XIV. NETWORK SYNTHESIS

Prof. E. A. Guillemin
Prof. P. M. Lewis II
Dr. M. V. Cerrillo
E. F. Bolinder
H. B. Lee
J. C. Pinson
T. G. Stockham

A. SOME GENERALIZATIONS OF LINEAR NETWORK ANALYSIS THAT ARE USEFUL WHEN ACTIVE AND/OR NONBILATERAL ELEMENTS ARE INVOLVED

Procedures for the analysis of linear networks are ordinarily confined to a consideration of configurations in which the branches or elements are two-terminal devices, such as the familiar resistances, inductances, and capacitances that characterize passive circuits. When active elements, such as vacuum tubes or transistors, are included, this simple topological picture no longer applies, since these elements are characterized schematically as boxes from which three or more terminals emanate. A network containing such elements, as well as the ordinary two-terminal elements, is shown in Fig. XIV-1. We shall show that only a minor addition to the existing analysis procedure is needed to accommodate this more general type of network.

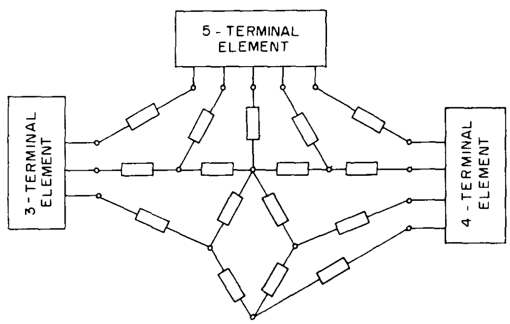

Fig. XIV-1. Network containing multiterminal elements, as well as the usual two-terminal elements.

It is well known that a box with p + 1 terminals is uniquely characterized by p voltages and p currents. The volt-ampere relations for the multiterminal device may be written

$$\begin{aligned} v_1 &= r_{11}j_1 + \dots + r_{1p}j_p \\ &\dots\dots\dots\dots \\ v_p &= r_{p1}j_1 + \dots + r_{pp}j_p \end{aligned} \tag{1}$$

or in inverse form,

$$\begin{aligned} j_1 &= g_{11}v_1 + \dots + g_{1p}v_p \\ &\dots\dots\dots\dots \\ j_p &= g_{p1}v_1 + \dots + g_{pp}v_p \end{aligned} \tag{2}$$

in which $j_1 \dots j_p$ are currents that enter terminals $1, 2, \dots p$, and return through the common terminal 0 (see Fig. XIV-2a), and $v_1 \dots v_p$ are voltage drops measured from

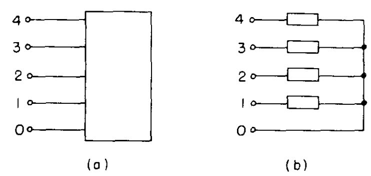

Fig. XIV-2. (a) Five-terminal element. (b) Topological equivalent of (a).

the corresponding terminals to the common datum.

Topologically, the device may be replaced by the set of ordinary branches, as shown in Fig. XIV-2b. This schematic is topologically adequate for the representation of the multiterminal device, and relations 1 or 2 are analytically adequate. The procedure for setting up equilibrium equations on either a node or loop basis follows well-established channels (1).

Figure XIV-3 is a version of the circuit of Fig. XIV-1 that treats the multiterminal elements in the manner shown in Fig. XIV-2. To this network graph we apply the familiar methods of choosing and algebraically defining an appropriate set of current or voltage variables. These defining equations, together with the pertinent Kirchhoff equations and volt-ampere relations for the branches, are combined to obtain the desired equilibrium equations. If the multiterminal device incorporates any current or voltage constraints, then, in the corresponding volt-ampere relations, the appropriate quantities v_k and j_k are replaced by $(v_k + e_{sk})$ and $(j_k + i_{sk})$, just as they are with the ordinary two-terminal elements. Sources may thus be dealt with in normal fashion.

If the volt-ampere relations (Eq. 1 or Eq. 2) exhibit a dissymmetrical matrix, the multiterminal device is nonbilateral; if this matrix defines a nonpositive quadratic form, the device is active. It may be either, or both, or neither.

If a given linear network contains active and/or nonbilateral elements, it may, for all analysis and synthesis purposes, be replaced by a passive, bilateral, reference network in which either the excitation or response quantities (for example, source voltages or loop currents) are subjected to an appropriate real nonsingular transformation. In order to demonstrate the truth of this statement, and to show how a pertinent transformation can be constructed, it is necessary to recall some fundamental analysis theory developed by the writer a long time ago and published in the form of class notes (2). Although these ideas are merely of collateral interest as far as a discussion of passive bilateral network analysis is concerned, they give the key to the application of transformation theory to the analysis and synthesis of linear active and/or nonbilateral networks. A condensed presentation of this approach to the analysis problem is therefore essential at this point.

The definition of loop currents is given by writing ℓ equations that express these currents as any desired linear combinations of the b branch currents. Since the matrix of these equations is not square, we cannot, by the simple process of inversion, obtain the corresponding expressions for the branch currents in terms of the loop currents defined by the matrix. This fundamental dilemma in the usual approach to network

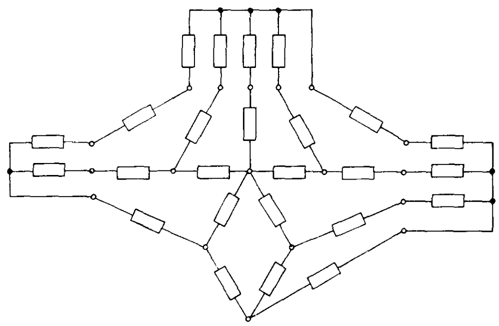

Fig. XIV-3. Topological equivalent for the network of Fig. XIV-1.

analysis is resolved here by observing that n additional equations of the same type as the ℓ defining equations for the loop currents are provided by the Kirchhoff relations that express source currents applied to node pairs as linear combinations of branch currents in the pertinent cut sets. When these are also written, we have a set of b equations with a square matrix which is nonsingular if the loop currents and the cut sets are properly chosen. The procedure for obtaining inverse relations expressing the branch currents in terms of the loop and source currents is now straightforward, and there is no ambiguity about the uniqueness of this relationship, as there seems to be in the usual approach to this problem.

We recognize, moreover, that there is a kinship between loop currents and source currents. The latter are loop currents also, for they circulate upon contours that can be selected in the same forthright manner that characterizes the familiar choice of loop currents when they are defined topologically rather than algebraically. This fact may seem a bit puzzling at first, since a source current applied to a node pair actually distributes itself throughout all branches of the network. Nevertheless, arbitrary circulating paths can be assumed for the source currents, since circulating currents automatically fulfill Kirchhoff's current law, which is the only law that currents need to fulfill.

This kinship between loop and source currents renders the distinction between the two kinds of currents rather flexible. In fact, we may say that there is no distinction between them except that the source currents are the currents whose values we know, and the loop currents are the currents whose values we do not know. We do not even have to commit ourselves at the outset as to which is which.

Another rather interesting result follows from this attitude. Since the number of loops ℓ equals the number of loop currents, and the number of node pairs n equals the number of source currents, we see that although $\ell + n = b$ remains fixed, the split as to the integer assigned to ℓ and that assigned to n remains flexible. If we decide that the value of a certain source current is unknown and allot that current to the group of loop currents, then the pertinent source becomes a voltage and we must concede that we now know one of the node-pair voltages. (A source, in contrast with a passive element for which a volt-ampere relation exists, is a branch for which either the current is fixed and the voltage remains arbitrary or the voltage is fixed and the current remains arbitrary.) We have one more unknown on the loop basis, and one less on the node basis.

The complete set of relations expressing loop and source currents in terms of branch currents reads

$$
\begin{aligned}
&\alpha_{11}j_1 + \alpha_{12}j_2 + \dots + \alpha_{1b}j_b = i_1 \\
&\dots\dots\dots\dots\dots\dots \\
&\alpha_{\ell 1}j_1 + \alpha_{\ell 2}j_2 + \dots + \alpha_{\ell b}j_b = i_\ell \\
&\alpha_{\ell+1,1}\, j_1 + \alpha_{\ell+1,2}\, j_2 + \dots + \alpha_{\ell+1,b}\, j_b = i_{s1} \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
&\alpha_{b1}j_1 + \alpha_{b2}j_2 + \dots + \alpha_{bb}j_b = i_{sn}
\end{aligned}
\tag{3}
$$

If we define the column matrices

$$
j = \begin{bmatrix} j_1 \\ \vdots \\ j_b \end{bmatrix} \quad \text{and} \quad i = \begin{bmatrix} i_1 \\ \vdots \\ i_\ell \\ i_{s1} \\ \vdots \\ i_{sn} \end{bmatrix}
\tag{4}
$$

and denote the matrix of 3 by α, then the matrix equivalent of 3 reads

$$\alpha j = i \tag{5}$$

and the inverse reads

$$\alpha^{-1} i = j \tag{6}$$

The first ℓ equations in the set 3 define the loop currents; the remaining n equations are Kirchhoff current-law equations. The quantities $i_{s1} \dots i_{sn}$ appearing on the right-hand sides of these equations are current sources that feed the pertinent node pairs. The last n rows of the matrix α yield the cut-set schedule, as usually defined.

If a tree is chosen and the link currents, numbered from 1 to ℓ, are identified with loop currents, then the α matrix assumes the special form

$$
\alpha = \begin{bmatrix}
1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
a_{\ell+1,1} & a_{\ell+1,2} & \cdots & a_{\ell+1,\ell} & 1 & 0 & \cdots & 0 \\
a_{\ell+2,1} & a_{\ell+2,2} & \cdots & a_{\ell+2,\ell} & 0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{b1} & a_{b2} & \cdots & a_{b\ell} & 0 & 0 & \cdots & 1
\end{bmatrix} \tag{7}
$$

which may be abbreviated as

$$
\alpha = \left[\begin{array}{c|c} u_\ell & 0 \\ \hline a_{n\ell} & u_n \end{array}\right] \tag{8}
$$

in which the submatrices u_ℓ and u_n are unit matrices of order ℓ and n, respectively, the zero in the upper right-hand corner is an $\ell \times n$ null matrix, and $a_{n\ell}$ is a submatrix of n rows and ℓ columns. Its elements are ± 1 or zero, according to whether the branch in question is or is not contained in the pertinent cut set. Since the tree branches, numbered $\ell + 1$ to b, are contained singly in the respective cut sets, the last n columns and the last n rows of α form a unit matrix of order n (the submatrix u_n in Eq. 8).

Analogously, on a voltage basis, we define node-pair voltage variables as linear combinations of the branch voltages. Algebraically, we obtain n equations which may be arbitrarily written, subject only to the condition that they be independent of each other and of the equations in a set of ℓ Kirchhoff voltage-law relations that express source voltages as linear combinations of branch voltages. The complete set of b equations thus obtained may be inverted to yield the branch voltages in terms of the node-pair and source voltages.

The situation, again, is clearly unambiguous and unique, although no a priori distinction need be made between source voltages and node-pair voltages, which are alike in kind and differ only in that the former are presumably known and the latter unknown. Again, the division of the integer b into its additive components ℓ and n may be revised according to what we wish to regard as known and unknown. The same kinship exists between the source voltages that act around loops and the node-pair voltages that exist, on a current basis, between the loop and the source currents.

In writing the complete set of b equations in this case, we write the Kirchhoff equations first and the defining equations for the node-pair voltages last. Thus we have

$$\begin{aligned}
&\beta_{11}v_1 + \beta_{12}v_2 + \dots + \beta_{1b}v_b = e_{s1} \\
&\dots\dots\dots\dots\dots\dots \\
&\beta_{\ell 1}v_1 + \beta_{\ell 2}v_2 + \dots + \beta_{\ell b}v_b = e_{s\ell} \\
&\beta_{\ell+1,1}v_1 + \beta_{\ell+1,2}v_2 + \dots + \beta_{\ell+1,b}v_b = e_1 \\
&\dots\dots\dots\dots\dots\dots \\
&\beta_{b1}v_1 + \beta_{b2}v_2 + \dots + \beta_{bb}v_b = e_n
\end{aligned} \tag{9}$$

If we define the column matrices

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_b \end{bmatrix} \quad \text{and} \quad e = \begin{bmatrix} e_{s1} \\ \vdots \\ e_{s\ell} \\ e_1 \\ \vdots \\ e_n \end{bmatrix} \tag{10}$$

and denote matrix 9 by β, then these equations in matrix form read

$$\beta v = e \tag{11}$$

with the inverse

$$\beta^{-1}e = v \tag{12}$$

The quantities $e_{s1} \dots e_{s\ell}$ are voltage sources acting upon the contours of loops for which the Kirchhoff voltage-law equations are written. The first ℓ rows of the matrix β yield the tie-set schedule, as usually defined.

If a tree is chosen and the tree-branch voltages, numbered from $\ell + 1$ to b, are identified with node-pair voltages, then the β matrix assumes the special form

$$\beta = \begin{bmatrix}
1 & 0 & \dots & 0 & \beta_{1,\ell+1} & \beta_{1,\ell+2} & \dots & \beta_{1b} \\
0 & 1 & \dots & 0 & \beta_{2,\ell+1} & \beta_{2,\ell+2} & \dots & \beta_{2b} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & 1 & \beta_{\ell,\ell+1} & \beta_{\ell,\ell+2} & \dots & \beta_{\ell b} \\
0 & \dots & \dots & 0 & 1 & 0 & \dots & 0 \\
0 & \dots & \dots & 0 & 0 & 1 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & \dots & \dots & 0 & 0 & 0 & \dots & 1
\end{bmatrix} \tag{13}$$

which may be abbreviated as

$$\beta = \left[\begin{array}{c|c} u_{\ell} & \beta_{\ell n} \\ \hline 0 & u_n \end{array}\right] \tag{14}$$

Its interpretation is analogous to that discussed for the abbreviated matrix 8. Elements in the submatrix $\beta_{\ell n}$ are ± 1 or zero, according to whether the branch in question is or is not part of the pertinent tie set. Since the links, numbered 1 to ℓ, are contained singly in the respective tie sets, the first ℓ columns and the first ℓ rows of β form a unit matrix of order ℓ (the submatrix u_{ℓ} in Eq. 14).

We recall that the columns of a cut-set schedule yield the coefficients in a set of equations that expresses the branch voltages in terms of the node-pair voltages; and that the columns of a tie-set schedule yield the coefficients in a set of equations that expresses the branch currents in terms of the loop currents. The latter may be regarded as expressed by Eq. 6 and the former by Eq. 12, since the matrices α and β may be interpreted as representing cut-set and tie-set schedules in which the normal loop currents are augmented by including the source currents, and the normal node-pair voltages are augmented by including the source voltages. If the closed paths for which Kirchhoff voltage-law equations are written are the same as those used to define loop currents, and if the cut sets for which the Kirchhoff current-law equations are written pertain to the same node pairs that are used to define node-pair voltages, then we observe that the columns in the matrix β yield coefficients in the relations expressed by Eq. 6, while columns in the matrix α yield coefficients in the relations expressed by Eq. 12. Under these conditions, which we refer to as "consistency conditions" because the Kirchhoff equations are consistent with the defining equations for the variables, it follows that β is the inverse transpose (or the reciprocal) of α and vice versa. That is,

$$\beta = \alpha_t^{-1} = \alpha^* \quad \text{and} \quad \alpha = \beta_t^{-1} = \beta^* \tag{15}$$

It follows from Eqs. 8 and 14 that in this case $\beta_{\ell n}$ is the negative of the transpose of $\alpha_{n\ell}$,

$$\beta_{\ell n} = -(\alpha_{n\ell})_t \tag{16}$$

which may be readily verified, since the matrix 8 multiplied by the transpose of matrix 14 (or vice versa) must yield a unit matrix of order b.

Matrix Eqs. 5 and 11 (or their inverses, Eqs. 6 and 12) contain the information about Kirchhoff relations, as well as the definition of voltage or current variables. All that need be added in order to construct the equilibrium equations, on either a voltage or current basis, is the set of volt-ampere relations linking the branch currents with the branch voltages. In matrix form (3) these read

$$v = Dj \quad \text{or} \quad j = D^{-1}v \tag{17}$$

in which the operator matrix D contains the branch inductance, resistance, and elastance parameter matrices. If an active and/or nonbilateral multiterminal device is involved, the branch resistance matrix (and in D^{-1} the branch conductance matrix) is no longer diagonal but contains some nondiagonal elements. However, the formation of D or of D^{-1} follows precisely the same pattern as for the ordinary linear passive two-terminal elements and needs no further elaboration.

If we want the equilibrium equations on a loop basis, we begin with Eq. 11 whose first ℓ rows are the Kirchhoff voltage-law equations. Since we want these expressed in terms of the loop currents, we express v in Eq. 11 in terms of j by means of Eq. 17, and subsequently express j in terms of i by means of Eq. 6. This double substitution yields

$$\beta \times D \times \alpha^{-1} \times i = e \tag{18}$$

If we want the equilibrium equations on a node basis, we begin with Eq. 5 whose last n rows are the Kirchhoff current-law equations. Since we want these expressed in terms of the node-pair voltages, we express j in Eq. 5 in terms of v by means of Eq. 17, and subsequently express v in terms of e by means of Eq. 12. This double substitution yields

$$\alpha \times D^{-1} \times \beta^{-1} \times e = i \tag{19}$$

which we recognize as the inverse of Eq. 18. In this approach the equilibrium equations on the loop and node bases are contained in matrix equations that are mutually inverse.

Equations 18 and 19 contain, besides the desired equilibrium equations, some rather interesting by-products which we shall place in evidence by partitioning the rows and columns of the resultant matrices into groups of ℓ and n. We shall assume henceforth that consistency conditions 15 are fulfilled, so that the resultant matrices in Eqs. 18 and 19 become symmetrical. We then have for the loop basis,

$$\beta \times D \times \beta_t \times i = e \tag{20}$$

and for the node basis,

$$\alpha \times D^{-1} \times \alpha_t \times e = i \tag{21}$$

Suppose that we let

$$\beta \times D \times \beta_t = Z \tag{22}$$

and

$$\alpha \times D^{-1} \times \alpha_t = Y = Z^{-1} \tag{23}$$

and partition these resultant matrices as follows:

$$Z = \begin{bmatrix} Z_{\ell\ell} & Z_{\ell n} \\ Z_{n\ell} & Z_{nn} \end{bmatrix} \tag{24}$$

$$Y = \begin{bmatrix} Y_{\ell\ell} & Y_{\ell n} \\ Y_{n\ell} & Y_{nn} \end{bmatrix} \tag{25}$$

If, correspondingly, we partition the column matrices e and i,

$$e = \begin{bmatrix} e_s \\ \cdots \\ e_v \end{bmatrix} \quad \text{and} \quad i = \begin{bmatrix} i_v \\ \cdots \\ i_s \end{bmatrix} \tag{26}$$

in which e_s and i_s represent source quantities, and e_v and i_v represent variables, then Eqs. 20 and 21 become

$$\begin{aligned} Z_{\ell\ell} i_v + Z_{\ell n} i_s &= e_s \\ Z_{n\ell} i_v + Z_{nn} i_s &= e_v \end{aligned} \tag{27}$$

and

$$\begin{aligned} Y_{\ell\ell} e_s + Y_{\ell n} e_v &= i_v \\ Y_{n\ell} e_s + Y_{nn} e_v &= i_s \end{aligned} \tag{28}$$

Equilibrium, on the loop basis, is expressed by the first of the matrix Eqs. 27 in which the second term effects the conversion of current sources into equivalent voltage sources. The second of Eqs. 28, similarly, expresses equilibrium on the node basis, and the conversion of voltage sources to equivalent current sources is given by the first term. The second equation in Eqs. 27 or the first in Eqs. 28 relates the node-pair voltages and the loop currents. These relations may be useful, for example, if for some reason it is desirable to express equilibrium in terms of some loop currents and of some node-pair voltages, on a sort of mixed basis. Since the relations between loop currents and node-pair voltages are available, we can trade variables of one sort for variables of another.

Now if an active and/or nonbilateral element is embedded in an otherwise passive bilateral network, then the branch resistance or conductance matrix has embedded in it a submatrix like that pertaining to either Eq. 1 or Eq. 2, which does not define a positive quadratic form and may be dissymmetrical. The operator matrix D in Eq. 20 or D^{-1} in Eq. 21 exhibits corresponding deviations from the form for passive bilateral networks. (We shall refer to the corresponding network as being non-P and/or non-B, as distinguished from the designation PB in the passive bilateral case.)

By well-known methods, it is always possible to find a real nonsingular matrix t of order b so that $D^{-1} \times t$ has PB character. Equation 21 is rewritten in the form

$$\alpha(D^{-1}t)\, \alpha_t \alpha_t^{-1} t^{-1} \alpha_t e = i \tag{29}$$

Let

$$\alpha_t^{-1} t^{-1} \alpha_t = \tau \tag{30}$$

and

$$\tau e = e' \tag{31}$$

Then Eq. 21 or Eq. 29 becomes

$$\alpha(D^{-1}t)\, \alpha_t e' = i \tag{32}$$

Since $D^{-1}t$ is PB, and transformation 31 is real and nonsingular, Eq. 32 represents the equilibrium, on a node basis, of a linear, passive, bilateral, reference network in which the voltage variables are uniquely and reversibly related to those in the original non-P, non-B network. The matrix τ which yields this relationship is easily obtainable from the transformation matrix t by the collinear transformation expressed by Eq. 30.

Observe that these manipulations are not possible unless the cut-set matrix α possesses an inverse! That is why the modified approach to the analysis problem given here is essential.

By analogy with Eq. 23, we can write

$$\alpha(D^{-1}t)\, \alpha_t = \hat{Y} \tag{33}$$

and regard $\hat{Y}$ as the node-admittance matrix of the PB reference network. Equations 21, 23, 31, and 32 then yield

$$Y = \hat{Y} \tau \tag{34}$$

and if we partition $\hat{Y}$ and τ in the manner shown for Y in Eq. 25, we find that

$$
\begin{aligned}
Y_{\ell\ell} &= \hat{Y}_{\ell\ell}\,\tau_{\ell\ell} + \hat{Y}_{\ell n}\,\tau_{n\ell} \\
Y_{\ell n} &= \hat{Y}_{\ell\ell}\,\tau_{\ell n} + \hat{Y}_{\ell n}\,\tau_{nn} \\
Y_{n\ell} &= \hat{Y}_{n\ell}\,\tau_{\ell\ell} + \hat{Y}_{nn}\,\tau_{n\ell} \\
Y_{nn} &= \hat{Y}_{n\ell}\,\tau_{\ell n} + \hat{Y}_{nn}\,\tau_{nn}
\end{aligned} \tag{35}
$$

If we are interested in the set of driving-point and transfer impedances of a passive, bilateral network with embedded active nonbilateral elements, then we are interested in the quantities e_v relative to i_s in the second of Eqs. 28 for $e_s \equiv 0$; that is to say, we are interested in the inverse of the submatrix Y_{nn} given by the last of Eqs. 35. Even if the passive, bilateral elements are restricted to R's and C's, we see that the determinant of Y_{nn} may have zeros anywhere in the complex s-plane, and hence the elements of Y_{nn}^{-1} (the open-circuit, driving-point, and transfer impedances) can have zeros and poles anywhere in the frequency plane.

If we wish, we can express the non-P, non-B system as a PB reference system in which the currents, instead of the voltages, are subjected to a real nonsingular transformation. To obtain this result we manipulate Eq. 21 as follows:

$$\alpha t^{-1}\alpha^{-1}\alpha t D^{-1}\alpha_t e = i \tag{36}$$

Then, we let

$$\alpha t \alpha^{-1} = \tau \tag{37}$$

and

$$i' = \tau i \tag{38}$$

whereupon premultiplication of Eq. 36 by τ yields

$$\alpha(TD^{-1})\alpha_t e = i' \tag{39}$$

The matrix t is here chosen so that tD^{-1} is PB. The matrix

$$\alpha(tD^{-1})\alpha_t = \hat{Y} \tag{40}$$

is then the node-admittance matrix of a PB reference network for which the equilibrium equations are contained in

$$\hat{Y} e = i' \tag{41}$$

and we now have, in contrast to Eq. 34

$$Y = \tau^{-1}\hat{Y} \tag{42}$$

Conclusions drawn from this result are similar to those just given with regard to Eqs. 34 and 35.

These methods make it possible to express the response functions of active and/or nonbilateral networks as linear or bilinear combinations of the response functions of a passive, bilateral, reference network. The application of this technique to synthesis problems will be discussed in later reports.

E. A. Guillemin

References

1. Cf. E. A. Guillemin, Introductory Circuit Theory (John Wiley and Sons, Inc., New York, 1953), Chap. X.
2. E. A. Guillemin, Notes for Principles of Electrical Communications 6.30 (Massachusetts Institute of Technology, Cambridge, Massachusetts, 1941), Chap. 3, pp. 9-18. (Out of print.)
3. E. A. Guillemin, Introductory Circuit Theory, op. cit., pp. 494-495.

B. DERIVATION OF HILBERT TRANSFORMS AND THE PALEY-WIENER CRITERION FROM CAUCHY'S INTEGRAL FORMULA

The following discussion will show how these relations can be extracted directly from the Cauchy integral formula. Let $\gamma(s)$ be analytic in the right half of the s-plane, inclusive of the j-axis. Physically, $\gamma(s)$ may be a propagation function, in which case we have $\gamma = \alpha + j\beta$ with α and β equal to the loss and phase functions, or it may be a driving-point or transfer impedance function. In either case, we can apply Cauchy's integral formula to the region bounded by the contour shown in Fig. XIV-4, which consists of the j-axis and a semicircular arc of arbitrarily large (but finite) radius that lies entirely in the right half-plane. If s denotes an internal point, and λ a point on the boundary, we have

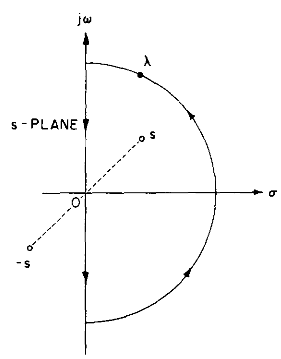

Fig. XIV-4. Contour appropriate to the evaluation of several integrals.

$$\gamma(s) = \frac{1}{2\pi j} \oint \frac{\gamma(\lambda)\, d\lambda}{\lambda - s} \tag{1}$$

and for the external point -s the Cauchy integral law yields

$$0 = \frac{1}{2\pi j} \oint \frac{\gamma(\lambda)\, d\lambda}{\lambda + s} \tag{2}$$

By addition or subtraction of Eqs. 1 and 2, we obtain

$$\gamma(s) = \frac{1}{2\pi j} \oint \left(\frac{1}{\lambda - s} \pm \frac{1}{\lambda + s}\right) \gamma(\lambda)\, d\lambda$$

That is to say, either

$$\gamma(s) = \frac{s}{j\pi} \oint \frac{\gamma(\lambda)\, d\lambda}{\lambda^2 - s^2} \tag{3}$$

or

$$\gamma(s) = \frac{1}{j\pi} \oint \frac{\lambda\, \gamma(\lambda)\, d\lambda}{\lambda^2 - s^2} \tag{4}$$

In either of these integrals, the path of integration may be abridged to the j-axis if the contribution from the arc is zero. If the asymptotic behavior of $\gamma(\lambda)$ is described by $\gamma(\lambda) \to \lambda^a$ for $\lambda \to \infty$, then in Eq. 3 the contribution from the arc is zero if $a < 1$, and in Eq. 4 if $a < 0$. Under these conditions, we can write integrals 3 and 4 in the forms

$$\gamma(s) = \frac{s}{j\pi} \int_{-j\infty}^{j\infty} \frac{\gamma(\lambda)\, d\lambda}{s^2 - \lambda^2} \tag{5}$$

and

$$\gamma(s) = \frac{1}{j\pi} \int_{-j\infty}^{j\infty} \frac{\lambda\, \gamma(\lambda)\, d\lambda}{s^2 - \lambda^2} \tag{6}$$

where the change in algebraic sign of the integrand is the result of traversing the j-axis in the opposite direction from that shown in Fig. XIV-4. Since λ is now restricted to the j-axis, we shall write $\lambda = j\xi$ and consistently have

$$\gamma(\lambda) = \gamma(j\xi) = \alpha(\xi) + j\,\beta(\xi) \tag{7}$$

where $\alpha(\xi)$ and $\beta(\xi)$ are the loss and phase functions along the j-axis, as usually defined. Since these functions are even and odd, respectively, integrals 5 and 6 become

$$\gamma(s) = \frac{s}{\pi} \int_{-\infty}^{\infty} \frac{\alpha(\xi)\, d\xi}{s^2 + \xi^2} \tag{8}$$

$$\gamma(s) = \frac{-1}{\pi} \int_{-\infty}^{\infty} \frac{\beta(\xi)\, \xi\, d\xi}{s^2 + \xi^2} \tag{9}$$

These results may be regarded as a slightly generalized version of the Hilbert transforms that enables computation of the complex propagation function γ for any right half-plane point, as well as for points on the j-axis, since $s = \sigma + j\omega$ and σ may have any finite positive value, as well as the value zero. Specifically, for the interpretation of Eqs. 8 and 9 with $s = j\omega$, we must recognize that

$$\left(\frac{s}{s^2 + \xi^2}\right)_{s\to j\omega} = \frac{1}{2}\left(\frac{1}{s - j\xi} + \frac{1}{s + j\xi}\right)_{s\to j\omega} = \frac{\pi}{2}\left[u_o(\omega-\xi) + u_o(\omega+\xi)\right] - \frac{j\omega}{\omega^2 - \xi^2} \tag{10}$$

and

$$\left(\frac{\xi}{s^2 + \xi^2}\right)_{s\to j\omega} = \frac{1}{2j}\left(\frac{1}{s - j\xi} - \frac{1}{s + j\xi}\right)_{s\to j\omega} = \frac{-j\pi}{2}\left[u_o(\omega-\xi) - u_o(\omega+\xi)\right] - \frac{\xi}{\omega^2 - \xi^2} \tag{11}$$

in which $u_o(\omega)$ is the unit impulse function occurring at $\omega = 0$. In other words, we must not overlook the j-axis impulses that result when the left-hand functions in Eqs. 10 and 11 are evaluated along the j-axis of the s-plane.

For integral 8, we thus obtain

$$\alpha(\omega) + j\beta(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} \left[u_o(\omega-\xi) + u_o(\omega+\xi)\right] \alpha(\xi)\, d\xi + \frac{j\omega}{\pi}\int_{-\infty}^{\infty} \frac{\alpha(\xi)\, d\xi}{\xi^2 - \omega^2} \tag{12}$$

The first of these integrals yields $\alpha(\omega)$, and, therefore, we have

$$\beta(\omega) = \frac{\omega}{\pi}\int_{-\infty}^{\infty} \frac{\alpha(\xi)\, d\xi}{\xi^2 - \omega^2} = \frac{2\omega}{\pi}\int_{0}^{\infty} \frac{\alpha(\xi)\, d\xi}{\xi^2 - \omega^2} \tag{13}$$

Similarly, integral 9 with Eq. 11 substituted gives

$$\alpha(\omega) + j\beta(\omega) = \frac{j}{2}\int_{-\infty}^{\infty} \left[u_o(\omega-\xi) - u_o(\omega+\xi)\right] \beta(\xi)\, d\xi - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\beta(\xi)\, \xi\, d\xi}{\xi^2 - \omega^2} \tag{14}$$

and, since the first of these integrals yields $j\beta(\omega)$, we have

$$\alpha(\omega) = \frac{-1}{\pi} \int_{-\infty}^{\infty} \frac{\beta(\xi)\,\xi\,d\xi}{\xi^2 - \omega^2} = \frac{-2}{\pi} \int_{0}^{\infty} \frac{\beta(\xi)\,\xi\,d\xi}{\xi^2 - \omega^2} \tag{15}$$

Equations 13 and 15 are the familiar Hilbert transforms in the form in which the even and odd character of $\alpha(\omega)$ and $\beta(\omega)$ are taken into account. In Eq. 13 we observe that if the asymptotic behavior of $\alpha(\xi)$ is indicated by writing $\alpha(\xi) \rightarrow \xi^a$ for $\xi \rightarrow \infty$, then a finite value for this integral can result only if $a < 1$. Essentially, this condition is the Paley-Wiener criterion. If we consider integral 8 whence Eq. 13 stems, we can say that the asymptotic character of $\alpha(\xi)$ must be such that this integral has a finite value for all finite values of s in the right half-plane. For this to be so, it is obviously sufficient that the integral have a finite value for $s = 1$ or that

$$\int_{0}^{\infty} \frac{\alpha(\xi)\,d\xi}{1 + \xi^2} \tag{16}$$

be finite, which is the more common form in which the Paley-Wiener criterion is stated.

A related question is: How must the behavior of $\beta(\xi)$ in Eq. 15 be restricted for $\xi \rightarrow \infty$ if this integral is to have a finite value? Here we find that if we write $\beta(\xi) \rightarrow \xi^a$ for $\xi \rightarrow \infty$, we must require $a < 0$. In fact, the conditions $a < 1$ for $\alpha(\xi)$ and $a < 0$ for $\beta(\xi)$ are just the conditions assumed initially for the asymptotic behavior of $\gamma(\lambda)$ in Eqs. 3 and 4, in order to render the contributions from the semicircular arcs zero, so that the subsequent derivation of the Hilbert transforms can be carried out. The resulting restriction upon $\alpha(\xi)$ we accept without reservation because it coincides with that imposed by the Paley-Wiener criterion; but the restriction just stated with respect to $\beta(\xi)$ is somewhat puzzling because we know that phase functions of minimum phase-shift networks are not so restricted. The condition $a < 0$ states that $\beta(\xi)$ must become zero for $\xi \rightarrow \infty$; and we know that in networks whose transfer functions have zeros at $s = \infty$, the phase approaches a constant nonzero asymptote. In fact we can conceive of minimum phase-shift networks with continuously increasing phase, for which, therefore, $\beta(\xi) \rightarrow \xi$ for $\xi \rightarrow \infty$. What is the explanation of this seeming inconsistency?

It is simply that $\alpha(\omega)$ in Eq. 15 (unlike $\beta(\omega)$ in Eq. 13) is determined only within an arbitrary additive constant; and the value of this constant is sometimes infinite. It is infinite, for example, in any situation in which the phase $\beta(\omega)$ approaches a constant asymptote. In such a case we should not try to compute $\alpha(\omega)$ from $\beta(\omega)$ but instead find

$$\alpha(\omega) - \alpha(0) = \frac{-2\omega^2}{\pi} \int_{0}^{\infty} \frac{\beta(\xi)\,d\xi}{\xi(\xi^2 - \omega^2)} \tag{17}$$

in which any additive constant in the function $\alpha(\omega)$ drops out. In other words, integral 17 yields the functional variation in $\alpha(\omega)$, which is really all that we are interested in. For its determination according to integral 17, we see that the asymptotic character described by $\beta(\xi) \rightarrow \xi^a$ for $\xi \rightarrow \infty$ allows a to be larger than unity as long as $a < 2$. This, then, is the true restriction upon the phase function.

As an example of the use of Eq. 17 we readily compute for the specification

$$\left.\begin{aligned} \beta(\xi) &= n\pi\xi \quad \text{for } -1 < \xi < 1 \\ \beta(\xi) &= n\pi \quad \text{for } |\xi| > 1 \end{aligned}\right\} \tag{18}$$

the solution

$$\alpha(\omega) - \alpha(0) = n \ln \left|1 - \omega^2\right| - n\omega \ln \left|\frac{1 - \omega}{1 + \omega}\right| \tag{19}$$

which checks with the solution to this situation obtained by other methods.

Again, for $\beta(\xi) = \xi$, Eq. 17 gives $\alpha(\omega) - \alpha(0) \equiv 0$, which agrees with the well-known physical fact that a linear phase shift over the infinite spectrum is associated with a constant attenuation, albeit an infinite attenuation if we conceive of obtaining this result with a minimum phase-shift network.

It is generally thought that the usefulness of Hilbert transforms is restricted to situations in which the given real or imaginary part is either graphically or analytically specified over intervals of the j-axis, as, for example, in specification 18, or in the computation of the phase associated with an attenuation function defined by confluent straight-line segments or arcs. Computation of the imaginary part from Eq. 13, when the real part is given as a rational function of the frequency variable s, is regarded as leading to an integral that is difficult to evaluate; and so such problems are usually solved by other methods (the Bode or Gewertz or Miyata methods). Through use of Eq. 5 or Eq. 8 (equivalent to Eq. 13) together with methods of complex integration, we can use the Hilbert transforms for problems of this sort with the same facility as the algebraic methods. In fact, we shall show that use of the Hilbert transform (Eq. 13) or of Eq. 8 leads directly to Bode's form of the algebraic solution.

Let us do this for an impedance Z(s), for which we write $Z(j\omega) = R(\omega) + jX(\omega)$, and have Eq. 13 in the form

$$X(\omega) = \frac{2\omega}{\pi} \int_0^\infty \frac{R(\xi)\, d\xi}{\xi^2 - \omega^2} \tag{20}$$

The j-axis real part of the impedance Z(s) is expressible in the familiar form

$$R(\omega) = \frac{1}{2}\left[Z(s) + Z(-s)\right]_{s=j\omega} = \left[R(-s^2)\right]_{s=j\omega} \tag{21}$$

in which $R(-s^2)$ is the even part of Z(s). Since we want to use the methods of complex integration for the evaluation of Eq. 20, it must be converted into a contour integral. This process is obviously just the reverse of the above derivation of Eq. 13 from Eqs. 5 and 8. If we use Eq. 5, and close the path of integration by adding the semicircular arc, as shown in Fig. XIV-4, we obtain

$$Z(s) = \frac{1}{j\pi} \oint \frac{s\,R(-\lambda^2)\,d\lambda}{\lambda^2 - s^2} \tag{22}$$

where traversal of the path is again counterclockwise. Since the even part of any physical impedance must be regular at infinity, the conditions for zero contribution from the semicircular arc are obviously fulfilled.

If the left half-plane poles of Z(s) are denoted by λ_i, and the residues of Z(s) in these poles are k_i, then a partial-fraction expansion of $R(-\lambda^2)$ reads

$$R(-\lambda^2) = \frac{1}{2} \sum_i \left(\frac{k_i}{\lambda - \lambda_i} - \frac{k_i}{\lambda + \lambda_i} \right) \tag{23}$$

because the residues of $R(-\lambda^2)$ in its left half-plane poles are (according to Eq. 21) equal to $k_i/2$, while those in the right half-plane poles are the negatives of these values, as is clear from the quadrantal symmetry of the pole-zero pattern of $R(-\lambda^2)$.

The partial-fraction expansion of the integrand in Eq. 22 is then seen to be

$$\frac{s\,R(-\lambda^2)}{\lambda^2 - s^2} = \frac{R(-s^2)}{2(\lambda - s)} - \frac{R(-s^2)}{2(\lambda + s)} + \frac{1}{2} \sum_i \frac{sk_i}{\lambda_i^2 - s^2} \left(\frac{1}{\lambda - \lambda_i} - \frac{1}{\lambda + \lambda_i} \right) \tag{24}$$

By Cauchy's residue theorem the value of the integral in Eq. 22 is equal to $2\pi j$ multiplied by the sum of the residues of the integrand in those poles enclosed by the contour of Fig. XIV-4. These poles are at $\lambda = s$ and $\lambda = -\lambda_i$. The residues are evident in Eq. 24, and so, for the evaluation of Eq. 22, we have

$$Z(s) = R(-s^2) - \sum_i \frac{sk_i}{\lambda_i^2 - s^2} \tag{25}$$

which, with Eq. 23, becomes

$$Z(s) = \frac{1}{2} \sum_i \left(\frac{k_i}{s - \lambda_i} - \frac{k_i}{s + \lambda_i} \right) + \frac{1}{2} \sum_i \left(\frac{k_i}{s - \lambda_i} + \frac{k_i}{s + \lambda_i} \right)$$

or

$$Z(s) = \sum_i \frac{k_i}{s - \lambda_i} \tag{26}$$

Hence the impedance generated by the rational even-part function $R(-\lambda^2)$ is found as readily through use of the Hilbert transform and complex integration as it is by Bode's algebraic method. In fact, the computational work is exactly the same in the two procedures.

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C. AN INDEPENDENT PROOF OF THE DARLINGTON THEOREM

Darlington showed that any positive real rational function (realizable driving-point impedance) can be constructed as the impedance on the input side of a lossless two

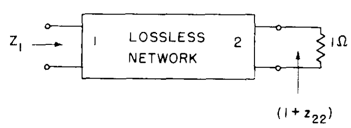

Fig. XIV-5. Network for relating open-circuit transients to impedance functions.

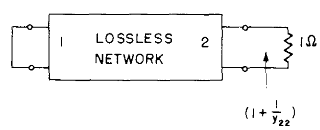

Fig. XIV-6. Network for relating short-circuit transients to impedance functions.

terminal-pair network terminated in a 1-ohm resistance. The following independent proof of this theorem makes use of well-known properties of Hurwitz polynomials and reactance functions, together with the fact that the open- and short-circuit natural frequencies of a network are the poles and zeros of its driving-point impedance.

Let the impedance Z_1 in Fig. XIV-5 be indicated by

$$Z_1 = \frac{m_1 + n_1}{m_2 + n_2} = \frac{P(s)}{Q(s)} \tag{1}$$

in which, as usual, m and n denote the even and odd parts of either polynomial P(s) or Q(s). If the network in Fig. XIV-5 is excited somehow (by having a little boy throw coulombs at the capacitors), its natural frequencies are the poles of $Z_1(s)$ or the zeros of $(1 + z_{22})$, and hence are zeros of $Q(s) = m_2 + n_2$. Here z_{22} is the open-circuit impedance of the lossless network at its terminal pair 2, and $(1 + z_{22})$, therefore, is the impedance that would result from cutting into the output mesh, as indicated in Fig. XIV-5. It follows that we must have

Case A: $$1 + z_{22} = \frac{m_2 + n_2}{n_2} = 1 + \frac{m_2}{n_2} \tag{2}$$

or

Case B: $$1 + z_{22} = \frac{m_2 + n_2}{m_2} = 1 + \frac{n_2}{m_2} \tag{3}$$

If the network in Fig. XIV-6 is excited, the natural frequencies are zeros of $Z_1(s)$, and hence zeros of $P(s) = m_1 + n_1$; but they must also be zeros of $1 + (1/y_{22})$, where y_{22} is the short-circuit admittance of the lossless network at its terminal pair 2. We have

Case A: $$1 + \frac{1}{y_{22}} = \frac{m_1 + n_1}{m_1} = 1 + \frac{n_1}{m_1} \tag{4}$$

or

Case B: $$1 + \frac{1}{y_{22}} = \frac{m_1 + n_1}{n_1} = 1 + \frac{m_1}{n_1} \tag{5}$$

Let z_{11} denote the open-circuit impedance of the lossless network at its terminal pair 1. We may then say that the poles of z_{11} are poles of z_{22}, and zeros of z_{11} are poles of $1/y_{22}$. From Eqs. 2, 3, 4, and 5 it follows that

For case A: $$z_{11} = m_1/n_2 \tag{6}$$

or

For case B: $$z_{11} = n_1/m_2 \tag{7}$$

Let y_{11} denote the short-circuit admittance of the lossless network at its terminal pair 1. The poles of y_{11} are the poles of y_{22}, and the zeros of y_{11} are poles of $1/z_{22}$. These conclusions (as well as those in the previous paragraph) follow from the consideration that the critical frequencies in question are natural frequencies of the lossless network under identical terminal constraints. For example, for the poles of y_{11}, the constraints are: terminal pair 1 shorted, terminal pair 2 shorted; the same holds for the poles of y_{22}. For the zeros of y_{11} or for the poles of $1/z_{22}$, the constraints are: terminal pair 1 open, terminal pair 2 shorted. Thus, we have

For case A: $$y_{11} = m_2/n_1 \tag{8}$$

or

For case B: $$y_{11} = n_2/m_1 \tag{9}$$

From the fact that reactance or susceptance functions must be ratios of two polynomials, one of which is even and the other odd, it follows, incidentally, that cases A and B are mutually exclusive.

From the separation property of the zeros and poles of reactance functions, and of the even and odd parts of Hurwitz polynomials, it follows that P(s) and Q(s), as well as $P^*(s) = m_1 + n_2$ and $Q^*(s) = m_2 + n_1$, are Hurwitz polynomials. We may conclude that the so-called double-alternance (1) property holds for the polynomials m_1m_2, and n_1n_2. As Reza has shown (1), this double-alternance property is a necessary (not sufficient) condition to ensure the positive real character of $Z_1(s)$. He also showed that a rational function with this property can be made to fulfil positive real conditions merely through assigning to either m_1m_2 or n_1n_2 an appropriate constant multiplier, which in our situation amounts to scaling the impedance level of one end of the lossless network. It follows, therefore, that the physical representation in Fig. XIV-5 enables the construction of any given positive real impedance function $Z_1(s)$. Q.E.D.

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References

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