,(2\,T - L))$$

$$- 360\, C_{4,4}\, H^2\, P^2\, Q^2 \cos(2\,(2\,T - L))$$

$$- 8\, C_{4,4}\, K^2\, Q^2 \cos(2\,(2\,T - L))$$

$$- 8\, C_{4,4}\, H^2\, Q^2 \cos(2\,(2\,T - L)) - 8\, C_{4,4}\, Q^2 \cos(2\,(2\,T - L))$$

$$+ 240\, S_{4,4}\, K^2\, P^3\, Q \cos(2\,(2\,T - L))$$

$$+ 480\, C_{4,4}\, H\, K\, P^3\, Q \cos(2\,(2\,T - L))$$

$$- 240\, S_{4,4}\, H^2\, P^3\, Q \cos(2\,(2\,T - L))$$

$$- 16\, S_{4,4}\, K^2\, P\, Q \cos(2\,(2\,T - L))$$

$$- 16\, S_{4,4}\, H^2\, P\, Q \cos(2\,(2\,T - L))$$

$$- 16\, S_{4,4}\, P\, Q \cos(2\,(2\,T - L))$$

$$- 60\, C_{4,4}\, K^2\, P^4 \cos(2\,(2\,T - L))$$

```
                        4
+ 120 S      H K P  COS(2 (2 T - L))
       4, 4

                  2  4
+ 60 C      H  P  COS(2 (2 T - L))
      4, 4

                 2  2
+ 8 C      K  P  COS(2 (2 T - L))
     4, 4

                 2  2                             2
+ 8 C      H  P  COS(2 (2 T - L)) + 8 C      P  COS(2 (2 T - L))
     4, 4                                 4, 4

             2
- C       K  COS(2 (2 T - L)) - 2 S      H K COS(2 (2 T - L))
   4, 4                            4, 4

             2                            5   2    2     4
+ C       H  COS(2 (2 T - L)))/(2 A  (Q  + P  + 1) )
   4, 4
```

To solve this problem, sin(2(2T-L)) and cos(2(2T-L)) were declared the main factorization variables in (D8) via RATVARS, (C10). The resulting factorization, using RAT, yielded the full potential, $(U_{4,4})_{FULL}$ (the potential truncated to the third power of h and k), (D11).

```
(C9) RATFAC:TRUE;

(D9)                          TRUE

(C10) RATVARS(SIN(2*(2*T-L)),COS(2*(2*T-L)));

(D10)          [SIN(2 (2 T - L)), COS(2 (2 T - L))]

(C11) U[4,4][FULL] = RAT(D8);
```

$$(D11)/R/\ (U_{4,4})_{FULL} = -105\,(((60\,S_{4,4}\,K^{2} + 120\,C_{4,4}\,H\,K$$

$$-\,60\,S_{4,4}\,H^{2})\,Q^{4} + (-\,240\,C_{4,4}\,K^{2} + 480\,S_{4,4}\,H\,K$$

$$+\,240\,C_{4,4}\,H^{2})\,P\,Q^{3} + ((-\,360\,S_{4,4}\,K^{2} - 720\,C_{4,4}\,H\,K$$

(This space purposely left blank. Expression continued on next page)

$$+ 360 S_{4,4} H^{2}) P^{2} + 8 S_{4,4} K^{2} + 8 S_{4,4} H^{2} + 8 S_{4,4}) Q^{2}$$

$$+ ((240 C_{4,4} K^{2} - 480 S_{4,4} H K - 240 C_{4,4} H^{2}) P^{3}$$

$$+ (- 16 C_{4,4} K^{2} - 16 C_{4,4} H^{2} - 16 C_{4,4}) P) Q$$

$$+ (60 S_{4,4} K^{2} + 120 C_{4,4} H K - 60 S_{4,4} H^{2}) P^{4}$$

$$+ (- 8 S_{4,4} K^{2} - 8 S_{4,4} H^{2} - 8 S_{4,4}) P^{2} + S_{4,4} K^{2}$$

$$- 2 C_{4,4} H K - S_{4,4} H^{2}) \mathrm{SIN}(2 (2 T - L))$$

$$+ ((- 60 C_{4,4} K^{2} + 120 S_{4,4} H K + 60 C_{4,4} H^{2}) Q^{4}$$

$$+ (- 240 S_{4,4} K^{2} - 480 C_{4,4} H K + 240 S_{4,4} H^{2}) P Q^{3}$$

$$+ ((360 C_{4,4} K^{2} - 720 S_{4,4} H K - 360 C_{4,4} H^{2}) P^{2} - 8 C_{4,4} K^{2}$$

$$- 8 C_{4,4} H^{2} - 8 C_{4,4}) Q^{2} + ((240 S_{4,4} K^{2} + 480 C_{4,4} H K$$

$$- 240 S_{4,4} H^{2}) P^{3} + (- 16 S_{4,4} K^{2} - 16 S_{4,4} H^{2} - 16 S_{4,4}) P)$$

$$Q + (- 60 C_{4,4} K^{2} + 120 S_{4,4} H K + 60 C_{4,4} H^{2}) P^{4}$$

$$+ (8\, C_{4,4}\, K^2 + 8\, C_{4,4}\, H^2 + 8\, C_{4,4})\, P^2 - C_{4,4}\, K^2$$

$$- 2\, S_{4,4}\, H\, K + C_{4,4}\, H^2)\, \mathrm{COS}(2\, (2\, T - L)))\, MU\, RE^4$$

$$/(2\, (Q^2 + P^2 + 1)^4\, A^5)$$

The potential further truncated to the first power of h and k was constructed via (C12) - (C14) by substituting 0, wherever h^2, k^2, and hk appeared. Employing a factoring procedure similar to that above, $(U_{4,4})_{TRUNC}$ resulted, (D16).

```
(C12) RATSUBST(0,H^2,D7)$

(C13) RATSUBST(0,K^2,D12)$

(C14) RATSUBST(0,H*K,D13)$

(C15) FACTORSUM(D14);
```

$$\text{(D15)} \quad - 420\, MU\, RE^4\, (S_{4,4}\, Q^2\, \mathrm{SIN}(2\, (2\, T - L))$$

$$- 2\, C_{4,4}\, P\, Q\, \mathrm{SIN}(2\, (2\, T - L)) - S_{4,4}\, P^2\, \mathrm{SIN}(2\, (2\, T - L))$$

$$- C_{4,4}\, Q^2\, \mathrm{COS}(2\, (2\, T - L)) - 2\, S_{4,4}\, P\, Q\, \mathrm{COS}(2\, (2\, T - L))$$

$$+ C_{4,4}\, P^2\, \mathrm{COS}(2\, (2\, T - L)))/(A^5\, (Q^2 + P^2 + 1)^4)$$

```
(C16) U[4,4][TRUNC] = RAT(D15);
```

(D16)/R/
$$(U_{4,4})_{TRUNC} = -420\,((S_{4,4}\,Q^2 - 2\,C_{4,4}\,P\,Q - S_{4,4}\,P^2)\,SIN(2\,(2\,T - L)) + (-C_{4,4}\,Q^2 - 2\,S_{4,4}\,P\,Q + C_{4,4}\,P^2)\,COS(2\,(2\,T - L)))\,MU\,RE^4/((Q^2 + P^2 + 1)^4\,A^5)$$

What follows is an example of how the VOP equations were formed from the resulting potential. The file (SKC614, 1, DSK, SKC), loaded in statement (C17) contained a set of VOP equations truncated to the first power of h and k. (D18) is the expression for the semi-major axis rate, where N is the mean motion and R_L represents the derivative of the potential with respect to the mean longitude L.

```
(C17) LOADFILE(SKC614,1,DSK,SKC);

(D17)                           DONE

(C18) DA/DT[4,4] = RHS(DIFFA);
```

(D18)
$$\left(\frac{DA}{DT}\right)_{4,4} = \frac{2\,R_L}{A\,N}$$

(D20) results after substituting the definition of the mean motion, (D19)

```
(C19) N = SQRT(MU/A^3);

                              SQRT(MU)
(D19)                     N = --------
                                 3/2
                                A

(C20) SUBST([D19],D18);

                                   2 SQRT(A) R
                     DA                       L
(D20)                (--)       = ------------
                     DT 4, 4        SQRT(MU)
```

MACSYMA possesses the capability to take partial derivatives of an expression with respect to a specified argument. Using DIFF, with L as the argument in (C23), R_L can be formed from the potential, (D16). Note that it was more convenient to differentiate (D15) rather than (D16).

```
(C23) R[L] = DIFF(D15,L,1);

                        4           2
(D23) R   = - 420 MU RE  (- 2 C      Q  SIN(2 (2 T - L))
       L                       4, 4

                                                2
 - 4 S      P Q SIN(2 (2 T - L)) + 2 C      P  SIN(2 (2 T - L))
      4, 4                            4, 4

             2
 - 2 S      Q  COS(2 (2 T - L)) + 4 C      P Q COS(2 (2 T - L))
      4, 4                           4, 4

             2                        5   2     2     4
 + 2 S      P  COS(2 (2 T - L)))/(A   (Q  + P  + 1) )
      4, 4
```

After substitution and factorization, a rate truncated to the first powers of h and k is seen in (D26).

```
(C24) SUBST([D23],D20);

       DA                                4
(D24)  (--)       = - 840 SQRT(MU) RE
       DT 4, 4

             2
 (- 2 C      Q  SIN(2 (2 T - L)) - 4 S      P Q SIN(2 (2 T - L))
       4, 4                          4, 4

            2                             2
 + 2 C      P  SIN(2 (2 T - L)) - 2 S      Q  COS(2 (2 T - L))
      4, 4                          4, 4

                                                2
 + 4 C       P Q COS(2 (2 T - L)) + 2 S       P  COS(2 (2 T - L)))
      4, 4                             4, 4

   9/2   2    2     4
/(A     (Q  + P  + 1) )

(C26) DA/DT[4,4] = RAT(RHS(D24));

            DA                        2                        2
(D26)/R/  (--)       = 1680 ((C       Q  + 2 S      P Q - C      P )
            DT 4, 4              4, 4          4, 4         4, 4

                                 2                         2
 SIN(2 (2 T - L)) + (S       Q  - 2 C      P Q - S      P )
                     4, 4            4, 4          4, 4

                                  4    2    2     4         9
 COS(2 (2 T - L))) SQRT(MU) RE /((Q  + P  + 1)  SQRT(A) )
```

If the rate (D18) had contained derivatives with respect to h and k, the potential $(U_{4,4})_{FULL}$ would be used since $(U_{4,4})_{TRUNC}$ could not contribute any first power terms. All other rates follow in similar fashion.

One thing that was not done, was to mechanize, on MACSYMA, the recursions governing the computation of the V, S and Y functions. If this were implemented, construction of potential and rates would be greatly facilitated.

The remaining harmonics were computed with the following blocks

(2,2) Harmonic

$$V^2_{2,-2} = 3 \qquad S_4^{(2,-2)} = \frac{(p-jq)^4}{(1+p^2+q^2)^2}$$

$$V^2_{2,0} = -1 \qquad S_4^{(2,0)} = \frac{6(p-jq)^2}{(1+p^2+q^2)^2}$$

$$V^2_{2,2} = 3 \qquad S_4^{(2,2)} = \frac{1}{(1+p^2+q^2)^2}$$

$$Y_1^{-3,-2} = \frac{1}{48}(k - jh)^3$$

$$Y_1^{-3,0} = (k - jh)\left[\frac{3}{2} + \frac{27}{16}(h^2 + k^2)\right]$$

$$Y_1^{-3,2} = (k + jh)\left[-\frac{1}{2} + \frac{1}{16}(h^2 + k^2)\right]$$

(3,2) Harmonic

$V^2_{3,-1} = 3$

$S_6^{(2,-1)} = \frac{5(2-p^2-q^2)}{(1+p^2+q^2)^3} (p - jq)^3$

$V^2_{3,1} = -3$

$S_6^{(2,1)} = \frac{5(1-2p^2-2q^2)}{(1+p^2+q^2)^3} (p - jq)$

$V^2_{3,3} = 15$

$S_6^{(2,3)} = - \frac{p + jq}{(1+p^2+q^2)^3}$

$Y_1^{-4,-1} = \frac{11}{8} (k - jh)^2$

$Y_1^{-4,1} = 1 + 2(h^2 + k^2)$

$Y_1^{-4,3} = \frac{1}{8} (k + jh)^2$

(4,2) Harmonic

$$V^2_{4,-2} = -\frac{15}{2} \qquad S_8^{(2,-2)} = \frac{(p-jq)^4}{(1+p^2+q^2)^2}\,(7\gamma^2 + 7\gamma + 1)$$

$$V^2_{4,0} = \frac{9}{2} \qquad S_8^{(2,0)} = \frac{5}{2}\,\frac{(p-jq)^2}{(1+p^2+q^2)^2}(7\gamma^2 - 1)$$

$$V^2_{4,2} = -\frac{15}{2} \qquad S_8^{(2,2)} = \frac{7\gamma^2 - 7\gamma + 1}{(1+p^2+q^2)^2}$$

$$V^2_{4,4} = \frac{105}{2} \qquad S_8^{(2,4)} = \frac{(p + jq)^2}{(1+p^2+q^2)^4}$$

$$\gamma = \frac{1-p^2-q^2}{1+p^2+q^2}$$

$$Y_1^{-5,-2} = \frac{49}{8}\,(k-jh)^3$$

$$Y_1^{-5,0} = \frac{5}{2}(k - jh)\left[1 + \frac{27}{8}\,(h^2 + k^2)\right]$$

$$Y_1^{-5,2} = \frac{1}{2}\,(k + jh)\left[1 + \frac{33}{8}\,(h^2 + k^2)\right]$$

$$Y_1^{-5,4} = -\frac{1}{48}\,(k + jh)^3$$

MACSYMA allocates only so much list space for each user upon which expansions, factorizations, etc. may be performed. When this is exceeded the function cannot be performed. In the case of (4,2) which had many more terms than the other harmonics, factorizations of $\dot{p}$, $\dot{q}$ and $\dot{\lambda}$ could not be performed for lack of list space. As a consequence they are not as compact as desired. The following table gives the page numbers within the Appendix where the potential and rates for each harmonic may be found.

	U_{FULL}	U_{TRUNC}	$\dot{a}$	$\dot{h}$	$\dot{k}$	$\dot{\lambda}$	$\dot{p}$	$\dot{q}$
(2,2)	100	103	104	105	106	107	108	109
(3,2)	110	112	113	114	116	118	119	120
(4,2)	121	126	128	130	131	132	138	142
(4,4)	146	148	149	150	151	152	153	154

$$(U_{2,2})_{FULL} = - MU\, RE^2 (((S_{2,2} K^3 + 3 C_{2,2} H K^2$$

$$- 3 S_{2,2} H^2 K - C_{2,2} H^3) Q^4 + (- 4 C_{2,2} K^3 + 12 S_{2,2} H K^2$$

$$+ 12 C_{2,2} H^2 K - 4 S_{2,2} H^3) P Q^3$$

$$+ ((- 6 S_{2,2} K^3 - 18 C_{2,2} H K^2 + 18 S_{2,2} H^2 K + 6 C_{2,2} H^3)$$

$$P^2 + 162 S_{2,2} K^3 + 162 C_{2,2} H K^2$$

$$+ (162 S_{2,2} H^2 + 144 S_{2,2}) K + 162 C_{2,2} H^3 + 144 C_{2,2} H) Q^2$$

$$+ ((4 C_{2,2} K^3 - 12 S_{2,2} H K^2 - 12 C_{2,2} H^2 K + 4 S_{2,2} H^3) P^3$$

$$+ (- 324 C_{2,2} K^3 + 324 S_{2,2} H K^2$$

$$+ (- 324 C_{2,2} H^2 - 288 C_{2,2}) K + 324 S_{2,2} H^3 + 288 S_{2,2} H)$$

$$P) Q + (S_{2,2} K^3 + 3 C_{2,2} H K^2 - 3 S_{2,2} H^2 K - C_{2,2} H^3) P^4$$

$$+ (- 162 S_{2,2} K^3 - 162 C_{2,2} H K^2$$

$$+ (- 162 S_{2,2} H^2 - 144 S_{2,2}) K - 162 C_{2,2} H^3 - 144 C_{2,2} H)$$

$$P^2 + 3 S_{2,2} K^3 - 3 C_{2,2} H K^2 + (3 S_{2,2} H^2 - 24 S_{2,2}) K$$

$$- 3 C_{2, 2} H^{3} + 24 C_{2, 2} H) \mathrm{SIN}(2 T - L)$$

$$+ ((- C_{2, 2} K^{3} + 3 S_{2, 2} H K^{2} + 3 C_{2, 2} H^{2} K - S_{2, 2} H^{3}) Q^{4}$$

$$+ (- 4 S_{2, 2} K^{3} - 12 C_{2, 2} H K^{2} + 12 S_{2, 2} H^{2} K + 4 C_{2, 2} H^{3}) P$$

$$Q^{3} + ((6 C_{2, 2} K^{3} - 18 S_{2, 2} H K^{2} - 18 C_{2, 2} H^{2} K + 6 S_{2, 2} H^{3})$$

$$P^{2} - 162 C_{2, 2} K^{3} + 162 S_{2, 2} H K^{2}$$

$$+ (- 162 C_{2, 2} H^{2} - 144 C_{2, 2}) K + 162 S_{2, 2} H^{3} + 144 S_{2, 2} H)$$

$$Q^{2} + ((4 S_{2, 2} K^{3} + 12 C_{2, 2} H K^{2} - 12 S_{2, 2} H^{2} K - 4 C_{2, 2} H^{3})$$

$$P^{3} + (- 324 S_{2, 2} K^{3} - 324 C_{2, 2} H K^{2}$$

$$+ (- 324 S_{2, 2} H^{2} - 288 S_{2, 2}) K - 324 C_{2, 2} H^{3} - 288 C_{2, 2} H)$$

$$P) Q + (- C_{2, 2} K^{3} + 3 S_{2, 2} H K^{2} + 3 C_{2, 2} H^{2} K - S_{2, 2} H^{3}) P^{4}$$

$$+ (162 C_{2, 2} K^{3} - 162 S_{2, 2} H K^{2} + (162 C_{2, 2} H^{2} + 144 C_{2, 2}) K$$

$$- 162 S_{2, 2} H^{3} - 144 S_{2, 2} H) P^{2} - 3 C_{2, 2} K^{3} - 3 S_{2, 2} H K^{2}$$

$$+ (24 C_{2,2} - 3 C_{2,2} H^{2}) K - 3 S_{2,2} H^{3} + 24 S_{2,2} H)$$

$$COS(2 T - L))/(16 A^{3} (Q^{2} + P^{2} + 1)^{2})$$

$$(U_{2,2})_{TRUNC} = -3\,MU\,RE^2\,(((6\,S_{2,2}\,K + 6\,C_{2,2}\,H)\,Q^2$$

$$+ (12\,S_{2,2}\,H - 12\,C_{2,2}\,K)\,P\,Q + (-6\,S_{2,2}\,K - 6\,C_{2,2}\,H)\,P^2$$

$$- S_{2,2}\,K + C_{2,2}\,H)\,SIN(2\,T - L)$$

$$+ ((6\,S_{2,2}\,H - 6\,C_{2,2}\,K)\,Q^2 + (-12\,S_{2,2}\,K - 12\,C_{2,2}\,H)\,P\,Q$$

$$+ (6\,C_{2,2}\,K - 6\,S_{2,2}\,H)\,P^2 + C_{2,2}\,K + S_{2,2}\,H)\,COS(2\,T - L))$$

$$/(2\,A^3\,(Q^2 + P^2 + 1)^2)$$

$$
\left(\frac{DA}{DT}\right)_{2,2} = 3\Big(\big((6\,C_{2,2}\,K - 6\,S_{2,2}\,H)\,Q^2 + (12\,S_{2,2}\,K + 12\,C_{2,2}\,H)\,P\,Q + (-6\,C_{2,2}\,K + 6\,S_{2,2}\,H)\,P^2
$$

$$
- C_{2,2}\,K - S_{2,2}\,H\big)\,\mathrm{SIN}(2\,T - L)
$$

$$
+ \big((6\,S_{2,2}\,K + 6\,C_{2,2}\,H)\,Q^2 + (-12\,C_{2,2}\,K + 12\,S_{2,2}\,H)\,P\,Q
$$

$$
+ (-6\,S_{2,2}\,K - 6\,C_{2,2}\,H)\,P^2 - S_{2,2}\,K + C_{2,2}\,H\big)
$$

$$
\mathrm{COS}(2\,T - L)\Big)\,\mathrm{SQRT}(MU)\,RE^2/\left((Q^2 + P^2 + 1)^2\,\mathrm{SQRT}(A)^5\right)
$$

$$\left(\frac{DH}{DT}\right)_{2,2} = -3\left(\left(6\,S_{2,2}\,Q^{2} - 12\,C_{2,2}\,P\,Q - 6\,S_{2,2}\,P^{2} - S_{2,2}\right)\mathrm{SIN}(2\,T - L) + \left(-6\,C_{2,2}\,Q^{2} - 12\,S_{2,2}\,P\,Q + 6\,C_{2,2}\,P^{2} + C_{2,2}\right)\mathrm{COS}(2\,T - L)\right)\mathrm{SQRT}(MU)\,RE^{2} / \left(2\,(Q^{2} + P^{2} + 1)^{2}\,\mathrm{SQRT}(A)^{7}\right)$$

$$\left(\frac{DK}{DT}\right)_{2,2} = 3\left(\left(6 C_{2,2} Q^2 + 12 S_{2,2} P Q - 6 C_{2,2} P^2 + C_{2,2}\right) \sin(2 T - L) + \left(6 S_{2,2} Q^2 - 12 C_{2,2} P Q - 6 S_{2,2} P^2 + S_{2,2}\right) \cos(2 T - L)\right) \sqrt{MU}\, RE^2 / \left(2 \left(Q^2 + P^2 + 1\right)^2 \sqrt{A}^7\right)$$

$$\left(\frac{DL}{DT}\right)_{2,2} = 3\,(((12\,S_{2,2}\,K + 12\,C_{2,2}\,H)\,Q^{4}$$

$$+ (-24\,C_{2,2}\,K + 24\,S_{2,2}\,H)\,P\,Q^{3}$$

$$+ (-94\,S_{2,2}\,K - 86\,C_{2,2}\,H)\,Q^{2}$$

$$+ ((-24\,C_{2,2}\,K + 24\,S_{2,2}\,H)\,P^{3}$$

$$+ (180\,C_{2,2}\,K - 180\,S_{2,2}\,H)\,P)\,Q$$

$$+ (-12\,S_{2,2}\,K - 12\,C_{2,2}\,H)\,P^{4} + (86\,S_{2,2}\,K + 94\,C_{2,2}\,H)\,P^{2}$$

$$+ 13\,S_{2,2}\,K - 13\,C_{2,2}\,H)\,\mathrm{SIN}(2\,T - L)$$

$$+ ((-12\,C_{2,2}\,K + 12\,S_{2,2}\,H)\,Q^{4}$$

$$+ (-24\,S_{2,2}\,K - 24\,C_{2,2}\,H)\,P\,Q^{3}$$

$$+ (94\,C_{2,2}\,K - 86\,S_{2,2}\,H)\,Q^{2} + ((-24\,S_{2,2}\,K - 24\,C_{2,2}\,H)\,P^{3}$$

$$+ (180\,S_{2,2}\,K + 180\,C_{2,2}\,H)\,P)\,Q$$

$$+ (12\,C_{2,2}\,K - 12\,S_{2,2}\,H)\,P^{4} + (-86\,C_{2,2}\,K + 94\,S_{2,2}\,H)\,P^{2}$$

$$- 13\,C_{2,2}\,K - 13\,S_{2,2}\,H)\,\mathrm{COS}(2\,T - L))\,\mathrm{SQRT}(MU)\,RE^{2}$$

$$/(4\,(Q^{2} + P^{2} + 1)^{2}\,\mathrm{SQRT}(A)^{7})$$

$$
\begin{aligned}
\left(\frac{DP}{DT}\right)_{2,2} = {} & 3\,(((3\,S_{2,2}\,K + 3\,C_{2,2}\,H)\,Q^{3} \\
& + (-\,9\,C_{2,2}\,K + 9\,S_{2,2}\,H)\,P\,Q^{2} \\
& + ((-\,9\,S_{2,2}\,K - 9\,C_{2,2}\,H)\,P^{2} - 4\,S_{2,2}\,K - 2\,C_{2,2}\,H)\,Q \\
& + (3\,C_{2,2}\,K - 3\,S_{2,2}\,H)\,P^{3} + (4\,C_{2,2}\,K - 2\,S_{2,2}\,H)\,P) \\
& \mathrm{SIN}(2\,T - L) + ((-\,3\,C_{2,2}\,K + 3\,S_{2,2}\,H)\,Q^{3} \\
& + (-\,9\,S_{2,2}\,K - 9\,C_{2,2}\,H)\,P\,Q^{2} \\
& + ((9\,C_{2,2}\,K - 9\,S_{2,2}\,H)\,P^{2} + 4\,C_{2,2}\,K - 2\,S_{2,2}\,H)\,Q \\
& + (3\,S_{2,2}\,K + 3\,C_{2,2}\,H)\,P^{3} + (4\,S_{2,2}\,K + 2\,C_{2,2}\,H)\,P) \\
& \mathrm{COS}(2\,T - L))\,\mathrm{SQRT}(MU)\,RE^{2}/(2\,(Q^{2} + P^{2} + 1)\,\mathrm{SQRT}(A)^{7})
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{DQ}{DT}\right)_{2,2} = -3\,(((&3\,C_{2,2}\,K - 3\,S_{2,2}\,H)\,Q^{3} \\
&+ (9\,S_{2,2}\,K + 9\,C_{2,2}\,H)\,P\,Q^{2} + ((-9\,C_{2,2}\,K + 9\,S_{2,2}\,H)\,P^{2} \\
&+ 2\,C_{2,2}\,K - 4\,S_{2,2}\,H)\,Q + (-3\,S_{2,2}\,K - 3\,C_{2,2}\,H)\,P^{3} \\
&+ (2\,S_{2,2}\,K + 4\,C_{2,2}\,H)\,P)\,\mathrm{SIN}(2\,T - L) \\
&+ ((3\,S_{2,2}\,K + 3\,C_{2,2}\,H)\,Q^{3} + (-9\,C_{2,2}\,K + 9\,S_{2,2}\,H)\,P\,Q^{2} \\
&+ ((-9\,S_{2,2}\,K - 9\,C_{2,2}\,H)\,P^{2} + 2\,S_{2,2}\,K + 4\,C_{2,2}\,H)\,Q \\
&+ (3\,C_{2,2}\,K - 3\,S_{2,2}\,H)\,P^{3} + (-2\,C_{2,2}\,K + 4\,S_{2,2}\,H)\,P) \\
&\mathrm{COS}(2\,T - L))\,\mathrm{SQRT}(MU)\,RE^{2}/(2\,(Q^{2} + P^{2} + 1)\,\mathrm{SQRT}(A)^{7})
\end{aligned}
$$

$$(U_{3,2\,FULL}) = -15\,MU\,RE^{3}\,(((11\,C_{3,2}\,K^{2} - 22\,S_{3,2}\,H\,K$$

$$- 11\,C_{3,2}\,H^{2})\,Q^{5} + (33\,S_{3,2}\,K^{2} + 66\,C_{3,2}\,H\,K - 33\,S_{3,2}\,H^{2})\,P$$

$$Q^{4} + ((-\,22\,C_{3,2}\,K^{2} + 44\,S_{3,2}\,H\,K + 22\,C_{3,2}\,H^{2})\,P^{2}$$

$$+ 10\,C_{3,2}\,K^{2} + 44\,S_{3,2}\,H\,K + 54\,C_{3,2}\,H^{2} + 16\,C_{3,2})\,Q^{3}$$

$$+ ((22\,S_{3,2}\,K^{2} + 44\,C_{3,2}\,H\,K - 22\,S_{3,2}\,H^{2})\,P^{3}$$

$$+ (-\,34\,S_{3,2}\,K^{2} - 132\,C_{3,2}\,H\,K + 98\,S_{3,2}\,H^{2} + 16\,S_{3,2})\,P)\,Q^{2}$$

$$+ ((-\,33\,C_{3,2}\,K^{2} + 66\,S_{3,2}\,H\,K + 33\,C_{3,2}\,H^{2})\,P^{4}$$

$$+ (98\,C_{3,2}\,K^{2} - 132\,S_{3,2}\,H\,K - 34\,C_{3,2}\,H^{2} + 16\,C_{3,2})\,P^{2}$$

$$- 15\,C_{3,2}\,K^{2} + 2\,S_{3,2}\,H\,K - 17\,C_{3,2}\,H^{2} - 8\,C_{3,2})\,Q$$

$$+ (-\,11\,S_{3,2}\,K^{2} - 22\,C_{3,2}\,H\,K + 11\,S_{3,2}\,H^{2})\,P^{5}$$

$$+ (54\,S_{3,2}\,K^{2} + 44\,C_{3,2}\,H\,K + 10\,S_{3,2}\,H^{2} + 16\,S_{3,2})\,P^{3}$$

$$+ (-\,17\,S_{3,2}\,K^{2} + 2\,C_{3,2}\,H\,K - 15\,S_{3,2}\,H^{2} - 8\,S_{3,2})\,P)$$

$$SIN(2\,T - L) + (((11\,S_{3,2}\,K^{2} + 22\,C_{3,2}\,H\,K - 11\,S_{3,2}\,H^{2})\,Q^{5}$$

$$+ (- 33\, C_{3,2}\, K^2 + 66\, S_{3,2}\, H\, K + 33\, C_{3,2}\, H^2)\, P\, Q^4$$

$$+ ((- 22\, S_{3,2}\, K^2 - 44\, C_{3,2}\, H\, K + 22\, S_{3,2}\, H^2)\, P^2 + 10\, S_{3,2}\, K^2$$

$$- 44\, C_{3,2}\, H\, K + 54\, S_{3,2}\, H^2 + 16\, S_{3,2})\, Q^3$$

$$+ ((- 22\, C_{3,2}\, K^2 + 44\, S_{3,2}\, H\, K + 22\, C_{3,2}\, H^2)\, P^3$$

$$+ (34\, C_{3,2}\, K^2 - 132\, S_{3,2}\, H\, K - 98\, C_{3,2}\, H^2 - 16\, C_{3,2})\, P)\, Q^2$$

$$+ ((- 33\, S_{3,2}\, K^2 - 66\, C_{3,2}\, H\, K + 33\, S_{3,2}\, H^2)\, P^4$$

$$+ (98\, S_{3,2}\, K^2 + 132\, C_{3,2}\, H\, K - 34\, S_{3,2}\, H^2 + 16\, S_{3,2})\, P^2$$

$$- 15\, S_{3,2}\, K^2 - 2\, C_{3,2}\, H\, K - 17\, S_{3,2}\, H^2 - 8\, S_{3,2})\, Q$$

$$+ (11\, C_{3,2}\, K^2 - 22\, S_{3,2}\, H\, K - 11\, C_{3,2}\, H^2)\, P^5$$

$$+ (- 54\, C_{3,2}\, K^2 + 44\, S_{3,2}\, H\, K - 10\, C_{3,2}\, H^2 - 16\, C_{3,2})\, P^3$$

$$+ (17\, C_{3,2}\, K^2 + 2\, S_{3,2}\, H\, K + 15\, C_{3,2}\, H^2 + 8\, C_{3,2})\, P)$$

$$COS(2\, T - L))/(8\, A^4\, (Q^2 + P^2 + 1)^3)$$

$$(U_{3,2})_{TRUNC} = -15\ MU\ (2\ (Q^2 + P^2) - 1)\ RE^3$$

$$((C_{3,2}\ Q + S_{3,2}\ P)\ SIN(2\ T - L)$$

$$+ (S_{3,2}\ Q - C_{3,2}\ P)\ COS(2\ T - L))/(A^4\ (Q^2 + P^2 + 1)^3)$$

$$\left(\frac{DA}{DT}\right)_{3,2} = -30\,\left(\left(S_{3,2}\,Q - C_{3,2}\,P\right)\,\mathrm{SIN}(2\,T - L) + \left(-C_{3,2}\,Q - S_{3,2}\,P\right)\,\mathrm{COS}(2\,T - L)\right)\,\left(2\,Q^2 + 2\,P^2 - 1\right)\,\mathrm{SQRT}(MU)\,RE^3 / \left(\left(Q^2 + P^2 + 1\right)^3\,\mathrm{SQRT}(A)^7\right)$$

$$\left(\frac{DH}{DT}\right)_{3,2} = 15\ (((C_{3,2}\ K + 11\ S_{3,2}\ H)\ Q^{5}$$

$$+ (-\ 21\ S_{3,2}\ K - 33\ C_{3,2}\ H)\ P\ Q^{4}$$

$$+ ((46\ C_{3,2}\ K - 22\ S_{3,2}\ H)\ P^{2} - 32\ C_{3,2}\ K - 18\ S_{3,2}\ H)\ Q^{3}$$

$$+ ((2\ S_{3,2}\ K - 22\ C_{3,2}\ H)\ P^{3} + (12\ S_{3,2}\ K + 62\ C_{3,2}\ H)\ P)\ Q^{2}$$

$$+ ((45\ C_{3,2}\ K - 33\ S_{3,2}\ H)\ P^{4}$$

$$+ (-\ 120\ C_{3,2}\ K + 70\ S_{3,2}\ H)\ P^{2} + 17\ C_{3,2}\ K - 3\ S_{3,2}\ H)\ Q$$

$$+ (23\ S_{3,2}\ K + 11\ C_{3,2}\ H)\ P^{5} + (-\ 76\ S_{3,2}\ K - 26\ C_{3,2}\ H)\ P^{3}$$

$$+ (19\ S_{3,2}\ K + C_{3,2}\ H)\ P)\ SIN(2\ T - L)$$

$$+ ((S_{3,2}\ K - 11\ C_{3,2}\ H)\ Q^{5} + (21\ C_{3,2}\ K - 33\ S_{3,2}\ H)\ P\ Q^{4}$$

$$+ ((46\ S_{3,2}\ K + 22\ C_{3,2}\ H)\ P^{2} - 32\ S_{3,2}\ K + 18\ C_{3,2}\ H)\ Q^{3}$$

$$+ ((-\ 2\ C_{3,2}\ K - 22\ S_{3,2}\ H)\ P^{3}$$

$$+ (-\ 12\ C_{3,2}\ K + 62\ S_{3,2}\ H)\ P)\ Q^{2}$$

$$+ ((45\ S_{3,2}\ K + 33\ C_{3,2}\ H)\ P^{4}$$

$$+ (- 120\ S_{3, 2}\ K - 70\ C_{3, 2}\ H)\ P^{2} + 17\ S_{3, 2}\ K + 3\ C_{3, 2}\ H)\ Q$$

$$+ (- 23\ C_{3, 2}\ K + 11\ S_{3, 2}\ H)\ P^{5} + (76\ C_{3, 2}\ K - 26\ S_{3, 2}\ H)\ P^{3}$$

$$+ (- 19\ C_{3, 2}\ K + S_{3, 2}\ H)\ P)\ COS(2\ T - L))\ SQRT(MU)\ RE^{3}$$

$$/(4\ (Q^{2} + P^{2} + 1)^{3}\ SQRT(A)^{9})$$

$$\left(\frac{DK}{DT}\right)_{3,2} = - 15 (((11 S_{3,2} K + 23 C_{3,2} H) Q^{5}$$

$$+ (- 33 C_{3,2} K + 45 S_{3,2} H) P Q^{4}$$

$$+ ((- 22 S_{3,2} K + 2 C_{3,2} H) P^{2} - 26 S_{3,2} K - 76 C_{3,2} H) Q^{3}$$

$$+ ((- 22 C_{3,2} K + 46 S_{3,2} H) P^{3}$$

$$+ (70 C_{3,2} K - 120 S_{3,2} H) P) Q^{2}$$

$$+ ((- 33 S_{3,2} K - 21 C_{3,2} H) P^{4} + (62 S_{3,2} K + 12 C_{3,2} H) P^{2}$$

$$+ S_{3,2} K + 19 C_{3,2} H) Q + (11 C_{3,2} K + S_{3,2} H) P^{5}$$

$$+ (- 18 C_{3,2} K - 32 S_{3,2} H) P^{3} + (- 3 C_{3,2} K + 17 S_{3,2} H) P)$$

$$\mathrm{SIN}(2 T - L) + ((- 11 C_{3,2} K + 23 S_{3,2} H) Q^{5}$$

$$+ (- 33 S_{3,2} K - 45 C_{3,2} H) P Q^{4}$$

$$+ ((22 C_{3,2} K + 2 S_{3,2} H) P^{2} + 26 C_{3,2} K - 76 S_{3,2} H) Q^{3}$$

$$+ ((- 22 S_{3,2} K - 46 C_{3,2} H) P^{3}$$

$$+ (70 S_{3,2} K + 120 C_{3,2} H) P) Q^{2}$$

$$+ ((33\, C_{3,2} K - 21\, S_{3,2} H) P^4 + (- 62\, C_{3,2} K + 12\, S_{3,2} H) P^2$$

$$- C_{3,2} K + 19\, S_{3,2} H) Q + (11\, S_{3,2} K - C_{3,2} H) P^5$$

$$+ (- 18\, S_{3,2} K + 32\, C_{3,2} H) P^3 + (- 3\, S_{3,2} K - 17\, C_{3,2} H) P)$$

$$COS(2\, T - L))\, SQRT(MU)\, RE^3 / (4\, (Q^2 + P^2 + 1)^3\, SQRT(A)^9)$$

$$
\left(\frac{DL}{DT}\right)_{3,2} = \Big((90\, C_{3,2}\, A^4\, MU\, Q^5 + 90\, S_{3,2}\, A^4\, MU\, P\, Q^4
$$

$$
+ (180\, C_{3,2}\, A^4\, MU\, P^2 - 645\, C_{3,2}\, A^4\, MU)\, Q^3
$$

$$
+ (180\, S_{3,2}\, A^4\, MU\, P^3 - 645\, S_{3,2}\, A^4\, MU\, P)\, Q^2
$$

$$
+ (90\, C_{3,2}\, A^4\, MU\, P^4 - 645\, C_{3,2}\, A^4\, MU\, P^2 + 255\, C_{3,2}\, A^4\, MU)\, Q
$$

$$
+ 90\, S_{3,2}\, A^4\, MU\, P^5 - 645\, S_{3,2}\, A^4\, MU\, P^3 + 255\, S_{3,2}\, A^4\, MU\, P)
$$

$$
RE^3\, SIN(2\, T - L) + (90\, S_{3,2}\, A^4\, MU\, Q^5 - 90\, C_{3,2}\, A^4\, MU\, P\, Q^4
$$

$$
+ (180\, S_{3,2}\, A^4\, MU\, P^2 - 645\, S_{3,2}\, A^4\, MU)\, Q^3
$$

$$
+ (645\, C_{3,2}\, A^4\, MU\, P - 180\, C_{3,2}\, A^4\, MU\, P^3)\, Q^2
$$

$$
+ (90\, S_{3,2}\, A^4\, MU\, P^4 - 645\, S_{3,2}\, A^4\, MU\, P^2 + 255\, S_{3,2}\, A^4\, MU)\, Q
$$

$$
- 90\, C_{3,2}\, A^4\, MU\, P^5 + 645\, C_{3,2}\, A^4\, MU\, P^3 - 255\, C_{3,2}\, A^4\, MU\, P)
$$

$$
RE^3\, COS(2\, T - L)\Big)/\left(2\, A^{17/2}\, SQRT(MU)\, (Q^2 + P^2 + 1)^3\right)
$$

$$\left(\frac{DP}{DT}\right)_{3,\,2} = 15\,((6\,C_{3,\,2}\,Q^4 + 12\,S_{3,\,2}\,P\,Q^3 - 11\,C_{3,\,2}\,Q^2$$

$$+ (12\,S_{3,\,2}\,P^3 - 12\,S_{3,\,2}\,P)\,Q - 6\,C_{3,\,2}\,P^4 + C_{3,\,2}\,P^2 + C_{3,\,2})$$

$$\mathrm{SIN}(2\,T - L) + (6\,S_{3,\,2}\,Q^4 - 12\,C_{3,\,2}\,P\,Q^3 - 11\,S_{3,\,2}\,Q^2$$

$$+ (-\,12\,C_{3,\,2}\,P^3 + 12\,C_{3,\,2}\,P)\,Q - 6\,S_{3,\,2}\,P^4 + S_{3,\,2}\,P^2$$

$$+ S_{3,\,2})\,\mathrm{COS}(2\,T - L))\,\mathrm{SQRT}(MU)\,RE^3/(4\,(Q^2 + P^2 + 1)^2\,\mathrm{SQRT}(A)^9)$$

$$
\left(\frac{DQ}{DT}\right)_{3,2} = 15 \Big(\big(6 S_{3,2} Q^{4} - 12 C_{3,2} P Q^{3} - S_{3,2} Q^{2} + (-12 C_{3,2} P^{3} + 12 C_{3,2} P) Q - 6 S_{3,2} P^{4} + 11 S_{3,2} P^{2} - S_{3,2}\big) \mathrm{SIN}(2 T - L) + \big(-6 C_{3,2} Q^{4} - 12 S_{3,2} P Q^{3} + C_{3,2} Q^{2} + (-12 S_{3,2} P^{3} + 12 S_{3,2} P) Q + 6 C_{3,2} P^{4} - 11 C_{3,2} P^{2} + C_{3,2}\big) \mathrm{COS}(2 T - L)\Big) \mathrm{SQRT}(MU)\, RE^{3} / \left(4 (Q^{2} + P^{2} + 1)^{2}\, \mathrm{SQRT}(A)^{9}\right)
$$

$$(U_{4,\,2\ FULL}) = -\ MU\ RE^{4}\ (S_{4,\,2}$$

$$(((245\ H^{3} - 735\ H\ K^{2})\ Q^{8} + (980\ K^{3} - 2940\ H^{2}\ K)\ P\ Q^{7}$$

$$+ ((2940\ H\ K^{2} - 980\ H^{3})\ P^{2} - 9405\ H\ K^{2} - 21165\ H^{3} - 5400\ H)\ Q^{6}$$

$$+ ((980\ K^{3} - 2940\ H^{2}\ K)\ P^{3} + (24690\ K^{3} + (71730\ H^{2} + 10800)\ K)$$

$$P)\ Q^{5} + ((7350\ H\ K^{2} - 2450\ H^{3})\ P^{4}$$

$$+ (-\ 62325\ H\ K^{2} - 3525\ H^{3} - 5400\ H)\ P^{2} + 45000\ H\ K^{2} + 59700\ H^{3}$$

$$+ 16200\ H)\ Q^{4} + ((2940\ H^{2}\ K - 980\ K^{3})\ P^{5}$$

$$+ (72900\ K^{3} + (72900\ H^{2} + 21600)\ K)\ P^{3}$$

$$+ ((-\ 141300\ H^{2} - 28800)\ K - 82500\ K^{3})\ P)\ Q^{3}$$

$$+ ((2940\ H\ K^{2} - 980\ H^{3})\ P^{6} + (-\ 25875\ H\ K^{2} + 32925\ H^{3} + 5400\ H)$$

$$P^{4} + (81000\ H\ K^{2} - 7200\ H^{3} + 3600\ H)\ P^{2} - 24270\ H\ K^{2} - 24130\ H^{3}$$

$$-\ 6840\ H)\ Q^{2} + ((2940\ H^{2}\ K - 980\ K^{3})\ P^{7}$$

$$+ (48210\ K^{3} + (1170\ H^{2} + 10800)\ K)\ P^{5}$$

$$+ ((-\ 53100\ H^{2} - 28800)\ K - 111900\ K^{3})\ P^{3}$$

$$+ (36520\ K^{3} + (36240\ H^{2} + 10800)\ K)\ P)\ Q$$

$$+ (245\ H^{3} - 735\ H\ K^{2})\ P^{8} + (27045\ H\ K^{2} + 15285\ H^{3} + 5400\ H)\ P^{6}$$

$$+ (-\ 52200\ H\ K^{2} - 37500\ H^{3} - 12600\ H)\ P^{4}$$

$$+ (12390\ H\ K^{2} + 12250\ H^{3} + 3960\ H)\ P^{2} + 495\ H\ K^{2} + 495\ H^{3}$$

$$+ 120\ H)\ \cos(2\ T - L) - ((245\ K^{3} - 735\ H^{2}\ K)\ Q^{8}$$

$$+ (2940\ H\ K^{2} - 980\ H^{3})\ P\ Q^{7} + ((2940\ H^{2}\ K - 980\ K^{3})\ P^{2}$$

$$+ 15285\ K^{3} + (27045\ H^{2} + 5400)\ K)\ Q^{6}$$

$$+ ((2940\ H\ K^{2} - 980\ H^{3})\ P^{3} + (1170\ H\ K^{2} + 48210\ H^{3} + 10800\ H)\ P)$$

$$Q^{5} + ((7350\ H^{2}\ K - 2450\ K^{3})\ P^{4} + (32925\ K^{3}$$

$$+ (5400 - 25875\ H^{2})\ K)\ P^{2} - 37500\ K^{3} + (- 52200\ H^{2} - 12600)\ K)$$

$$Q^{4} + ((980\ H^{3} - 2940\ H\ K^{2})\ P^{5} + (72900\ H\ K^{2} + 72900\ H^{3}$$

$$+ 21600\ H)\ P^{3} + (- 53100\ H\ K^{2} - 111900\ H^{3} - 28800\ H)\ P)\ Q^{3}$$

$$+ ((2940\ H^{2}\ K - 980\ K^{3})\ P^{6} + ((- 62325\ H^{2} - 5400)\ K - 3525\ K^{3})$$

$$P^{4} + ((81000\ H^{2} + 3600)\ K - 7200\ K^{3})\ P^{2} + 12250\ K^{3}$$

$$+ (12390\ H^{2} + 3960)\ K)\ Q^{2} + ((980\ H^{3} - 2940\ H\ K^{2})\ P^{7}$$

$$+ (71730\ H\ K^{2} + 24690\ H^{3} + 10800\ H)\ P^{5}$$

$$+ (- 141300\ H\ K^{2} - 82500\ H^{3} - 28800\ H)\ P^{3}$$

$$+ (36240\ H\ K^{2} + 36520\ H^{3} + 10800\ H)\ P)\ Q$$

$$+ (245\ K^{3} - 735\ H^{2}\ K)\ P^{8} + ((- 9405\ H^{2} - 5400)\ K - 21165\ K^{3})\ P^{6}$$

$$+ (59700\ K^{3} + (45000\ H^{2} + 16200)\ K)\ P^{4}$$

```
                2                    3   2      3
+ ((- 24270 H  - 6840) K - 24130 K ) P  + 495 K

           2
+ (495 H  + 120) K) SIN(2 T - L))

                3           2   8          3         2       7
+ C      (((245 H  - 735 H K ) Q  + (980 K  - 2940 H  K) P Q
   4, 2

                2        3   2              2          3            6
+ ((2940 H K  - 980 H ) P  - 9405 H K  - 21165 H  - 5400 H) Q

       3            2     3             3            2
+ ((980 K  - 2940 H  K) P  + (24690 K  + (71730 H  + 10800) K)

    5             2          3   4
P) Q  + ((7350 H K  - 2450 H ) P

                   2          3             2            2          3
+ (- 62325 H K  - 3525 H  - 5400 H) P  + 45000 H K  + 59700 H

            4           2          3   5
+ 16200 H) Q  + ((2940 H  K - 980 K ) P

          3            2                  3
+ (72900 K  + (72900 H  + 21600) K) P

                   2                     3      3
+ ((- 141300 H  - 28800) K - 82500 K ) P) Q

                2        3   6                2          3
+ ((2940 H K  - 980 H ) P  + (- 25875 H K  + 32925 H  + 5400 H)

 4              2         3          2             2            3
P  + (81000 H K  - 7200 H  + 3600 H) P  - 24270 H K  - 24130 H

           2          2          3   7
- 6840 H) Q  + ((2940 H  K - 980 K ) P

          3           2                 5
+ (48210 K  + (1170 H  + 10800) K) P

                 2                      3   3
+ ((- 53100 H  - 28800) K - 111900 K ) P

          3            2
+ (36520 K  + (36240 H  + 10800) K) P) Q

        3            2   8              2           3           6
+ (245 H  - 735 H K ) P  + (27045 H K  + 15285 H  + 5400 H) P
```

$$+ (- 52200\ H\ K^{2} - 37500\ H^{3} - 12600\ H)\ P^{4}$$

$$+ (12390\ H\ K^{2} + 12250\ H^{3} + 3960\ H)\ P^{2} + 495\ H\ K^{2} + 495\ H^{3}$$

$$+ 120\ H)\ SIN(2\ T - L) + ((245\ K^{3} - 735\ H^{2}\ K)\ Q^{8}$$

$$+ (2940\ H\ K^{2} - 980\ H^{3})\ P\ Q^{7} + ((2940\ H^{2}\ K - 980\ K^{3})\ P^{2}$$

$$+ 15285\ K^{3} + (27045\ H^{2} + 5400)\ K)\ Q^{6}$$

$$+ ((2940\ H\ K^{2} - 980\ H^{3})\ P^{3} + (1170\ H\ K^{2} + 48210\ H^{3} + 10800\ H)\ P)$$

$$Q^{5} + ((7350\ H^{2}\ K - 2450\ K^{3})\ P^{4} + (32925\ K^{3}$$

$$+ (5400 - 25875\ H^{2})\ K)\ P^{2} - 37500\ K^{3} + (- 52200\ H^{2} - 12600)\ K)$$

$$Q^{4} + ((980\ H^{3} - 2940\ H\ K^{2})\ P^{5} + (72900\ H\ K^{2} + 72900\ H^{3}$$

$$+ 21600\ H)\ P^{3} + (- 53100\ H\ K^{2} - 111900\ H^{3} - 28800\ H)\ P)\ Q^{3}$$

$$+ ((2940\ H^{2}\ K - 980\ K^{3})\ P^{6} + ((- 62325\ H^{2} - 5400)\ K - 3525\ K^{3})$$

$$P^{4} + ((81000\ H^{2} + 3600)\ K - 7200\ K^{3})\ P^{2} + 12250\ K^{3}$$

$$+ (12390\ H^{2} + 3960)\ K)\ Q^{2} + ((980\ H^{3} - 2940\ H\ K^{2})\ P^{7}$$

$$+ (71730\ H\ K^{2} + 24690\ H^{3} + 10800\ H)\ P^{5}$$

$$+ (- 141300\ H\ K^{2} - 82500\ H^{3} - 28800\ H)\ P^{3}$$

$$+ (36240\ H\ K^{2} + 36520\ H^{3} + 10800\ H)\ P)\ Q$$

$$+ (245 K^3 - 735 H^2 K) P^8 + ((- 9405 H^2 - 5400) K - 21165 K^3) P^6$$

$$+ (59700 K^3 + (45000 H^2 + 16200) K) P^4$$

$$+ ((- 24270 H^2 - 6840) K - 24130 K^3) P^2 + 495 K^3$$

$$+ (495 H^2 + 120) K) COS(2 T - L)))/(32 A^5 (Q^2 + P^2 + 1)^4)$$

$$(U_{4,2})_{TRUNC} = - MU\, RE^{4} (S_{4,2}$$

$$((- 5400 H Q^{6} + 10800 K P Q^{5} + (16200 H - 5400 H P^{2}) Q^{4}$$

$$+ (21600 K P^{3} - 28800 K P) Q^{3} + (5400 H P^{4} + 3600 H P^{2} - 6840 H)$$

$$Q^{2} + (10800 K P^{5} - 28800 K P^{3} + 10800 K P) Q + 5400 H P^{6}$$

$$- 12600 H P^{4} + 3960 H P^{2} + 120 H) COS(2 T - L)$$

$$- (5400 K Q^{6} + 10800 H P Q^{5} + (5400 K P^{2} - 12600 K) Q^{4}$$

$$+ (21600 H P^{3} - 28800 H P) Q^{3} + (- 5400 K P^{4} + 3600 K P^{2}$$

$$+ 3960 K) Q^{2} + (10800 H P^{5} - 28800 H P^{3} + 10800 H P) Q$$

$$- 5400 K P^{6} + 16200 K P^{4} - 6840 K P^{2} + 120 K) SIN(2 T - L))$$

$$+ C_{4,2} ((- 5400 H Q^{6} + 10800 K P Q^{5} + (16200 H - 5400 H P^{2}) Q^{4}$$

$$+ (21600 K P^{3} - 28800 K P) Q^{3} + (5400 H P^{4} + 3600 H P^{2} - 6840 H)$$

$$Q^{2} + (10800 K P^{5} - 28800 K P^{3} + 10800 K P) Q + 5400 H P^{6}$$

$$- 12600 H P^{4} + 3960 H P^{2} + 120 H) SIN(2 T - L)$$

$$+ (5400 K Q^{6} + 10800 H P Q^{5} + (5400 K P^{2} - 12600 K) Q^{4}$$

$$+ (21600 H P^{3} - 28800 H P) Q^{3} + (- 5400 K P^{4} + 3600 K P^{2}$$

$$+ 3960 K) Q^{2} + (10800 H P^{5} - 28800 H P^{3} + 10800 H P) Q$$

$$- 5400\ K\ P^{6} + 16200\ K\ P^{4} - 6840\ K\ P^{2} + 120\ K)\ COS(2\ T - L)))$$

$$/(32\ A^{5}\ (Q^{2} + P^{2} + 1)^{4})$$

$$\left(\frac{DA}{DT}\right)_{4,2} = -15\,(((45\,C_{4,2}\,K - 45\,S_{4,2}\,H)\,Q^{6}$$

$$+ (90\,S_{4,2}\,K + 90\,C_{4,2}\,H)\,P\,Q^{5}$$

$$+ ((45\,C_{4,2}\,K - 45\,S_{4,2}\,H)\,P^{2} - 105\,C_{4,2}\,K + 135\,S_{4,2}\,H)\,Q^{4}$$

$$+ ((180\,S_{4,2}\,K + 180\,C_{4,2}\,H)\,P^{3}$$

$$+ (-240\,S_{4,2}\,K - 240\,C_{4,2}\,H)\,P)\,Q^{3}$$

$$+ ((-45\,C_{4,2}\,K + 45\,S_{4,2}\,H)\,P^{4} + (30\,C_{4,2}\,K + 30\,S_{4,2}\,H)\,P^{2}$$

$$+ 33\,C_{4,2}\,K - 57\,S_{4,2}\,H)\,Q^{2} + ((90\,S_{4,2}\,K + 90\,C_{4,2}\,H)\,P^{5}$$

$$+ (-240\,S_{4,2}\,K - 240\,C_{4,2}\,H)\,P^{3}$$

$$+ (90\,S_{4,2}\,K + 90\,C_{4,2}\,H)\,P)\,Q$$

$$+ (-45\,C_{4,2}\,K + 45\,S_{4,2}\,H)\,P^{6}$$

$$+ (135\,C_{4,2}\,K - 105\,S_{4,2}\,H)\,P^{4}$$

$$+ (-57\,C_{4,2}\,K + 33\,S_{4,2}\,H)\,P^{2} + C_{4,2}\,K + S_{4,2}\,H)$$

$$\mathrm{SIN}(2\ T - L) + ((45\ S_{4,\ 2}\ K + 45\ C_{4,\ 2}\ H)\ Q^{6}$$

$$+ (-\ 90\ C_{4,\ 2}\ K + 90\ S_{4,\ 2}\ H)\ P\ Q^{5}$$

$$+ ((45\ S_{4,\ 2}\ K + 45\ C_{4,\ 2}\ H)\ P^{2} - 105\ S_{4,\ 2}\ K - 135\ C_{4,\ 2}\ H)\ Q^{4}$$

$$+ ((-\ 180\ C_{4,\ 2}\ K + 180\ S_{4,\ 2}\ H)\ P^{3}$$

$$+ (240\ C_{4,\ 2}\ K - 240\ S_{4,\ 2}\ H)\ P)\ Q^{3}$$

$$+ ((-\ 45\ S_{4,\ 2}\ K - 45\ C_{4,\ 2}\ H)\ P^{4} + (30\ S_{4,\ 2}\ K - 30\ C_{4,\ 2}\ H)\ P^{2}$$

$$+ 33\ S_{4,\ 2}\ K + 57\ C_{4,\ 2}\ H)\ Q^{2} + ((-\ 90\ C_{4,\ 2}\ K + 90\ S_{4,\ 2}\ H)\ P^{5}$$

$$+ (240\ C_{4,\ 2}\ K - 240\ S_{4,\ 2}\ H)\ P^{3}$$

$$+ (-\ 90\ C_{4,\ 2}\ K + 90\ S_{4,\ 2}\ H)\ P)\ Q$$

$$+ (-\ 45\ S_{4,\ 2}\ K - 45\ C_{4,\ 2}\ H)\ P^{6}$$

$$+ (135\ S_{4,\ 2}\ K + 105\ C_{4,\ 2}\ H)\ P^{4}$$

$$+ (-\ 57\ S_{4,\ 2}\ K - 33\ C_{4,\ 2}\ H)\ P^{2} + S_{4,\ 2}\ K - C_{4,\ 2}\ H)$$

$$\mathrm{COS}(2\ T - L))\ \mathrm{SQRT}(MU)\ RE^{4} / (2\ (Q^{2} + P^{2} + 1)^{4}\ \mathrm{SQRT}(A)^{9})$$

$$\left(\frac{DH}{DT}\right)_{4,2} = 15 \, ((45 \, S_{4,2} \, Q^{6} - 90 \, C_{4,2} \, P \, Q^{5}$$

$$+ (45 \, S_{4,2} \, P^{2} - 105 \, S_{4,2}) \, Q^{4} + (- 180 \, C_{4,2} \, P^{3} + 240 \, C_{4,2} \, P)$$

$$Q^{3} + (- 45 \, S_{4,2} \, P^{4} + 30 \, S_{4,2} \, P^{2} + 33 \, S_{4,2}) \, Q^{2}$$

$$+ (- 90 \, C_{4,2} \, P^{5} + 240 \, C_{4,2} \, P^{3} - 90 \, C_{4,2} \, P) \, Q - 45 \, S_{4,2} \, P^{6}$$

$$+ 135 \, S_{4,2} \, P^{4} - 57 \, S_{4,2} \, P^{2} + S_{4,2}) \, \mathrm{SIN}(2 \, T - L)$$

$$+ (- 45 \, C_{4,2} \, Q^{6} - 90 \, S_{4,2} \, P \, Q^{5}$$

$$+ (- 45 \, C_{4,2} \, P^{2} + 105 \, C_{4,2}) \, Q^{4}$$

$$+ (- 180 \, S_{4,2} \, P^{3} + 240 \, S_{4,2} \, P) \, Q^{3}$$

$$+ (45 \, C_{4,2} \, P^{4} - 30 \, C_{4,2} \, P^{2} - 33 \, C_{4,2}) \, Q^{2}$$

$$+ (- 90 \, S_{4,2} \, P^{5} + 240 \, S_{4,2} \, P^{3} - 90 \, S_{4,2} \, P) \, Q + 45 \, C_{4,2} \, P^{6}$$

$$- 135 \, C_{4,2} \, P^{4} + 57 \, C_{4,2} \, P^{2} - C_{4,2}) \, \mathrm{COS}(2 \, T - L)) \, \mathrm{SQRT}(MU) \, RE^{4}$$

$$/(4 \, (Q^{2} + P^{2} + 1)^{4} \, \mathrm{SQRT}(A)^{11})$$

$$
\begin{aligned}
\left(\frac{DK}{DT}\right)_{4,2} = {} & -15\Big(\big(45 C_{4,2} Q^{6} + 90 S_{4,2} P Q^{5} \\
& + (45 C_{4,2} P^{2} - 135 C_{4,2}) Q^{4} + (180 S_{4,2} P^{3} - 240 S_{4,2} P) Q^{3} \\
& + (-45 C_{4,2} P^{4} - 30 C_{4,2} P^{2} + 57 C_{4,2}) Q^{2} \\
& + (90 S_{4,2} P^{5} - 240 S_{4,2} P^{3} + 90 S_{4,2} P) Q - 45 C_{4,2} P^{6} \\
& + 105 C_{4,2} P^{4} - 33 C_{4,2} P^{2} - C_{4,2}\big) \mathrm{SIN}(2 T - L) \\
& + \big(45 S_{4,2} Q^{6} - 90 C_{4,2} P Q^{5} + (45 S_{4,2} P^{2} - 135 S_{4,2}) Q^{4} \\
& + (-180 C_{4,2} P^{3} + 240 C_{4,2} P) Q^{3} \\
& + (-45 S_{4,2} P^{4} - 30 S_{4,2} P^{2} + 57 S_{4,2}) Q^{2} \\
& + (-90 C_{4,2} P^{5} + 240 C_{4,2} P^{3} - 90 C_{4,2} P) Q - 45 S_{4,2} P^{6} \\
& + 105 S_{4,2} P^{4} - 33 S_{4,2} P^{2} - S_{4,2}\big) \mathrm{COS}(2 T - L)\Big) \mathrm{SQRT}(MU)\, RE^{4} \\
& / \left(4 \left(Q^{2} + P^{2} + 1\right)^{4} \mathrm{SQRT}(A)^{11}\right)
\end{aligned}
$$

$$\left(\frac{DL}{DT}\right)_{4,\,2} = -\,15\ \mathrm{SQRT}(MU)\ RE^{4}$$

$$(90\ S_{4,\,2}\ K\ Q^{8}\ \mathrm{SIN}(2\ T - L) + 90\ C_{4,\,2}\ H\ Q^{8}\ \mathrm{SIN}(2\ T - L)$$

$$-\ 180\ C_{4,\,2}\ K\ P\ Q^{7}\ \mathrm{SIN}(2\ T - L) + 180\ S_{4,\,2}\ H\ P\ Q^{7}\ \mathrm{SIN}(2\ T - L)$$

$$+\ 180\ S_{4,\,2}\ K\ P^{2}\ Q^{6}\ \mathrm{SIN}(2\ T - L)$$

$$+\ 180\ C_{4,\,2}\ H\ P^{2}\ Q^{6}\ \mathrm{SIN}(2\ T - L) - 1635\ S_{4,\,2}\ K\ Q^{6}\ \mathrm{SIN}(2\ T - L)$$

$$-\ 1755\ C_{4,\,2}\ H\ Q^{6}\ \mathrm{SIN}(2\ T - L) - 540\ C_{4,\,2}\ K\ P^{3}\ Q^{5}\ \mathrm{SIN}(2\ T - L)$$

$$+\ 540\ S_{4,\,2}\ H\ P^{3}\ Q^{5}\ \mathrm{SIN}(2\ T - L)$$

$$+\ 3390\ C_{4,\,2}\ K\ P\ Q^{5}\ \mathrm{SIN}(2\ T - L)$$

$$-\ 3390\ S_{4,\,2}\ H\ P\ Q^{5}\ \mathrm{SIN}(2\ T - L)$$

$$-\ 1515\ S_{4,\,2}\ K\ P^{2}\ Q^{4}\ \mathrm{SIN}(2\ T - L)$$

$$-\ 1875\ C_{4,\,2}\ H\ P^{2}\ Q^{4}\ \mathrm{SIN}(2\ T - L) + 2823\ S_{4,\,2}\ K\ Q^{4}\ \mathrm{SIN}(2\ T - L)$$

$$+\ 3717\ C_{4,\,2}\ H\ Q^{4}\ \mathrm{SIN}(2\ T - L) - 540\ C_{4,\,2}\ K\ P^{5}\ Q^{3}\ \mathrm{SIN}(2\ T - L)$$

$$+\ 540\ S_{4,\,2}\ H\ P^{5}\ Q^{3}\ \mathrm{SIN}(2\ T - L)$$

$$+ 6780\, C_{4,2}\, K\, P^{3}\, Q^{3}\, \mathrm{SIN}(2\, T - L)$$

$$- 6780\, S_{4,2}\, H\, P^{3}\, Q^{3}\, \mathrm{SIN}(2\, T - L)$$

$$- 6540\, C_{4,2}\, K\, P\, Q^{3}\, \mathrm{SIN}(2\, T - L)$$

$$+ 6540\, S_{4,2}\, H\, P\, Q^{3}\, \mathrm{SIN}(2\, T - L)$$

$$- 180\, S_{4,2}\, K\, P^{6}\, Q^{2}\, \mathrm{SIN}(2\, T - L)$$

$$- 180\, C_{4,2}\, H\, P^{6}\, Q^{2}\, \mathrm{SIN}(2\, T - L)$$

$$+ 1875\, S_{4,2}\, K\, P^{4}\, Q^{2}\, \mathrm{SIN}(2\, T - L)$$

$$+ 1515\, C_{4,2}\, H\, P^{4}\, Q^{2}\, \mathrm{SIN}(2\, T - L)$$

$$- 894\, S_{4,2}\, K\, P^{2}\, Q^{2}\, \mathrm{SIN}(2\, T - L)$$

$$+ 894\, C_{4,2}\, H\, P^{2}\, Q^{2}\, \mathrm{SIN}(2\, T - L) - 751\, S_{4,2}\, K\, Q^{2}\, \mathrm{SIN}(2\, T - L)$$

$$- 1319\, C_{4,2}\, H\, Q^{2}\, \mathrm{SIN}(2\, T - L) - 180\, C_{4,2}\, K\, P^{7}\, Q\, \mathrm{SIN}(2\, T - L)$$

$$+ 180\, S_{4,2}\, H\, P^{7}\, Q\, \mathrm{SIN}(2\, T - L) + 3390\, C_{4,2}\, K\, P^{5}\, Q\, \mathrm{SIN}(2\, T - L) \quad .$$

$$- 3390 S_{4,2} H P^{5} Q \sin(2T - L)$$

$$- 6540 C_{4,2} K P^{3} Q \sin(2T - L)$$

$$+ 6540 S_{4,2} H P^{3} Q \sin(2T - L) + 2070 C_{4,2} K P Q \sin(2T - L)$$

$$- 2070 S_{4,2} H P Q \sin(2T - L) - 90 S_{4,2} K P^{8} \sin(2T - L)$$

$$- 90 C_{4,2} H P^{8} \sin(2T - L) + 1755 S_{4,2} K P^{6} \sin(2T - L)$$

$$+ 1635 C_{4,2} H P^{6} \sin(2T - L) - 3717 S_{4,2} K P^{4} \sin(2T - L)$$

$$- 2823 C_{4,2} H P^{4} \sin(2T - L) + 1319 S_{4,2} K P^{2} \sin(2T - L)$$

$$+ 751 C_{4,2} H P^{2} \sin(2T - L) - 21 S_{4,2} K \sin(2T - L)$$

$$+ 21 C_{4,2} H \sin(2T - L) - 90 C_{4,2} K Q^{8} \cos(2T - L)$$

$$+ 90 S_{4,2} H Q^{8} \cos(2T - L) - 180 S_{4,2} K P Q^{7} \cos(2T - L)$$

$$- 180 C_{4,2} H P Q^{7} \cos(2T - L) - 180 C_{4,2} K P^{2} Q^{6} \cos(2T - L)$$

$$+ 180 S_{4,2} H P^{2} Q^{6} \cos(2T - L) + 1635 C_{4,2} K Q^{6} \cos(2T - L)$$

$$- 1755 S_{4,2} H Q^{6} \cos(2T - L) - 540 S_{4,2} K P^{3} Q^{5} \cos(2T - L)$$

$$- 540\ C_{4,2}\ H\ P^{3}\ Q^{5}\ COS(2\ T - L)$$

$$+ 3390\ S_{4,2}\ K\ P\ Q^{5}\ COS(2\ T - L)$$

$$+ 3390\ C_{4,2}\ H\ P\ Q^{5}\ COS(2\ T - L)$$

$$+ 1515\ C_{4,2}\ K\ P^{2}\ Q^{4}\ COS(2\ T - L)$$

$$- 1875\ S_{4,2}\ H\ P^{2}\ Q^{4}\ COS(2\ T - L) - 2823\ C_{4,2}\ K\ Q^{4}\ COS(2\ T - L)$$

$$+ 3717\ S_{4,2}\ H\ Q^{4}\ COS(2\ T - L) - 540\ S_{4,2}\ K\ P^{5}\ Q^{3}\ COS(2\ T - L)$$

$$- 540\ C_{4,2}\ H\ P^{5}\ Q^{3}\ COS(2\ T - L)$$

$$+ 6780\ S_{4,2}\ K\ P^{3}\ Q^{3}\ COS(2\ T - L)$$

$$+ 6780\ C_{4,2}\ H\ P^{3}\ Q^{3}\ COS(2\ T - L)$$

$$- 6540\ S_{4,2}\ K\ P\ Q^{3}\ COS(2\ T - L)$$

$$- 6540\ C_{4,2}\ H\ P\ Q^{3}\ COS(2\ T - L)$$

$$+ 180\ C_{4,2}\ K\ P^{6}\ Q^{2}\ COS(2\ T - L)$$

$$- 180 S_{4,2} H P^{6} Q^{2} \cos(2 T - L)$$

$$- 1875 C_{4,2} K P^{4} Q^{2} \cos(2 T - L)$$

$$+ 1515 S_{4,2} H P^{4} Q^{2} \cos(2 T - L)$$

$$+ 894 C_{4,2} K P^{2} Q^{2} \cos(2 T - L)$$

$$+ 894 S_{4,2} H P^{2} Q^{2} \cos(2 T - L) + 751 C_{4,2} K Q^{2} \cos(2 T - L)$$

$$- 1319 S_{4,2} H Q^{2} \cos(2 T - L) - 180 S_{4,2} K P^{7} Q \cos(2 T - L)$$

$$- 180 C_{4,2} H P^{7} Q \cos(2 T - L) + 3390 S_{4,2} K P^{5} Q \cos(2 T - L)$$

$$+ 3390 C_{4,2} H P^{5} Q \cos(2 T - L)$$

$$- 6540 S_{4,2} K P^{3} Q \cos(2 T - L)$$

$$- 6540 C_{4,2} H P^{3} Q \cos(2 T - L) + 2070 S_{4,2} K P Q \cos(2 T - L)$$

$$+ 2070 C_{4,2} H P Q \cos(2 T - L) + 90 C_{4,2} K P^{8} \cos(2 T - L)$$

$$- 90 S_{4,2} H P^{8} \cos(2 T - L) - 1755 C_{4,2} K P^{6} \cos(2 T - L)$$

$$+ 1635 S_{4,2} H P^{6} \cos(2 T - L) + 3717 C_{4,2} K P^{4} \cos(2 T - L)$$

$$- 2823\, S_{4,2}\, H\, P^{4} \cos(2T - L) - 1319\, C_{4,2}\, K\, P^{2} \cos(2T - L)$$

$$+ 751\, S_{4,2}\, H\, P^{2} \cos(2T - L) + 21\, C_{4,2}\, K \cos(2T - L)$$

$$+ 21\, S_{4,2}\, H \cos(2T - L))/(8\, A^{11/2} (Q^{2} + P^{2} + 1)^{4})$$

$$\left(\frac{DP}{DT}\right)_{4,2} = - 15\ \mathrm{SQRT}(MU)\ RE^{4}$$

$$(45\ S_{4,2}\ K\ Q^{7}\ \mathrm{SIN}(2\ T - L) + 45\ C_{4,2}\ H\ Q^{7}\ \mathrm{SIN}(2\ T - L)$$

$$- 135\ C_{4,2}\ K\ P\ Q^{6}\ \mathrm{SIN}(2\ T - L) + 135\ S_{4,2}\ H\ P\ Q^{6}\ \mathrm{SIN}(2\ T - L)$$

$$- 45\ S_{4,2}\ K\ P^{2}\ Q^{5}\ \mathrm{SIN}(2\ T - L) - 45\ C_{4,2}\ H\ P^{2}\ Q^{5}\ \mathrm{SIN}(2\ T - L)$$

$$- 345\ S_{4,2}\ K\ Q^{5}\ \mathrm{SIN}(2\ T - L) - 405\ C_{4,2}\ H\ Q^{5}\ \mathrm{SIN}(2\ T - L)$$

$$- 225\ C_{4,2}\ K\ P^{3}\ Q^{4}\ \mathrm{SIN}(2\ T - L)$$

$$+ 225\ S_{4,2}\ H\ P^{3}\ Q^{4}\ \mathrm{SIN}(2\ T - L) + 795\ C_{4,2}\ K\ P\ Q^{4}\ \mathrm{SIN}(2\ T - L)$$

$$- 855\ S_{4,2}\ H\ P\ Q^{4}\ \mathrm{SIN}(2\ T - L) - 225\ S_{4,2}\ K\ P^{4}\ Q^{3}\ \mathrm{SIN}(2\ T - L)$$

$$- 225\ C_{4,2}\ H\ P^{4}\ Q^{3}\ \mathrm{SIN}(2\ T - L)$$

$$+ 210\ S_{4,2}\ K\ P^{2}\ Q^{3}\ \mathrm{SIN}(2\ T - L) + 90\ C_{4,2}\ H\ P^{2}\ Q^{3}\ \mathrm{SIN}(2\ T - L)$$

$$+ 309\ S_{4,2}\ K\ Q^{3}\ \mathrm{SIN}(2\ T - L) + 441\ C_{4,2}\ H\ Q^{3}\ \mathrm{SIN}(2\ T - L)$$

$$- 45\ C_{4,2}\ K\ P^{5}\ Q^{2}\ \mathrm{SIN}(2\ T - L) + 45\ S_{4,2}\ H\ P^{5}\ Q^{2}\ \mathrm{SIN}(2\ T - L)$$

$$+ 690\ C_{4,2}\ K\ P^{3}\ Q^{2}\ \mathrm{SIN}(2\ T - L)$$

$$- 810\, S_{4,2}\, H\, P^{3}\, Q^{2}\, \mathrm{SIN}(2\,T - L) - 651\, C_{4,2}\, K\, P\, Q^{2}\, \mathrm{SIN}(2\,T - L)$$

$$+ 699\, S_{4,2}\, H\, P\, Q^{2}\, \mathrm{SIN}(2\,T - L) - 135\, S_{4,2}\, K\, P^{6}\, Q\, \mathrm{SIN}(2\,T - L)$$

$$- 135\, C_{4,2}\, H\, P^{6}\, Q\, \mathrm{SIN}(2\,T - L) + 555\, S_{4,2}\, K\, P^{4}\, Q\, \mathrm{SIN}(2\,T - L)$$

$$+ 495\, C_{4,2}\, H\, P^{4}\, Q\, \mathrm{SIN}(2\,T - L) - 291\, S_{4,2}\, K\, P^{2}\, Q\, \mathrm{SIN}(2\,T - L)$$

$$- 159\, C_{4,2}\, H\, P^{2}\, Q\, \mathrm{SIN}(2\,T - L) - 29\, S_{4,2}\, K\, Q\, \mathrm{SIN}(2\,T - L)$$

$$- 61\, C_{4,2}\, H\, Q\, \mathrm{SIN}(2\,T - L) + 45\, C_{4,2}\, K\, P^{7}\, \mathrm{SIN}(2\,T - L)$$

$$- 45\, S_{4,2}\, H\, P^{7}\, \mathrm{SIN}(2\,T - L) - 105\, C_{4,2}\, K\, P^{5}\, \mathrm{SIN}(2\,T - L)$$

$$+ 45\, S_{4,2}\, H\, P^{5}\, \mathrm{SIN}(2\,T - L) - 51\, C_{4,2}\, K\, P^{3}\, \mathrm{SIN}(2\,T - L)$$

$$+ 99\, S_{4,2}\, H\, P^{3}\, \mathrm{SIN}(2\,T - L) + 43\, C_{4,2}\, K\, P\, \mathrm{SIN}(2\,T - L)$$

$$- 47\, S_{4,2}\, H\, P\, \mathrm{SIN}(2\,T - L) - 45\, C_{4,2}\, K\, Q^{7}\, \mathrm{COS}(2\,T - L)$$

$$+ 45\, S_{4,2}\, H\, Q^{7}\, \mathrm{COS}(2\,T - L) - 135\, S_{4,2}\, K\, P\, Q^{6}\, \mathrm{COS}(2\,T - L)$$

$$- 135\, C_{4,2}\, H\, P\, Q^{6}\, \mathrm{COS}(2\,T - L) + 45\, C_{4,2}\, K\, P^{2}\, Q^{5}\, \mathrm{COS}(2\,T - L)$$

$$- 45 S_{4,2} H P^{2} Q^{5} \cos(2T - L) + 345 C_{4,2} K Q^{5} \cos(2T - L)$$

$$- 405 S_{4,2} H Q^{5} \cos(2T - L) - 225 S_{4,2} K P^{3} Q^{4} \cos(2T - L)$$

$$- 225 C_{4,2} H P^{3} Q^{4} \cos(2T - L) + 795 S_{4,2} K P Q^{4} \cos(2T - L)$$

$$+ 855 C_{4,2} H P Q^{4} \cos(2T - L) + 225 C_{4,2} K P^{4} Q^{3} \cos(2T - L)$$

$$- 225 S_{4,2} H P^{4} Q^{3} \cos(2T - L)$$

$$- 210 C_{4,2} K P^{2} Q^{3} \cos(2T - L) + 90 S_{4,2} H P^{2} Q^{3} \cos(2T - L)$$

$$- 309 C_{4,2} K Q^{3} \cos(2T - L) + 441 S_{4,2} H Q^{3} \cos(2T - L)$$

$$- 45 S_{4,2} K P^{5} Q^{2} \cos(2T - L) - 45 C_{4,2} H P^{5} Q^{2} \cos(2T - L)$$

$$+ 690 S_{4,2} K P^{3} Q^{2} \cos(2T - L)$$

$$+ 810 C_{4,2} H P^{3} Q^{2} \cos(2T - L) - 651 S_{4,2} K P Q^{2} \cos(2T - L)$$

$$- 699 C_{4,2} H P Q^{2} \cos(2T - L) + 135 C_{4,2} K P^{6} Q \cos(2T - L)$$

$$- 135 S_{4,2} H P^{6} Q \cos(2T - L) - 555 C_{4,2} K P^{4} Q \cos(2T - L)$$

$$+ 495 S_{4,2} H P^{4} Q \cos(2T - L) + 291 C_{4,2} K P^{2} Q \cos(2T - L)$$

$$- 159 S_{4,2} H P^2 Q \cos(2 T - L) + 29 C_{4,2} K Q \cos(2 T - L)$$

$$- 61 S_{4,2} H Q \cos(2 T - L) + 45 S_{4,2} K P^7 \cos(2 T - L)$$

$$+ 45 C_{4,2} H P^7 \cos(2 T - L) - 105 S_{4,2} K P^5 \cos(2 T - L)$$

$$- 45 C_{4,2} H P^5 \cos(2 T - L) - 51 S_{4,2} K P^3 \cos(2 T - L)$$

$$- 99 C_{4,2} H P^3 \cos(2 T - L) + 43 S_{4,2} K P \cos(2 T - L)$$

$$+ 47 C_{4,2} H P \cos(2 T - L))/(8 A^{11/2} (Q^2 + P^2 + 1)^3)$$

$$\left(\frac{DQ}{DT}\right)_{4,2} = 15\ MU\ RE^{4}\ (45\ C_{4,2}\ K\ Q^{7}\ SIN(2\ T - L)$$

$$- 45\ S_{4,2}\ H\ Q^{7}\ SIN(2\ T - L) + 135\ S_{4,2}\ K\ P\ Q^{6}\ SIN(2\ T - L)$$

$$+ 135\ C_{4,2}\ H\ P\ Q^{6}\ SIN(2\ T - L) - 45\ C_{4,2}\ K\ P^{2}\ Q^{5}\ SIN(2\ T - L)$$

$$+ 45\ S_{4,2}\ H\ P^{2}\ Q^{5}\ SIN(2\ T - L) - 45\ C_{4,2}\ K\ Q^{5}\ SIN(2\ T - L)$$

$$+ 105\ S_{4,2}\ H\ Q^{5}\ SIN(2\ T - L) + 225\ S_{4,2}\ K\ P^{3}\ Q^{4}\ SIN(2\ T - L)$$

$$+ 225\ C_{4,2}\ H\ P^{3}\ Q^{4}\ SIN(2\ T - L) - 495\ S_{4,2}\ K\ P\ Q^{4}\ SIN(2\ T - L)$$

$$- 555\ C_{4,2}\ H\ P\ Q^{4}\ SIN(2\ T - L) - 225\ C_{4,2}\ K\ P^{4}\ Q^{3}\ SIN(2\ T - L)$$

$$+ 225\ S_{4,2}\ H\ P^{4}\ Q^{3}\ SIN(2\ T - L)$$

$$+ 810\ C_{4,2}\ K\ P^{2}\ Q^{3}\ SIN(2\ T - L)$$

$$- 690\ S_{4,2}\ H\ P^{2}\ Q^{3}\ SIN(2\ T - L) - 99\ C_{4,2}\ K\ Q^{3}\ SIN(2\ T - L)$$

$$+ 51\ S_{4,2}\ H\ Q^{3}\ SIN(2\ T - L) + 45\ S_{4,2}\ K\ P^{5}\ Q^{2}\ SIN(2\ T - L)$$

$$+ 45\ C_{4,2}\ H\ P^{5}\ Q^{2}\ SIN(2\ T - L) - 90\ S_{4,2}\ K\ P^{3}\ Q^{2}\ SIN(2\ T - L)$$

$$- 210\ C_{4,2}\ H\ P^{3}\ Q^{2}\ SIN(2\ T - L) + 159\ S_{4,2}\ K\ P\ Q^{2}\ SIN(2\ T - L)$$

$$+ 291\, C_{4,2}\, H\, P\, Q^{2} \sin(2T - L) - 135\, C_{4,2}\, K\, P^{6}\, Q \sin(2T - L)$$

$$+ 135\, S_{4,2}\, H\, P^{6}\, Q \sin(2T - L) + 855\, C_{4,2}\, K\, P^{4}\, Q \sin(2T - L)$$

$$- 795\, S_{4,2}\, H\, P^{4}\, Q \sin(2T - L) - 699\, C_{4,2}\, K\, P^{2}\, Q \sin(2T - L)$$

$$+ 651\, S_{4,2}\, H\, P^{2}\, Q \sin(2T - L) + 47\, C_{4,2}\, K\, Q \sin(2T - L)$$

$$- 43\, S_{4,2}\, H\, Q \sin(2T - L) - 45\, S_{4,2}\, K\, P^{7} \sin(2T - L)$$

$$- 45\, C_{4,2}\, H\, P^{7} \sin(2T - L) + 405\, S_{4,2}\, K\, P^{5} \sin(2T - L)$$

$$+ 345\, C_{4,2}\, H\, P^{5} \sin(2T - L) - 441\, S_{4,2}\, K\, P^{3} \sin(2T - L)$$

$$- 309\, C_{4,2}\, H\, P^{3} \sin(2T - L) + 61\, S_{4,2}\, K\, P \sin(2T - L)$$

$$+ 29\, C_{4,2}\, H\, P \sin(2T - L) + 45\, S_{4,2}\, K\, Q^{7} \cos(2T - L)$$

$$+ 45\, C_{4,2}\, H\, Q^{7} \cos(2T - L) - 135\, C_{4,2}\, K\, P\, Q^{6} \cos(2T - L)$$

$$+ 135\, S_{4,2}\, H\, P\, Q^{6} \cos(2T - L) - 45\, S_{4,2}\, K\, P^{2}\, Q^{5} \cos(2T - L)$$

$$- 45\, C_{4,2}\, H\, P^{2}\, Q^{5} \cos(2T - L) - 45\, S_{4,2}\, K\, Q^{5} \cos(2T - L)$$

$$- 105\, C_{4,2}\, H\, Q^{5} \cos(2T - L) - 225\, C_{4,2}\, K\, P^{3} Q^{4} \cos(2T - L)$$

$$+ 225\, S_{4,2}\, H\, P^{3} Q^{4} \cos(2T - L) + 495\, C_{4,2}\, K\, P\, Q^{4} \cos(2T - L)$$

$$- 555\, S_{4,2}\, H\, P\, Q^{4} \cos(2T - L) - 225\, S_{4,2}\, K\, P^{4} Q^{3} \cos(2T - L)$$

$$- 225\, C_{4,2}\, H\, P^{4} Q^{3} \cos(2T - L)$$

$$+ 810\, S_{4,2}\, K\, P^{2} Q^{3} \cos(2T - L)$$

$$+ 690\, C_{4,2}\, H\, P^{2} Q^{3} \cos(2T - L) - 99\, S_{4,2}\, K\, Q^{3} \cos(2T - L)$$

$$- 51\, C_{4,2}\, H\, Q^{3} \cos(2T - L) - 45\, C_{4,2}\, K\, P^{5} Q^{2} \cos(2T - L)$$

$$+ 45\, S_{4,2}\, H\, P^{5} Q^{2} \cos(2T - L) + 90\, C_{4,2}\, K\, P^{3} Q^{2} \cos(2T - L)$$

$$- 210\, S_{4,2}\, H\, P^{3} Q^{2} \cos(2T - L) - 159\, C_{4,2}\, K\, P\, Q^{2} \cos(2T - L)$$

$$+ 291\, S_{4,2}\, H\, P\, Q^{2} \cos(2T - L) - 135\, S_{4,2}\, K\, P^{6} Q \cos(2T - L)$$

$$- 135\, C_{4,2}\, H\, P^{6} Q \cos(2T - L) + 855\, S_{4,2}\, K\, P^{4} Q \cos(2T - L)$$

$$+ 795\, C_{4,2}\, H\, P^{4} Q \cos(2T - L) - 699\, S_{4,2}\, K\, P^{2} Q \cos(2T - L)$$

$$- 651\, C_{4,2}\, H\, P^{2} Q \cos(2T - L) + 47\, S_{4,2}\, K\, Q \cos(2T - L)$$

$$+ 43 C_{4, 2} H Q \cos(2 T - L) + 45 C_{4, 2} K P^{7} \cos(2 T - L)$$

$$- 45 S_{4, 2} H P^{7} \cos(2 T - L) - 405 C_{4, 2} K P^{5} \cos(2 T - L)$$

$$+ 345 S_{4, 2} H P^{5} \cos(2 T - L) + 441 C_{4, 2} K P^{3} \cos(2 T - L)$$

$$- 309 S_{4, 2} H P^{3} \cos(2 T - L) - 61 C_{4, 2} K P \cos(2 T - L)$$

$$+ 29 S_{4, 2} H P \cos(2 T - L))/(8 A^{7} N (Q^{2} + P^{2} + 1)^{3})$$

$$(U_{4,4})_{FULL} = - 105\ MU\ RE^{4} (((60 S_{4,4} K^{2} + 120 C_{4,4} H K$$

$$- 60 S_{4,4} H^{2}) Q^{4} + (- 240 C_{4,4} K^{2} + 480 S_{4,4} H K$$

$$+ 240 C_{4,4} H^{2}) P Q^{3} + ((- 360 S_{4,4} K^{2} - 720 C_{4,4} H K$$

$$+ 360 S_{4,4} H^{2}) P^{2} + 8 S_{4,4} K^{2} + 8 S_{4,4} H^{2} + 8 S_{4,4}) Q^{2}$$

$$+ ((240 C_{4,4} K^{2} - 480 S_{4,4} H K - 240 C_{4,4} H^{2}) P^{3}$$

$$+ (- 16 C_{4,4} K^{2} - 16 C_{4,4} H^{2} - 16 C_{4,4}) P) Q$$

$$+ (60 S_{4,4} K^{2} + 120 C_{4,4} H K - 60 S_{4,4} H^{2}) P^{4}$$

$$+ (- 8 S_{4,4} K^{2} - 8 S_{4,4} H^{2} - 8 S_{4,4}) P^{2} + S_{4,4} K^{2}$$

$$- 2 C_{4,4} H K - S_{4,4} H^{2}) SIN(2 (2 T - L))$$

$$+ ((- 60 C_{4,4} K^{2} + 120 S_{4,4} H K + 60 C_{4,4} H^{2}) Q^{4}$$

$$+ (- 240 S_{4,4} K^{2} - 480 C_{4,4} H K + 240 S_{4,4} H^{2}) P Q^{3}$$

$$+ ((360 C_{4,4} K^{2} - 720 S_{4,4} H K - 360 C_{4,4} H^{2}) P^{2} - 8 C_{4,4} K^{2}$$

$$- 8 C_{4,4} H^{2} - 8 C_{4,4}) Q^{2} + ((240 S_{4,4} K^{2} + 480 C_{4,4} H K$$

$$- 240\, S_{4,4}\, H^{2})\, P^{3} + (- 16\, S_{4,4}\, K^{2} - 16\, S_{4,4}\, H^{2} - 16\, S_{4,4})\, P)$$

$$Q + (- 60\, C_{4,4}\, K^{2} + 120\, S_{4,4}\, H\, K + 60\, C_{4,4}\, H^{2})\, P^{4}$$

$$+ (8\, C_{4,4}\, K^{2} + 8\, C_{4,4}\, H^{2} + 8\, C_{4,4})\, P^{2} - C_{4,4}\, K^{2}$$

$$- 2\, S_{4,4}\, H\, K + C_{4,4}\, H^{2})\, \mathrm{COS}(2\, (2\, T - L)))$$

$$/(2\, A^{5}\, (Q^{2} + P^{2} + 1)^{4})$$

$$(U_{4,4})_{TRUNC} = -420\ MU\ RE^4$$

$$((S_{4,4} Q^2 - 2 C_{4,4} P Q - S_{4,4} P^2) SIN(2 (2 T - L))$$

$$+ (- C_{4,4} Q^2 - 2 S_{4,4} P Q + C_{4,4} P^2) COS(2 (2 T - L)))$$

$$/(A^5 (Q^2 + P^2 + 1)^4)$$

$$\left(\frac{DA}{DT}\right)_{4,4} = 1680\,\Big(\big(C_{4,4}\,Q^{2} + 2\,S_{4,4}\,P\,Q - C_{4,4}\,P^{2}\big)\,SIN(2\,(2\,T - L)) + \big(S_{4,4}\,Q^{2} - 2\,C_{4,4}\,P\,Q - S_{4,4}\,P^{2}\big)\,COS(2\,(2\,T - L))\Big)\,SQRT(MU)\,RE^{4}/\big((Q^{2} + P^{2} + 1)^{4}\,SQRT(A)^{9}\big)$$

$$\left(\frac{DH}{DT}\right)_{4,4} = -105\,(((48\,S_{4,4}\,K + 60\,C_{4,4}\,H)\,Q^{4}$$

$$+ (-216\,C_{4,4}\,K + 240\,S_{4,4}\,H)\,P\,Q^{3}$$

$$+ ((-360\,S_{4,4}\,K - 360\,C_{4,4}\,H)\,P^{2} + 12\,S_{4,4}\,K + 4\,C_{4,4}\,H)\,Q^{2}$$

$$+ ((264\,C_{4,4}\,K - 240\,S_{4,4}\,H)\,P^{3}$$

$$+ (-24\,C_{4,4}\,K + 8\,S_{4,4}\,H)\,P)\,Q + (72\,S_{4,4}\,K + 60\,C_{4,4}\,H)\,P^{4}$$

$$+ (-12\,S_{4,4}\,K - 4\,C_{4,4}\,H)\,P^{2} + S_{4,4}\,K - C_{4,4}\,H)$$

$$\mathrm{SIN}(2\,(2\,T - L)) + ((-48\,C_{4,4}\,K + 60\,S_{4,4}\,H)\,Q^{4}$$

$$+ (-216\,S_{4,4}\,K - 240\,C_{4,4}\,H)\,P\,Q^{3}$$

$$+ ((360\,C_{4,4}\,K - 360\,S_{4,4}\,H)\,P^{2} - 12\,C_{4,4}\,K + 4\,S_{4,4}\,H)\,Q^{2}$$

$$+ ((264\,S_{4,4}\,K + 240\,C_{4,4}\,H)\,P^{3}$$

$$+ (-24\,S_{4,4}\,K - 8\,C_{4,4}\,H)\,P)\,Q$$

$$+ (-72\,C_{4,4}\,K + 60\,S_{4,4}\,H)\,P^{4} + (12\,C_{4,4}\,K - 4\,S_{4,4}\,H)\,P^{2}$$

$$- C_{4,4}\,K - S_{4,4}\,H)\,\mathrm{COS}(2\,(2\,T - L)))\,\mathrm{SQRT}(MU)\,RE^{4}$$

$$/((Q^{2} + P^{2} + 1)^{4}\,\mathrm{SQRT}(A)^{11})$$

$$
\begin{aligned}
\left(\frac{DK}{DT}\right)_{4,4} = 105\,\Big(\big(&(60\,C_{4,4}\,K - 72\,S_{4,4}\,H)\,Q^{4} \\
&+ (240\,S_{4,4}\,K + 264\,C_{4,4}\,H)\,P\,Q^{3} \\
&+ ((-\,360\,C_{4,4}\,K + 360\,S_{4,4}\,H)\,P^{2} - 4\,C_{4,4}\,K + 12\,S_{4,4}\,H)\,Q^{2} \\
&+ ((-\,240\,S_{4,4}\,K - 216\,C_{4,4}\,H)\,P^{3} \\
&+ (-\,8\,S_{4,4}\,K - 24\,C_{4,4}\,H)\,P)\,Q + (60\,C_{4,4}\,K - 48\,S_{4,4}\,H)\,P^{4} \\
&+ (4\,C_{4,4}\,K - 12\,S_{4,4}\,H)\,P^{2} - C_{4,4}\,K - S_{4,4}\,H\big) \\
&\mathrm{SIN}(2\,(2\,T - L)) + \big((60\,S_{4,4}\,K + 72\,C_{4,4}\,H)\,Q^{4} \\
&+ (-\,240\,C_{4,4}\,K + 264\,S_{4,4}\,H)\,P\,Q^{3} \\
&+ ((-\,360\,S_{4,4}\,K - 360\,C_{4,4}\,H)\,P^{2} - 4\,S_{4,4}\,K - 12\,C_{4,4}\,H)\,Q^{2} \\
&+ ((240\,C_{4,4}\,K - 216\,S_{4,4}\,H)\,P^{3} + (8\,C_{4,4}\,K - 24\,S_{4,4}\,H)\,P) \\
&Q + (60\,S_{4,4}\,K + 48\,C_{4,4}\,H)\,P^{4} + (4\,S_{4,4}\,K + 12\,C_{4,4}\,H)\,P^{2} \\
&- S_{4,4}\,K + C_{4,4}\,H\big)\,\mathrm{COS}(2\,(2\,T - L))\Big)\,\mathrm{SQRT}(MU)\,RE^{4} \\
&/((Q^{2} + P^{2} + 1)^{4}\,\mathrm{SQRT}(A)^{11})
\end{aligned}
$$

$$\left(\frac{DL}{DT}\right)_{4,\,4} = 420\,\Big((S_{4,\,4}\,Q^{2} - 2\,C_{4,\,4}\,P\,Q - S_{4,\,4}\,P^{2})\,\mathrm{SIN}(2\,(2\,T - L)) + (-\,C_{4,\,4}\,Q^{2} - 2\,S_{4,\,4}\,P\,Q + C_{4,\,4}\,P^{2})\,\mathrm{COS}(2\,(2\,T - L))\Big)\,(3\,Q^{2} + 3\,P^{2} - 11)\,\mathrm{SQRT}(MU)\,RE^{4} / \left((Q^{2} + P^{2} + 1)^{4}\,\mathrm{SQRT}(A)^{11}\right)$$

$$
\left(\frac{DP}{DT}\right)_{4,4} = 210\,\Big(\big(3 S_{4,4} Q^{3} - 9 C_{4,4} P Q^{2} + (-9 S_{4,4} P^{2} - S_{4,4}) Q + 3 C_{4,4} P^{3} + C_{4,4} P\big) \sin(2\,(2\,T - L)) + \big(-3 C_{4,4} Q^{3} - 9 S_{4,4} P Q^{2} + (9 C_{4,4} P^{2} + C_{4,4}) Q + 3 S_{4,4} P^{3} + S_{4,4} P\big) \cos(2\,(2\,T - L))\Big) \sqrt{MU}\, RE^{4} \Big/ \left((Q^{2} + P^{2} + 1)^{3} \sqrt{A}^{11}\right)
$$

$$\left(\frac{DQ}{DT}\right)_{4,4} = -210\,\Big(\big(3\,C_{4,4}\,Q^{3} + 9\,S_{4,4}\,P\,Q^{2} + (-9\,C_{4,4}\,P^{2} + C_{4,4})\,Q - 3\,S_{4,4}\,P^{3} + S_{4,4}\,P\big)\,\mathrm{SIN}(2\,(2\,T - L)) + \big(3\,S_{4,4}\,Q^{3} - 9\,C_{4,4}\,P\,Q^{2} + (-9\,S_{4,4}\,P^{2} + S_{4,4})\,Q + 3\,C_{4,4}\,P^{3} - C_{4,4}\,P\big)\,\mathrm{COS}(2\,(2\,T - L))\Big)\,\mathrm{SQRT}(MU)\,RE^{4} / \left((Q^{2} + P^{2} + 1)^{3}\,\mathrm{SQRT}(A)^{11}\right)$$

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