X. ANALOG COMPUTER RESEARCH

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A. THE OPERATION OF PRESENT COMPUTERS

1. The Macnee Differential Analyzer

The Macnee machine was used for preliminary exploratory work in an investigation of the general second-order servomechanism problem. The final results were then obtained on a Reeve computer.

T. E. Stern

B. THE DESIGN OF COMPUTING ELEMENTS

1. A Fourier Transformer

An initial unit consisting of ten synchronized, sinusoidal, harmonically related generators has been constructed for performing Fourier transforms, as described in the Quarterly Progress Report, April 15, 1953. The sinusoids are phased on an oscilloscope, and their amplitudes are adjusted by potentiometers. Transforms have been taken of a number of simple functions. The results obtained compare favorably with the theoretical values.

An evaluation of the synchronization process, on both a theoretical and experimental basis, has been attempted.

For a description of this device see J. Petrishen: An Electronic Computer for Fourier Transforms, Master's Thesis, Department of Electrical Engineering, M.I.T. 1953.

J. Petrishen

2. Power Supply Design

Some problems associated with power supply design are being investigated. These problems include: (a) output voltage and current limits, (b) output ripple voltage, (c) short-term stability, (d) long-term stability, (e) pulse loading, (f) size, weight, and cost.

Several novel designs are being evaluated. One design has proved useful in increasing the voltage range of a power supply that does not possess a negative bias voltage. Since this increase is achieved with a series tube combination, additional tubes are not required. The method is as follows: Assume that the standard regulator circuit consists of a number of regulator tubes in parallel, as shown in Fig. X-1. If these are placed in cascade, as shown by Fig. X-2, it is possible to increase the voltage drop across the series tubes. Variations of this scheme have been attempted.

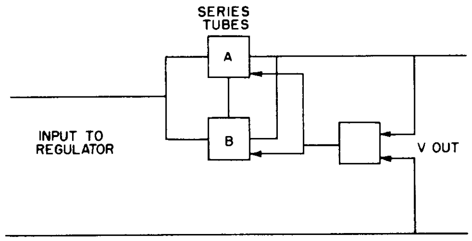

Fig. X-1
Regulator tubes in parallel.

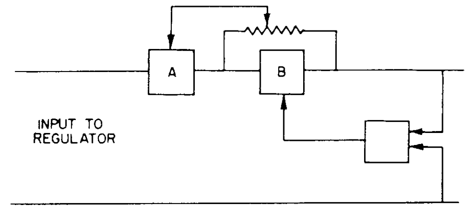

Fig. X-2
Regulator tubes in series.

Therefore, with a standard power supply, a lower regulated voltage can be obtained from the same unit for a specified input voltage. Conversely, the same regulated voltage can be obtained for a higher input voltage than the range obtainable with the original supply. Successive cascading can be accomplished in the same manner.

S. Fine

3. Two-Dimensional Visual Display

Present visual displays are not ideal for several reasons. The mechanical displays are too slow, and a cathode ray tube is too inaccurate. Furthermore, in common with all angularly deflected display devices the cathode ray tube has an adverse ratio of volume to indicator surface.

Preliminary work has been done on a new display device of a different type. The new device depends on two electronic units. The first unit is a voltage-to-linear-scale converter. The operation performed is the conversion of a voltage amplitude into a linear displacement along a nonlinear artificial transmission line. The location of the signal is characterized by the peak of a triangular voltage distribution along the line. Procedures have been developed for determining the parameter values to be used in the line and in the driving source. Because the output voltage can be read only at the nodes of the line, a process of digitalization is implicit in the method.

The second component of the proposed system is a multiple-element gaseous discharge tube, which will be used with one or more of the voltage-to-linear-scale converters to give a visual indication of the position of the voltage peak and hence of the amplitude of the given input voltage.

Two models of the gas tube have been constructed and the results obtained are promising. Further research, however, must be done to produce a really practical device. The voltage-to-linear-scale converter already works very well and can be applied to analog-to-digital conversion, coding, multiple-decision switching, and electronic control of multiple-unit systems.

W. A. Koelsch, Jr.

C. APPLIED NETWORK THEORY

1. Approximations on Network Synthesis

A procedure of obtaining transfer functions for RC-filter networks through a method of interpolation to known functions was presented previously (1). A more general method for determining transfer impedance functions with arbitrary poles anywhere in the left half of the s-plane to give any predescribed modulus-frequency response is under investigation. The method is to be based on an interpolation procedure.

The main reasons for undertaking this study are the following: First, there exist methods of approximation which, with rational functions in the complex plane, achieve very satisfactory results as far as the mathematical problem of approximation is concerned. However, they lack the freedom of preassigning the location of the poles. This disadvantage is very important for the engineering problem, inasmuch as it is the location of the poles that determines the character of the network to be synthesized. For instance, none of the existing standard methods of approximation result in a transfer impedance of RC character. Second, most of the standard approximation methods, in order to obtain a high "Q" circuit, result in a transfer function with at least a pair of poles close to the j-axis. To realize this network, coils of very high Q are often required. However, it is well known that it is possible to design high-Q circuits with low-Q elements. It is true that the latter may require more elements, but the question as to which one is the most economical is one of engineering judgement.

a. Mathematical development

The following is a short presentation of a sequence of orthogonal functions useful in approximating transfer functions by means of rational functions, realizable as passive linear networks, in the complex s-plane.

Let the function f(z) be analytic on and within c: $|z| = 1$, and let the function r(z) be of the form

$$r(z) = \frac{c_o z^{n-1} + c_1 z^{n-2} + \dots c_{n-1}}{\prod^{n} (z - \alpha_k)}, \quad z = x + jy \tag{1}$$

interpolating f(z) at the points 0, $\beta_1, \dots, \beta_{n-1}$. To construct r(z), let

$$\left.\begin{aligned} f(\beta_1) &= \left[\frac{a_1 z}{z - \alpha_1}\right]_{z=\beta_1} \\ f(\beta_2) &= \left\{\frac{a_1 z}{z - \alpha_1} + \frac{a_2\left[1 - (1/\beta_1 z)\right]}{(z - \alpha_1)(z - \alpha_2)}\right\}_{z=\beta_2} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}\right\} \tag{2}$$

which suggest that

$$r(z) = a_o + \frac{a_1 z}{z - \alpha_1} + \frac{a_2\left[1 - (1/\beta_1 z)\right]}{(z - \alpha_1)(z - \alpha_2)}$$

$$+ \frac{a_3\left[1 - (1/\beta_1 z)\right]\left[1 - (1/\beta_2 z)\right]}{(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)} + \dots \tag{3}$$

indeed interpolates the function f(z) at the given points β_k. The general form of expansion of f(z) is

$$f_1(z) = K_1 C_1(z) + K_2 C_2(z) + \dots$$

where $C_k(z)$ is a sequence in the variable z. If $f_1(z)$ is to be expanded in accordance with the least-mean-square-error method, $C_k(z)$ must be a set of orthogonal functions. Hence it is required that

$$\int_c C_k(z)\, \overline{C_n(z)}\, |dz| = 0, \quad k \neq n \tag{4}$$

If we identify $f_1(z)$ with r(z),

$$C_o(z) = 1$$

$$C_1(z) = \frac{z}{z - \alpha_1}$$

$$C_2(z) = \frac{z\left[1 - (1/\beta_1) z\right]}{(z - \alpha_1)(z - \alpha_2)}$$

$$\dots\dots\dots\dots$$

and it is sufficient for the orthogonality condition if $\beta_1 = 1/\bar{\alpha}_1$

$$\left.\begin{aligned} C_o(z) &= 1 \\ C_1(z) &= \frac{z}{z - \alpha_1} \\ C_2(z) &= \frac{z(1 - \bar{\alpha}_1 z)}{(z - \alpha_1)(z - \alpha_2)} \\ &\dots\dots\dots\dots \end{aligned}\right\} \tag{5}$$

The coefficients a_k are theoretically given by

$$a_k = \frac{\alpha_k \bar{\alpha}_k - 1}{2\pi j} \int_c f(z) \frac{\prod_{m=1}^{k-1} (z - \alpha_m)}{z \prod_{m=1}^{k} (1 - \bar{\alpha}_m z)}\, dz, \quad k \neq 0$$

However, a much more convenient way by which complex integration can be avoided is to observe that

$$r(z) = a_o + \frac{a_1 z}{z - \alpha_1} + \frac{a_2 z(1 - \bar{\alpha}_1 z)}{(z - \alpha_1)(z - \alpha_2)} + \frac{a_3 z(1 - \bar{\alpha}_1 z)(1 - \bar{\alpha}_2 z)}{(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)} + \dots \tag{6}$$

is equal to f(z) at the point $1/\bar{\alpha}_k$. Therefore

$$\left.\begin{aligned}
a_o &= f(o) \\
a_1 &= 1 - \alpha_1\bar{\alpha}_1 \left[f\left(\frac{1}{\bar{\alpha}_1}\right) - a_o\right] \\
a_2 &= \frac{1 - \alpha_2\bar{\alpha}_2}{\bar{\alpha}_2 - \bar{\alpha}_1} \left\{(1 - \alpha_1\bar{\alpha}_2)\left[f\left(\frac{1}{\bar{\alpha}_2}\right) - a_o\right] - a_1\right\} \\
a_3 &= \frac{1 - \alpha_3\bar{\alpha}_3}{\bar{\alpha}_3 - \bar{\alpha}_2}\left(\frac{1 - \alpha_2\bar{\alpha}_3}{\bar{\alpha}_3 - \bar{\alpha}_1}\left\{(1 - \alpha_1\bar{\alpha}_3)\left[f\left(\frac{1}{\bar{\alpha}_2}\right) - a_o\right] - a_1\right\} - a_2\right)
\end{aligned}\right\} \tag{7}$$

. . . .

A complete table can easily be made. At this point it will be necessary to investigate the convergence of the sequence of Eq. 5. Consider the N-th term

$$C_N(z) = \frac{z(1 - \bar{\alpha}_1 z)(1 - \bar{\alpha}_2 z) \dots}{(z - \alpha_1)(z - \alpha_2)(z - \alpha_3) \dots} = \frac{z}{z - \alpha_{N+1}} \prod_{i=1}^{N} \frac{1 - \bar{\alpha}_i z}{z - \bar{\alpha}_i}$$

or

$$\frac{z - \alpha_{N+1}}{z} C_N(z) = \prod_{i=1}^{N} \frac{1 - \bar{\alpha}_i z}{z - \alpha_i} = \prod_{i=1}^{N} \frac{\bar{\alpha}_i\left[(1/\bar{\alpha}_1) - z\right]}{\alpha_i\left[z(1/\bar{\alpha}_1) - 1\right]}$$

Now let $1/\bar{\alpha}_i = \beta_i$

$$\frac{z - \alpha_{N+1}}{z} C_N(z) = \prod_{i=1}^{N} \frac{\bar{\beta}_i}{\beta_i} \cdot \frac{(\beta_i - z)}{(z\bar{\beta}_i - 1)} = \prod_{i=1}^{N} \frac{\bar{\beta}_i\bar{\beta}_i}{\bar{\beta}_i\beta_i} \cdot \frac{(\beta_i - z)}{(\bar{\beta}_2 z - z)}$$

$$= \prod_{i=1}^{N} \frac{\bar{\beta}_i^2}{|\beta_i|^2} \cdot \frac{(\beta_i - z)}{(z\bar{\beta}_i - 1)}$$

The above expression is identified as the Blaschke product corresponding to the sequence β_i. It is known that*

$$\lim_{N\to\infty} \prod_{i=1}^{N} \frac{\beta_i}{|\beta_i|} \frac{(\beta_i - z)}{(z\bar{\beta}_i - 1)} = 0$$

on c: $|z| = 1$, provided $|\beta_i| < 1$ and the sequence

$$\prod_{i=1}^{N} |\beta_i|$$

converge. Consequently, it has been proved that the function

$$r_N(z) = a_o + a_1 \frac{z}{z - \alpha_1} + a_2 \frac{z(1 - \bar{\alpha}_1 z)}{(z - \alpha_1)(z - \alpha_2)}$$

$$+ a_3 \frac{z(1 - \bar{\alpha}_1 z)(1 - \bar{\alpha}_2 z)}{(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)} + \dots$$

converges on the circumference c, if the points $1/\bar{\alpha}_i$ lie interior to c: $|z| = 1$

b. Application to network theory

In this section the mathematical ideas developed previously will be applied directly to the approximation problem of network synthesis. The problem as it stands is to find a rational function R(s) of the complex variable s ($s = \sigma + j\omega$) to meet certain desired magnitude and phase characteristics along the imaginary axis, within a specified tolerance, while some freedom is permitted pertaining to the location of the poles. Usually the desired characteristics are given as plots of the real frequency ω. It will be assumed that a function F(s) has been found to satisfy the magnitude and phase requirements. The function F(s) does not necessarily need to be rational though it must be analytic throughout the right half-plane of the s-plane. References 4 and 5 elaborate on this specific problem.

Consider the function

$$z = \frac{s - 1}{s + 1}$$

or (8)

$$s = \frac{1 + z}{1 - z}$$

*There is a discontinuity for $z = \pm 1$, but the function is to be approached along the unit circle and not along the real axis.

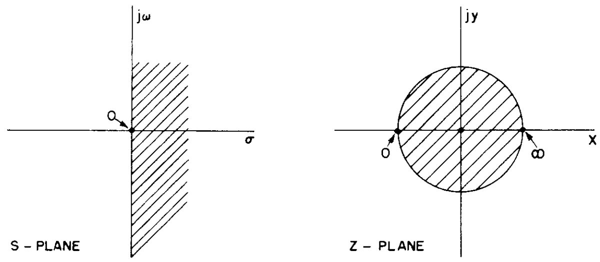

Fig. X-3
A transformation of the s-plane.

This is a well-known transformation through which the right half-plane of the s-plane corresponds to the unit circle in the z-plane (see Fig. X-3). Furthermore, the negative and positive real axis of the z-plane represents the negative real axis segments, 0 to -1 and -1 to $-\infty$, of the s-plane. Let the function F(s) be of the form

$$F(s) = A \frac{\prod_{k=1}^{m} (s + \zeta_k)(s + \bar{\zeta}_k)}{\prod_{i=1}^{n} (s + \zeta_i)(s + \bar{\zeta}_i)} \tag{9}$$

then its transform on the z-plane is

$$f(z) = F\left(\frac{s-1}{s+1}\right) = K \frac{\prod_{i=1}^{2(n-m)} (1-z) \prod_{k=1}^{n} (z + \xi_k)(z + \bar{\xi}_k)}{\prod_{i=1}^{n} (z + \xi_i)(z + \bar{\xi}_i)} \tag{10}$$

where

$$\xi_i = \frac{1 + \zeta_i}{1 - \zeta_i}$$

$$K = A \frac{\prod_{k=1}^{m} |1 - \zeta_k|^2}{\prod_{i=1}^{n} |1 - \zeta_i|^2}$$

It is obvious that the function f(z) is analytic inside the unit circle. Therefore, it can be approximated on c: $|z| = 1$ by the series

$$r(z) = a_o + a_1 \frac{z}{z - \alpha_1} + a_2 \frac{z(1 - \bar{\alpha}_1 z)}{(z - \alpha_1)(z - \alpha_2)} + \dots \tag{11}$$

The sequence of α's is not to be arbitrary. Further study of the convergence of the product

$$\prod_{k=1}^{N} \left|\frac{1}{\alpha_k}\right| \tag{12}$$

is required in order to establish a measure of the degree of convergence of the series in Eq. 11. Furthermore, α's must be selected in conjugate pairs in order for r(z) to have real coefficients. After the necessary algebraic steps, r(z) can be set in the form

$$r(z) = \frac{q_o + q_1 z + \dots q_\ell z^\ell}{\prod\limits_{k=1}^{\ell} (z + \alpha_k)(z + \bar{\alpha}_k)} \tag{13}$$

or

$$R(s) = \frac{Q_o + Q_1 s + \dots Q_\ell s^\ell}{\prod\limits_{k=1}^{\ell} (s + \delta_k)(s + \bar{\delta}_k)} \tag{14}$$

where

$$\delta_k = \frac{\alpha_k - 1}{\alpha_k + 1} \tag{15}$$

If δ_k's are preassigned, the equation

$$\alpha_k = \frac{1 + \delta_k}{1 - \delta_k}$$

is used to determine the α_k of the series in Eq. 11.

c. Measure of the approximation

According to the least-mean-square-error criterion

$$e = \int_c \left| f(z) - \sum_0^n a'_k C_k \right|^2 |dz| = \int_c \left[f(z) - \sum_0^n a'_k C_k \right] \left[f(\bar{z}) + \sum_0^n \bar{a}'_k \bar{C}_k \right] |dz|$$

or

$$e = \int_c f(z)\, f(\bar{z})\, |dz| - \sum_0^n \bar{a}'_k \int_c f(z)\, \bar{C}_k\, |dz| + \sum_0^n a'_k \int_c f(\bar{z})\, C_k\, |dz| - \sum_0^n a'_k \bar{a}'_k$$

But:

$$\int_c f(z)\, \bar{C}_k\, |dz| = a'_k$$

and

$$\int_c f(\bar{z})\, C_k\, |dz| = \bar{a}'_k$$

Therefore

$$e = \int_c f(z)\, f(\bar{z})\, |dz| - \sum_0^n a'_k \bar{a}'_k \tag{16}$$

As a result of this criterion, e is minimum on c: $|z| = 1$ for the choice of α's indicated. It must be observed that

$$\int_c f(z)\, f(\bar{z})\, |dz| = \int_c |f(z)|^2\, |dz| = \int_{-\infty}^{+\infty} |F(j\omega)|\, d\omega$$

or the area of the amplitude vs real frequency ω characteristic curve.

N. DeClaris

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2. Potential Analogs

A potential analog method has been developed (1) to determine the natural modes of a network which will have a prescribed gain function $\alpha(\omega)$ over the range $\omega_1 < \omega < \omega_2$ of the jω-axis of the s-plane. With the transformation

$$s^2 = -\omega_1^2 + \frac{\omega_2^2 - \omega_1^2}{4}\left(z - \frac{1}{z}\right)^2 \tag{1}$$

the interval of interest is transformed into the unit circle ($|z| = 1$) of the z-plane, and the gain $\bar{\alpha}(1, \phi)$ is analytically continued over the z-plane. The problem of the arbitrary gain is then transformed to a filter problem. This is achieved by construction of a function in z which, solely over the interval of interest, has the negative values of $\bar{\alpha}(1, \phi)$. A criterion is then developed for the location of the natural modes on the z-plane. This criterion is similar to that given by Darlington (2). It states that the natural modes should lie on a contour defined by the real part of the equation

$$2n \log z + \sum_{k=0}^{\infty} \frac{A_{2k}}{2}\left[z^{2k} + \frac{1}{z^{2k}}\right] = \log M \tag{2}$$

where $\mathrm{Re}\left[\log M\right] = \text{constant}$. The distribution of poles on the contour is determined by the corresponding imaginary part of the above equation. The coefficients A_{2k} are the Fourier series expansion coefficients of $\bar{\alpha}(z)$ on $|z| = 1$, and n is the number of desired poles.

The simple case of the gain $\bar{\alpha}(\omega) = -J_2\omega^2$ has been thoroughly investigated and the extent to which M and n influence the approximation is discussed. For the general case the expansion:

$$\bar{\alpha}(\omega) = \sum_{n=0}^{\infty} J_{2n}\omega^{2n} \tag{3}$$

is used and an example is given in reference 1. From further investigation it was concluded that from the practical point of view it is preferable to plot the gain as a function of ϕ and then to determine the Fourier coefficients A_{2k} of Eq. 2, by numerical methods. For a gain varying linearly with ϕ, the Fourier expansion used is

$$\bar{\alpha}(1, \phi) = -C\left[\frac{\cos 2\phi}{1} + \frac{\cos 6\phi}{9} + \frac{\cos 10\phi}{25} + \dots\right] \tag{4}$$

For numerical results, Eq. 4 is truncated. The effect of truncating the series at the n-th term (poles = number of terms) is being studied.

M. Macrakis

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