XVII. MICROWAVE THEORY

E. F. Bolinder

A. USE OF 4 × 4 REAL MATRICES IN MICROWAVE THEORY

In the Quarterly Progress Report of April 15, 1956, page 123, it was pointed out that impedance transformations through bilateral two terminal-pair networks can be performed by using 4 × 4 real matrices that belong to the G_+ subgroup of the proper Lorentz group. The 4 × 4 matrix corresponding to a given network can be obtained by two different methods, which will be briefly outlined. A more extensive treatment will be given in reference 1.

Using the notations of Fig. XVII-1, we can write

$$\begin{pmatrix} v' \\ i' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v \\ i \end{pmatrix} \tag{1}$$

If we let $v'/i' = Z'$, $v/i = Z$, then we obtain

$$Z' = \frac{aZ + b}{cZ + d} \tag{2}$$

where, for bilateral networks, $ad - bc = 1$.

The simplest way of deriving a 4 × 4 real matrix from the 2 × 2 complex matrix in Eq. 1 is to map the complex impedance plane (Z-plane) stereographically on the Riemann unit sphere. A point (R, X) in the Z-plane will correspond to a point (x, y, z) on the surface of the sphere, where

$$\left.\begin{aligned} x &= \frac{2R}{R^2 + X^2 + 1} \\ y &= \frac{2X}{R^2 + X^2 + 1} \\ z &= \frac{R^2 + X^2 - 1}{R^2 + X^2 + 1} \end{aligned}\right\} \tag{3}$$

After some simple calculations have been made, it can be shown that the linear

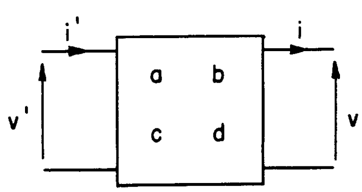

Fig. XVII-1. Two terminal-pair network notations.

fractional transformation, Eq. 2, on the surface of the sphere corresponds to

$$\rho \begin{pmatrix} x_1' \\ y_1' \\ z_1' \\ w_1' \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix} \tag{4}$$

where (x, y, z, w) is the point (x, y, z) expressed in projective coordinates, e.g., $x = x_1/w_1$, $y = y_1/w_1$, and $z = z_1/w_1$. The 16 elements of the 4×4 real matrix (Eq. 4) are all expressed in the real and imaginary parts of the complex coefficients a, b, c, d. Because the points (x, y, z) and (x', y', z') always lie on the surface of the sphere, there are 10 relations between the elements of the 4×4 matrix, which fact, together with the fact that the determinant is +1, makes it belong to the G_+ subgroup of the proper Lorentz group.

Instead of using a relation between the output and the input of the network in the form of Eq. 2, an impedance equation, we can use a power relation. This can be done by using the associated vector P of a spinor – a vector in three-dimensional Euclidean space. The spinor is assumed to consist of the complex voltage v and the complex current i (2, 3). Then, we obtain

$$\left.\begin{aligned} P_1 &= \mathrm{Re}(vi^*) \\ P_2 &= \mathrm{Im}(vi^*) \\ P_3 &= \tfrac{1}{2}\left(|v|^2 - |i|^2\right) \\ P_o &= \tfrac{1}{2}\left(|v|^2 + |i|^2\right) \end{aligned}\right\} \tag{5}$$

where P_1, P_2, and P_3 are the components of the vector P. If no noise is present, P_o is the magnitude of the vector P:

$$P_o^2 = P_1^2 + P_2^2 + P_3^2 \tag{6}$$

The four quantities of Eq. 5 can be used to form a 4-vector that is analogous to the Stokes vector used in optics. After some algebraic operations (4) the 4-vector at the input can be expressed in the output vector by means of a 4×4 real matrix, which is analogous to the 4×4 matrix that is used in optics by Perrin, Soleillet, and Mueller. For the simple case of bilateral networks, the point (x, y, z) on the Riemann unit sphere can be written

$$x = P_1/P_o, \quad y = P_2/P_o, \quad z = P_3/P_o$$

so that the 4×4 real matrix is the same as the one in Eq. 4.

Of the two methods described, the impedance method and the power method, the second is more general, because it allows the treatment of noisy active networks.

E. F. Bolinder

References

1. E. F. Bolinder, Technical Report 312, Research Laboratory of Electronics, M.I.T. (to be published).
2. W. T. Payne, Am. J. Phys. 20, 253-262 (May 1952).
3. W. T. Payne, J. Math. Phys. 32, 19-33 (April 1953).
4. F. D. Murnaghan, The Theory of Group Representations (John Hopkins University Press, Baltimore, 1938).

B. CASCADING TWO TERMINAL-PAIR NETWORKS BY THE ISOMETRIC CIRCLE METHOD

It was shown in the Quarterly Progress Report of April 15, 1956, page 123, that the isometric circles of a linear fractional transformation can be used for a graphical method of impedance transformation. Let us assume that we know the complex constants a, b, c, d in the expression

$$Z' = \frac{aZ + b}{cZ + d}, \quad ad - bc = 1 \tag{1}$$

for two given networks. Then we know the isometric circles C_d and C_i (centers at $O_d = -d/c$ and $O_i = a/c$; radii $R_c = 1/|c|$) and the rotation angle $-2 \arg (a+d)$ for the two transformations of the two networks. We now want to find what the corresponding quantities are for the resultant network when the two networks are cascaded. The simplest way of finding these quantities is by using analytical formulas that have been partly worked out in the theory of automorphic functions (1).

If we indicate the quantities of the two given networks by index numbers 1 and 2 and leave the quantities of the resultant network without indices, a simple calculation reveals that the resultant isometric circles are characterized by

$$\left.\begin{aligned} O_d &= -\frac{d}{c} = O_{d_2} + r_{c_2}^2/(O_{i_2} - O_{d_1}) \\ O_i &= \frac{a}{c} = O_{i_1} - r_{c_1}^2/(O_{i_2} - O_{d_1}) \\ r_c &= \frac{1}{c} = (r_{c_1}\, r_{c_2})/(O_{i_2} - O_{d_i}) \end{aligned}\right\} \tag{2}$$

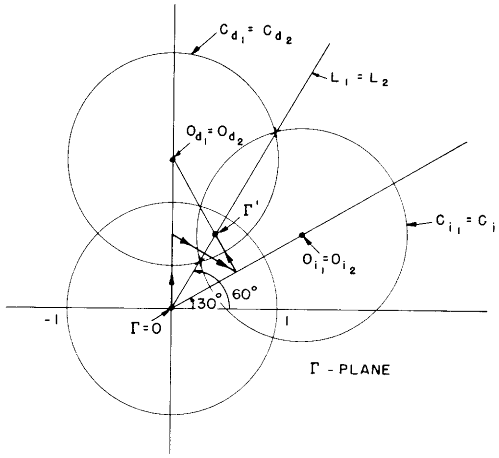

Fig. XVII-2. Cascading of two equal lossless networks.

where $r_c = 1/c = R_c \exp(-j\phi c)$; so that $R_c = |r_c|$. The constants a, b, c, d are easily obtained from Eq. 2, and the relation $ad - bc = 1$ is found.

Example. For a simple example, let us transform the reflection coefficient $\Gamma = 0$ through two equal lossless networks corresponding to

$$\Gamma' = \frac{\Gamma \sqrt{2}\, e^{j60^\circ} + e^{-j30^\circ}}{\Gamma e^{j30^\circ} + \sqrt{2}\, e^{-j60^\circ}}$$

We have

$$O_{d_1} = O_{d_2} = j\sqrt{2}$$

$$O_{i_1} = O_{i_2} = \sqrt{2}\, e^{j30^\circ}$$

$$r_{c_1} = r_{c_2} = e^{-j30^\circ} \qquad R_{c_1} = R_{c_2} = 1$$

The graphical transformations, performed in Fig. XVII-2, give the resultant reflection coefficient $\Gamma' = \left[(6)^{1/2}/3\right] \exp(j60^\circ)$. From Eq. 2, for the resultant network, we obtain

$$O_d = \frac{6^{1/2}}{2} e^{j60^\circ} = O_i$$

$$R_c = \frac{1}{\sqrt{2}}$$

The transformation of $\Gamma = 0$ yields $\Gamma' = \left[(6)^{1/2}/3\right] \exp(j60^\circ)$, as is seen from

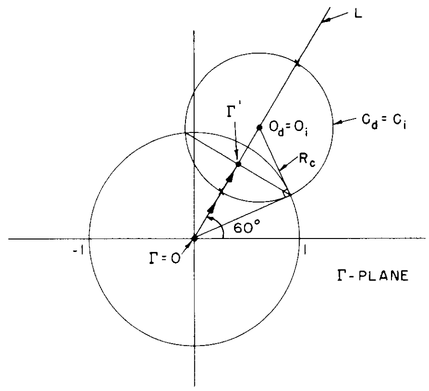

Fig. XVII-3. Transformation through the resultant network.

Fig. XVII-3. In this special case the two isometric circles are coincident.

The fixed points are the same in Figs. XVII-2 and 3. This is understandable because the transformation is elliptic ($a + d = \sqrt{2}$ is real and <2), so that, if the Γ-plane is mapped on the sphere, the transformations through equal lossless networks correspond to non-Euclidean rotations around an axis through the fixed points.

E. F. Bolinder

References

1. L. R. Ford, Automorphic Functions (Chelsea Publishing Co., New York, 1951).

C. IMPEDANCE TRANSFORMATIONS OF THE NONLOXODROMIC TYPE

Impedance transformations through bilateral two terminal-pair networks can be classified as loxodromic and nonloxodromic (1). In order to study the nonloxodromic transformations let us select the simple network in Fig. XVII-4. We have

$$\begin{pmatrix} v' \\ i' \end{pmatrix} = \begin{pmatrix} 1 + Z_1 Y_2 & Z_1 \\ Y_2 & 1 \end{pmatrix} \begin{pmatrix} v \\ i \end{pmatrix} \tag{1}$$

The trace of the matrix in Eq. 1 is real for nonloxodromic transformations, so that $Z_1 Y_2$ is real. Therefore, $Z_1 = |Z_1| \exp(j\phi)$, and $Y_2 = |Y_2| \exp(-j\phi)$. For simplicity, let us assume that $\phi = 0$, so that $Z_1 = R_1$, $Y_2 = G_2 = 1/R_2$. If we vary R_2, then $-\infty \leqslant R_2 \leqslant \infty$; the fixed points of the network move along two perpendicular straight lines in the complex impedance plane (Z-plane), as shown in Fig. XVII-5.

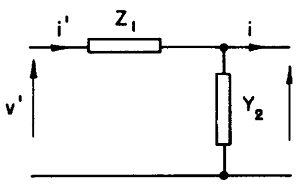

Fig. XVII-4. Simple two terminal-pair network.

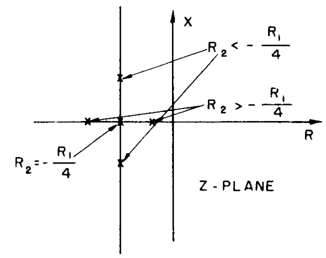

Fig. XVII-5. Positions of the fixed points in the Z-plane.

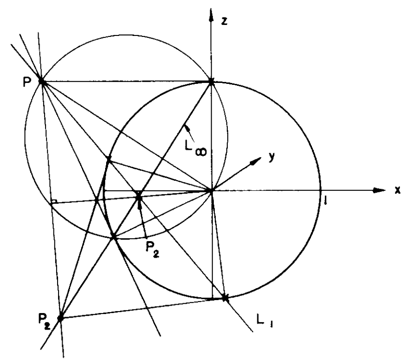

Fig. XVII-6. Positions of the fixed points on the Riemann sphere.

We now map the Z-plane stereographically on the Riemann unit sphere and project the sphere orthographically on the xz-plane. See Fig. XVII-6. The straight line $R = -R_1/4$ in the Z-plane corresponds to a circle through the top of the sphere and to a straight line L_∞ in the xz-plane. The fixed points are obtained as the points where a straight line L_1 through the polar P of L_∞ cuts the unit circle. If we vary R_2, the line L_1 rotates around P. If $R_2 > -R_1/4$, then L_1 cuts the unit circle in two real points, and the hyperbolic case is obtained (trace of the matrix in Eq. 1 > 2). The polar of L_1, P_2, is situated on L_∞. If $R_2 = -R_1/4$, then L_1 is tangent to the unit circle so that the two fixed points coincide and the parabolic case is obtained (trace = -2). If $R_2 < -R_1/4$, then L_1 is exterior to the unit circle; the fixed points are obtained as the points where the polar of L_1 (perpendicular to the xz-plane) through P_2 cuts the sphere, and the elliptic case (trace < 2) is obtained. Finally, if $R_2 = 0$, another parabolic case (trace = 2) is obtained, with the coinciding fixed points at (0, 1) in the xz-plane. Obviously, when L_1 rotates, a nonloxodromic cycle (hyperbolic – parabolic – elliptic – parabolic – hyperbolic) is described.

If the angle ϕ is varied, the constructions in Fig. XVII-6 are rotated an angle ϕ around the z-axis. If $\phi = 90°$, Z_1 and Z_2 are both reactances and the constructions are performed in the yz-plane.

E. F. Bolinder

References

1. E. F. Bolinder, Quarterly Progress Report, Research Laboratory of Electronics, M.I.T., April 15, 1956, p. 123.