XIX. ANALOG COMPUTER RESEARCH

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RESEARCH OBJECTIVES

The Analog Computer group is engaged in a number of basic investigations of analog and special-purpose computers. The emphasis is on underlying concepts rather than on the design and operation of a large computing facility.

An analog computer can be viewed as an active, nonlinear network. At the present time our research includes both general active networks, and general nonlinear networks. In active network synthesis our effort is directed toward general theorems that can be used in analyzing present computer configurations and in designing new ones. The work in nonlinear network synthesis is directed toward piecewise linear and piecewise planar devices. The end in view is the construction of multiple variable function generators. These devices are useful as elements of computers and in many instances they can be used to replace conventional computers.

A third area which is being considered is the use of distributed parameter elements in analog computers and the converse problem of solving partial differential equations by analog computers.

R. E. Scott

A. ACTIVE NETWORK SYNTHESIS

1. A General Theorem on the Transfer Functions of Three-Pair Terminal Networks

Consider a linear network N with three-pair terminals, 1-1', 2-2', and 3-3' (Fig. XIX-1). The only restriction we shall impose upon N is that it shall contain lumped parameter linear elements; that is, one can write the following set of equations

$$\left.e_i\right] = \left[z_{ij}(s)\right] \left.i_j\right] \qquad \begin{matrix} i = 1, 2, 3 \\ j = 1, 2, 3 \end{matrix} \tag{1}$$

Of course, there exists another set of equations on the basis of admittance functions.

The constraints upon Q are expressed in the fact that the functions $z_{ij}(s)$ are all rational functions in s. We are interested in the response at the terminals 2-2' from a signal at the terminals 1-1', as affected by the behavior of terminals 3-3'. Let

$$\left.\begin{aligned} e_2 &= -i_2 z_2 \\ e_3 &= -i_3 z_3 \end{aligned}\right\} \tag{2}$$

Thus the matrix $\left[z_{ij}\right]$ becomes

$$[Z] = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} + z_2 & z_{23} \\ z_{31} & z_{32} & z_{33} + z_3 \end{bmatrix} \tag{3}$$

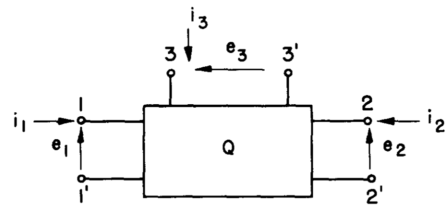


Fig. XIX-1
General three-pair terminal network.

The transfer impedance from terminals 1-1' to 2-2', then, is (1)

$$\left.\begin{aligned} Z_{12} &= \frac{e_2}{i_2} = \frac{\Delta}{\Delta_{12}} \\ Z_{12} &= \frac{\Delta' + (z_{33} + z_3)\,\Delta_{33}}{\Delta'_{12} + (z_{33} + z_3)\,\Delta_{33\,12}} = \frac{\Delta^{o} + z_3\Delta_{33}}{\Delta^{o}_{12} + z_3\Delta_{33\,12}} \end{aligned}\right\} \qquad (4)$$

where Δ is the determinant of $[Z]$, Δ_{ij} is a minor of Δ, $\Delta' = \Delta\big|_{z_{33}+z_3=0}$, and $\Delta^{o} = \Delta\big|_{z_3=0}$. However

$$\left.\begin{aligned} \frac{\Delta^{o}}{\Delta^{o}_{12}} &= Z_{12}\Big|_{Z_3=0} = Z^{o}_{12} \\ \frac{\Delta^{o}_{33}}{\Delta_{33\,12}} &= Z_{12}\Big|_{z_3=\infty} = Z^{\infty}_{12} \\ \frac{\Delta^{o}_{12}}{\Delta_{33\,12}} &= z_{\infty} \qquad Z_{12}\Big|_{z_3=-z_{\infty}} = \infty \end{aligned}\right\} \qquad (5)$$

Hence Eq. 4 can be written as

$$Z_{12} = \frac{Z^{\infty}_{12} z_3 + Z^{o}_{12}\, z_{\infty}}{z_3 + z_{\infty}} \qquad (6)$$

Following the same argument, one can arrive at the following general result.

Theorem. Given a linear lumped-parameter system (it could be active), as shown in Fig. XIX-2, the response R(s), voltage or current, resulting from some excitation F(s) current or voltage, is related to an element M(s), impedance or admittance (all not necessarily located at the same place in the system) in the following manner

$$T = \frac{T^{\infty}M + T^{o}M_{\infty}}{T_{\infty} + T} \tag{7}$$

where $T = R(s)/F(s)$, and $(-M_{\infty})$ is the value of M which makes the system nonstable. The superscript refers to the value of M at which T is evaluated.

We shall investigate M_{∞}. Consider first the case in which F(s) is a voltage source, and regard the network P as a two-pair terminal network, terminated to $z = -M_{\infty}$ (see Fig. XIX-3). It is obvious that since $T = \infty$, then $z_{mm'}$, the driving-point impedance at the terminals m-m' with the terminals f-f' terminated to $-M_{\infty}$, must necessarily be zero. However, it can be shown (2) that

$$Z_{mm'} = Z^{oc}_{mm'} \frac{Z^{sc}_{ff'} + z}{Z^{oc}_{ff'} + z} \tag{8}$$

where Z^{oc} and Z^{sc} are the driving-point impedances when the subscript terminals are open-circuited and short-circuited, respectively.

Therefore

$$N = -Z^{sc}_{ff'}$$

or

$$M_{\infty} = Z^{sc}_{ff'} \tag{9}$$

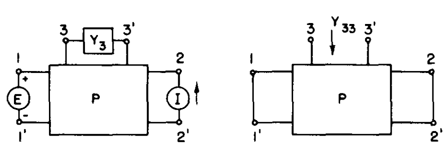
Similarly, if F(s) is a current source, Z_{ff} must be infinite, or

$$M_{\infty} = Z^{oc}_{ff'} \tag{10}$$

Corollary. The instability impedance, that is, the impedance that will make the network unstable when terminated at the terminals m-m', is equal to the negative impedance when one looks from those terminals with the excitation and response terminals short- or open-circuited (depending on whether the excitation is a voltage source or a current source).

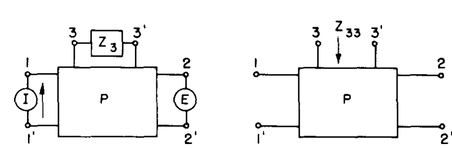
These results are summarized as follows:

Case 1. $F(s)$ = Voltage; $R(s)$ = Current



$$Y_{12} = \frac{Y_3 Y_{12}^{\infty} + Y_{33} Y_{12}^{o}}{Y_3 + Y_{33}}$$

Case 2. $F(s)$ = Current; $R(s)$ = Voltage



$$Z_{12} = \frac{Z_{12}^{\infty} Z_3 + Z_{12}^{o} Z_{33}}{Z_{33} + Z_3}$$

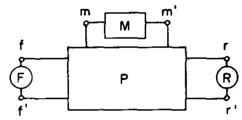


Fig. XIX-2

General three-pair terminal network with termination.

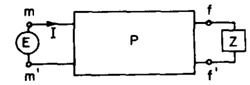


Fig. XIX-3

Equivalent two-pair terminal network.

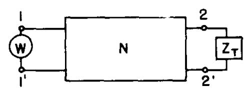


Fig. XIX-4

Two-pair terminal network with termination.

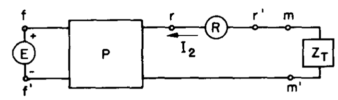


Fig. XIX-5

Two-pair terminal network regarded as three-pair terminal.

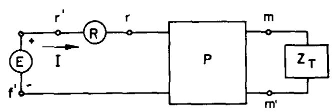


Fig. XIX-6

Equivalent of one-pair terminal network.

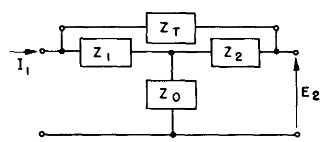


Fig. XIX-7

General bridge-T network.

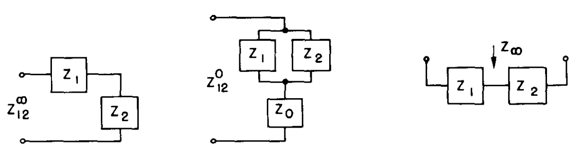


Fig. XIX-8

Simplified equivalent networks.

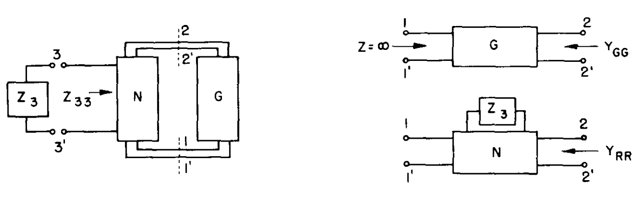


Fig. XIX-9

A general active network. (a) construction for negative impedance; (b) components of the general active network.

2. Application to Two-Pair Terminal Networks

By applying the foregoing theorem we shall find relations between the terminated and unterminated functions of two-pair terminal networks (Fig. XIX-4). We shall represent the functions of the terminated network by capital letters, the unterminated characteristics by lower case letters.

a. Transfer functions

Obviously, for this case, terminals m-m' and r-r' are not independent (see Fig. XIX-2). Thus, if W of Fig. XIX-4 is a voltage source, to find the transfer admittance of N (Fig. XIX-2), we must redraw P as shown in Fig. XIX-5. The following relations are then easily obtained

$$T^o = \left.\frac{I_2}{E_1}\right|_{\substack{m-m' \\ \text{open}}} = 0, \qquad T^\infty = y_{12}, \qquad M_\infty = y_{22}$$

Thus

$$Y_{12} = \frac{y_{12}y_T}{y_{22} + y_T} \tag{11}$$

In a similar manner

$$Z_{12} = \frac{z_{12}z_T}{z_{22} + z_T} \tag{12}$$

b. Driving-point functions

For this case, terminals f-f' and r-r' are related. Thus, for the impedance function we obtain Fig. XIX-6, and we have

$$T^\infty = z_{11}, \qquad T^o = \frac{1}{y_{11}}, \qquad M_\infty = z_{22}$$

Thus

$$Z_{11} = \frac{z_{11}z_T + \dfrac{z_{22}}{y_{11}}}{z_{22} + z_T} = z_{11}\,\frac{z_T + \dfrac{z_{22}}{y_{11}z_{11}}}{z_{22} + z_T}$$

But

$$\frac{z_{22}}{y_{11}z_{11}} = \frac{1}{y_{22}}$$

Therefore

$$Z_{11} = z_{11} \frac{\frac{1}{y_{22}} + z_T}{z_{22} + z_T} \tag{13}$$

3. Passive Bridge-T Networks

With the help of the previously stated theorem one can easily obtain the transfer functions of bridge-T circuits. Figure XIX-7 represents the general form of an unterminated bridge-T network for which we shall compute the transfer impedance $Z_{12} = (E_2/I_1)$. This example corresponds to case 2. Figure XIX-8 is used to derive the following functions

$$Z_{12}^{\infty} = z_o, \qquad Z_{12}^{o} = \frac{z_1 z_2}{z_1 + z_2} + z_o, \qquad Z_{\infty} = z_1 + z_2 \tag{14}$$

Thus

$$Z_{12} = \frac{z_o z_T + z_1 z_2 + z_o(z_1 + z_2)}{z_1 + z_2 + z_T}$$

or

$$Z_{12} = z_o + \frac{z_1 z_2}{z_1 + z_2 + z_T} \tag{15}$$

4. Negative Driving-Point Impedances

Now we shall consider the problem of synthesizing a negative driving-point impedance. Obviously, the system must contain at least one amplifier. The general form of the network is shown in Fig. XIX-9, where N contains only linear passive and bilateral elements. Let the transfer admittances of the networks in Fig. XIX-9(b) be G and y_{12}; then the transfer admittance $Y_{12}^{(A)}$ of the network in Fig. XIX-9(a) is

$$Y_{12}^{(A)} = \frac{y_{12} y_{GG}}{y_{GG} + y_{RR}} - \frac{G y_{RR}}{y_{GG} + y_{RR}} \tag{16}$$

Now let z_3 be such that $Y_{12}^{(A)}$ equals zero. Therefore

$$y_{12} y_{GG} = G y_{RR} \tag{17}$$

and from Eq. 7

$$Y_{33} = -Y_3 \frac{Y_{12}^{(A)\infty}}{Y_{12}^{(A)o}} \tag{18}$$

with the introduced quantities as defined in the previous section. A more explicit form of Eq. 18 is

$$Y_{33} = -Y_3 \frac{y_{12}^{\infty} y_{GG} - G y_{RR}}{y_{12}^{o} y_{GG} - G y_{RR}} \frac{y_{GG} + y_{RR}^{o}}{y_{GG} + y_{RR}^{\infty}} \tag{19}$$

Equations 17 and 18 are the essential relations in synthesizing negative driving-point functions. Of course, if G, y_{GG}, y_{12}, and y_{RR} are all resistive terms, the synthesis problem is greatly simplified. It should be realized that the quantity $y_{12}^{(A)\infty}/y_{12}^{(A)o}$ could, indeed, be always positive, since, with reference to Fig. XIX-9(b),

$$y_{33} = y_3 \frac{y_{12}^{\infty} - y_{12}}{y_{12} - y_{12}^{o}} \tag{20}$$

is always positive, and G could be negative.

N. DeClaris

References

1. H. W. Bode, Network Analysis and Feedback Amplifier Design (D. Van Nostrand and Company, Inc., New York, 1945).
2. N. DeClaris, Quarterly Progress Report, Research Laboratory of Electronics, M.I.T., April 15, 1954, p. 84.

B. NETWORK SYNTHESIS WITH DISTRIBUTED ELEMENTS

Theoretical, experimental, and analog results have been obtained for a new method of designing distributed parameter filters in the time domain. The method is based upon the open-loop impulse response of the system, and it yields the closed-loop transient response and the desired compensating network. Details of the method will be given in forthcoming Technical Report 288.

Y. C. Ho

C. PIECEWISE LINEAR NETWORK THEORY

Substantial progress has been made in the application of an algebra of piecewise linear functions to both analysis and synthesis problems. This algebra is based upon the transformations defined in the Quarterly Progress Report, July 15, 1954, page 87. The algebra has been developed to a point where it affords a systematic mathematical method of analyzing and synthesizing piecewise linear networks that do not contain energy storage elements.

A selection of some of the basic definitions and theorems follows. These are given in purely abstract form. However, in the applications, the variables mentioned below become currents and voltages, while the constants are voltage or current sources, resistances, or conductances.

The elements of the algebra are known as scalars and vectors. (There are many similarities between this algebra and the algebra of vector spaces.) They will be formed from the elements of an ordered field, F (the real number system).

Definition 1. A scalar is any member of an ordered field, F. (Scalars will be denoted by small English letters or numbers.)

Definition 2. A vector is any proper subset of F. The elements of the vectors, and the scalars, are both members of the same field. A vector will be denoted by a single Greek letter, ξ, to indicate the whole set of elements, or by (a, b, ..., n) to enumerate each element. The elements of a vector are scalars. Note that the order in which the elements of ξ appear is immaterial.

A scalar can either be a constant, (a), a variable, (x), or a function of a variable, (a + bx). A vector can also contain members which are any of these three.

It should be observed that according to Definitions 1 and 2, a single element standing alone may be either a vector or a scalar. In the derivations that follow, single elements will be treated as either vectors or scalars interchangeably, but their status at any given time will be clear from the context.

Definition 3. Scalar multiplication. The product of a scalar, c, and a vector, $\lambda = (\ell_1, \ell_2, \ldots, \ell_n)$, is denoted by $c\lambda$, where $c\lambda = (c\ell_1, c\ell_2, \ldots, c\ell_n)$.

Definition 4. Vector addition. The sum of two vectors, $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_m)$, is denoted by $\alpha \oplus \beta$ where $\alpha \oplus \beta$ is the set of all scalars, $a_p + b_q$, $a_p \in \alpha$, and $b_q \in \beta$.

Example: Let $\alpha = (0, 3, 3 - 2x)$, $\beta = (0, -2x)$. Then $\alpha \oplus \beta = (0, 3, 3 - 2x, -2x, 3 - 4x)$.

The transformations, ϕ^+ and ϕ^-, were defined in the Quarterly Progress Report of July 15, 1954, page 87. The following simple examples serve to illustrate the usefulness of these transformations in representing piecewise linear functions.

Example 1. Conventional representation: Symbolism:

$$\left.\begin{array}{ll} y = 2x & x \leqslant 1 \\ y = x + 1 & 1 < x < 2 \\ y = 3 & x \geqslant 2 \end{array}\right\} \ldots\ldots \quad y = (2x,\ x + 1,\ 3)\phi^-$$

Example 2: Conventional representation: Symbolism:

$$\left.\begin{array}{ll} y = -1 & x < -1 \\ y = x & -1 \leqslant x \leqslant 1 \\ y = 1 & x > 1 \end{array}\right\} \ldots\ldots \quad y = \left[(x,\ 1)\phi^-,\ -1\right]\phi^+$$

Some theorems which have wide application are:

Theorem 1. $(c\alpha)\phi^{\pm} = c(\alpha\phi^{\pm}) \qquad c \geqslant 0$

$$(c\alpha)\phi^{\pm} = c(\alpha\phi^{\mp}) \qquad c \leqslant 0$$

or, equivalently,

$$(c\alpha)\phi^{\pm} = (0, c)\phi^{+}(\alpha\phi^{\pm}) + (0, c)\phi^{-}\ (\alpha\phi^{\mp})$$

A special case of Theorem 1 is

$$\alpha\phi^{\pm} = -\left[(-\alpha)\phi^{\mp}\right]$$

Theorem 2. Let $\alpha = (a)$. Then

$$\alpha\phi^{+} = \alpha\phi^{-} = a$$

Theorem 3. $(\alpha \oplus \beta)\phi^{\pm} = \alpha\phi^{\pm} + \beta\phi^{\pm}$

Theorem 4. Inversion theorem.

Let $y = F(x) = \left[f_1(x), f_2(x), \ldots, f_n(x)\right] \phi^{\pm}$. If, for each function, $F, f_1, f_2, \ldots, f_n$, there exists an inverse function, f_p^{-1}, such that

$$y = f_p\left[f_p^{-1}(y)\right] \qquad \text{and} \qquad y = f_p^{-1}\left[f_p(y)\right] \qquad \text{for all } y$$

then

$$x = F^{-1}(y) = \left[f_1^{-1}(y), f_2^{-1}(y), \ldots, f_n^{-1}(y)\right] \phi^{\mp}$$

(Note the reversal of the positions of the plus and minus signs.)

In general, the condition that the inverse of each function shall exist is satisfied if and only if the function is strictly monotonic and continuous. (That is, the function is a 1-1 transformation.) It should be observed that Theorem 4 establishes sufficient conditions for the inversion of a function. This does not imply that an equation which does not satisfy the above conditions cannot be inverted.

A wide variety of analysis and synthesis problems deals with monotonic functions that do not fulfill the conditions of being strictly monotonic and continuous; that is, they have regions of either zero or infinite derivative. The simplest example of this is the volt-ampere characteristic of an ideal diode, which has one region of zero slope and another of infinite slope. It is useful to be able to deal analytically with these functions. To this end the following two functions will be defined.

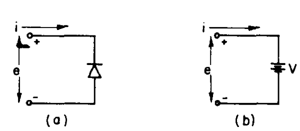


Fig. XIX-10
Diode and voltage source.

Definition 5. The function, $y = 0(x)$, is defined as

$$y = \lim_{n\to\infty} \left(\frac{x}{n}\right) = 0 \qquad \text{for all } x$$

Definition 6. The function, $y = \infty(x)$, is defined as

$$y = \lim_{n\to\infty} (nx) = \begin{cases} +\infty & x > 0 \\ 0 & x = 0 \\ -\infty & x < 0 \end{cases}$$

It is clear that for any finite value of n, no matter how large, these functions are inverses. The question of whether they remain inverses in the limit is more difficult and will not be discussed here. The important point is that for the purpose of this discussion they behave as inverses and will be considered as such. Some of their properties are

(a) $0(x) = \infty^{-1}(x)$

(b) $\infty(x) = 0^{-1}(x)$

(c) $0(x) + f(x) = f(x)$ for any f

(d) $\infty(x) + f(x) = \begin{cases} \infty(x) & x \neq 0 \\ f(x) & x = 0 \end{cases}$ for any finite f

Example 1. From Definitions 5 and 6, the impedance of the diode of Fig. XIX-10(a) is

$$e = z(i) = \left[0(i), \infty(i)\right] \phi^{+}$$

Its admittance is

$$i = y(e) = z^{-1}(e) = \left[\infty(e), 0(e)\right] \phi^{-}$$

Example 2. The impedance of the voltage source of Fig. XIX-10(b) is

$$e = z(i) = V$$

Its admittance is

$$i = y(e) = z^{-1}(e)$$

But

$$z(i) = V = V + 0(i) = e$$

$$0(i) = e - V$$

$$i = \infty(e-V)$$

The following simple example serves to illustrate an application of the algebra to

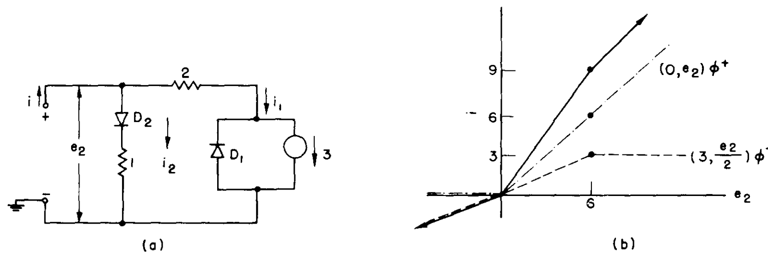


Fig. XIX-11
Diode network and its driving-point admittance.

the analysis of diode circuits. Each step in the analysis is stated explicitly with a reference to the theorem being used.

Example. Consider the circuit of Fig. XIX-11(a). The i vs. e relationship for D_1 is $i = \left[0(e_1), \infty(e_1)\right] \phi^-$. Thus

$$i_1 = 3 + \left[0(e_1), \infty(e_1)\right] \phi^- = \left[3, \infty(e_1)\right] \phi^- = y_1(e_1) \quad \text{(Theorems 2, 3, Def. 4)}$$

Then

$$e_1 = y_1^{-1}(i_1) = z_1(i_1) = \left[\infty(i_1 - 3), 0(i_1)\right] \phi^+ \quad \text{(Theorem 4)}$$

$$e_2 = 2i_1 + \left[\infty(i_1 - 3), 0(i_1)\right] \phi^+ = \left[\infty(i_1 - 3), 2i_1\right] \phi^+ = z_2(i_1) \quad \text{(Theorems 2, 3, Def. 4)}$$

$$i_1 = z_2^{-1}(e_2) = (3, e_2/2)\,\phi^- \quad \text{(Theorem 4)}$$

$$i_2 = \left[0(e_2), e_2\right] \phi^+$$

$$i = i_1 + i_2 = (3, e_2/2)\,\phi^- + (0, e_2)\,\phi^+ \quad \text{(See Fig. XIX-11(b).)}$$

T. E. Stern