The Overconvergent de Rham-Witt Complex

by

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Abstract

We define the overconvergent de Rham-Witt complex $W^\dagger\Omega_{\overline{C}}$ for a smooth affine variety over a perfect field in characteristic p. We show that, after tensoring with $\mathbb{Q}$, its cohomology agrees with Monsky-Washnitzer cohomology. If $\dim \overline{C} < p$, we have an isomorphism integrally. One advantage of our construction is that it does not involve a choice of lift to characteristic zero.

To prove that the cohomology groups are the same, we first define a comparison map

$$t_F : \Omega_{C^\dagger} \to W^\dagger\Omega_{\overline{C}}.$$

(See Section 4.1 for the notation.) We cover our smooth affine $\overline{C}$ with affines $\overline{B}$ each of which is finite, étale over a localization of a polynomial algebra. For these particular affines, we decompose $W^\dagger\Omega_{\overline{B}}$ into an integral part and a fractional part and then show that the integral part is isomorphic to the Monsky-Washnitzer complex and that the fractional part is acyclic. We deduce our result from a homotopy argument and the fact that our complex is a Zariski sheaf with sheaf cohomology equal to zero in positive degrees. (For the latter, we lift the proof from [4] and include it as an appendix.)

We end with two chapters featuring independent results. In the first, we reinterpret several rings from p-adic Hodge theory in such a way that they admit analogues which use big Witt vectors instead of p-typical Witt vectors. In this generalization we check that several familiar properties continue to be valid. In the second, we offer a proof that the Frobenius map on $W(\mathscr{O}_{\mathbb{C}_p})$ is not surjective for $p > 2$.

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Acknowledgments

If you spent much time around me during the past five years, you probably heard me complain about the Boston area. I didn't particularly like the food, the weather, the traffic... But hopefully I made it equally clear that I had a great time during these five years, both academically and personally, and I want to take this chance to thank the people responsible.

First, I want to thank my adviser, Kiran Kedlaya. Literally this entire thesis has benefited from the guidance, suggestions, and in certain cases outright solutions that he offered. He suggested the problem, and while working on it I consulted him regularly, whether to discuss successes or failures.

The main result here appears in joint work with Andreas Langer and Thomas Zink. They invited me to spend a month with them in England and Germany where I met with them daily. My understanding of the material increased dramatically during that time.

This thesis has also benefited from numerous less obvious sources. Early on, the direction of our project was heavily influenced by meetings with Lars Hesselholt. For instance, he suggested phrasing Theorem 4.1.6 in terms of chain homotopies, now an essential part of our argument. More recently, Abhinav Kumar served on my thesis committee and provided a very careful reading. He offered simplifications, suggestions, and helped me create a more polished finished product. Along with Professor Kumar, I thank Clark Barwick for serving on my committee. I also was helped both within this thesis and throughout my graduate student career by my mathematical "siblings" Ruochuan Liu and Liang Xiao. And thanks to Peter Lee and Daniel Mellis for stylistic help. Finally, I thank the mathematics department itself for enabling each one of these opportunities.

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Thanking my family really means thanking my mom, Dr. Bonnie Rothman Davis. The list of ways my life would be different without her support and guidance is basically infinite. She cares about education, and I have spent the first twenty-seven years of my life in school. She cares about food, and I spend far too much of my time thinking about it... These are just two of the ways; I thank her for them all.

Contents

Chapter 1

Introduction

We begin in the next chapter by recalling some concepts which we will use repeatedly. Our goal is to provide an alternative construction of the Monsky-Washnitzer cohomology groups of a smooth affine variety, and in Section 2.1 we recall the current construction. We point out the drawback of the current construction, namely, its reliance on a non-functorial choice of lift to characteristic zero (despite which the cohomology groups are functorial).

In Section 2.2 we recall the map t_F which will be the basis for our comparison map between Monsky-Washnitzer cohomology and overconvergent de Rham-Witt cohomology. We also recall the notion of basic Witt differentials, which are a sort of basis for the de Rham-Witt complex of a polynomial algebra in characteristic p. We make the sense in which they are a basis precise in Proposition 2.2.8, both for the full de Rham-Witt complex and for its finite level analogues.

In Chapter 3 we define the overconvergent Witt vectors of radius C for any smooth affine ring (always of characteristic p). We define a Witt vector to be *overconvergent* if it is C-overconvergent for some radius C. The notion of a fixed radius is important because it enables us to work with finite-level Witt vectors. (The general notion of overconvergence is meaningless for finite-level Witt vectors, because every finite-level Witt vector is overconvergent of some radius.) There is a different definition of overconvergence in [4], and we check that the two notions agree.

As the overconvergent Witt vectors form a subring of the Witt vectors, we define the overconvergent de Rham-Witt complex as a subcomplex of the de Rham-Witt complex. Because the de Rham-Witt complex contains elements which are infinite sums of basic differentials, our definition must take into account specific radii. (An earlier definition we considered involved defining the overconvergent de Rham-Witt complex as the sub-differential graded algebra generated in degree zero by overconvergent Witt vectors. This complex is a proper subcomplex of the overconvergent de Rham-Witt complex as we define it.)

Our strategy for both the overconvergent Witt vectors and for the overconvergent de Rham-Witt complex is to define the notion carefully for the case of a polynomial algebra, and then use the functoriality of our constructions to extend the notion to the case of general (smooth) affines. (See Definitions 3.0.5 and 3.0.6, respectively.)

In Section 4.1 we define our comparison map between the Monsky-Washnitzer complex and the overconvergent de Rham-Witt complex. We check that it is has nice properties with respect to radii both for the case of a polynomial algebra and for the case of a finite étale extension of a polynomial algebra, though the latter result is significantly more subtle.

This section also contains one of the more important results of the paper, Theorem 4.1.6. We adapt an argument from Monsky and Washnitzer's original paper [18] to establish both the functoriality of our construction and the independence of our comparison map from the choice of Frobenius lift, at least after tensoring with $\mathbb{Q}$. We describe to what extent our results also hold integrally.

In Section 4.2 we use an analogue of the basic Witt differentials described in Section 2.2 to check that our comparison map is a quasi-isomorphism. We sacrifice the independence property that the basic Witt differentials possess, but in exchange our description is valid for a much larger class of rings. Our description holds for every ring $\overline{B}$ which is finite étale over a localization of a polynomial algebra. For such a ring, we decompose $W^\dagger\Omega_{\overline{B}}$ into an integral part and a fractional part and then show that the integral part is isomorphic to the Monsky-Washnitzer complex and that the fractional part is acyclic. This argument is inspired by the argument of Section 3.3 in [14], which compares de Rham-Witt cohomology with crystalline cohomology.

In Section 4.3 we check that the special affines we have been working with suffice to cover any smooth affine variety. This allows us to extract from our work in the previous sections statements concerning general smooth affine varieties. The section concludes with the following, which represents our main theorem.

Theorem 4.3.3. *Let $\overline{C}$ denote a smooth variety over a perfect field in characteristic p. Let $\kappa = \lfloor \log_p(\dim \overline{C}) \rfloor$. Let $C^\dagger$ denote a lift as in Section 2.1. Fix a Frobenius lift F on $C^\dagger$. Let t_F denote the comparison map described in Section 4.1.*

(a) Then after multiplying by p^κ the induced map on cohomology

$$p^\kappa t_F : H^\cdot(\Omega_{C^\dagger}) \to H^\cdot(W^\dagger\Omega_{\overline{C}})$$

is independent of F.

(b) It is functorial in the sense that for any map of smooth affines $\overline{g} : \overline{C} \to \overline{D}$, Frobenius lift F' on $D^\dagger$, and lift $g : C^\dagger \to D^\dagger$, the two induced compositions

$$p^\kappa t_{F'} \circ g : H^\cdot(\Omega_{C^\dagger}) \to H^\cdot(\Omega_{D^\dagger}) \to H^\cdot(W^\dagger \Omega_{\overline{D}})$$

and

$$\overline{g} \circ p^\kappa t_F : H^\cdot(\Omega_{C^\dagger}) \to H^\cdot(W^\dagger \Omega_{\overline{C}}) \to H^\cdot(W^\dagger \Omega_{\overline{D}})$$

are equal.

(c) Rationally, the map $p^\kappa t_F$ is a quasi-isomorphism. If $\dim \overline{C} < p$, we have an integral isomorphism.

The proof of this theorem relies on the fact that the overconvergent de Rham-Witt complex is a Zariski sheaf with sheaf cohomology equal to zero in positive degrees; for that we reference [4]. For the convenience of the reader, we lift the proof in op. cit. and include it, essentially verbatim, in our Appendix B.

The following two chapters, the final two before the appendices, contain significantly different material, although the hope is that the notion of overconvergent Witt vectors (if not the full complex) will eventually play a role here as well. In Chapter 5, we begin by reformulating several rings from p-adic Hodge theory in a way that makes essential use of the ring of p-typical Witt vectors. In particular, we show that $\tilde{A}^+$ may be reinterpreted as an inverse limit of rings of Witt vectors, with transition maps the Witt vector Frobenius. We verify several known properties of this ring. In the next section, we replace the p-typical Witt vectors W in the earlier constructions with the ring of big Witt vectors $\mathbb{W}$. We mimic our proofs from the previous section to deduce analogous properties of these new rings.

Chapter 6 is a note explaining a result that surprised us during our study of the material in Chapter 5. Explicitly, we prove there that the Witt vector Frobenius

$$F : W(\mathscr{O}_{\mathbb{C}_p}) \to W(\mathscr{O}_{\mathbb{C}_p})$$

is not surjective for $p > 2$ (though it is surjective on finite levels).

Our paper contains two appendices. The first is a collection of computations involving Witt vectors which are necessary for some of our arguments but are not particularly enlightening. It is the author's belief that the proofs of some of these technical lemmas could be simplified through the use of Gauss norms as in [3] and [4].

Finally, as already stated, our Appendix B reproduces the proof in [4] that the overconvergent de Rham-Witt complex is a Zariski sheaf.

Chapter 2

Preliminary Material

2.1 Monsky-Washnitzer cohomology

This section serves the dual purposes of reviewing Monsky-Washnitzer cohomology and of establishing some notation. Let k denote a perfect field of characteristic p. Let $W(k)$ denote the ring of p-typical Witt vectors with coefficients in k. Consider any $a \in W(k)\langle x \rangle$, the ring of convergent power series in one variable over $W(k)$. We can write

$$a = \sum_{k=0}^{\infty} p^k \sum_{j=0}^{n_k} a_{jk} x^j,$$

where we force: each $a_{jk} \in W(k)^* \cup 0$; $a_{n_k,k} \neq 0$ unless $n_k = 0$; and for fixed j, $a_{jk} \neq 0$ for at most one value of k. Under these restrictions, this form is unique. Then we say an element is *overconvergent* if there exists a C such that $n_k \leq C(k+1)$ for $k \geq 0$. If we wish to be specific, we call such an element C-overconvergent. The collection of all overconvergent series is a ring which we denote $W(k)\langle x \rangle^\dagger$.

We can generalize this to the multivariable case. For this, the second series should be over a (bounded) multi-index, and the criterion becomes $|n_k| \leq C(k+1)$, where the norm is $|(a_1, \ldots, a_n)| = \sum a_i$. This agrees with the usual notion of overconvergence (see for instance [11] for that notion). Our description enables us to write the ring $W(k)\langle x_1, \ldots, x_n \rangle^\dagger$ as a union of C-overconvergent modules, and we have defined the radius C in such a way that when we make the analogous construction in the ring of Witt vectors, the notions of radius will match.

Convention. Unless otherwise stated, $\overline{A}$ denotes the polynomial algebra in n variables $k[\overline{x_1}, \ldots, \overline{x_n}]$; A denotes the lift $W(k)[x_1, \ldots, x_n]$; A_n denotes the polynomial algebra over the truncated Witt vectors $W_n(k)[x_1, \ldots, x_n]$; $A^\dagger$ denotes the ring of overconvergent elements just described, and $A^{\dagger,C}$ denotes the submodule of C-overconvergent elements.

We must also extend the notion of overconvergence to quotients of polynomial algebras. For a smooth affine variety $\operatorname{Spec} \overline{B}$ over k, there exists a smooth lift of the form $B = W(k)[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ (see [20], p. 35). We then set

$$B^\dagger := W(k)\langle x_1, \ldots, x_n\rangle^\dagger/(f_1, \ldots, f_m).$$

For this fixed presentation, we say an element x is C-overconvergent if there exists a C-overconvergent element of $W(k)\langle x_1, \ldots, x_n\rangle^\dagger$ projecting onto x. The notion for a particular C depends on the presentation, but the union over all C will be independent.

One important property of these overconvergent lifts is that we can also lift maps, though not uniquely (see Theorem 2.4.4 (ii) of [20]). In particular, there exist lifts of Frobenius.

The previous paragraphs explained how to associate a ring $B^\dagger$ to a smooth affine ring $\overline{B} = k[x_1, \ldots, x_n]/I$. Let $\Omega_{B^\dagger/W(k)}$ denote the module of *continuous* differentials of $B^\dagger$ relative to $W(k)$. The Monsky-Washnitzer cohomology groups of $\overline{B}$ are defined to be the cohomology groups of the complex

$$\Omega_{B^\dagger/W(k)} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The key point is that it is not obvious if these groups are independent of the choice of lift or if the construction is functorial. Both of these questions have affirmative answers, see [20] p. 37. The alternative construction we propose will not depend on any choices and will be obviously functorial.

These notions are also well-defined and functorial integrally. This is implied in Monsky and Washnitzer's paper [18], p. 205.

2.2 The de Rham-Witt complex

In this section we review the de Rham-Witt complex. Our focus will be on the results required for what follows, and in particular our treatment will be by no means complete. We will focus in this section on two main details: a homomorphism t_F which injects $B^\dagger$ into $W(\overline{B})$, and the notion of basic Witt differentials for the de Rham-Witt complex over a polynomial algebra $\overline{A}$.

For a smooth affine $\overline{B}$ and a choice of Frobenius lift $F : B^\dagger \to B^\dagger$, we have a map

$$s_F : B^\dagger \to W(B^\dagger) \tag{2.2.1}$$

where, if we denote $s_F(x) = (s_0, s_1, \ldots)$, the s_i are the unique solutions to the equations $s_0^{p^n} + ps_1^{p^{n-1}} + \cdots + p^n s_n = F^n(x)$ for every $n \geq 0$ (see p. 508 of [10]). The important map for us will be t_F, the composite of s_F with projection to $W(\overline{B})$:

$$t_F : B^\dagger \xrightarrow{s_F} W(B^\dagger) \to W(\overline{B}).$$

Proposition 2.2.2. *The map t_F satisfies the following properties: (i) It is injective. (ii) It is p-adically continuous. (iii) It is functorial in the pair $B^\dagger, F$. (iv) If $F(x) = x^p$ for some x, then $t_F(x) = [\overline{x}]$, the Teichmüller lift of the reduction of x.*

Proof. It is clear from the definition that if $p \nmid x$, then $\overline{s_0} \neq 0$. Because t_F is a homomorphism and $W(\overline{B})$ is p-torsion free, property (i) holds. For property (ii), note that the length-n truncated Witt vector $t_F(x)|_n$ depends only on $x \mod p^{n+1}$. Properties (iii) and (iv) are stated explicitly in [10]. □

We now define the de Rham-Witt complex. Continue to let $\overline{B}$ denote an affine ring in characteristic p, although the following few results come from [9], where they consider in fact arbitrary $\mathbb{Z}_{(p)}$-algebras.

Definition 2.2.3. *A* Witt complex *over $\overline{B}$ is the following:*

1. *A pro-differential graded ring $E^*_\cdot$ and a strict map of pro-rings $\lambda : W_\cdot(\overline{B}) \to E^0_\cdot$.*

2. *A strict map (Frobenius) of pro-rings $F : E^*_\cdot \to E^*_{\cdot-1}$ such that $F\lambda = \lambda F$ (where the second F is the usual Witt vector Frobenius) and $Fd[b] = [b]^{p-1}d[b]$. (Here $[b]$ denote the image of the usual Teichmüller lift of b under λ.)*

3. *A strict map of graded $E^*_\cdot$-modules $V : F_* E^*_{\cdot-1} \to E^*_\cdot$ such that $\lambda V = V\lambda$, $FdV = d$ and $FV = p$. (The notation $F_* E^*_{\cdot-1}$ indicates that we think of $E^*_{\cdot-1}$ here as an $E^*_\cdot$-module via the Frobenius map.)*

A map of Witt complexes is a strict map of pro-differential graded rings commuting with the three maps λ, F, and V.

The reason for this definition is the following.

Definition 2.2.4. *The de Rham-Witt complex over $\overline{B}$ is the initial object in the category of Witt complexes over $\overline{B}$. It is denoted $W_\cdot\Omega_{\overline{B}}$.*

(Of course, it must be shown that this initial object exists. See for instance [9], where this is proven more generally for Witt complexes over any $\mathbb{Z}_{(p)}$-algebra. In the characteristic p case, it should be noted, we have not only $FV = p$ but also $VF = p$.) From now on, when we are not speaking of the de Rham-Witt complex in finite levels, we omit the "." subscript. The following proposition (Theorem A of [9]) gives us a way to represent elements of $W\Omega_{\overline{B}}$, though not uniquely.

Proposition 2.2.5. *The canonical map $\Omega_{W_n(\overline{B})} \to W_n\Omega_{\overline{B}}$ induced by λ in degree zero, is surjective.*

We now review some of the tools for studying the de Rham-Witt complex that will be used in our analysis. They will give us a unique way to write each element of the de Rham-Witt complex, though only over a polynomial algebra $\overline{A}$. Most of these definitions and results come from [14].

Definition 2.2.6. *A* weight *k is a function with values in $\mathbb{Z}[\frac{1}{p}]$ defined on the integers in a certain interval, $k : [1, n] \to \mathbb{Z}[\frac{1}{p}]$. The weight is called* integral *if each component is integral.*

The support of k, denoted Supp k, are those integers i for which $k(i) \neq 0$. We fix a numbering of the elements in the support,

$$\text{Supp } k = \{i_1, \dots, i_r\},$$

in such a way that

$$\text{ord}_p\, k_{i_1} \leq \cdots \leq \text{ord}_p\, k_{i_r}.$$

We use I to denote an interval of Supp k with the prescribed order,

$$I = \{i_l, i_{l+1}, \dots, i_{l+m}\}.$$

We let k_I denote the restriction of k to I.

We denote by $\mathcal{P}$ a partition of Supp k into disjoint intervals

$$\text{Supp } k = I_0 \sqcup I_1 \sqcup \ldots \sqcup I_l,$$

where the order of the intervals I_m agrees with the chosen order of Supp k. The interval I_0 may be empty, but the intervals $I_1, \dots, I_l$ may not be. Denote by $r \in [0, l-1]$ the first index such that $k_{I_{r+1}}$ is integral. If I_l is not integral, set $r = l$.

Let $t(k)$ denote the smallest integer so that $p^{t(k)}k$ is integral, and we let $u(k) := \max\{0, t(k)\}$. We can think of $u(k)$ as the denominator of k. Make similar definitions for k_{I_m}, for intervals I_m.

Definition 2.2.7. *A basic Witt differential $e := e(\xi, k, \mathcal{P})$, for k and $\mathcal{P}$ as above and $\xi \in V^{u(I)}W(k)$, is a special element of $W\Omega^l_{\overline{A}}$ defined as follows. Set $\xi = V^{u(I)}\eta$.*

1. *If $I_0 \neq \emptyset$, then*

$$e := V^{u(I_0)}\left(\eta X^{p^{u(I_0)}k_{I_0}}\right)\left(dV^{u(I_1)}X^{p^{u(I_1)}k_{I_1}}\right)\cdots\left(dV^{u(I_r)}X^{p^{u(I_r)}k_{I_r}}\right)\cdot$$
$$\left(F^{-t(I_{r+1})}dX^{p^{t(I_{r+1})}k_{I_{r+1}}}\right)\cdots\left(F^{-t(I_l)}dX^{p^{t(I_l)}k_{I_l}}\right);$$

2. *if $I_0 = \emptyset$ and k is not integral, then*

$$e := \left(dV^{u(I_1)}\left(\eta X^{p^{u(I_1)}k_{I_1}}\right)\right)\cdots\left(dV^{u(I_r)}X^{p^{u(I_r)}k_{I_r}}\right)\left(F^{-t(I_{r+1})}dX^{p^{t(I_{r+1})}k_{I_{r+1}}}\right)\cdots$$
$$\left(F^{-t(I_l)}dX^{p^{t(I_l)}k_{I_l}}\right);$$

3. *if $I_0 = \emptyset$ and k is integral, then*

$$e := \eta\left(F^{-t(I_1)}dX^{p^{t(I_1)}k_{I_1}}\right)\cdots\left(F^{-t(I_l)}dX^{p^{t(I_l)}k_{I_l}}\right).$$

Proposition 2.2.8. *Each element of $W\Omega_{\overline{A}}$ can be written uniquely as a (possibly infinite) sum of basic Witt differentials, where for any m all but finitely many of the ξ terms are in $V^m(W(k))$. Furthermore, each element of $W_n\Omega_{\overline{A}}$ can be written uniquely as a finite sum of basic Witt differentials with weights k having denominators at most p^{n-1}.*

Proof. These results are Theorem 2.8 and Proposition 2.17 of [14]. □

From the above proposition, it makes sense to consider the additive subset of $W\Omega_{\overline{A}}$ consisting of elements whose terms are sums of basic Witt differentials of fixed weight m. Proposition 2.6 of [14] tells us that this is a differential graded $W(k)$-module. The same is true if we only fix the weight modulo 1. Let us denote the differential graded module corresponding to a fixed weight $m \mod 1$ by $W^m\Omega_{\overline{A}}$. If our weight is fractional and we wish to emphasize this, we may write $W_n^{\text{frac},m}\Omega_{\overline{A}}$.

The weight zero modulo 1 module is precisely what Langer and Zink call the integral part. They call the complementary module the fractional part. They show that $W_n^{int}\Omega_{\overline{A}} \cong \Omega_{A_n}$ and that $W_n^{frac}\Omega_{\overline{A}}$ is acyclic (p. 74). Our strategy is to show that

$$\varinjlim_C \varprojlim_n W_n^{\dagger,C,int}\Omega_{\overline{B}} \cong \varinjlim_C \Omega_{B^{\dagger,C}}$$

and that $W_n^{\dagger,C,m}\Omega_{\overline{A}}$ is acyclic for fixed non-zero weight m modulo 1, for every n, and for C sufficiently large.

Proposition 2.2.9. *Let k denote a fixed fractional weight mod 1, and let F denote a lift of Frobenius to $A^\dagger$ which maps $x_i \mapsto x_i^p$. The group $W_n^{\mathrm{frac},k}\Omega_{\overline{A}}$ is an $A^\dagger$-module, where the multiplication is defined by $aw := t_F(a)w$.*

Proof. We will follow Langer and Zink's convention and let X_i denote $[x_i]$. We need to check that $W_n^{\mathrm{frac},k}\Omega_{\overline{A}}$ is preserved under multiplication by $t_F(a)$, and it suffices to check this for a monic monomial a, in which case $t_F(a) = X^m$ for some integral weight m (here we used our assumption that the Frobenius lift maps x_i to x_i^p). We consider aw for w a basic Witt differential of fractional weight k mod 1. There are two cases, corresponding to the first two cases on p. 40 of [14].

In the first case,

$$w = V^{u_0}(\eta X^{p^{u_0}k_0})(dV^{u_1}X^{p^{u_1}k_1})\cdots(F^{-t_l}dX^{p^{t_l}k_l}),$$

where the important thing point is that $u_0 \geq u_i$ for all i. (There is no such inequality comparing u_0 to the t_j's.) We can write aw as

$$aw = V^{u_0}(\eta X^{p^{u_0}(k_0+m)}F^{u_0-u_1}(dX^{p^{u_1}k_1})\cdots F^{u_0-t_l}(dX^{p^{t_l}k_l})) =: V^{u_0}(\tilde{w}).$$

In the notation of [14] p. 43, each degree one term of $\tilde{w}$ has the form

$$d(\frac{X^{p^{u_0}k_i}}{p^{u_0-u_i}}) \quad \text{or} \quad d(\frac{X^{p^{u_0}k_j}}{p^{u_0-t_j}}).$$

Then Langer and Zink's Proposition 2.11 shows that $\tilde{w}$ is a sum of basic Witt differentials of weight $p^{u_0}(k+m)$. Hence the result follows from their Proposition 2.5.

In the second case $u_0 = 0$, so

$$aw = X^m(dV^{u_1}\eta X^{p^{u_1}k_1})\cdots(F^{-t_l}dX^{p^{t_l}k_l}),$$

where now $u_1 \leq u_i$ for all i. Using the Leibniz rule and ignoring signs, this can be expressed as a sum of a term

$$V^{u_1}\eta X^{p^{u_1}k_1}\cdots(F^{-t_l}dX^{p^{t_l}k_l})F^{-t}(dX^{p^t m})$$

and a term

$$d(V^{u_1}\eta X^{p^{u_1}(k_1+m)})\cdots(F^{-t_l}dX^{p^{t_l}k_l}) = d\left(V^{u_1}\eta X^{p^{u_1}(k_1+m)}\cdots(F^{-t_l}dX^{p^{t_l}k_l}\right).$$

The first of these summands can be treated as in the previous paragraph, and the second is d applied to a term as in the previous paragraph. Proposition 2.6 assures that d does not change the weight, so we are done. □

Chapter 3

The Overconvergent de Rham-Witt Complex

Our goal is to define a subring $W^\dagger(\overline{B}) \subseteq W(\overline{B})$ with $\overline{B}$ a smooth affine ring over a perfect field of characteristic p. Our strategy is to define it carefully for the case of a polynomial algebra, and then allow functoriality to provide the definition in general. For the purposes of the following proofs, we define first $W^\dagger(R[x_1, \ldots, x_n]) \subseteq W(R[x_1, \ldots, x_n])$ for R a perfect field of characteristic p and also for R an arbitrary ring of characteristic zero.

As a subset, we define $W^\dagger(R[x_1, \ldots, x_n]) \subseteq R[x_1, \ldots, x_n]^{\mathbb{N}}$ as those sequences of polynomials $(f_0, f_1, \ldots)$ such that there exists some constant C for which $\deg(f_i) \leq C(i+1)p^i$ for $i \geq 0$. Call such a sequence overconvergent, or, if we wish to be more specific, "C-overconvergen"t or "overconvergent of radius C". In the next lemma, we will use the same definition of overconvergent whether we are talking about elements in the ring $W(R[x_1, \ldots, x_n])$ or in the ring $R[x_1, \ldots, x_n]^{\mathbb{N}}$.

Lemma 3.0.1. *Assume R has characteristic zero. A sequence is C-overconvergent in the sense above if and only if its image under the ghost map is.*

Proof. $\Rightarrow$ If we begin with the sequence $(f_0, f_1, \ldots)$, then its image under the ghost map is $(f_0, f_0^p + pf_1, f_0^{p^2} + pf_1^p + p^2 f_2, \ldots)$. Clearly this new sequence is overconvergent, with the same constant C. Of course, this did not use that our ring was characteristic zero.

$\Leftarrow$ Here we do use the hypothesis of characteristic zero. Trivially $\deg(f_0) \leq \deg(f_0)$, $\deg(f_1) \leq \max(\deg(f_0^p), \deg(f_0^p + pf_1))$ since $p \neq 0$, and so on. Hence the latter sequence being overconvergent implies that the former sequence is overconvergent. $\square$

We are now ready to prove the following:

Proposition 3.0.2. *In the case in which R is a perfect field of characteristic p, and in the case where R has characteristic zero, the subset of overconvergent Witt vectors $W^\dagger(R[x_1, \ldots, x_n]) \subseteq W(R[x_1, \ldots, x_n])$ is in fact a subring.*

Proof. We note now that in $R[x_1, \ldots, x_n]^{\mathbb{N}}$ (with componentwise addition and multiplication), the sum and product of two overconvergent sequences is overconvergent. For addition, we simply take the larger of the two constants. For multiplication, it suffices to add the two constants.

This fact together with the lemma and the fact that the ghost map is a ring homomorphism already implies that $W^\dagger(R[x_1, \ldots, x_n]) \subseteq W(R[x_1, \ldots, x_n])$ is a subring when R has characteristic zero. For $R = k$ a perfect field of characteristic p, we begin with the obvious surjection $W(k)[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$. We've just remarked that the overconvergent Witt vectors of the left hand side form a subring. The image of an overconvergent series must be overconvergent (because the degree can only go down after killing the terms divisible by p), and hence because the induced map on Witt vectors is a ring homomorphism, we've proven the result in general. (It should also be remarked that we can find a preimage of any overconvergent sequence in $k[x_1, \ldots, x_n]$ that is an overconvergent sequence in $W(k)[x_1, \ldots, x_n]$. But this is easy, because we may simply replace each coefficient with its Teichmüller lift without affecting the degree.) $\square$

The above proof yields the following corollary, which will be useful later:

Corollary 3.0.3. *The C-overconvergent Witt vectors form a $W(k)$-submodule of the ring $W^\dagger(k[x_1, \ldots, x_n])$.*

It will be important to extend the notion to quotients of polynomial algebras. Given a presentation $k[x_1, \ldots, x_n]/I$, it is obvious what the analogue should be: If $w = (w_0, w_1, \ldots) \in W(k[x_1, \ldots, x_n]/I)$, we say w is overconvergent of radius C if there exists $v \in W(k[x_1, \ldots, x_n])$ which is overconvergent of radius C projecting onto w. We cannot expect the number C to be independent of the presentation, but we do have the following:

Lemma 3.0.4. *The ring $W^\dagger(k[x_1, \ldots, x_n]/I) = \cup_C W^{\dagger, C}(k[x_1, \ldots, x_n]/I)$ is independent of presentation.*

Proof. Assume we have an isomorphism

$$\varphi : k[x_1, \ldots, x_n]/I \to k[y_1, \ldots, y_m]/J.$$

Let d_i denote the minimal degree among representatives in $k[y_1, \ldots, y_m]$ of $\varphi(x_i)$. Let d denote the maximum of the d_i's. Then it's clear that an element of radius C maps to an element of radius dC. □

The preceding lemma ensures us that the following definition makes sense.

Definition 3.0.5. *For a smooth affine $\overline{C} \cong k[x_1, \ldots, x_n]/I$, the overconvergent Witt vectors with coefficients in $\overline{C}$ are defined to be the image of $W^\dagger(k[x_1, \ldots, x_n])$ under the map $W(k[x_1, \ldots, x_n]) \to W(\overline{C})$.*

We use the above definitions to define the degree zero part of an overconvergent de Rham-Witt complex. Within the de Rham-Witt complex (not yet the overconvergent de Rham-Witt complex), we extend the notion of C-overconvergence to higher degrees as follows. We declare that: dw is C-overconvergent if w is; the sum of C_α-overconvergent elements for varying C_α is $\sup C_\alpha$-overconvergent, if that supremum is finite; and the product of a C_1-overconvergent element and a C_2-overconvergent element is $C_1 + C_2$-overconvergent. Of course, elements in $W\Omega_{k[x_1,\ldots,x_n]}$ will have many representations, and we say an element is overconvergent of radius C if at least one of those representations is overconvergent of radius C under the above formula. Note that in degree zero, our declarations concerning products and sums have already been verified.

Definition 3.0.6. *The overconvergent de Rham-Witt Complex of a polynomial algebra $k[x_1, \ldots, x_n]$ is defined as the sub-differential graded algebra of $W\Omega_{k[x_1,\ldots,x_n]}$ consisting of those elements which are C-overconvergent for some C, as described in the previous paragraph. It is denoted by $W^\dagger\Omega_{k[x_1,\ldots,x_n]}$. For $\overline{B}$ an affine ring in characteristic p, we define $W^\dagger\Omega_{\overline{B}}$ as the image of $W^\dagger\Omega_{k[x_1,\ldots,x_n]}$ under the map induced by some presentation $k[x_1, \ldots, x_n] \twoheadrightarrow \overline{B}$.*

Lemma 3.0.7. *The complex $W^\dagger\Omega_{\overline{B}}$ does not depend on presentation.*

Proof. We know that the degree zero part of the complex does not depend on presentation thanks to Lemma 3.0.4. Now consider a term $\omega_0 d\omega_1 \ldots d\omega_m$, with each $\omega_i \in W^\dagger\Omega^0_{\overline{B}} = W^\dagger(\overline{B})$ which has radius C as specified by one presentation. By the lemma, each ω_i has some radius $\tilde{C}$ depending on C as specified by another presentation. Because the number of terms $m+1$ is bounded by $\dim \overline{B} + 1$, we see that our notion of overconvergence does not depend on presentation. □

We now show that the definition of overconvergent Witt vectors defined here matches the notion from [4]. We begin by recalling that notion.

Definition 3.0.8. *An element $\omega \in W(k[x_1, \ldots, x_n])$ is called* overconvergent *if there exist $\varepsilon > 0$ and $N > -\infty$ such that for every basic Witt differential $e(\xi_{k,\mathcal{P}}, k, \mathcal{P})$ appearing in the decomposition of ω we have $\operatorname{ord}_V(\xi_{k,\mathcal{P}}) - \varepsilon|k| \geq N$. (Here $|k| := k_1 + \cdots + k_n$ and $\operatorname{ord}_V(w)$ is defined to be the largest number m such that $w \in V^m W(k[x_1, \ldots, x_n])$.)*

We restate the above definition using the concept of Gauss norm from [4].

Definition 3.0.9. *For $\omega = \sum_{k,\mathcal{P}} e(\xi_{k,\mathcal{P}}, k, \mathcal{P})$ in $W\Omega_{k[x_1,\ldots,x_n]}$ and $\epsilon > 0$, we define the Gauss norm of ω:*

$$\gamma_\epsilon(\omega) = \inf_{k,\mathcal{P}}\{\operatorname{ord}_V \xi_{k,\mathcal{P}} - \epsilon|k|\}.$$

With this terminology, ω is overconvergent if $\gamma_\epsilon(\omega) > -\infty$ for some $\epsilon > 0$.

Remark 3.0.10. *We should point out that in our definition, radius C_0 is stronger than radius C_1 for $C_0 < C_1$. In the definition of [4], radius ϵ_1 is stronger than radius ϵ_0 for $\epsilon_0 < \epsilon_1$. From a geometric viewpoint, their terminology is preferable.*

Lemma 3.0.11. *Our notion of overconvergence agrees with that from [4].*

Proof. We show first that our notions agree in degree zero. Assume that we have an element $e(\xi_{k,\mathcal{P}}, k, \mathcal{P})$ satisfying $\operatorname{ord}_V(\xi_{k,\mathcal{P}}) - \varepsilon|k| \geq N$. The first $u(k) + \operatorname{ord}_V(\eta) = \operatorname{ord}_V(\xi_{k,\mathcal{P}})$ components of this element are zero, and the next component has degree $p^{\operatorname{ord}_V(\xi_{k,\mathcal{P}})}|k|$. The later components may be ignored. To say that such an element is overconvergent under our definition, we must find a C such that

$$(\operatorname{ord}_V(\xi_{k,\mathcal{P}}) + 1)p^{\operatorname{ord}_V(\xi_{k,\mathcal{P}})}C \geq p^{\operatorname{ord}_V(\xi_{k,\mathcal{P}})}|k|.$$

We can indeed find such a C: if $N \geq 0$, we may take $C = \frac{1}{\epsilon}$; if $N < 0$, we may take $C = \frac{1}{|N|\epsilon}$. It's essential that these expressions for C depend only on ϵ and N.

The other direction is easier. Assume we have Witt components $(f_0, f_1, \ldots)$ of an element satisfying $\deg f_i \leq (i+1)p^i C$ for some C. We claim that such an element is overconvergent under the notion of [4] with $\epsilon = \frac{1}{C}$ and $N = -1$. Consider (a monomial in) the term f_0. The basic Witt differential to which it corresponds has $|k| \leq C$ and $\operatorname{ord}_V(\xi_{k,\mathcal{P}}) = 0$. We subtract off monomials in f_0 until we reach $f_0 = 0$.

Inductively, assume we have reached $(0, \ldots, 0, f_i, f_{i+1}, \ldots)$. Consider now (a monomial in) the term f_i. Now $\operatorname{ord}_V(\xi_{k,\mathcal{P}}) = i$ and $|k| \leq (i+1)C$. This completes the induction.

We show now that our notions agree in positive degrees. First we show that an element which is overconvergent under our definition is overconvergent under the definition of [4]. Note that their definition of radius behaves identically to ours with respect to sums (p. 5). Their definition also behaves the same as ours with respect to products, at least if the radii are the same (p. 5 again), and we can arrange that the radii are the same because we need products of at most $\dim \overline{A} + 1$ elements. Finally, from [14] Proposition 2.6, their radius can only improve after differentiation.

It remains to show that an overconvergent element under their definition is overconvergent under ours. Because our notions behave the same with respect to sums, it suffices to show that a basic Witt differential with constants ϵ and N corresponds to an overconvergent Witt differential under our definition with radius C depending only on ϵ and N.

We start with the first case within Definition 2.2.7. With our usual notation, together with the notation of that definition, we are assuming that

$$\mathrm{ord}_V(\eta) + u(I_0) - \epsilon|k_{I_0}| - \cdots - \epsilon|k_{I_l}| \geq N.$$

The key is to use the formula $V(x)dy = V(xF(dy)) = V(xy^{p-1}dy) = V(xy^{p-1})dVy$, where y is a Teichmüller, using the equations (V2) and (V3) on p. 543 of [10]. For instance, if $l = 1 = r$, (where $r = 1$ is a concise way of saying that I_0 and I_1 are both fractional), we can rewrite the basic Witt differential as

$$\begin{aligned} V^{u(I_0)}\left(\eta X^{p^{u(I_0)}k_{I_0}}\right)\left(dV^{u(I_1)}X^{p^{u(I_1)}k_{I_1}}\right) &= \\ V^{u(I_1)}\left(V^{u(I_0)-u(I_1)}\left(\eta X^{p^{u(I_0)}k_{I_0}}\right)dX^{p^{u(I_1)}k_{I_1}}\right) &= \\ V^{u(I_1)}\left(V^{u(I_0)-u(I_1)}\left(\eta X^{p^{u(I_0)}k_{I_0}+p^{u(I_0)}k_{I_1}-p^{u(I_1)}k_{I_1}}\right)dV^{u(I_0)-u(I_1)}X^{p^{u(I_1)}k_{I_1}}\right) &= \\ V^{u(I_0)}\left(\eta X^{p^{u(I_0)}k_{I_0}+p^{u(I_0)}k_{I_1}-p^{u(I_1)}k_{I_1}}\right)dV^{u(I_0)}\left(X^{p^{u(I_1)}k_{I_1}}\right). \end{aligned}$$

It is easy to see that this is C-overconvergent under our definition, for some C depending only on ϵ and N. The other cases may be treated similarly. □

Chapter 4

Overconvergent de Rham-Witt Cohomology

4.1 The comparison map

The construction in this section will make explicit the connection between the overconvergent Witt vectors and the Monsky-Washnitzer algebra. Abbreviate as before $\overline{A} = k[x_1, \dots, x_n]$ and $A^\dagger = W(k)\langle x_1, \dots, x_n\rangle^\dagger$. Fix a Frobenius lift F on $A^\dagger$ which sends $x_i \to x_i^p$.

In Section 2.2 we described the map

$$t_F : A^\dagger \to W(\overline{A}).$$

We claim that its image actually lands in $W^\dagger(\overline{A})$. Write

$$a = \sum_{k=0}^{\infty} p^k \sum_{j=0}^{n_k} a_{jk} x^j$$

subject to the conventions of Section 2.1. Assume a is a C-overconvergent series. We will compute the Witt vector components of $t_F(a)$ and check that they are polynomials with degrees satisfying certain bounds, which will ensure that $t_F(a)$ is overconvergent, and in fact C-overconvergent for the same C.

Proposition 4.1.1. *Write*

$$(c_0, c_1, c_2, \dots) = t_F(a) \in W(\overline{A}),$$

with a as above. Then the polynomials c_i satisfy $\deg c_i \leq C(i+1)p^i$. *Hence t_F maps*

$A^{\dagger,C}$ *into* $W^{\dagger,C}(\overline{A})$.

Proof. Our goal is to compute $t_F(\sum_{k=0}^{\infty} p^k \sum_{j=0}^{n_k} a_{jk}x^j)$. We already noted that our map is a p-adically continuous homomorphism. So our goal is to compute

$$\sum_{k=0}^{\infty} p^k \sum_{j=0}^{n_k} t_F(a_{jk}) t_F(x)^j.$$

We always have $VF = p$, and for the Witt vectors of an $\mathbb{F}_p$-algebra, we also have $FV = p$ (see [10], p. 507). In particular, in our case F and V commute and $p^k = V^k F^k$. So our goal is to compute

$$\sum_{k=0}^{\infty} V^k F^k \sum_{j=0}^{n_k} t_F(a_{jk}) t_F(x)^j.$$

Next, note that for $a_{jk} \in W(k)$, $t_F(a_{jk}) = a_{jk}$. This is true for Teichmüller lifts because, as k is perfect, every lift of Frobenius to $W(k)$ sends $[b]$ to $[b^p] = [b]^p$. Hence it is true for any a_{jk}, because we can write

$$a_{jk} = \sum_{i=0}^{\infty} p^i [b_i].$$

Because of our requirement that $F(x_i) = x_i^p$, we have that $t_F(x) = [x]$.

So finally we have reduced to calculating the Witt vector components $(c_0, c_1, \ldots)$ of

$$\sum_{k=0}^{\infty} V^k \sum_{j=0}^{n_k} F^k(a_{jk})[x^{jp^k}].$$

Write $(h_{0k}, h_{1k}, h_{2k}, \ldots) = \sum_{j=0}^{n_k} F^k(a_{jk})[x^{jp^k}]$. Write $F^k(a_{jk}) = (a_{0jk}, a_{1jk}, \ldots)$. Then by [10] p. 505,

$$F^k(a_{jk})[x^{jp^k}] = (a_{0jk}x^{jp^k}, a_{1jk}x^{jp^{k+1}}, a_{2jk}x^{jp^{k+2}}, \ldots).$$

Then it will be an immediate corollary of Lemma A.0.1 in the appendix that deg $h_{ik} \leq p^{i+k}|n_k|$. (Here it is important that we took our norm on multi-indices to be the sum of the indices, not the maximum of the indices.) Then another immediate application of the lemma, together with our bounds on $|n_k|$, will yield our proposition. □

We have just shown that $t_F : A^{\dagger,C} \to W^{\dagger,C}(\overline{A})$. We will now show moreover that if an element of $W^{\dagger,C}$ is in the image of $A^\dagger$ under the map t_F, then it is in the image of $A^{\dagger,C}$. This will require a more careful calculation than did the bound in the other direction.

Proposition 4.1.2. *Assume $a \in A^\dagger$ is not in $A^{\dagger,C}$. Then its image $t_F(a)$ is not in $W^{\dagger,C}$.*

Proof. In terms of Witt vector components, write $t_F(a) = (c_0, c_1, c_2, \ldots)$. We will show that if $n_0 > C$, then deg $c_0 > C$. We will show that if $|n_0| \leq C$ and $|n_1| > 2C$, then deg $c_1 > 2pC$. We will show that if $|n_0| \leq C$ and $|n_1| \leq 2C$ and $|n_2| > 3C$, then deg $c_2 > 3Cp^2$. And so on.

We have represented a in such a way that for each k, if $|n_k| > 0$, then $a_{n_k k} \neq 0$. Recall that this implies $a_{n_k k}$ is a unit, i.e., is not divisible by p. Thus, if $n_k > (k+1)C$,we can assume $a_{0 n_k k} \neq 0$ (recall the notation from the previous proof, that $F^k(a_{jk}) = (a_{0jk}, a_{1jk}, \ldots)$.) This will be important, because it enables us to get a lower bound on the first term of at least one element

$$F^k(a_{jk})[x^{jp^k}] = (a_{0jk}x^{jp^k}, a_{1jk}x^{jp^{k+1}}, a_{2jk}x^{jp^{k+2}}, \ldots).$$

Our base case is the case $|n_0| > C$. Here it is clear that deg $c_0 > C$. In general, we assume $|n_i| \leq (i+1)C$ for $0 \leq i \leq k-1$, and $|n_k| > (k+1)C$. From the previous proposition, we know deg $c_i \leq C(i+1)p^i$ for $0 \leq C \leq k-1$. The first term of $F^k(a_{n_k k})[x^{n_k p^k}]$ has degree $> (k+1)Cp^k$.

If we write

$$\sum_{j=0}^{n_k} F^k(a_{jk})[x^{jp^k}] = (b_{0k}, b_{1k}, \ldots),$$

then we know deg $b_{0k} > C(k+1)p^k$ (because addition is simple in the first Witt vector component). Then considering the definition of addition in terms of ghost components, we know

$$c_0^{p^k} + \cdots + p^k c_k = p^k b_{0k} + \cdots .$$

All of the unwritten terms on the right hand side have degree $\leq C(k+1)p^k$. Hence deg $c_k > C(k+1)p^k$, proving the proposition. □

We should also extend our results to quotients of polynomial algebras. Let $\overline{B}$ denote a smooth affine, let B denote a lift to characteristic zero as in Section 2.1, and let $B^{\dagger,C}$ denote the C-overconvergent submodule corresponding to this presentation.

For a given element $b \in B^{\dagger,C}$, choose a series

$$a = \sum_{k=0}^{\infty} p^k \sum_{j=0}^{n_k} a_{jk} x^j \in W(k)\langle x_1, \ldots, x_n \rangle^\dagger = A^\dagger$$

which is C-overconvergent mapping to b. Let F denote a lift of Frobenius to $B^\dagger$. We can find some F' on $A^\dagger$ which induces F, but it may not send $x_i \mapsto x_i^p$. Nonetheless, from Proposition A.0.3 in the appendix, $t_{F'}(a)$ is $(C + D)$-overconvergent, where D depends only on F'. Because t_F commutes with projection, we have that $t_F(b)$ is $(C+D)$-overconvergent, and in general C-overconvergent elements map into $(C+D)$-overconvergent elements (as long as we pick consistent presentations).

Proving a partial converse will take more work. We restrict to affines as in the following definition. Our terminology is taken from [19].

Definition 4.1.3. *A* standard étale affine *is a ring of the form* $\overline{B} = (\overline{A}_h[z]/P(z))_g$ *with* $\overline{A}$ *a polynomial algebra as before,* $h \in \overline{A}$*,* $\overline{A}_h$ *the localization,* $P(z)$ *monic,* $g \in \overline{A}_h[z]/P(z)$*,* $(\overline{A}_h[z]/P(z))_g$ *again the localization, and* $P'(z)$ *invertible in* $\overline{B}$*.*

For instance, an obvious choice is to take $h = P'(z)$, and thus force the invertibility. As the terminology suggests, a standard étale affine is étale over affine space (because of our constraints on P). In the context of these special affines, we can get a partial converse to the previous results.

Proposition 4.1.4. *Let* $\overline{B}$ *denote a standard étale affine, and fix a presentation*

$$\overline{B} = k[x_1, \ldots, x_n, y, z, w]/(yh - 1, P(z), wg - 1).$$

Let

$$\overline{\omega} \in W^{\dagger,C}(\overline{B})$$

be in the image of t_F*. Then it is in the image of* $t_F|_{B^{\dagger,C+D}}$*, where* D *is a constant depending only on* $\overline{B}$*.*

Proof. To prove the proposition, we must study the degree of lifts of pth roots.

Lemma 4.1.5. *Let* $\overline{w} \in \overline{B}$ *denote a* p*th power which has a lift to* $\overline{A}$ *of degree* C*. Then its* p*th root has a lift to* A *of degree at most* $\lfloor \frac{C}{p} \rfloor + D$*, where* D *is a constant depending only on* $\overline{B}$*.*

Proof. First note that $dx_1, \ldots, dx_n$ are free generators for $\Omega^1_{\overline{B}}$: This is true of course for a polynomial algebra, and hence also for its étale extension $\overline{B}$. (An étale extension is formally étale (p. 30 of [17]), so by [7] Corollary 20.5.8 we have an isomorphism $\Omega^1_{\overline{A}/k} \otimes_{\overline{A}} \overline{B} \cong \Omega^1_{\overline{B}/k}$.)

We claim that $x_1^{e_1} \cdots x_n^{e_n}$ with $0 \leq e_i \leq p-1$ are free generators for $\overline{B}$ as a $\overline{B}^p$-module. It is clear that the module generated by these monomials over $\overline{B}^p$ is in fact a ring which contains $x_1, \ldots, x_n$ and the (perfect) field k. To prove that they generate all of $\overline{B}$, we refer to Proposition 21.1.7 of [7]. It asserts that because $\Omega^1_{\overline{B}/k}$ is generated by $dx_1, \ldots, dx_n$, and because $\overline{B}$ is a finite type $k[\overline{B}^p]$ algebra, then the given monomials are indeed generators. It remains to check that they are free generators.

Assuming they are not free generators, pick an expression for zero in the form

$$\sum c_I x_1^{i_1} \ldots x_n^{i_n},\ c_I \text{ not all zero}$$

which is minimal in the following sense: after ordering the monomials lexicographically, the largest monomial appearing in the expression with a non-zero coefficient is smallest. Because $dx_1, \ldots, dx_n$ are free generators for $\Omega^1_{\overline{B}}$, the formula

$$d(\sum c_I x_1^{i_1} \ldots x_n^{i_n}) = 0$$

yields n new expressions for zero, at least one of which has a non-zero coefficient, and each of which is smaller under the ordering described above, which is a contradiction. (For instance, the expression we deduce from the coefficient of dx_1 is

$$\sum i_1 c_I x_1^{i_1-1} \ldots x_n^{i_n},$$

since c_I is a pth power and hence its partial derivatives vanish.)

From the above results, given any element in $\overline{B}$, we can associate a unique p^n-tuple of pth powers in $\overline{B}$, namely, the coefficients of the monomials $x_1^{i_1} \ldots x_n^{i_n}$. We can find a constant D such that each element in the p^n-tuples corresponding to y^i, z^i, and w^i for $1 \leq i \leq p-1$ has pth root with a lift of degree at most D.

We now carry out the following steps to prove the lemma. Write a lift w of $\overline{w}$ as a polynomial of degree at most C in $\overline{A}$. It can be written as a sum of terms which are pth powers (of elements of degree $\lfloor \frac{C}{p} \rfloor$) multiplied by sums of terms of the form $x_1^{i_1} \cdots x_n^{i_n} y^{i_{n+1}} z^{i_{n+2}} w^{i_{n+3}}$, with $0 \leq i_j \leq p-1$. The elements of the p^n-tuple corresponding to a term $y^{i_{n+1}} z^{i_{n+2}} w^{i_{n+3}}$ have pth roots of degree at most $3D$. Thus the elements of the p^n-tuple corresponding to $x_1^{i_1} \cdots x_n^{i_n} y^{i_{n+1}} z^{i_{n+2}} w^{i_{n+3}}$ have pth roots of degree at most $3D+n$. In particular the coefficient of 1 for $\overline{w}$ has pth root of degree at most $\lfloor \frac{C}{p} \rfloor + 3D + n$ and the other coefficients must be zero (from our assumption that $\overline{w}$ is a pth power). This proves the lemma. □

Choose D_1 such that $t_F(x)$ is D_1-overconvergent for every degree one $x \in A$. Let D denote the max of $2D_1$ and the D of the previous lemma.

Now we assume we are given an element $\omega \in W^{\dagger,C}(\overline{A})$ mapping to our element $\overline{\omega}$. We can of course find an element $\nu_0 \in A$ of degree C projecting to $\omega_0 \in \overline{A}$. The element $w_0 := \omega - t_F(\nu_0)$ has jth component with degree at most $(j+1)p^jC+(j+1)p^jD$ by Lemma A.0.10. Its projection $\overline{w_0}$ to $W(\overline{B})$ is the difference of two elements in the image of t_F, and hence its projection is in the image of t_F. This together with the fact that its zeroth component $w_{00} = 0$ implies we can write $\overline{w_0} = p\overline{v_1}$. From the previous lemma, the jth component of $\overline{v_1}$ has a lift of degree at most $(j+2)p^jC + (j+3)p^jD$.

Inductively, assume that at the kth stage of this process we have an element v_k whose jth component has degree at most $(j+1+k)p^jC + (j+2k)p^jD$. Using the previously cited lemma, killing off the zeroth component yields w_k whose jth component has degree at most $(j+1+k)p^jC+(j+2k)p^jD$. As before, its projection to $W(\overline{B})$ is in the image of t_F and has zeroth component equal to zero, hence can be written as $p\overline{v_{k+1}}$, where, by the previous lemma, the jth component has degree $(j+1+k+1)p^jC + (j+2k+2)p^jD$. This completes the induction.

We have shown that $\overline{w}$ can be written as $t_F(\nu)$, where for ν we have $n_k \leq (k+1)C + (2k)D$ for $k \geq 1$. In particular, ν is $C+2D$-overconvergent. Replacing D with $2D$, we are done. □

We now extend t_F to a map of complexes. This yields our desired comparison map:

$$t_F : \Omega_{B^\dagger} \to W\Omega^\dagger_{\overline{B}}.$$

As indicated by the notation, it still depends on our choice of Frobenius lift F. The following theorem will show as a corollary that the induced map on cohomology does not depend on F, at least after tensoring with $\mathbb{Q}$. It will also establish the functoriality of our comparison map over $\mathbb{Q}$.

We will closely follow the argument on pages 205-206 of [18]. Let $\overline{B}$ and $\overline{C}$ denote standard étale affines.

Theorem 4.1.6. *Let $\psi_1, \psi_2 : B^\dagger \to W^\dagger(\overline{C})$ denote two ring homomorphisms such that for every $b \in B^\dagger$, $\psi_2(b) - \psi_1(b) = V(w)$ for some w. Then the induced maps on differential graded algebras*

$$p^\kappa\psi_1, p^\kappa\psi_2 : \Omega_{B^\dagger} \to W^\dagger\Omega_{\overline{C}}$$

are chain homotopic, where $\kappa = \lfloor \log_p(\dim \overline{B}) \rfloor$.

The chain homotopy we produce will factor through the following algebra.

Definition 4.1.7. *Denote by* $D'(\overline{C})$ *the differential graded algebra with ith graded piece:*

$$D'(\overline{C})^i = W\Omega^i_{\overline{C}}[\frac{1}{p}][[T]] \oplus W\Omega^{i-1}_{\overline{C}}[\frac{1}{p}][[T]] \wedge dT.$$

Fix a presentation $B^\dagger = W(k)\langle x_1, \ldots, x_n, y, z, w\rangle^\dagger/(yh-1, P(z), wg-1)$ and with respect to this presentation define a homomorphism of differential graded $W(k)$-algebras

$$\begin{aligned} \varphi : \Omega_{B^\dagger} &\to D'(\overline{C}) \\ x_i &\mapsto \psi_1(x_i) + \frac{T(\psi_2(x_i) - \psi_1(x_i))}{p}. \end{aligned}$$

Denote by h_0 and h_p reduction modulo (T) and $(T-p)$ respectively. The key point of course is that $h_0 \circ \varphi = \psi_1$ and $h_p \circ \varphi = \psi_2$.

We prove that φ extends to all of $\Omega_{B^\dagger}$. From the universal property of the de Rham complex, it suffices to prove that φ extends to a map $B^\dagger \to W(\overline{C})[\frac{1}{p}][[T]]$.

Considering first the polynomial algebra case, we have

$$\sum_{k=0}^{\infty} p^k \sum_{j=0}^{n_k} a_j x^j \mapsto \sum_{k=0}^{\infty} p^k \sum_{j=0}^{n_k} a_j(\psi_1(x_1) + \frac{T}{p}(\psi_2(x_1) - \psi_1(x_1)))^{j_1} \cdots .$$

We claim that the right hand side can be written as

$$\sum_{m=0}^{\infty} (\frac{T}{p})^m w_m,$$

where $w_m \in W(\overline{C})$, for all m. But clearly, for fixed k,

$$\sum_{j=0}^{n_k} a_j(\psi_1(x_1) + \frac{T}{p}(\psi_2(x_1) - \psi_1(x_1)))^{j_1} \cdots$$

can be written in this way, and so the claim follows from the fact that $W(\overline{C})$ is p-adically complete.

From this representation of $\varphi(f)$, we see that the image of $A^\dagger$ does indeed land in $D'(\overline{C})$, i.e., that for a fixed power of T, we do not have arbitrarily large powers of p

in the denominator. It shows moreover that in fact the power of p is bounded by the power of T.

From the previous argument, $\varphi(h) = \sum \frac{T^i}{p^i} w_i$ for some w_i. Also, note that $w_0 = \psi_1(h)$ must be invertible. Hence

$$\varphi(y) = \frac{\frac{1}{w_0}}{1 - \left(-(\sum \frac{T^i}{p^i} \frac{w_i}{w_0})\right)},$$

the sum being over $i \geq 1$. Repeat the exact same argument as above, this time for

$$\sum_{k=0}^{\infty} p^k \sum_{j=0}^{n_k} a_j x^j y^l \mapsto \sum_{k=0}^{\infty} p^k \sum_{j=0}^{n_k} a_j (\psi_1(x_1) + \frac{T}{p}(\psi_2(x_1) - \psi_1(x_1)))^{j_1} \cdots (\varphi(y))^l.$$

Now we identify the image of z under φ. Write

$$P(z) = z^r + f_1 z^{r-1} + \ldots + f_{r-1} z + f_r,$$

where each f_r lies in $W(k)\langle x_1, \ldots, x_n, y\rangle$ (subject to certain conditions to ensure the extension is étale). We guess

$$\varphi(z) = \sum c_i \frac{T^i}{p^i}$$

and solve for the c_i's. The term c_0 must be a root of the polynomial

$$x^r + \psi_1(f_1)x^{r-1} + \cdots + \psi_1(f_r),$$

and so we set $c_0 = \psi_1(z)$.

Considering the coefficient of T, we find that

$$\psi_1(P)'(c_0)c_1 = \varphi(f_1)_1 c_0^{r-1} + \cdots + \varphi(f_r)_1,$$

where $\varphi(f)_i$ denotes the coefficient of $\frac{T^i}{p^i}$ in $\varphi(f)$. This is solvable because our extension is étale. (Specifically, from p. 25 of [17], our extension being étale implies that the resultant res (P, P') of the polynomial P is a unit. Immediately from the definition of the resultant, see for instance p. 119 of [2], if α is a root of $P(z)$ then $P'(\alpha)$ divides the resultant and hence it too is a unit. Finally applying the homomorphism ψ_1 yields that $\psi_1(P)'(c_0)$ is invertible.) We are able to get an equation for every c_i in this way, and solving these yields our choice of $\varphi(z)$. Checking that such an expression satisfies appropriate convergence conditions (described below) will take some care.

Finally, we extend to w in the exact same way that we extended to y.

To prove the homotopy asserted in our theorem, we define "maps" from $D'(\overline{C})^i \to W^\dagger\Omega^{i-1}_{\overline{C}}$ which will serve as a homotopy between our maps $p^\kappa h_0$ and $p^\kappa h_p$, where the word "maps" is in quotes because the maps will actually only be defined on the image of $p^\kappa\varphi$. Let $\omega_i \in W\Omega^i_{\overline{B}}[\frac{1}{p}]$. Then set

$$L(T^j\omega_i) = 0$$

and

$$L(T^j dT \wedge \omega_i) = \frac{p^{j+1}}{j+1}\omega_i.$$

The right-hand term lies in $W\Omega_{\overline{B}}[\frac{1}{p}]$, and for a general element $\delta \in D'(\overline{C})^i$, $L(\delta)$ could be an infinite sum of such terms. We must show that L actually maps the image of $p^\kappa\varphi$ into $W^\dagger\Omega_{\overline{B}}$. The next lemma assures us everything but the overconvergence.

Lemma 4.1.8. *The map L sends the image of $p^\kappa\varphi$ into $W\Omega_{\overline{B}}$.*

Proof. For an arbitrary $x \in \Omega_{B^\dagger}$, assume $\varphi(x) = \cdots + T^j dT \wedge \omega_i + \cdots$. From the definition of φ we see that $p^{j+1}\omega_i \in W\Omega^i_{\overline{B}}$. Furthermore, ω_i is a product of $j+1$ terms of the form $V(w)$ or $dV(w)$ with $w \in W\Omega^0_{\overline{B}}$. The terms $dV(w)$ correspond to terms $dx \in \Omega_{\overline{B}}$, hence there are no more than $\dim \overline{B}$ of them. But the term dT also corresponds to some $dx \in \Omega_{\overline{B}}$, hence there are no more than $\dim \overline{B} - 1$ of the terms $dV(w)$. Because of equation 1.3.12 on p. 508 of [10], a term of the form

$$\prod_{i=1}^{k} V(w_i)$$

is divisible by p^{k-1}. In our specific case $k \geq j + 1 - (\dim \overline{B} - 1)$. So for $p^\kappa \frac{p^{j+1}}{j+1}\omega_i$ to have no p in the denominator, it suffices that $-\kappa$ be a lower bound on the quantity $\lceil \log_p(\frac{p^{j-\dim \overline{B}+1}}{j+1})\rceil$ in the case $j > \dim \overline{B} - 1$ and on the quantity $\lceil \log_p(\frac{1}{j+1})\rceil$ in the case $j \leq \dim \overline{B} - 1$. In both cases, this is clear. $\square$

We now prove that the image of L is in fact overconvergent. This will rely on the following definition.

Definition 4.1.9. *We say an element $T^j \frac{w_i}{p^j}$ is overconvergent of radius C if w_i is overconvergent of radius C. Radii for sums and differentials are defined as before.*

This definition is useful because of the following.

Proposition 4.1.10. *(a) If $x \in \Omega_{B^\dagger}$ is overconvergent, then $p^\kappa\varphi(x)$ is overconvergent.*

(b) If $y = p^\kappa\varphi(x)$ is overconvergent, then $L(y)$ is overconvergent.

Proof. (a) We first set some notation. For each i, let w_i be such that $\psi_2(x_i) - \psi_1(x_i) = V(w_i)$. Let D be such that each $\psi_j(x_i)$ is D-overconvergent. Note that $V(w_i)$ is also D-overconvergent. Next, note that if two elements of $D'(\overline{C})$ are C_1 and C_2-overconvergent, then their product is $C_1 + C_2$ overconvergent, and their sum is $\max(C_1, C_2)$-overconvergent.

Consider $a \in A^\dagger$ which is overconvergent of radius C. After fixing k, consider the quantity

$$p^k \sum_{j=0}^{n_k} a_j(\psi_1(x_1) + \frac{T}{p}V(w_1))^{j_1} \cdots (\psi_1(x_n) + \frac{T}{p}V(w_n))^{j_n}.$$

We can rewrite this as

$$p^k \sum_{i=0}^{|n_k|} \frac{T^i}{p^i} v_i,$$

where each v_i is a sum of degree at-most $|n_k|$ products of D-overconvergent Witt vectors. From Lemma A.0.4 in the appendix, $p^k v_i$ is then $D(C+1)$ overconvergent for each i.

To complete the degree zero case, we must account for terms containing $y = \frac{1}{h}$, z, and $w = \frac{1}{g}$.

From the previous argument, $\varphi(h) = \sum \frac{T^i}{p^i} w_i$ for some w_i, all overconvergent of some radius D_1. Also, note that $w_0 = \psi_1(h)$ must be invertible. After possibly increasing D_1, we may assume that it too is overconvergent of radius D_1. Write

$$\varphi(y) = \frac{\frac{1}{w_0}}{1 - \left(-(\sum \frac{T^i}{p^i}\frac{w_i}{w_0})\right)},$$

the sum being over $i \geq 1$. The key point is that $\frac{w_i}{w_0}$ is $2D_1$-overconvergent and has zeroth component of degree zero. Hence from lemma A.0.7, the element $\varphi(y)$ is overconvergent of radius $5D_1$. We may now apply the same argument as in the previous paragraphs.

We now consider the terms including powers of $\varphi(z)$. From the above results it is clear that every c_i (in the notation above) will be overconvergent. We must verify that they are all overconvergent for a common radius. Choose D such that the following are all overconvergent of radius D: $(\psi_1'(c_0))^{-1}$, $\varphi(f_i)$ for all $1 \leq i \leq r$, and c_0^j for $0 \leq j \leq r-1$. From the formula already given, this immediately implies that c_1 is overconvergent of radius $(r+1)D$ and is in the image of Verschiebung, and hence is extraconvergent of radius $2(r+1)D$ (see the appendix).

We inductively assume that c_i is in the image of V^i and has jth component with degree at most $jp^j2(r+1)D + ip^i(r+1)D$. This is in particular $4(r+1)D$-extraconvergent (because of our assumption that the first i components are zero), and lemma A.0.8 in the appendix implies that c_{i+1} also satisfies our inductive assumption, and so we are done.

Again, we can use the same proof for $w = \frac{1}{g}$ that we used for $y = \frac{1}{h}$, now that we know $\varphi(g)$ can be written as $\sum \frac{T^i}{p^i} w_i$ for some w_i, all overconvergent of some radius D_1.

The extension of this proof to higher degrees is automatic, because each element in $\Omega_{B^\dagger}$ is a finite combination (under the operations of differentiation, multiplication, and addition) of terms in degree zero. (Note that this proof makes no mention of p^κ.)

(b) Because our map is additive and because of our definition of radius, it suffices to show that if ω_i is C-overconvergent, then $L(T^j \frac{\omega_i}{p^{j+1}} \wedge dT) = \frac{\omega_i}{j+1}$ is lC-overconvergent, where l is some number depending only on $\overline{B}$.

We use equation 1.3.12 on p. 508 of [10] again. It implies that the element ω_i is in V^m, where $m \geq j - \dim \overline{B} + \kappa$ and the element $\omega_i/(j+1)$ is in $V^{m'}$, where $m' \geq j - \dim \overline{B} + \kappa - \log_p(j+1)$. There exists an N depending only on $\overline{B}$ such that for $j \geq N$, $m' \geq \log_p(j+1)$.

Set $v = \lfloor \log_p(j+1) \rfloor$. For any element $\omega \in V^v$ such that $(j+1)\omega$ is overconvergent of radius C, we claim that ω was overconvergent of radius $2C$. Here is the argument: Write $\omega = (0, \ldots, 0, f_v, f_{v+1}, \ldots)$, and write $d_k = \deg f_k$. We are interested in d_k when $k \geq v$. Let $d = v_p(j+1)$. Then $p^d\omega$ is C-overconvergent by assumption, which translates to $p^d d_k \leq (d+k)p^{d+k}C$ for every $k \geq v$ Hence $d_k \leq (d+k+1)p^kC \leq 2(k+1)p^kC$, where the last inequality follows from the assumptions that $k \geq v \geq d$.

For the constant l, we may take $\max(2, N)$: we have covered all the cases except for the finitely many in which $j < N$, and in those cases we are dividing by no more than a factor of N, which can increase the radius by no more than N (and in fact, no more than a factor of $\lfloor \log_p(N) \rfloor$). □

It is trivial to check that L is indeed a chain homotopy. For the convenience of the reader, we state explicitly the applicable sign convention:

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^i \omega \wedge d\eta,$$

where ω is of degree i. Our notation is taken directly from [13], p. 748.

We have two corollaries to the theorem of this section.

Corollary 4.1.11. *Let F, F' denote two lifts of Frobenius to $B^\dagger$. Then the induced maps*

$$p^\kappa t_F, p^\kappa t_{F'} : \Omega_{B^\dagger} \to W^\dagger \Omega_{\overline{B}}$$

are chain homotopic.

Corollary 4.1.12. *Let $\overline{g} : \overline{B} \to \overline{C}$ denote a morphism of localizations of polynomial algebras. Let F_1, F_2 denote Frobenius lifts on $B^\dagger, C^\dagger$ respectively. Fix a lift $g : B^\dagger \to C^\dagger$. Let $\overline{g}$ also denote the induced map $W^\dagger\Omega_{\overline{B}} \to W^\dagger\Omega_{\overline{C}}$. Then the two maps*

$$p^\kappa t_{F_2} \circ g,\, p^\kappa \overline{g} \circ t_{F_1} : \Omega_{B^\dagger} \to W^\dagger \Omega_{\overline{C}}$$

are chain homotopic.

The best case occurs when $\kappa = 0$, or, equivalently, when $\dim \overline{B} < p$. In this case we bound the degrees of non-zero terms occurring in $\Omega_{B^\dagger}$. The following result is attained by applying the above argument to terms in degrees $p-1$ and below. Because we our argument now applies to only some of the terms in the complex, the result can no longer be phrased in terms of a chain homotopy.

Corollary 4.1.13. *Let F, F' denote two lifts of Frobenius to $B^\dagger$. Then the induced maps on cohomology*

$$t_F, t_{F'} : H^i(\Omega_{B^\dagger}) \to H^i(W^\dagger\Omega_{\overline{B}})$$

are the same for $i \leq p-2$.

Proof. For $i \leq p-1$, we can emulate the above argument to find maps $h : \Omega^i_{B^\dagger} \to W\Omega^{i-1}_{\overline{B}}$ such that $dh + hd = t_F - t_{F'}$. Hence, the induced maps on cohomology are the same for $i \leq p-2$. □

Remark 4.1.14. *We thank Liang Xiao for pointing out that such a result might hold.*

4.2 The induced map on cohomology

In this section we prove that the comparison map of the previous section is in fact a quasi-isomorphism. Our basic strategy follows that of [14], where they compare de Rham-Witt cohomology with Crystalline cohomology. They decompose $W\Omega_{\overline{A}}$ into an integral part and a fractional part and demonstrate an isomorphism between the integral part in finite levels and Ω_{A_n} and show that the fractional part is acyclic. We mimic this, although we work with our standard étale affines $\overline{B}$.

We know from [14] that

$$W\Omega_{k[x_1,\ldots,x_n]} \cong W^{int}\Omega_{k[x_1,\ldots,x_n]} \oplus W^{frac}\Omega_{k[x_1,\ldots,x_n]}.$$

This decomposition comes from the weights k in the basic Witt differentials. Using these same weights we induce a direct sum decomposition

$$W^{\dagger}\Omega_{k[x_1,\ldots,x_n]} \cong W^{\dagger,int}\Omega_{k[x_1,\ldots,x_n]} \oplus W^{\dagger,frac}\Omega_{k[x_1,\ldots,x_n]}.$$

In finite levels, from Section 3.3 of [14], we have an isomorphism

$$\Omega_{A_n} \xrightarrow{\sim} W_n^{int}\Omega_{k[x_1,\ldots,x_n]}.$$

Because inverse limits are left exact, and because B/A is flat, we know at least that $\Omega_{B^{\infty}} \to W^{int}\Omega_{\overline{B}}$ is injective, where B^{∞} denotes the p-adic completion of B.

We claim that the map

$$\varinjlim_{C} \varprojlim_{n} \Omega_{B_n^{\dagger,C}} \to \varinjlim_{C} \varprojlim_{n} W_n^{int,\dagger,C}\Omega_{\overline{B}}$$

is an isomorphism. Its injectivity follows from the injectivity of the maps $\Omega_{B^{\dagger,C}} \to W^{int,\dagger,C+D}\Omega_{\overline{B}}$ (described in the previous paragraph) and the injectivity of the transition maps in the direct limit. The surjectivity follows from Proposition 4.1.4.

Our next objective is to show that for C sufficiently large, $W_n^{frac,\dagger,C}\Omega_{\overline{B}}$ is acyclic. We keep the notation from the preceding section. Using the basic Witt differentials of [14] we showed in Section 2.2 a finite direct sum decomposition

$$W_n^{frac}\Omega_{\overline{A}} \cong \oplus_m W_n^m\Omega_{\overline{A}},$$

where the sum is over nonzero weights mod 1. We also checked that $W_n^m\Omega_{\overline{A}}$ is an $A^{\dagger}$-module for any (possibly zero) weight mod 1. Using our corollary on p. 70 in the appendix, we deduce

$$W_n^{frac}\Omega_{\overline{B}} \cong B^{\dagger} \otimes_{A^{\dagger}} W_n^{frac}\Omega_{\overline{A}} \cong B^{\dagger} \otimes_{A^{\dagger}} (\oplus_m W_n^m\Omega_{\overline{A}}) = \oplus_m \left(B^{\dagger} \otimes_{A^{\dagger}} W_n^m\Omega_{\overline{A}}\right).$$

(Tensor products of modules commute with direct sums, see [13], p. 608.)

Using basic properties of differentials and the above isomorphism, we see that any element of $W_n^{frac}\Omega_{\overline{B}}$ can be written as a finite (because we are in a finite level) sum of elements of the form $f w_0 dw_1 \cdots dw_m$ satisfying the following conditions:

1. Each w_i is a monic monomial with (possibly fractional) exponents < 2.
2. No x_i appears in two of the w_i's.
3. Some x_i appears to a fractional exponent.
4. The term f corresponds to an element of $B^\dagger$.

If we want to restrict further to a specific weight $k \mod 1$ (this is as specific as we can get in this context), we also force:

5. The term $f w_0 dw_1 \cdots dw_m$ has weight $k \mod 1$, as read off from the exponents of the monomials.

To summarize, f corresponds to a degree zero integral element, the fractional part is contained in the w_i's, and we have put strong restrictions on the exponents which can occur in the w_i's so that if we wish to compute the radius, this part may essentially be ignored. The monomials are required to be monic so that p-power coefficients will affect the radius of f.

To show that $W_n^k \Omega_{\overline{B}}$ is acyclic, without yet considering radius, we give a recipe g which, given a cocycle, produces a preimage of it. (We suspect that g is in fact a homotopy between the maps 0 and 1 on $W_n^k \Omega_{\overline{B}}$, but checking that g is well-defined with respect to the relations which hold in the de Rham-Witt complex seems difficult.)

Continue to have the weight k fixed and non-zero. Without loss of generality we may assume that the x_1 exponent is fractional and appears in either the w_0 or w_1 term (but not both by property 2 above). Let $1/a$ denote the exponent of x_1. So in particular, $p|a$ in $\mathbb{Z}_{(p)}$.

We define g on a given term $f w_0 dw_1 \cdots dw_n$ and extend it additively; g should be thought of as integration with respect to x_1. For our purposes, checking that the map g is independent of such a presentation is unnecessary.

If x_1 appears in w_0, then put $g(f w_0 dw_1 \cdots dw_n) = 0$. If x_1 appears in w_1, then put

$$g(f w_0 dw_1 \cdots dw_n) = w_0 w_1 (1 - a(x_1 \frac{\partial}{\partial x_1}) + a^2 (x_1 \frac{\partial}{\partial x_1})^2 - \cdots) f dw_2 \ldots dw_n.$$

Note that the image of g has the same form as our original term: for instance, $w_0 w_1$ will play the role of w_0, which is allowed because of our hypothesis that there are no repeated variables, so no exponent becomes too large. This restriction on the form will be important when we wish to read off the radius of an element.

Proposition 4.2.1. *For a fixed presentation of an element w, $dg + gd = 1$. In particular, every cocycle is a coboundary.*

Proof. It clearly suffices to prove this for a single term $w = fw_0dw_1 \dots dw_n$. The two cases, as before, are w_0 possesses an x_1 term and w_1 possesses an x_1 term. In the first, $dg(w) = 0$ and

$$gd(w) = w_0(1 - \cdots)fdw_1 \dots dw_n + g(w_0dfdw_1 \dots dw_n).$$

Expand $df = \sum \partial f/\partial x_i dx_i$. The terms other than dx_1 get sent to zero under g. (Note that the weight is preserved throughout, since we are working with weights mod 1.) So

$$gd(w) = w_0(1 - \cdots)fdw_1 \dots dw_n + g(w_0\frac{\partial f}{\partial x_1}dx_1dw_1 \dots dw_n).$$

We need to merge the w_0 term and the dx_1 term, because they both have x_1 terms which is not allowed. This yields

$$\begin{aligned} gd &= w_0(1 - \cdots)fdw_1 \dots dw_n + g(ax_1\frac{\partial f}{\partial x_1}dw_0dw_1 \dots dw_n) \\ &= w_0(1 - \cdots)fdw_1 \dots dw_n + w_0(1 - a(x_1\frac{\partial}{\partial x_1}) + \cdots)ax_1\frac{\partial f}{\partial x_1}dw_1 \dots dw_n) \\ &= fw_0dw_1 \dots dw_n \end{aligned}$$

where the rearrangement is allowed because, as p divides a, the series is absolutely convergent.

The other case we must check is where w_1 possesses an x_1 term. Before proceeding we will make a simplifying assumption. We can assume that each w_i for $i \geq 1$ possesses only one term (using the product rule), and in particular w_1 has only x_1. This is admissible as long as we don't try to recursively apply our assertion to a term that is no longer in the proper form, which we will not. Then

$$\begin{aligned} d(w) &= -fdw_1dw_0 \cdots dw_n + w_0dfdw_1 \cdots dw_n \\ gd(w) &= w_1(1 - \cdots)fdw_0dw_2 \dots dw_n + g(w_0dfdw_1 \cdots dw_n) \\ &= -w_1(1 - \cdots)fdw_0dw_2 \dots dw_n - \sum w_0w_1(1 - \cdots)\frac{\partial f}{\partial x_i}dx_idw_2 \cdots dw_n \end{aligned}$$

(note that by our assumptions $dx_1 dw_1 = 0$ so the sum does not include $i = 1$) and

$$dg(w) = w_1(1 - \cdots) f dw_0 dw_2 \ldots dw_n + w_0(1 - \cdots) f dw_1 \cdots dw_n + \\ w_0 w_1 d((1 - \cdots) f) dw_2 \cdots dw_n.$$

So we have to show

$$w_0(1 - \cdots) f dw_1 \cdots dw_n + w_0 w_1 d((1 - \cdots) f) dw_2 \cdots dw_n - \\ \sum w_0 w_1 (1 - \cdots) \frac{\partial f}{\partial x_i} dx_i dw_2 \cdots dw_n = f w_0 dw_1 \cdots dw_n.$$

Write $\phi := (1 - \cdots) f$. Then ϕ has the property that $\phi + a x_1 \frac{\partial \phi}{\partial x_1} = f$. Additionally, for any $i \neq 1$, $\frac{\partial \phi}{\partial x_i} = (1 - \cdots) \frac{\partial f}{\partial x_i}$. Our assertion follows directly from these two facts. (As for the previous case, we replace $w_1 dx_1$ with $a x_1 dw_1$.) □

We must now incorporate radii into the discussion. We will actually define a new radius which takes advantage of our restrictions on the form $f w_0 dw_1 \cdots dw_n$. Fix a presentation.

Definition 4.2.2. *The radius of a term $f w_0 dw_1 \cdots dw_n$ is defined to be the radius of the term f. The radius of a sum of such terms is defined to be the supremum of the radii of the terms.*

Because the w_i terms can contribute at most $2n+2$ to the old definition of radius, and because n is bounded by the dimension of our variety, in the limit no new overconvergent terms are introduced. In other words, an element which is C-overconvergent in the new definition is $(C + 2n + 2)$-overconvergent in the old definition.

Lemma 4.2.3. *An overconvergent element under the old definition is overconvergent under the new definition.*

Proof. First consider the polynomial algebra case in degree zero. We can write any C-overconvergent Witt vector as

$$\sum_{n=0}^{\infty} \sum_J V^n(a_J x^J),$$

where each summand is C-overconvergent. Consider such an element $V^n(a_J x^J)$. Say $|J| = d$. In the old definition this corresponds to a term of radius $d/(n+1)p^n$. Write $x^J = x_I^{p^n} x_F$ where each variable of x_F appears to degree $< p^n$. Then $x =$

$p^n(x_I x_F^{1/p^n})$. In the notation from above, $f = p^n x_I$ where $d/p^n \geq \deg x_I \geq d/p^n - m$, where m is the number of variables. Then f has radius between $d/(n+1)p^n$ and $d/(n+1)p^n - m/(n+1)$. This shows that in the polynomial algebra case in degree zero, a term of radius C in the old definition will also be overconvergent of radius C in the new definition.

Next we consider the polynomial case in higher degrees. Write it as $v_0 dv_1 \dots dv_n$. Each v_i can be written as $f w_0 dw_1 \dots dw_n$. Then the result follows from the following two facts: (1) If $w = f w_0 dw_1 \dots dw_n$ is C-overconvergent (in the new definition), then dw is $(C+1)$-overconvergent. (2) If w, w' are C, C'-overconvergent respectively (in the new definition), then ww' is $(C + C' + M)$-overconvergent, where M depends only on the ring $\overline{B}$.

To extend to the general case, we call upon lemmas A.0.11 and A.0.12 from the appendix, which together with the above give it to us immediately. □

Let m_y denote the degree of $\frac{\partial y}{\partial x_1}$ and similarly for m_z and m_w. Let $m = m_y + m_z + m_w$. We will show that for $C \geq m$, g maps $W^{frac,\dagger,C,k}\Omega_{\overline{B}}$ into itself (with our new definition of radius).

Lemma 4.2.4. *For an element w of radius $C \geq m$ (under our new definition of radius), $g(w)$ is also overconvergent of radius C.*

Proof. Because of our definition of g and radius, (and because 0 is of course overconvergent of every radius), it suffices to prove

$$(1 - a(x_i \frac{\partial}{\partial x_1}) + a^2(x_i \frac{\partial}{\partial x_1})^2 - \cdots)f$$

is overconvergent of radius C.

Write $f = \sum c_i x^{i_1} y^{i_2} z^{i_3} w^{i_4}$, a representation of f for which

$$\frac{v_p(c_i)}{|i_1| + i_2 + i_3 + i_4} \geq \frac{1}{C}.$$

Because the series $(1 - \cdots)f$ converges, as already noted, it suffices to check that each term $a^j(x_1 \frac{\partial}{\partial x_1})^j f$ is overconvergent of radius C. By the same reasoning, it suffices to consider the case $f = c_i x^{i_1} y^{i_2} z^{i_3} w^{i_4}$. And finally, because the map $a^j(x_1 \frac{\partial}{\partial x_1})^j$ is a composite of maps $a(x_1 \frac{\partial}{\partial x_1})$, by induction it suffices to show the result for $j = 1$. Let

d_1 denote the exponent of x_1. Then we calculate:

$$a(x_1\frac{\partial}{\partial x_1})c_ix^{i_1}y^{i_2}z^{i_3}w^{i_4} = d_1c_iax^{i_1}y^{i_2}z^{i_3}w^{i_4} + i_2c_iax^{i_1+1}y^{i_2-1}\frac{\partial y}{\partial x_1}z^{i_3}w^{i_4}$$
$$+ i_3c_iax^{i_1+1}y^{i_2}z^{i_3-1}\frac{\partial z}{\partial x_1}w^{i_4} + i_4c_iax^{i_1+1}y^{i_2}z^{i_3}w^{i_4-1}\frac{\partial w}{\partial x_1}$$

To complete the claim, we need to show that all four of these terms are overconvergent of radius C. Because $p|a$, it suffices to check that

$$\frac{v_p(c_i)+1}{|i_1|+i_2+i_3+i_4+m} \geq \frac{1}{C}.$$

This is equivalent to checking

$$Cv_p(c_i) + C \geq |i_1| + i_2 + i_3 + i_4 + m,$$

which we know by our assumptions. □

4.3 The comparison map for smooth varieties

This section establishes that the special affines considered in this paper suffice to cover any smooth affine variety in characteristic p. It concludes with our main theorem.

Proposition 4.3.1. *Any smooth scheme has an open cover by standard étale affines. Moreover, we can choose this cover to consist of sets so that any finite intersection is again standard étale affine.*

Proof. Any smooth scheme is covered by opens which are étale over affine space. The first statement then follows from [19], Chapter 2, Theorem 1.1. Within the proof, we see that we may choose our cover to consist of distinguished opens of the form $D(f)$. It's clear from the definition of standard étale affine that a distinguished subset of one is again standard étale. Finally, the intersection of two distinguished opens is again a distinguished open. From this, the second statement follows. □

We must use the following external result from [4]. Its proof is reproduced in Appendix B.

Proposition 4.3.2. *Let* $\operatorname{Spec}\overline{C}$ *denote a smooth affine.*

(a) We denote by $f \in \overline{C}$ an arbitrary element. Let $d \in \mathbb{Z}$ be nonnegative. The presheaf defined on the basis $\{\overline{C}_f\}$ by

$$W^\dagger\Omega^d_{\operatorname{Spec}\overline{C}}(\operatorname{Spec}\overline{C}_f) := W^\dagger\Omega^d_{\overline{C}_f}$$

is a sheaf for the Zariski topology on $\operatorname{Spec}\overline{C}$.

(b) The Zariski cohomology of these sheaves vanishes in degrees $j > 0$, i.e.

$$H^j_{Zar}(\operatorname{Spec}\overline{C}, W^\dagger\Omega^d_{\operatorname{Spec}\overline{C}}) = 0.$$

We are ready to state the main result of our paper.

Theorem 4.3.3. *Let $\overline{C}$ denote a smooth variety over a perfect field in characteristic p. Let $\kappa = \lfloor \log_p(\dim \overline{C}) \rfloor$. Let $C^\dagger$ denote a lift as in Section 2.1. Fix a Frobenius lift F on $C^\dagger$. Let t_F denote the comparison map described in Section 4.1.*

(a) Then after multiplying by p^κ the induced map on cohomology

$$p^\kappa t_F : H^\cdot(\Omega_{C^\dagger}) \to H^\cdot(W^\dagger\Omega_{\overline{C}})$$

is independent of F.

(b) It is functorial in the sense that for any map of smooth affines $\overline{g} : \overline{C} \to \overline{D}$, Frobenius lift F' on $D^\dagger$, and lift $g : C^\dagger \to D^\dagger$, the two induced compositions

$$p^\kappa t_{F'} \circ g : H^\cdot(\Omega_{C^\dagger}) \to H^\cdot(\Omega_{D^\dagger}) \to H^\cdot(W^\dagger\Omega_{\overline{D}})$$

and

$$\overline{g} \circ p^\kappa t_F : H^\cdot(\Omega_{C^\dagger}) \to H^\cdot(W^\dagger\Omega_{\overline{C}}) \to H^\cdot(W^\dagger\Omega_{\overline{D}})$$

are equal.

(c) Rationally, the map $p^\kappa t_F$ is a quasi-isomorphism. If $\dim \overline{C} < p$, we have an integral isomorphism.

Proof. Because our complex is a sheaf, it suffices to check (a) and (b) locally. Proposition 4.3.1 shows that we may then reduce to the special affines considered in the previous sections. For such affines, these properties were asserted in the corollaries on p. 38.

In Section 4.2, we checked that $t_F : \Omega_{B^\dagger} \to W^{int,\dagger}\Omega_{\overline{B}}$ was an isomorphism, and that $W_n^{frac,\dagger,C}\Omega_{\overline{B}}$ was acyclic for C sufficiently large. (For $\overline{B}$ of the specific form in question.) We wish to say

$$\varinjlim_{C} \varprojlim_{n} W_n^{frac,\dagger,C}\Omega_{\overline{B}} = W^{frac,\dagger}\Omega_{\overline{B}}$$

is also acylic. For fixed n and C, $W_n^{frac,\dagger,C}\Omega_{\overline{B}}$ is finite length over $W(k)$, so cohomology commutes with inverse limits in our case. Cohomology always commutes with direct limits, so we have succeeded in showing that t_F is a quasi-isomorphism for our special affines. Hence, by parts (a) and (b), for any smooth affine $\overline{C}$, the induced map on cohomology

$$p^\kappa t_F : H^\cdot(\Omega_{C^\dagger}) \to H^\cdot(W^\dagger\Omega_{\overline{C}})$$

locally has the form $p^\kappa\varphi$, with φ an isomorphism.

Cover $\operatorname{Spec}\overline{C}$ by our special affines as in Proposition 4.3.1. Call the special affines $\operatorname{Spec}\overline{B_i}$; we need only finitely many because $\operatorname{Spec}\overline{C}$ is compact. For some tuple of indices I, let $\operatorname{Spec}\overline{B_I}$ denote $\bigcap_{i\in I}\operatorname{Spec}\overline{B_i}$. From the proposition we know that these are also special affines. For the following spectral sequence, let H denote either Monsky-Washnitzer cohomology or overconvergent de Rham-Witt cohomology. From our Čech resolution we have a spectral sequence with $E_1^{pq} = H^q(\operatorname{Spec}\overline{B_I})$ where I is a p-tuple. Because the Zariski cohomology of our sheaf vanishes in positive degree, by [6], Proposition 0.11.4.5, this spectral sequence degenerates to $H^{p+q}(\operatorname{Spec}\overline{C})$. By Theorem 3.5 of [15], our local isomorphism thus determines a global isomorphism. □

Using Corollary 4.1.13, we can attain an integral quasi-isomorphism in low degrees.

Corollary 4.3.4. *The induced map on cohomology*

$$t_F : H^i(\Omega_{C^\dagger}) \to H^i(W^\dagger\Omega_{\overline{C}})$$

is an isomorphism for $i \leq p-2$.

Proof. We may use the same proof as for Theorem 4.3.3. The point is that the map t_F induces isomorphisms between the two spectral sequences in the regions $q \leq p-2$. (Here p stands for the characteristic, not the horizontal coordinate of the spectral sequence. We let p_1 denote this horizontal coordinate.) Because we are beginning at the sheet E_1, all differentials map from this region $q \leq p-2$ into itself. Our ultimate goal is an isomorphism between the components $E_\infty^{p_1q}$ for $p_1 + q \leq p-2$. No non-zero term from outside the region $q \leq p-2$ is mapped into the region $p_1 + q \leq p-2$ by a differential. So, our local isomorphism in low degrees provides a global isomorphism. □

We end this section by noting that the bound on i in Corollary 4.3.4 is not always sharp. In particular, if $\dim \overline{C} = p - 1$, then by Theorem 4.3.3 we have an integral isomorphism also for H^{p-1}. More specifically still, we always have an integral isomorphism in the case $\operatorname{Spec} \overline{C}$ is a curve, even if $p = 2$.

Chapter 5

An Approach to p-adic Hodge Theory

This chapter and the next are not closely connected to the previous chapters. In this chapter, we reinterpret some rings from p-adic Hodge theory in such a way that they admit "big" analogues. Most significantly, we do this for the ring $\tilde{A}^+$. In future work, we hope that our previous material will play an important role, in particular for the construction of p-adic Hodge theory's dagger rings. Bear in mind that where we write W, we mean p-typical Witt vectors, and where we write $\mathbb{W}$, we mean big Witt vectors.

5.1 The p-typical case

We start by recalling a result used in the proof of Lemma 1.1 in [8].

Lemma 5.1.1. *Let A denote a ring, and $a, b \in A$. If $a \equiv b \mod p$, then $a^{p^{v-1}} \equiv b^{p^{v-1}} \mod p^v$. More generally, if $a \equiv b \mod p^j$, then $a^{p^{v-1}} \equiv b^{p^{v-1}} \mod p^{v+j-1}$.*

Proof. The first statement is proved directly in [8], and the proof is easily adapted for the more general statement. □

We denote by $\mathscr{O}_{\overline{\mathbb{Q}_p}}$ the ring of integers of the algebraic closure of $\mathbb{Q}_p$ and by $\mathscr{O}_{\mathbb{C}_p}$ the completion of $\mathscr{O}_{\overline{\mathbb{Q}_p}}$.

Definition 5.1.2. *We write $\tilde{E}^+$ for $\varprojlim \mathscr{O}_{\mathbb{C}_p}$ with transition maps $x \mapsto x^p$, where the* *latter is equipped with a ring structure by declaring*

$$(x+y)_i = \lim_{j\to\infty}(x_{i+j}+y_{i+j})^{p^j} \text{ and}$$
$$(xy)_i = x_i y_i.$$

Proposition 5.1.3. *We have an isomorphism*

$$\varprojlim W(\mathscr{O}_{\mathbb{C}_p}) \xrightarrow{\sim} W(\varprojlim \mathscr{O}_{\mathbb{C}_p}) =: \tilde{A}^+,$$

where the maps in the first inverse system are the (p-typical) Witt vector Frobenius.

Proof. We will define three maps and show that each is an isomorphism. Their composition will be the isomorphism promised in the proposition.

We now define the following maps:

$$\varprojlim W(\mathscr{O}_{\mathbb{C}_p}) \xrightarrow{\pi} \varprojlim W(\mathscr{O}_{\mathbb{C}_p}/p) \xrightarrow{\alpha} W(\varprojlim \mathscr{O}_{\mathbb{C}_p}/p) \xrightarrow{\beta} W(\varprojlim \mathscr{O}_{\mathbb{C}_p}).$$

The map π is the obvious projection.

Next we explain the map α. Write $x = (x_1, x_p, \ldots)$ for an element of $\varprojlim W(\mathscr{O}_{\mathbb{C}_p}/p)$, so each $x_{p^i} \in W(\mathscr{O}_{\mathbb{C}_p}/p)$. Let $x_{p^ip^j}$ denote the jth Witt vector component of x_{p^i}. Write an element of $W(\varprojlim \mathscr{O}_{\mathbb{C}_p}/p)$ as $y = (y_1, y_p, \ldots)$, where each $y_{p^i} \in \varprojlim \mathscr{O}_{\mathbb{C}_p}/p$. Then we define $\alpha(x) = y$ with $y_{p^jp^i} = x_{p^ip^j}$ (i.e., the indices switch). This is well-defined because over a ring of characteristic p, the Witt vector Frobenius is induced by the map $x \mapsto x^p$. It is a ring homomorphism because $(x^{(1)} + x^{(2)})_{p^ip^j} = f(x^{(1)}_{p^i1}, x^{(2)}_{p^i1}, \ldots, x^{(1)}_{p^ip^j}, x^{(2)}_{p^ip^j})$, while $(y^{(1)} + y^{(2)})_{p^jp^i} = f(y^{(1)}_{1p^i}, y^{(2)}_{1p^i}, \ldots, y^{(1)}_{p^jp^i}, y^{(2)}_{p^jp^i})$, where f is the polynomial defining Witt vector addition, and similarly for multiplication.

Finally, we define the map β. It suffices to describe a map $\varprojlim \mathscr{O}_{\mathbb{C}_p}/p \to \varprojlim \mathscr{O}_{\mathbb{C}_p}$. Let $(\overline{a_1}, \overline{a_p}, \ldots) \in \varprojlim \mathscr{O}_{\mathbb{C}_p}/p$. Let a_{p^i} denote any lift to $\mathscr{O}_{\mathbb{C}_p}$. The map β sends this element to $(b_1, b_p, \ldots)$, where $b_{p^i} := \lim_{j\to\infty} a_{p^{i+j}}^{p^j}$. The limit exists by Lemma 5.1.1. The same lemma shows that the map does not depend on the choice of lifts. The map is clearly multiplicative. It is additive because of the definition of addition in this ring (Definition 5.1.2).

We now show that each of these ring homomorphisms is an isomorphism. The map π is injective because $v_p(x_{p^ip^j}) = pv_p(x_{p^{i+1}p^j})$, and so $\pi(x) = 0$ if and only if $v_p(x_{p^ip^j}) = \infty$ for every i, j. To see that it is surjective, we construct a preimage of $(\overline{x_1}, \overline{x_p}, \ldots)$, where each $\overline{x_{p^i}} \in W(\mathscr{O}_{\mathbb{C}_p}/p)$. Let x_{p^i} denote any lift. Then our preimage is defined to have p^ith component

$$\lim_{j\to\infty} F^j(x_{p^{i+j}}).$$

The map α is clearly injective and surjective. The only non-trivial part was checking that it was well-defined and a ring homomorphism, which we have already done.

Finally, the map $\varprojlim \mathscr{O}_{\mathbb{C}_p}/p \to \varprojlim \mathscr{O}_{\mathbb{C}_p}$ inducing β is an isomorphism because it has an obvious inverse given by projection. That this is a left inverse is completely trivial. To see that it is a right inverse, note that if we start with an element $(b_1, b_p, \ldots)$, then when we lift the projected elements $(\overline{b_1}, \overline{b_p}, \ldots)$, we may simply take the original elements again. $\square$

Recall, for instance from [1], the map $\theta : \tilde{A}^+ \to \mathscr{O}_{\mathbb{C}_p}$ which sends

$$\sum_{k=0}^{\infty} p^k [x_{p^k}] \mapsto \sum_{k=0}^{\infty} p^k x_{p^k 1},$$

where each $x_{p^k} \in \varprojlim \mathscr{O}_{\mathbb{C}_p}$. The left hand term can be written in terms of Witt vector components as $(x_1, x_p^p, \ldots)$. Because $\tilde{E}^+ = \varprojlim \mathscr{O}_{\mathbb{C}_p}$ is perfect, given the Witt vector components of $W(\varprojlim \mathscr{O}_{\mathbb{C}_p})$, we have a unique corresponding set of $x_1, x_p, \ldots$ as above. The p^nth root of some $y_{p^i} = (y_{p^i 1}, y_{p^i p}, \ldots) \in \tilde{E}^+$ is $(y_{p^i p^n}, y_{p^i p^{n+1}}, \ldots)$. Hence

$$\theta : (y_1, y_p, \ldots) \mapsto \sum_{k=0}^{\infty} p^k y_{p^k p^k}.$$

Proposition 5.1.4. *The map θ sends $x := (x_1, x_p, \ldots) \in \varprojlim W(\mathscr{O}_{\mathbb{C}_p})$ to x_{11}.*

Proof. Let $y := \alpha \circ \pi(x)$, so in particular $y_{p^j p^i} = \overline{x_{p^i p^j}}$. Our definition of β involves choosing lifts of the terms $y_{p^j p^i}$, but in this case we have a canonical such lift, namely $x_{p^i p^j}$. So from the definition,

$$(\beta \circ \alpha \circ \pi(x))_{p^j p^j} = \lim_{k\to\infty} x_{p^{j+k} p^j}^{p^k}.$$

Replacing k by $k - j$, which does not affect the limit, we have

$$(\beta \circ \alpha \circ \pi(x))_{p^j p^j} = \lim_{k\to\infty} x_{p^k p^j}^{p^{k-j}}.$$

Plugging this in to the definition of θ from above yields

$$(\theta \circ \beta \circ \alpha \circ \pi(x)) = \sum_{j=0}^{\infty} p^j \lim_{k\to\infty} x_{p^k p^j}^{p^{k-j}} = \sum_{j=0}^{\infty} \lim_{k\to\infty} p^j x_{p^k p^j}^{p^{k-j}} = \lim_{i\to\infty} \sum_{j=0}^{i} p^j x_{p^i p^j}^{p^{i-j}}.$$

Now note that $\sum_{j=0}^{i} p^j x_{p^i p^j}^{p^{i-j}}$ is precisely the p^ith ghost component of x_{p^i}. But by the definition of the Frobenius map, this term is independent of i. Taking $i = 0$ completes the proof. □

Note. The above proof is due to Ruochuan Liu. It is significantly more transparent than the author's original proof.

Note. From now on, unless otherwise noted, we will write $\tilde{A}^+$ to denote $\varprojlim W(\mathscr{O}_{\mathbb{C}_p})$. This will be the most useful of the various representations, because it translates most easily to the "big" setting.

Lemma 5.1.5. *We have a well-defined injective ring homomorphism, which we call the "ghost map",*

$$w : \tilde{A}^+ \to \prod_{i \in p^{\mathbb{Z}}} \mathscr{O}_{\mathbb{C}_p},$$

defined by $w_{p^i}(x) = w_{p^k}(x_{p^j})$ *for any* $k - j = i$ *where* w_{p^k} *denotes the* p^k*th component of the usual ghost map defined on* $W(\mathscr{O}_{\mathbb{C}_p})$.

Proof. This is clearly well-defined from the definition of the Witt vector Frobenius on ghost components. Because the usual ghost map is a ring homomorphism, so too is this extended ghost map. It is injective because the usual ghost map is injective, which is true because $\mathscr{O}_{\mathbb{C}_p}$ is p-torsion free. □

Note. Under the above definition, the map θ on $\tilde{A}^+$ corresponds to the first ghost component w_1.

Lemma 5.1.6. *For every* $r \in p^{\mathbb{Z}}$, *we have a Frobenius homomorphism* $F_r : \tilde{A}^+ \to \tilde{A}^+$ *defined as follows. Let* $x = (x_1, x_p, \ldots) \in \tilde{A}^+$. *If* $r = p^i$ *with* $i \geq 0$, $F_r(x) = (F^i(x_1), F^i(x_p), \ldots)$ *where* F *without a subscript denotes the usual (p-typical) Witt vector Frobenius. If* $r = p^i$ *with* $i \leq 0$, *then* $F_r(x) = (x_{p^i}, x_{p^{i+1}}, \ldots)$.

Proof. For the $i \geq 0$ case, it is clear that F_r is a ring homomorphism because the usual Witt vector Frobenius is. We attain the result for all values of r by noting that F_{p^i} is the inverse of $F_{p^{-i}}$. □

Within the previous proof we used the following result.

Corollary 5.1.7. *The Frobenii defined above are in fact automorphisms.*

Proposition 5.1.8. *Let $x \in \tilde{A}^+$ and suppose $p \mid w_{p^{-i}}(x)$ for every $i \geq 0$. Then $p \mid x$.*

Note. The author thanks Abhinav Kumar for showing him this short proof.

Proof. We show that if $p \mid w_{p^{-1}}(x)$, then $p \mid x_1$. The same argument will show that $p \mid x_{p^i}$ for every i. This, together with the fact that F is a homomorphism, implies $p \mid x$.

We simplify notation. Let $a, b \in W(\mathscr{O}_{\mathbb{C}_p})$ be (p-typical) Witt vectors with $F(b) = a$ and $p \mid b_1$. Then we will show a is divisible by p.

Write $b = p[\beta_0] + V\left(\tilde{b}\right)$. All we are using here is that multiplication by p behaves as expected on the first Witt vector component, and that the difference of two elements with the same first Witt vector component is in the image of verschiebung. Then

$$F(b) = pF([\beta_0]) + FV\left(\tilde{b}\right) = p([\beta_0^p] + \tilde{b}).$$

This completes the proof. □

Lemma 5.1.9. *The ring $\tilde{A}^+$ is separated for the p-adic topology.*

Proof. From Lemma 5.1.5 we know the ghost map is injective, so the result follows from the fact that $\mathscr{O}_{\mathbb{C}_p}$ is separated. □

Lemma 5.1.10. *We have a one-to-one correspondence between elements in $\tilde{A}^+$ and the tails of their ghost components $(\ldots, w_{p^{-2}}, w_{p^{-1}}, w_1, w_p, \ldots)$.*

Proof. We need only show that if $(\ldots, 0, 0, w_{p^i}, w_{p^{i+1}}, \ldots)$ is in the image of the ghost map, then it must be $(\ldots, 0, 0, 0, 0, \ldots)$. By applying a suitable Frobenius isomorphism, we may assume $w_{p^{-i}} = 0$ for all $-i \leq 0$. From Proposition 5.1.8, such an element would be infinitely divisible by p (because dividing by p corresponds to dividing each ghost component by p, and so this tail would remain unchanged). Because $\tilde{A}^+$ is separated by the previous lemma, we are done. □

Lemma 5.1.11. *Fix $x \in \tilde{A}^+$. If some $w_r(x) \neq 0$, then $w_{rp^{-i}}(x) \neq 0$ for all $-i << 0$.*

Proof. Because $\tilde{A}^+$ is p-torsion free, we may use Proposition 5.1.8 to assume that, after potentially changing r, $p \nmid w_r(x)$. Because the Frobenius maps are isomorphisms, we may assume $r = 1$.

From the definition of the Witt vector Frobenius, we have that $x_{p1}^p \equiv x_{11} \mod p$, and more generally $x_{p^i1}^{p^i} \equiv x_{11} \mod p$. Then because $x_{11} = w_1(x)$ and more generally $x_{p^i1} = w_{p^{-i}}(x)$, we have that $w_{p^{-i}} \not\equiv 0 \mod p$ for all $-i \leq 0$, and we are done. □

Proposition 5.1.12. *The ring $\tilde{A}^+$ is a domain.*

Proof. This follows from Lemma 5.1.11 and the fact that $\mathscr{O}_{\mathbb{C}_p}$ is a domain. $\square$

Proposition 5.1.13. *We have $(\tilde{A}^+)^{G_{\mathbb{Q}_p}} = \mathbb{Z}_p$.*

Proof. Such an invariant x will have to be in

$$\varprojlim W(\mathbb{Z}_p) \subseteq \tilde{A}^+.$$

We claim that all of the ghost components of x are equal. Assume two of them are not equal. This implies two adjacent terms are not equal, and after using a Frobenius map, we may assume $w_1(x) \neq w_p(x)$.

We now apply Dwork's Lemma 1.1 of [8] with $\phi_p = \text{id}$. Because these elements are in the image of x_1 under the ghost map, we have that $w_1(x) \equiv w_p(x) \mod p$. But we also have that $w_1(x)$ equals the p^ith ghost component of x_{p^i}, hence $w_1(x) \equiv w_p(x) \mod p^i$ for every i. This is a contradiction, and so the statement of the proposition holds. $\square$

Proposition 5.1.14. *We have $(\tilde{A}^+[\frac{1}{p}])^{G_{\mathbb{Q}_p}} = \mathbb{Q}_p$ and moreover $(\text{Frac }\tilde{A}^+)^{G_{\mathbb{Q}_p}} = \mathbb{Q}_p$.*

Proof. The first statement follows from the previous proposition and the fact that $\tilde{A}^+$ is p-torsion free. The bulk of the work lies in proving the second assertion.

The inclusion $\mathbb{Q}_p \subseteq (\text{Frac }\tilde{A}^+)^{G_{\mathbb{Q}_p}}$ is obvious. Conversely, let $\frac{x}{y} \in (\text{Frac }\tilde{A}^+)^{G_{\mathbb{Q}_p}}$. Because $\tilde{A}^+$ is p-torsion free, if $\frac{p^i a}{p^j b} \in (\text{Frac }\tilde{A}^+)^{G_{\mathbb{Q}_p}}$ then so too is $\frac{a}{b}$. Hence we will assume $p \nmid x$ and $p \nmid y$, and prove that $\frac{x}{y} \in \mathbb{Z}_p^*$. By the previous lemma, only the tails of the ghost components are important, and so we may assume $p \nmid x_{11}$ and $p \nmid y_{11}$. We know that $x_{p^i 1}^{p^i} \equiv x_{11} \mod p$ and similarly for y. In particular, $v_p(x_{p^i 1}) < \frac{1}{p^i}$ and similarly for y. Thus we may define the ghost components $w_{p^{-i}}$ for non-negative i as ratios of the corresponding ghost components of x and y; we will not be dividing by zero.

For our element to be invariant under $G_{\mathbb{Q}_p}$, each ghost component w_{p^i} must lie in $\mathbb{Q}_p$. From our assumptions on x and y, each ghost component must actually lie in $\mathbb{Z}_p^*$. After multiplying by an element in $\mathbb{Z}_p^*$, we may assume $w_1 = 1$, which means precisely that $x_{11} = y_{11}$. (About x_{11} itself, we know its p-adic valuation is less than 1, but it need not lie in $\mathbb{Z}_p$.)

We will prove by induction on i that $w_{p^{-i}} \equiv 1 \mod p$ for all $i \geq 0$. We have already asserted the base case. Inductively assume the result for a fixed value $i - 1$.

We have by definition $\frac{x_{p^i 1}}{y_{p^i 1}} = w_{p^{-i}}$ and $\frac{x_{p^i 1}^p + p x_{p^i p}}{y_{p^i 1}^p + p y_{p^i p}} = w_{p^{-i+1}}$. Combining these yields

$$(w_{p^{-i}}^p - w_{p^{-i+1}}) y_{p^i 1}^p = p(w_{p^{-i+1}} y_{p^i p} - x_{p^i p}).$$

Considering p-adic valuations, and using the fact that $p \nmid y_{p^i 1}^p$ as assumed in the first paragraph above, we see that $v_p(w_{p^{-i}}^p - w_{p^{-i+1}}) > 0$. Since it is in $\mathbb{Z}_p$, the valuation is at least one. Now, using our inductive assumption, we know $w_{p^{-i}} \equiv 1 \mod p$, which completes the induction.

We next claim that $x_{p^i p^j} \equiv y_{p^i p^j} \mod p$ for all $0 \leq j \leq i$. For fixed i we will prove this by induction on j, the case $j = 0$ having been proved above. Now assume the result for a fixed $j - 1$. We know

$$x_{p^i 1}^{p^j} + p x_{p^i p}^{p^{j-1}} + \cdots + p^j x_{p^i p^j} = w_{p^{-(i-j)}} (y_{p^i 1}^{p^j} + p y_{p^i p}^{p^{j-1}} + \cdots + p^j y_{p^i p^j}).$$

(Note that this formula only makes sense if $j \leq i$.) Using Lemma 5.1.1 and reducing both sides modulo p^{j+1} shows that $x_{p^i p^j} \equiv y_{p^i p^j} \mod p$, which completes the induction.

We will now show that $w_{p^{-(i-j)}} \equiv 1 \mod p^{j+1}$ for each $0 \leq j \leq i$. Then we will be done. We again use the formula

$$x_{p^i 1}^{p^j} + p x_{p^i p}^{p^{j-1}} + \cdots + p^j x_{p^i p^j} = w_{p^{-(i-j)}} (y_{p^i 1}^{p^j} + p y_{p^i p}^{p^{j-1}} + \cdots + p^j y_{p^i p^j}).$$

It is immediately clear from what we have proven that

$$x_{p^i 1}^{p^j} + p x_{p^i p}^{p^{j-1}} + \cdots + p^j x_{p^i p^j} \equiv y_{p^i 1}^{p^j} + p y_{p^i p}^{p^{j-1}} + \cdots + p^j y_{p^i p^j} \mod p^{j+1}.$$

Hence $w_{p^{-(i-j)}} \in \mathbb{Z}_p^*$ can be written as $1 + p^{j+1} \frac{a}{b}$ with $a, b \in \mathscr{O}_{\mathbb{C}_p}$ and $p \nmid b$. This completes the proof. $\square$

5.2 The big case

Let $\overline{\mathbb{Z}}$ denote the integral closure of $\mathbb{Z}$ in $\mathbb{C}$, and define $\tilde{\mathbb{A}}^+ := \varprojlim \mathbb{W}(\overline{\mathbb{Z}})$ where $\mathbb{W}$ denotes the big Witt vectors and with transition maps the big Witt vector Frobenii. Our goal in this section is to prove results for $\tilde{\mathbb{A}}^+$ analogous to those proved for $\tilde{A}^+$ in the previous section. For instance, the following corresponds to Lemma 5.1.5 in both content and proof.

Lemma 5.2.1. *We have a well-defined injective ring homomorphism, which we call* *the "ghost map",*

$$w : \tilde{\mathbb{A}}^+ \to \prod_{r \in \mathbb{Q}^+} \overline{\mathbb{Z}},$$

defined by $w_r(x) = w_b(x_a)$ *for any* $\frac{a}{b} = r$ *where* w_b *denotes the* b*th component of the usual ghost map defined on* $\mathbb{W}(\overline{\mathbb{Z}})$.

Proof. This is clearly well-defined from the definition of the Witt vector Frobenius on ghost components. Because the usual ghost map is a ring homomorphism, so too is this extended ghost map. It is injective because the usual ghost map is injective, which is true because $\overline{\mathbb{Z}}$ is torsion free. □

As in the p-typical case, we will be able to simplify certain arguments by using Frobenius isomorphisms defined on $\tilde{\mathbb{A}}^+$.

Lemma 5.2.2. *For every* $r \in \mathbb{Q}^+$, *we have a Frobenius homomorphism* $F_r : \tilde{\mathbb{A}}^+ \to \tilde{\mathbb{A}}^+$ *defined as follows. Let* $x = (x_1, x_2, \ldots) \in \tilde{\mathbb{A}}^+$. *If* $r \in \mathbb{Z}^+$, *we set* $F_r(x) = (F_r(x_1), F_r(x_2), \ldots)$, *where on the right hand side* F_r *denotes the usual* r*th Frobenius on big Witt vectors. Still assuming* $r \in \mathbb{Z}^+$, *we set* $F_{\frac{1}{r}}(x) = (x_r, x_{2r}, \ldots)$. *For general* $r = \frac{a}{b} \in \mathbb{Q}^+$, *we define* F_r *as the composition* $F_a \circ F_{\frac{1}{b}} = F_{\frac{1}{b}} \circ F_a$. *As in the p-typical case, these Frobenii are isomorphisms.*

Proof. It is clear for $r \in \mathbb{Z}^+$ that F_r is a homomorphism, and in fact an isomorphism with inverse $F_{\frac{1}{r}}$. This shows that $F_{\frac{1}{r}}$ is also a homomorphism. Thus so too are the composites. The morphisms for general rational r are well-defined by properties of the usual Frobenius morphisms. □

Proposition 5.2.3. *The ring* $\tilde{\mathbb{A}}^+$ *is a domain.*

Proof. Let x, y denote two non-zero elements of $\tilde{\mathbb{A}}^+$. It will suffice to show that $w_r(x) \neq 0$ and $w_r(y) \neq 0$ for some rational r. We know that there are some rational u, v such that $w_u(x) \neq 0$ and $w_v(y) \neq 0$. Let p denote a prime occurring in the factorization of u or v. From the p-typical case (we can use Frobenii to apply it) we know that there exists an N such that $w_{\frac{u}{p^n}}(x) \neq 0$ for all $n \geq N$, and similarly for y. In this way we can assume that u and v have the same p-adic valuation, and then, after applying Frobenius, that this valuation is zero. Because u and v have only finitely many prime factors, this process terminates. □

Proposition 5.2.4. *We have* $(\tilde{\mathbb{A}}^+)^{G_{\mathbb{Q}_p}} = \mathbb{Z}$.

Proof. The proof is the same as for Proposition 5.1.13. □

Proposition 5.2.5. *We have* $(\mathrm{Frac}\ \tilde{\mathbb{A}}^+)^{G_{\mathbb{Q}_p}} = \mathbb{Q}$.

Proof. Write $\frac{x}{y}$ with $x, y \in \tilde{\mathbb{A}}^+$ for a fixed element. Without loss of generality we may assume $w_1(y) \neq 0$. Fix any prime p. We have a Galois equivariant projection $\tilde{\mathbb{A}}^+ \longrightarrow \tilde{A}^+_{(p)}$, where the subscript is to remind us of the fixed p. Then from Proposition 5.1.14 we know that $\frac{w_{p^i}(x)}{w_{p^i}(y)} = \frac{w_1(x)}{w_1(y)}$ for every $i \in \mathbb{Z}$. The right hand side is independent of p, hence $\frac{w_{p^i}(x)}{w_{p^i}(y)} = \frac{w_{q^j}(x)}{w_{q^j}(y)}$ for any primes p, q and integers i, j.

The key point in the above was showing that ghost components with prime power index were equal to ghost components with index 1. But after applying Frobenius, we can show that ghost components with index a product of two prime powers equal ghost components with prime power index. Continuing in this manner we get that the ghost components are independent of index, which proves the proposition. $\square$

Chapter 6

Properties of the Frobenius map

We denote by $\mathcal{O}_{\overline{\mathbb{Q}_p}}$ the ring of integers of the algebraic closure of $\mathbb{Q}_p$ and by $\mathcal{O}_{\mathbb{C}_p}$ the completion of $\mathcal{O}_{\overline{\mathbb{Q}_p}}$. For arguments which are equally valid for both these rings, we denote the unspecified ring by R.

Lemma 6.0.1. *Fix an element $x_1 \in R$ and an integer $n \geq 1$. There exists an element $x = (x_1, x_p, \ldots, x_{p^n}) \in W_{n+1}(R)$ with $F(x) = 0 \in W_n(R)$ if and only if $v_p(x_1) \geq \frac{1}{p} + \cdots + \frac{1}{p^n}$.*

Proof. $\Leftarrow$ We will find $x_p, \ldots, x_{p^n}$ as in the theorem which moreover satisfy $v_p(x_{p^i}) \geq \frac{1}{p} + \cdots + \frac{1}{p^{n-i}}$ for all $i < n$. In order that $x = (x_1, x_p, \ldots, x_{p^n})$ be in the kernel of Frobenius, it is necessary and sufficient that

$$\sum_{i=0}^{j} p^i x_{p^i}^{p^{j-i}} = 0$$

for all $1 \leq j \leq n$. We will show that the equation with $j = 1$ is solvable by an element x_p satisfying our condition on valuation. Then inductively we will show that for $x_1, \ldots, x_{j-1}$ fixed and satisfying our valuation conditions, the jth equation is solvable by an element x_{p^j} satisfying our condition on valuation.

Base case: The equation $x_1^p + px_p = 0$ is solvable by an element of valuation $pv_p(x_1) - 1 = \frac{1}{p} + \cdots + \frac{1}{p^{n-1}}$.

Inductive step: We now consider the jth equation with $j < n$, having chosen elements $x_1, \ldots, x_{p^{j-1}}$ satisfying our first $j - 1$ equations as well as our valuation conditions. We are done if we show that for each $i \leq j - 1$, the element $p^i x_{p^i}^{p^{j-i}}$ has valuation at least $j + \frac{1}{p} + \cdots + \frac{1}{p^{n-j}}$ for $j < n$. This is just a calculation: $v_p(p^i x_{p^i}^{p^{j-i}}) \geq i + p^{j-i}(\frac{1}{p} + \cdots + \frac{1}{p^{n-i}})$.

Some new notation will help: Set $m = j - i \geq 1$ and $k = n - i \geq 2$. Note that $k \geq m + 1$. Then we are trying to show

$$p^m(\frac{1}{p} + \cdots \frac{1}{p^k}) \geq m + \frac{1}{p} + \cdots + \frac{1}{p^{k-m}}.$$

We can rewrite the left-hand side as $(p^{m-1} + \cdots + 1) + \frac{1}{p} + \cdots + \frac{1}{p^{k-m}}$, from which the proposed inequality is obvious.

It remains to show that the very last equation, i.e., the one for $j = n$, is solvable given fixed $x_1, \ldots, x_{p^{n-1}}$ of the promised valuations. (For x_{p^n} we do not have a valuation requirement.) So we must show $v_p(p^i x_{p^i}^{p^{n-i}}) \geq n$ for $i < n$. So we wish to show $i + p^{n-i} v_p(x_{p^i}) \geq n$. Using the notation $m = n - i$, we are done because $p^m(\frac{1}{p} + \cdots + \frac{1}{p^m}) \geq m$.

$\Rightarrow$ We continue to refer to the equations above. The equation for $j = 1$ shows $v_p(x_1) \geq \frac{1}{p}$. And inductively, the equation for fixed $j \leq n$ shows that $v_p(x_{p^{j-1}}) \geq \frac{1}{p}$.

Inductively, assume that for fixed $k \leq n - 1$ we have $v_p(x_{p^i}) \geq \frac{1}{p} + \cdots \frac{1}{p^k}$ for $0 \leq i \leq n - k$. The equation with $j = 1$ yields $v_p(x_1) \geq \frac{1}{p} + \cdots + \frac{1}{p^{k+1}}$. Inductively, to extract the result from fixed $j \leq n - k$, we verify $v_p(p^i x_{p^i}^{p^{j-i}}) \geq j + \frac{1}{p} + \cdots + \frac{1}{p^k}$ for $i \neq j - 1$. For $i = j$, this follows from the previous stage. For $i \leq j - 2$, we compute $v_p(p^i x_{p^i}^{p^{j-i}}) \geq i + p^{j-i}(\frac{1}{p} + \cdots + \frac{1}{p^{k+1}}) \geq j + \frac{1}{p} + \cdots + \frac{1}{p^k}$, where the last equality follows from the same considerations as above. So we deduce that $v_p(p^{j-1} x_{p^{j-1}}^p) \geq j + \frac{1}{p} + \cdots + \frac{1}{p^k}$. From this the result follows at once. □

Lemma 6.0.2. *Fix an element $x_1 \in R$. There exists an element $x = (x_1, x_p, \ldots) \in W(R)$ with $F(x) = 0$ if and only if $v_p(x_1) \geq \frac{1}{p-1}$.*

Proof. $\Rightarrow$ This direction follows immediately from the previous lemma.

$\Leftarrow$ From the previous lemma, the following elements are in the image of the ghost map: $(x_1, 0, *, \ldots), (x_1, 0, 0, *, \ldots)$, etc. Because the ghost map is injective (and because of the injectivity of its restriction to finite levels) the result follows. □

It turns out that the Frobenius map is not surjective on $W(\mathscr{O}_{\mathbb{C}_p})$ or $W(\mathscr{O}_{\overline{\mathbb{Q}_p}})$.

Proposition 6.0.3. *Assume $p > 2$.*

(i) The maps $F : W_{n+1}(R) \to W_n(R)$ are surjective.

(ii) The map $F : W(R) \to W(R)$ is not surjective.

Proof. (i) Every element in $W(R)$ may be written in the form $\sum_{i=0}^{\infty} V^i([x_i])$. We know that $F([x]) = [x^p]$. So from the formula $V^i(F^i([x_i])) = V^i([1])[x_i]$ we see that it suffices to show $V^i([1])$ is in the image of F.

Our strategy is to consider the image of $V^i([1])$ under the ghost map, which is $(0,\dots,0,p^i,p^i,\dots)$, and to show that there is some element $(a_i,0,\dots,0,p^i,p^i,\dots)$ in the image of the ghost map. Because $\mathscr{O}_{\mathbb{C}_p}$ is p-torsion free, the ghost map is injective. Hence showing the existence of such an element in the image of the ghost map is sufficient to show that $V^i([1])$ has a preimage.

We prove this by induction on i, starting with the case $i=1$. We have an element $(p,p,p,\dots)$ in the image of the ghost map: it is the image of $p[1]$. We also have an element $(p^{\frac{1}{p}},p,p^p,\dots)$ in the image of the ghost map: it is the image of $[p^{\frac{1}{p}}]$. Finally, as $p>2$, the Dwork Lemma applied to $W(\mathbb{Z})$ shows that $(0,0,p^p,\dots)$ is in the image of the ghost map. Combining these, we see that $(p-p^{\frac{1}{p}},0,p,p,\dots)$ is in the image of the ghost map. From our lemma on the kernel of F, we get that $(-p^{\frac{1}{p}},0,p,p,\dots)$ is in the image of the ghost map, and moreover, an element $(a_1,0,p,p,\dots)$ is in the image of the ghost map if and only if $a_1 \equiv -p^{\frac{1}{p}} \mod p^{\frac{1}{p-1}}$.

Now inductively assume that for a fixed value i, an element $(a_i,0,\dots,0,p^i,p^i,\dots)$ is in the image of the ghost map if and only if $a_i \equiv (-1)^i p^{\frac{1}{p}+\dots+\frac{1}{p^i}} \mod p^{\frac{1}{p-1}}$. Applying Verschiebung to this element we find an element

$$(0,(-1)^i p^{1+\frac{1}{p}+\dots+\frac{1}{p^i}},0,\dots,0,p^{i+1},p^{i+1},\dots) \tag{6.0.4}$$

in the image of the ghost map. We also have elements

$$(p^{\frac{1}{p}},p,0,0,\dots)$$

and

$$(p^{\frac{1}{p^2}+\dots+\frac{1}{p^{i+1}}},p^{\frac{1}{p}+\dots+\frac{1}{p^i}},p^{1+\frac{1}{p}+\dots+\frac{1}{p^{i-1}}},\dots)$$

and

$$((-1)^{i+1},(-1)^{i+1},\dots).$$

Multiplying these previous three elements together, and adding them to the element (6.0.4) above, completes the induction.

(ii) We consider whether or not an arbitrary element $\sum_{i=0}^{\infty} V^i([1])[y_{p^i}]$ is in the image of F. For its supposed preimage, write $(x_1,x_p,\dots)$. Considering the its first two ghost components, we get immediately that $x_1 \equiv y_1^{\frac{1}{p}} \mod p^{\frac{1}{p}}$. Considering the first three ghost components, we have

$$x_1 \equiv y_1^{\frac{1}{p}} - p^{\frac{1}{p}} y_p^{\frac{1}{p}} \mod p^{\frac{1}{p}+\frac{1}{p^2}}.$$

From the first four:

$$x_1 \equiv y_1^{\frac{1}{p}} - p^{\frac{1}{p}} y_p^{\frac{1}{p}} + p^{\frac{1}{p}+\frac{1}{p^2}} y_{p^2}^{\frac{1}{p}} \mod p^{\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}}.$$

We show that, for certain choices of y_{p^i}, no $x_1 \in \mathscr{O}_{\mathbb{C}_p}$ can simultaneously satisfy each of these conditions. In particular, no $x_1 \in R$ can either.

Consider the sequence $(\frac{1}{p}, \frac{1}{p} + \frac{1}{p^2}, \ldots)$. This may be written in the form

$$\frac{1}{p-1}(1 - \frac{1}{p}, 1 - \frac{1}{p^2}, \ldots).$$

W will describe how to pick a sequence of y_{p^i} as above with $y_1 = 0$ such that no such x_1 can exist.

Each of our choices of R contains the ring $W(\overline{\mathbb{F}_p})$. We restrict to the case where our coefficients $y_i \in R$ are Teichmüller lifts of elements in $\overline{\mathbb{F}_p}$. We use the results of the paper [12]. In its notation, the set $S_{a,b}$ contains $\frac{1}{p}$ only if $1 \leq a \leq p$. Which values are possible if we require it to contain the entire sequence? From our second formulation of the sequence, clearly $a = p - 1$ is possible. We claim it is the only possibility. This follows immediately if we write

$$(\frac{1}{p} + \cdots + \frac{1}{p^n})(p - c) = (1 - \frac{c-1}{p} - \cdots - \frac{c-1}{p^{n-1}} - \frac{c}{p^n}).$$

In order that the function $f : T_b \to \overline{\mathbb{F}_p}$ given by $f(i) = x_{\frac{i}{p-1}}$ be twist-recurrent, it is necessary that the sequence y_i be eventually periodic. Choosing a sequence for which this fails, we have that x_1 could not be in the ring $\mathscr{O}_{\mathbb{C}_p}$, and hence not in our ring R. □

Appendix A

Witt Vector Calculations

This appendix contains several necessary but perhaps unenlightening calculations involving Witt vectors. Its organization is not as linear as that of the previous sections.

Lemma A.0.1. *Let* $f, g \in W(\overline{A}^{\text{perf}})$ *and write* $f = (f_0, f_1, \ldots)$ *and similarly for* g*. Write* $d_i = \max(\deg f_i, \deg g_i)$*. If* $f + g = h$*, then*

$$\deg h_i \leq \max(p^i d_0, p^{i-1} d_1, \ldots, d_i).$$

If the components of f *and* g *have only integer exponents, then so too do the components of* h*.*

Proof. From the definition of addition in terms of ghost components, we see immediately that

$$\deg h_i \leq \max_{j,k}(p^j d_{i-j}, p^k h_{i-k})$$

for $0 \leq j \leq i$ and $1 \leq k \leq i$. The first claim follows by induction on i. The claim concerning integral exponents follows from the fact that $W(\overline{A}) \subseteq W(\overline{A}^{\text{perf}})$ is a subring. □

Lemma A.0.2. *Let* F *denote a lift of Frobenius to* $A^\dagger$*. Let* D *be such that* $F(x_i)$ *is overconvergent of radius* $D/2$ *for every* i*. Then*

$$F : A^{\dagger,C} \to A^{\dagger,pC+D}.$$

Proof. Write

$$g = \sum_{k=0}^{\infty} p^k \sum_{j=0}^{n_k} a_{kj} x^j$$

for an element of $A^{\dagger,C}$. This means in particular that $|n_k| \leq (k+1)C$. (Recall that j is a multi-index.) The map F is a p-adically continuous homomorphism, so

$$F(g) = \sum_{k=0}^{\infty} p^k \sum_{j=0}^{n_k} F(a_{kj})F(x^j).$$

For each $i = 1, \ldots, n$, write

$$F(x_i) = \sum_{k=0}^{\infty} p^k \sum_{j=0}^{n_{ik}} b_{ikj}x^j.$$

Because F is a lift of Frobenius, $|n_{i0}| = p$. We have assumed that each $F(x_i)$ is $D/2$-overconvergent; below we will need the slightly weaker bounds $D \geq p$ and $|n_{ik}| \leq kD$ for $k \geq 1$.

For $l \geq 1$, let $r_l : A^\dagger \to W_l(k)[x_1, \ldots, x_n]$ denote the natural restriction map. For $a \in A^\dagger$, let $d(l, a) := \deg r_l(F(a))$. We also define

$$d(l, n_k) := \max_{0 \leq |j| \leq |n_k|} d(l, x^j).$$

Then we have

$$\begin{aligned}
d(1, n_k) &\leq n_k p \\
d(2, n_k) &\leq (n_k - 1)p + D \\
d(3, n_k) &\leq (n_k - 2)p + 2D \\
&\cdots \\
d(n_k, n_k) &\leq 1p + (n_k - 1)D \\
d(n_k + 1, n_k) &\leq n_k D,
\end{aligned}$$

and so on. If we write $F(g) = \sum_{k=0}^{\infty} p^k \sum_{j=0}^{m_k} c_{kj}x^j$, with the usual conventions, then we find immediately

$$\begin{aligned}
|m_i| \leq \max_{l-1+k=i} d(l, n_k) &\leq n_k p + (l-1)D \\
&\leq (k+1)Cp + (l-1)D \\
&\leq (i+1)(Cp + D).
\end{aligned}$$

Thus $F(g)$ is overconvergent of radius $pC + D$, as claimed. □

Proposition A.0.3. *For any lift F' of the Frobenius map to $B^\dagger$, the induced map*

$$t_{F'} : \Omega_{B^\dagger} \to W\Omega_{\overline{B}}$$

has image in $W^\dagger\Omega_{\overline{B}}$.

Proof. First, note that we can restrict to the case of degree zero. This is because the left-hand side consists of finite combinations, under the operations of differentiation, addition, and multiplication, of elements in degree zero. Restricting then to degree zero, we start with the polynomial algebra case. Let $g \in A^\dagger$ have radius C. By Lemma A.0.2, $(F)^n(g)$ has radius at most $p^n(C+D)$. We will show $t_F(g)$ has radius at most $C+D$.

To fix notation, we write $s_F(g) = (s_0(g), s_1(g), \ldots) \in W(A^\dagger)$, where s_F is the map 2.2.1. Using the notation of the previous lemma, $t_F(g) = (r_1(s_0(g)), r_1(s_1(g)), \ldots)$, so we would like to prove $d(1, s_n(g)) \leq (n+1)p^n(C+D)$ for each n.

Our inductive proof requires bounds on $d(i, s_n(g))$ also for $i \neq 1$. In particular we prove $d(i, s_n(g)) \leq (n+i)p^n(C+D)$ for all $n \geq 0$, $i \geq 1$.

Base case: Because $s_0(g) = g$ is C-overconvergent, we know $d(i, s_0(g)) \leq iC \leq i(C+D)$, as required.

Now assume the result for $s_0(g), \ldots, s_{n-1}(g)$. By the definition of $s_n(g)$, we have

$$s_0(g)^{p^n} + \ldots + p^j s_j(g)^{p^{n-j}} + \ldots + p^n s_n(g) = F^n(g).$$

From the definition it is clear that $d(i,a) = d(n+i, p^n a)$ for any $a \in A^\dagger$. We thus complete the assumption if we show $(n+i)p^n(C+D)$ is an upper bound for $d(n+i, F^n(g))$ and $d(n+i-j, s_j(g)^{p^{n-j}})$. For the first, we apply Lemma A.0.2. For the second, we use the inductive assumption to note that $d(n+i-j, s_j(g)) \leq (n+i)p^j(C+D)$. The result then follows from the observation that $d(i, a^k) \leq kd(i,a)$.

This completes the proof in the case of a polynomial algebra for any lift of Frobenius. In general, for a smooth algebra $\overline{B}$, we start with a surjective map $\overline{A} \to \overline{B}$. We find a Frobenius on $A^\dagger$ lifting F', and use the functoriality of the map t_F asserted in Proposition 2.2.2. □

Lemma A.0.4. *Let $\psi : A^\dagger \to W^\dagger(\overline{A})$ denote a homomorphism and D denote a number such that $\psi(x_i)$ is D-overconvergent for each i. If x denotes a monomial of degree C, then $\psi(x) = (f_0, f_1, f_2, \ldots)$ with $\deg f_i \leq (i+1)p^iD + p^iDC$. Moreover, if $g \in A^\dagger$, not necessarily a monomial, has radius C, then $\psi(g)$ is $D(C+1)$-overconvergent.*

Proof. We prove this by induction on C. The base case $C = 1$ follows immediately from the definition of D. Assume the result for a fixed value of C. We must compute

$$(f_0, f_1, f_2, \ldots)(g_0, g_1, g_2, \ldots) = (h_0, h_1, h_2, \ldots)$$

where $(g_0, g_1, g_2, \ldots) = \psi(x_j)$ for some j and where $\deg f_i \leq (i+1)p^i D + p^i DC$. We use the definition of Witt vector multiplication in terms of ghost components. From this we find

$$(f_0, f_0^p + pf_1, \ldots)(g_0, g_1^p + pg_1, \ldots) = (h_0, h_0^p + ph_1, \ldots). \tag{A.0.5}$$

We wish to bound the degree of h_i. The formula for the ith ghost component involves terms $p^j f_j^{p^{i-j}} p^k g_k^{p^{i-k}}$ for $j, k \leq i$. But as we are interested in the part $p^i h_i$, we may further restrict to $j + k \leq i$. (Recall that, though these formulas are found in characteristic zero, we will eventually be projecting to characteristic p, so any p-power coefficients in our formula for h_i will vanish.)

So it suffices to study the degree of $f_j^{p^{i-j}} g_k^{p^{i-k}}$ for $j + k \leq i$. The degree of such a term is at most

$$p^{i-j}((j+1)p^j D + p^j DC) + p^{i-k}((k+1)p^k D \leq (j+k+1)p^i D + p^i D(C+1).$$

Because $j + k \leq i$, such a term satisfies the required bounds and completes the induction.

For the last assertion, let y denote a monic monomial of degree $C(m+1)$. Then $\psi(p^m y) = (f_0, f_1, \ldots)$ where $f_i = 0$ for $i < m$ and

$$\begin{aligned}\deg f_i \leq p^m((i-m+1)p^{i-m}D + p^{i-m}DC(m+1)) &= (i-m+1)p^i D + p^i DC(m+1) \\ &\leq (i+1)p^i D(C+1)\end{aligned}$$

for $i \geq m$. □

The following lemma is analogous in content to the previous one, but its proof is greatly simplified by a stronger definition of overconvergent, which requires in particular that the zeroth term have degree zero.

Definition A.0.6. *A Witt vector $w = (w_0, w_1, \ldots)$ is called C-extraconvergent if $\deg w_i \leq ip^i C$ for all i.*

Note that a C-extraconvergent Witt vector is C-overconvergent, and conversely, a C-overconvergent Witt vector with $\deg w_0 = 0$ is $2C$-extraconvergent.

Lemma A.0.7. *Let $(f_0, f_1, \ldots)$ and $(g_0, g_1, \ldots)$ be extraconvergent Witt vectors of radii C and D. Then their product $(h_0, h_1, \ldots)$ is $\max(C, D)$-extraconvergent.*

Proof. As in the proof of lemma A.0.4, to bound the degree of h_i it suffices to check the degree of $f_j^{p^{i-j}} g_k^{p^{i-k}}$ for $j + k \leq i$. This latter degree is

$$p^{i-j}(jp^j C) + p^{i-k}(kp^k D) \leq ip^i \max(C, D),$$

as required. □

Lemma A.0.8. *Let $(0, f_1, \ldots)$ be C-extraconvergent and $(g_0, g_1, \ldots)$ D-overconvergent, with $C \geq 2D$. Then their product $(h_0, h_1, \ldots)$ satisfies $\deg h_i \leq ip^i C + p^i D$.*

Proof. Similarly to the previous lemmas, we consider $p^{i-j}(jp^j C) + p^{i-k}((k+1)p^k D)$ and note that for $i \geq 1$,

$$\max_{j+k \leq i} jp^i C + (k+1)p^i D = ip^i C + p^i D.$$

□

Lemma A.0.9. *We have a sequence of expressions*

$$\begin{aligned} z &= \sum (z^{a_{1I}})^p f_{1I} \\ &= \sum (z^{a_{2I}})^{p^2} f_{2I} \\ &= \cdots . \end{aligned}$$

Furthermore, there exists a constant C for which we can find such expressions with a_{iI} bounded for all i, I, and with $\deg f_{iI} \leq Cp^i$ for all i.

Proof. We have already seen that $\overline{B}$ is generated as a $\overline{B}^p$-algebra by monomials $x_1^{e_1} \cdots x_n^{e_n}$ with $0 \leq e_i \leq p - 1$. We can recursively apply the result to find that $\overline{B}$ is generated as a $\overline{B}^{p^n}$-algebra by monomials $x_1^{e_1} \cdots x_n^{e_n}$ with $0 \leq e_i \leq p^n - 1$. (For example, we can replace an expression of the form $z = z^{2p}x_1^3 + z^p x_2$ with $z = (z^{2p}x_1^3 + z^p x_2)^{2p} x_1^3 + (z^{2p}x_1^3 + z^p x_2)^p x_2$. Use the fact that $(a+b)^{p^n} = a^{p^n} + b^{p^n}$.)

Because these expressions possess finitely many terms, the result is trivial for the first equation. Let r denote the degree of the monic polynomial $P(z)$ which z satisfies. Let C_1 denote a constant so that we can find an expression $z = \sum (z^{a_{1I}})^p f_{1I}$ with $a_{1I} < r$ and $\deg f_{1I} \leq pC_1$.

Let l denote the maximal degree among the coefficients of $P(z)$. Pick $C \geq C_1$ such that $(p-1)C \geq r^2lp + C_1rp$.

We have an expression for z with $a_{1I} \leq r-1$ and with $\deg f_{1I} \leq pC$. Inductively, assume we have written the nth expression for z in this way, with $a_{nI} \leq r-1$ and $\deg f_{nI} \leq p^nC$.

To simplify notation, it is harmless to drop the sums and consider a single expression

$$z = (z^{a_n})^{p^n} f_n.$$

Substituting in our 1st expression for z, with the sums still suppressed, we find

$$z = (z^{a_1p}f_1)^{a_np^n} f_n = z^{a_1a_np^{n+1}} f_1^{a_np^n} f_n.$$

Using $P(z)$, we can write

$$z^{a_1a_np^{n+1}} = \sum_{i=0}^{r-1} z^i g_i,$$

and from our bound l on the degrees of the coefficients of $P(z)$, we get

$$\deg g_i \leq a_1a_np^{n+1}l \leq r^2lp^{n+1}.$$

We are finished by induction because

$$p^{n+1}C \geq r^2lp^{n+1} + C_1rp^{n+1} + Cp^n.$$

□

Lemma A.0.10. *Let w denote a D-overconvergent Witt vector with zeroth component of degree one. Write $w^N = (f_{N0}, f_{N1}, \ldots)$. Then these terms satisfy the following bounds:*

$$\deg f_{Ni} \leq 2ip^iD + (N-i)p^i.$$

Proof. The proof is by induction on N. For $N=1$, the case $i=0$ is trivial, and the cases $i \geq 1$ follow from the assumption that w is D-overconvergent. We inductively assume the result for some fixed value of N and consider the case $N+1$. As in the previous proofs, our goal is to compute

$$(f_{10}, f_{11}, \ldots)(f_{N0}, f_{N1}, \ldots) = (f_{N+1,0}, f_{N+1,1}, \ldots).$$

Also as in the previous proofs, we find

$$\deg f_{N+1,i} \leq p^{i-j} \deg f_{1j} + p^{i-k} \deg f_{Nk}.$$

We are done if we show that

$$2(j+k)p^i D + (N+1-j-k)p^i \leq 2ip^i D + (N+1-i)p^i.$$

This follows from the inequality

$$(i-j-k)p^i \leq 2(i-j-k)p^i D,$$

which in turn follows from $j+k \leq i$ and $1 \leq D$ (the latter of which was implied by our assumptions on w). □

Lemma A.0.11. *Let $w = V^k(x_1^{e_1} \dots x_n^{e_n} y^e)$ be C-overconvergent. Then we can write w as as a sum of terms of the form*

$$t_F(f_i) V^{k+i}(x_1^{\tilde{e_1}} \dots x_n^{\tilde{e_n}}),$$

where each $\tilde{e}_j \leq p^{k+i}-1$ and where $p^{k+i} f_i$ is $\tilde{C}$-overconvergent, where $\tilde{C}$ depends only on C and $\overline{B}$, and in particular does not depend on k.

Proof. It is immediate that

$$w = [x_1^{a_1} \cdots x_n^{a_n}][y^{a+1}] V^k(x_1^{\tilde{e_1}} \cdots x_n^{\tilde{e_n}} h^{\tilde{e}}),$$

where $a_i, a \leq C(k+1)$ and $e_i \leq p^k - 1$. We checked in section 4.2 that $[x_1^{a_1} \cdots x_n^{a_n}]$ could be written in the stated form. Likewise the bounds on $\tilde{e}$ mean that

$$V^k(x_1^{\tilde{e_1}} \cdots x_n^{\tilde{e_n}} h^{\tilde{e}})$$

will pose no trouble. The difficulty is in accounting for the term $[y^{a+1}]$.

From our lemma A.0.10, we know

$$[y^{a+1}] = t_F(y^{a+1}) + \sum_{j=1}^{\infty} V^j([g_{0j}]),$$

where $\deg g_{0j} \leq 2jp^jD + (a+1)p^j$. Inductively, assume we have written

$$[y^{a+1}] = \sum_{j=0}^{k} t_F(g_j)V^j + \sum_{j=k}^{\infty} V^j([g_{kj}]),$$

where g_j has degree $2jD+a+1$ for all j and g_{jk} has degree $(2jD+a+1)p^k$ for all j, k, and where the notation V^j has suppressed terms of degree at most $n(p^j-1)+d(p^j-1)$. We have just checked the base case $k=0$.

To check the case $k+1$, we note that as in our first paragraph,

$$V^j([g_{kk}]) = [g_k]V^1,$$

where $\deg g_k \leq 2kD + a + 1$. We can rewrite $[g_k]$ as $t_F(g_k) + \sum V^j([h_{kj}])$ where $\deg h_{kj} \leq (2jD + 2kD + a + 1)p^j$, and this completes the induction.

Substituting in $(k+1)C$ for a, we see that we may let $\tilde{C} = C + 2D + 2$. $\square$

Lemma A.0.12. *Let $w = V^k(x_1^{e_1} \ldots x_n^{e_n} y^e z^{e'} w^{e''})$ be C-overconvergent. Then we can write w as as a sum of terms of the form*

$$t_F(f_i)V^{k+i}(x_1^{\tilde{e_1}} \ldots x_n^{\tilde{e_n}}),$$

where each $\tilde{e}_j \leq p^{k+i} - 1$ and where $p^{k+i}f_i$ is $\tilde{C}$-overconvergent, where $\tilde{C}$ depends only on C and $\overline{B}$, and in particular does not depend on k.

Proof. We begin with the equation

$$w = [x_1^{a_1} \cdots x_n^{a_n}][y^{a+1}][z^{a'}][g^{a''}]V^k(x_1^{\tilde{e_1}} \cdots x_n^{\tilde{e_n}} h^{\tilde{e}} z^{e'} w^{e''}),$$

where $a_i, a, a', a'' \leq C(k+1)$ and $\tilde{e}_i, \tilde{e}, e', e'' \leq p^k - 1$. Again, we may consider terms individually. For all but the last term, the proof goes through as before. (We don't even need to reduce high powers of z using its minimal polynomial.)

From the argument in lemma A.0.9, we see that we can write $z^{e'}$ as a sum of terms $(z^c)^{p^k} f_k$, where $c \leq r$ and $\deg f_k \leq Np^k + e'd \leq \tilde{N}p^k(k+1)C$. Thus

$$V^k(x_1^{\tilde{e_1}} \cdots x_n^{\tilde{e_n}} h^{\tilde{e}} z^{e'} g^{e''}) = [z^c]V^k[*].$$

As before, the term $[z^c]$ (here c is even bounded) poses no problem, and for the term $V^k[*]$, we may apply lemma A.0.11. $\square$

Corollary A.0.13. *Let $\overline{B}/\overline{A}$ denote a finite étale extension of a distinguished open in affine space. Then we have an isomorphism*

$$B^\dagger \otimes_{A^\dagger} W_n\Omega_{\overline{A}} \cong W_n\Omega_{\overline{B}}.$$

Proof. We use the following isomorphisms:

$$\begin{array}{rcll} W_n(\overline{B}) \otimes_{W_n(\overline{A})} W_n\Omega_{\overline{A}} & \cong & W_n\Omega_{\overline{B}} & \text{from [14], Proposition 1.7;} \\ W_n(\overline{A}) \otimes_{A_{n+1}} B_{n+1} & \cong & W_n(\overline{B}) & \text{from [14], p. 69;} \\ A_{n+1} \otimes_{A^\dagger} B^\dagger & \cong & B_{n+1} & \text{since } A^\dagger/(p^{n+1}) \cong A_{n+1}, \text{ and similarly for } B. \end{array}$$

□

Appendix B

The Overconvergent de Rham-Witt Complex is a Sheaf

In this appendix we reproduce Proposition 1.2 of [4], together with its proof, to show that our overconvergent de Rham-Witt complex is a Zariski sheaf. (It is, in fact an étale sheaf, see loc. cit. Theorem 1.8, but we will not need this fact.) The only difference between this appendix and what appears in [4] is that we have removed a few parts which are unnecessary for our purposes. In particular, the notation below is somewhat different from the rest of this paper. In it, A denotes a smooth affine over k, not necessarily a polynomial algebra. The map λ is a surjective map $k[T_1, \ldots, T_r] \twoheadrightarrow A$ as in our Definition 3.0.6.

Proposition B.0.1 (Proposition 1.2, [4]). *(a) We denote by $f \in A$ an arbitrary element. Let $d \in \mathbb{Z}$ be nonnegative. The presheaf*

$$W^\dagger\Omega^d_{\operatorname{Spec} A/k}(\operatorname{Spec} A_f) := W^\dagger\Omega^d_{A_f/k}$$

is a sheaf for the Zariski topology on $\operatorname{Spec} A$ *(compare [5],* **0**, *3.2.2).*

(b) The Zariski cohomology of these sheaves vanishes in degrees $j > 0$, i.e.

$$H^j_{Zar}(\operatorname{Spec} A, W^\dagger\Omega^d_{\operatorname{Spec} A/k}) = 0.$$

We fix generators $t_1, \ldots, t_r$ of A and denote by $[t_1], \ldots, [t_r]$ the Teichmüller representatives in $W(A)$. An elementary Witt differential in the variables $[t_1], \ldots, [t_r]$ is the image of a basic Witt differential in variables $[T_1], \ldots, [T_r]$ under the map λ.

Before we prove the proposition, we need a special description of an overconvergent element z in $W^\dagger\Omega^d_{A_f/k}$. Let $[f] \in W(A)$ be the Teichmüller representative. Hence

$\frac{1}{[f]} = \left[\frac{1}{f}\right]$ is the Teichmüller of $\frac{1}{f}$ in $W(A_f)$. For the element z we have the following description.

Proposition B.0.2. *The element $z \in W^\dagger\Omega^d_{A_f/k}$ can be written as a convergent series*

$$z = \sum_{l=0}^{\infty} \frac{1}{[f]^{r_l}} \overline{\eta}_l$$

where each $\overline{\eta}_l$ is a finite sum of elementary Witt differentials $\overline{\eta}_l^{(t)}$ in the variables $[t_1], \ldots, [t_r]$, images of basic Witt differentials $\eta_l^{(t)}$ in variables $[T_1], \ldots, [T_r]$ which have weights k_l^t satisfying the following growth condition:

$\exists C_1 > 0, C_2 \in \mathbb{R}$ such that for each summand $\eta_l^{(t)}$ we have

$$r_l + |k_l^t| \leq C_1 \, ord_p \eta_l^{(t)} + C_2.$$

Furthermore we require that for a given $K > 0$,

$$\min_t \; ord_p \eta_l^{(t)} > K \textit{ for almost all } l.$$

Proof. Using the composite map of de Rham-Witt complexes

$$\begin{array}{rcl rcl} W\Omega_{k[T_1,\ldots,T_r,Z_1,Z_2]/k} & \longrightarrow & W\Omega_{k[T_1,\ldots,T_r,Y,Y^{-1}]/k} & & \longrightarrow & W\Omega_{A_f/k} \\ T_i & \longmapsto & T_i & T_i & \longmapsto & t_i \\ Z_1 & \longmapsto & Y & Y & \longmapsto & f \\ Z_2 & \longmapsto & Y^{-1} & Y^{-1} & \longmapsto & f^{-1} \end{array}$$

we see that z_i is the image of an overconvergent sum of basic Witt differentials in $W\Omega_{k[T_1,\ldots,T_r,Y,Y^{-1}]/k}$.

We use here an extended version of basic Witt differentials to the localized polynomial algebra $k[T_1, \ldots, T_r, Y, Y^{-1}]$ (compare [10]): A basic Witt differential α in $W\Omega_{k[T_1,\ldots,T_r,Y,Y^{-1}]/k}$ has the following shape:

I) α is a classical basic Witt differential in variables $[T_1], \ldots, [T_r], [Y]$.

II) Let $e(\xi_{k,\mathcal{P}}, k, \mathcal{P})$ be a basic Witt differential in variables $[T_1], \ldots, [T_r]$. Then

II 1) $\alpha = e(\xi_{k,\mathcal{P}}, k, \mathcal{P}) \, d \, \log[Y]$

II 2) $\alpha = [Y]^{-r} e(\xi_{k,\mathcal{P}}, k, \mathcal{P})$ for some $r > 0, r \in \mathbb{N}$

II 3) $\alpha = F^s d[Y]^{-l} e(\xi_{k,\mathcal{P}}, k, \mathcal{P})$ for some $l > 0$, $p \nmid l$, $s \geq 0$.

III) $\alpha = {}^{V^u}\left(\xi[Y]^{p^u k_Y}[T]^{p^u k_{I_0}}\right) d^{V^{u(I_1)}}[T]^{p^{u(I_1)} k_{I_1}} \dots F^{-t(I_d)} d[T]^{p^{t(I_d)} k_{I_d}}$ (see [3], (2.15)).

In particular, k is a weight function on variables $[T_1], \dots, [T_r]$ with partition $I_0 \cup \dots \cup I_d = \mathcal{P}, u > 0, k_Y \in \mathbb{Z}\left[\frac{1}{p}\right]_{<0}$,

$$u(k_Y) \leq u = \max\{u(I_0), u(k_Y)\} \text{ (notations as in [3]).}$$

If $I_0 = \emptyset$, we require $u = \max\{u(I_1), u(k_Y)\}$.

IV) $\alpha = d\alpha'$ when α' is as in III).

An overconvergent sum of basic Witt differentials α in $W\Omega^\bullet_{k[T_1,\dots,T_r,Y,Y^{-1}]/k}$ is an infinite convergent sum ω such that there exists $\widetilde{C_1} > 0, \widetilde{C_2} \in \mathbb{R}$ with the following properties:

- If α of type I) or of type II 1) occurs as a summand in ω, we require

$$|k| \leq \widetilde{C_1} \operatorname{ord}_p \xi_{k,\mathcal{P}} + \widetilde{C_2}.$$

- If α is of type II 2) or II 3) occurs as a summand in ω then

$$r + |k| \leq \widetilde{C_1} \operatorname{ord}_p \xi_{k,\mathcal{P}} + \widetilde{C_2} \text{ (with } r = l \cdot p^s \text{ in case II 3).}$$

- If α is of type III) or IV), then

$$|k_Y| + \sum_{j=0}^{d} |k_{I_j}| \leq \widetilde{C_1} \operatorname{ord}_p (V^u \xi) + \widetilde{C_2}$$

(here, $|k_Y| = -k_Y, |k_{I_j}| = \sum_{i \in I_j} k_i$).

It is a straightforward exercise to show that the image of

$$W^\dagger\Omega_{k[T_1,\dots,T_r,Z_1,Z_2]/k} \longrightarrow W\Omega_{k[T_1,\dots,T_r,Y,Y^{-1}]/k}$$

consists exactly of sums ω described above.

In the situation of condition III) we consider the first factor $V^u\left(\xi[Y]^{p^u k_Y}[T]^{p^u k_{I_0}}\right)$. For simplicity we assume $I_0 = \emptyset$; this does not affect the following calculations.

Let $-k_Y = \dfrac{r}{p^u}$ and $l \leq \frac{r}{p^u} < l+1$ for an integer l. We have

$$\begin{aligned}
V^u\left(\xi[Y]^{p^u k_Y}\right) &= V^u\left(\xi\frac{1}{[Y]^r}\right) = V^u\left(\xi\frac{1}{[Y]^{lp^u}}\cdot\frac{1}{[Y]^{r-lp^u}}\right)\\
&= \frac{1}{[Y]^l}V^u\left(\xi\frac{1}{[Y]^{r-lp^u}}\right) = \frac{1}{[Y]^l}V^u\left(\xi\frac{[Y]^{p^u-r+lp^u}}{[Y]^{p^u}}\right)\\
&= \frac{1}{[Y]^{l+1}}V^u\left(\xi[Y]^{(l+1)p^u-r}\right).
\end{aligned}$$

Now consider the image of α in $W\Omega^d_{A_f/k}$ where

$$[Y]\to[f],\qquad [Y^{-1}]\to[f^{-1}],\qquad [T_i]\to[t_i].$$

The above factor $\frac{1}{[Y]^{l+1}}V^u\left(\xi[Y]^{(l+1)p^u-r}\right)$ is mapped to $\frac{1}{[f]^{l+1}}V^u\left(\xi[f]^{(l+1)p^u-r}\right)$.

Represent f as a polynomial of degree g in $t_1,\ldots,t_r$. Then it is easy to see that the image of α in $W\Omega^d_{A_f/k}$ is of the form $\frac{1}{[f]^{l+1}}\widetilde{\widetilde{\eta}}$ where $\widetilde{\widetilde{\eta}}$ is a (possibly infinite) sum of images of basic Witt differentials $\hat{\eta}^t$ in variables $[T_1],\ldots,[T_r]$ with weights k^t satisfying

$$\begin{aligned}
|k^t| &\leq g\left(l+1-\frac{r}{p^u}\right)+\sum_{j=0}^{d}|k_{I_j}|\\
&\leq g+\sum_{j=0}^{d}|k_{I_j}|.
\end{aligned}$$

The case $d\alpha$ (type IV) is deduced from the case III by applying d to α and the Leibniz rule to the image of $d\alpha$ in $W\Omega^d_{A_f/k}$. So if the image of α as above is $\frac{1}{[f]^{l+1}}\widetilde{\widetilde{\eta}}$ then the image of $d\alpha$ is

$$\begin{aligned}
\frac{1}{[f]^{l+1}}d\widetilde{\widetilde{\eta}}-\frac{1}{[f]^{l+2}}\cdot l\,d[f]\widetilde{\widetilde{\eta}} &= \frac{1}{[f]^{l+2}}\left([f]d\widetilde{\widetilde{\eta}}-l\,d[f]\widetilde{\widetilde{\eta}}\right)\\
&= \frac{1}{[f]^{l+2}}\widetilde{\widetilde{\widetilde{\eta}}},
\end{aligned}$$

where $\widetilde{\widetilde{\widetilde{\eta}}}$ is a sum of images of basic Witt differentials $\widetilde{\widetilde{\eta}}^t$ in variables $[T_1],\ldots,[T_r]$ with weights k^t satisfying

$$|k^t|\leq 2g+\sum_{j=0}^{d}|k_{I_j}|.$$

We can also compute the images of α in $W\Omega_{A_{f_i}/k}$ where α is of type I or II and obtain again a representation

$$\frac{1}{[f]^r}\tilde{\overline{\eta}} \text{ for } r \geq 0.$$

These cases are easier and omitted.

Now we return to the original element $z \in W^\dagger\Omega^d_{A_f/k}$. We may write z as a convergent sum

$$z = \sum_{m=0}^{\infty} \widetilde{\omega}_m,$$

where $\widetilde{\omega}_m$ is an elementary Witt differential being the image of a basic Witt differential α_m in $W\Omega_{k[T_1,\ldots,T_r,Y,Y^{-1}]/k}$ of type I, II, III or IV.

In all cases we have a representation

$$\widetilde{\omega}_m = \frac{1}{[f]^{r_m}}\tilde{\overline{\eta}}_m$$

where $\tilde{\overline{\eta}}_m$ is the sum of images of basic Witt differentials $\widetilde{\eta}^t_m$ in variables $[T_1], \ldots, [T_r]$ with weights k^t_m such that

$$r_m + |k^t_m| \leq \widetilde{C_1}\operatorname{ord}_p\left(\widetilde{\eta}^t_m\right) + \widetilde{C_2} + 2(g+1).$$

Now consider - for a given integer N - the element z modulo Fil^N, so the image $\overline{z}^{(N)}$ of z in

$$W_N\Omega^d_{A_f/k} = W_N\Omega^d_{A/k} \bigotimes_{W_N(A)} W_N(A)\left[\frac{1}{[f]}\right].$$

One then finds a lifting $z^{(N)}$ of $\overline{z}^{(N)}$ in $W\Omega_{A_f/k}$ such that $z^{(N)} = \sum_{m=0}^{b(N)} \omega_m$ is a finite sum, i.e.

$$\omega_m = \frac{1}{[f]^{r_m}}\overline{\eta}_m$$

where now $\overline{\eta}_m$ is a finite sum of images of basic Witt differentials η^t_m in variables $[T_1], \ldots, [T_r]$ satisfying the growth condition

$$r_m + |k^t_m| \leq C_1 \operatorname{ord}_p(\eta^t_m) + C_2$$

with $C_1 := \widetilde{C_1}, C_2 = \widetilde{C_2} + 2(g+1)$.

The elements $z^{(N)}$ can be chosen to be compatible for varying N and we have $z = \lim z^{(N)}$. It is clear that the second condition of the lemma is also satisfied, this finishes the proof of Proposition B.0.2. □

Now we are ready to prove Proposition B.0.1.

As $W\Omega^\bullet$ is a complex of Zariski sheaves we need to show–in order to prove part (a) of the proposition–the following claim:

Let $z \in W\Omega^d_{A/k}$ for some fixed d, let $\{f_i\}_i$ be a collection of finitely many elements in A that generate A as an ideal. Assume that for each i the image z_i of z in $W\Omega^d_{A_{f_i}/k}$ is already in $W^\dagger\Omega^d_{A_{f_i}/k}$. Then $z \in W^\dagger\Omega^d_{A/k}$.

Let $[f_i]$ be the Teichmüller of f_i with inverse $\frac{1}{[f_i]} = [\frac{1}{f_i}]$.

Lemma B.0.3. *There are elements $r_i \in W^\dagger(A)$ such that $\sum_{i=1}^{n} r_i[f_i] = 1$.*

Proof. Consider a relation $\sum_{i=1}^{n} a_i f_i = 1$ in A. Then $\sum_{i=1}^{n}[a_i][f_i] = 1 + {}^V\eta \in W^\dagger(A)$. By Lemma 2.25 in [3],

$$(1 + {}^V\eta)^{-1} \in W^\dagger(A).$$

Define $r_i = (1 + {}^V\eta)^{-1} \cdot [a_i]$. □

Lemma B.0.4. *For each t there are polynomials $Q_{i,t}[T_1, \dots, T_{2n}]$ in $2n$ variables such that*

(1) degree $Q_{i,t} \leq 3 \cdot nt$

(2) $\sum_{i=1}^{n} Q_{i,t}\left([f_1], \dots, [f_n], r_1, \dots, r_n\right)[f_i]^t = 1$.

For the proof of this Lemma, compare [16].

We know that $\operatorname{Spec} A = \cup_{i=1}^{n} D(f_i)$. For a tuple $1 \leq i_1 < \cdots < i_m \leq n$, let $\mathfrak{U}_{i_1 \dots i_m} = \cap_{j=1}^{m} D(f_{i_j})$. Fix $d \in \mathbb{N}$ and let

$$\begin{aligned} C^m &= C^m(\operatorname{Spec} A, W^\dagger\Omega^d_{A/k}) \\ &= \oplus_{1 \leq i_1 < \cdots < i_m \leq n} W^\dagger\Omega_{A_{f_{i_1} \cdots f_{i_m}}/k} \\ &= \oplus_{1 \leq i_1 < \cdots < i_m \leq n} \Gamma(\mathfrak{U}_{i_1 \dots i_m}, W^\dagger\Omega^d_{A/k}). \end{aligned}$$

Then consider the Čech complex

$$0 \to C^0 \to C^1 \to C^2 \to \cdots.$$

We have $C^0 = W^\dagger\Omega^d_{A/k}$ and $C^0 \to C^1$ is the restriction map $W^\dagger\Omega^d_{A/k} \to W\Omega_{A_{f_i}/k}$ for all i. It is then clear that Proposition B.0.1 follows from the following.

Proposition B.0.5. *The complex $C^\bullet$ is exact.*

Proof. The proof is very similar to the proof of Lemma 7 in [16]. We fix as before k-algebra generators $t_1, \dots, t_r$ of A. Suppose $\sigma \in C^m, m \geq 2$, is a cocycle. Then σ has components

$$\sigma_{i_1 \dots i_m} \in \Gamma(\mathfrak{U}_{i_1 \dots i_m}, W^\dagger \Omega^d_{\mathrm{Spec}\, A/k}) = W^\dagger \Omega^d_{A_{f_{i_1} \cdots f_{i_m}}/k}.$$

Applying Proposition B.0.2 we see that $\sigma_{i_1 \dots i_m}$ has a representation as an overconvergent sum of Witt differentials as follows: $\sigma_{i_1 \dots i_m} = \sum_{l=0}^{\infty} M_l^{i_1 \dots i_m}$ with

$$M_l^{i_1 \dots i_m} = \sum_j \frac{1}{[f_{i_1 \dots i_m}]^j} \overline{\eta}^{(j)}_{l i_1 \dots i_m} \text{ a finite sum}$$

where $[f_{i_1 \dots i_m}]^j := [f_{i_1}]^j \cdots [f_{i_m}]^j$, $\overline{\eta}^{(j)}_{l i_1 \dots i_m}$ is a sum of images of basic Witt differentials $\eta^{(j_t)}_{d i_1 \dots i_m}$ in variables $[T_1], \dots, [T_r]$, $(T_i \to t_i)$ and weights $k^{(j_t)}_{l i_1 \dots i_m}$ satisfying

i) $j + |k^{(j_t)}_{l i_1 \dots i_m}| \leq C(\mathrm{ord}_p\, \eta^{(j_t)}_{l i_1 \dots i_m} + 1)$

ii) $l \geq \mathrm{ord}_p\, \eta^{(j_t)}_{l i_1 \dots i_m} \geq l - 1$.

Notation: We say that $M_l^{i_1 \dots i_r}$ has degree $\leq C(l+1)$.

We shall construct a cochain τ so that $\partial \tau = \sigma$. The reduced complex

$$C^\bullet / \mathrm{Fil}^n C^\bullet = C^\bullet(\{D(f_i)\}_i, W_n \Omega^\bullet_{A/k})$$

is exact. We will inductively construct a sequence of cochains

$$\tau_k = \sum_{1 \leq i_1 < \cdots < i_{m-1} \leq n} \tau_{k i_1 \dots i_{m-1}}$$

such that the sum

$$\sum_{k=0}^{\infty} \tau_k$$

converges in C^{m-1} to a coboundary of σ. The τ_k are chosen to satisfy the following properties:

(1) $\partial(\sum_{k=0}^{l-1} \tau_k) = \sigma$ modulo $Fil^{2^l - 1} C^m$

(2) $\tau_{0 i_1 \dots i_{m-1}} \in W^\dagger \Omega_{A_{f_{i_1} \cdots f_{i_m}}/k}$, for $k \geq 1$ $\tau_{k i_1 \dots i_{m-1}} \in Fil^{2^k - 1} W^\dagger \Omega_{A_{f_{i_1} \cdots f_{i_m}}/k}$

(3) $\tau_{ki_1\ldots i_{m-1}} \in W\Omega^{fin}_{A/k}\left[[f_1],\ldots,[f_n],r_1,\ldots,r_n,\frac{1}{[f_{i_1\cdots i_{m-1}}]}\right]$ to be understood as a polynomial in the "variables" $[f_1],\ldots,[f_n],r_1,\ldots,r_n,\frac{1}{[f_{i_1\cdots i_{m-1}}]}$ with the coefficients being finite sums of elementary Witt differentials in $[t_1],\ldots,[t_r]$ such that the total degree (with $[t_1],\ldots,[t_r]$ contributing to the degree via possibly fractional weights) is bounded by $24nC2^k$. We write degree $\tau_{ki_1\ldots i_{m-1}} \leq 24nC2^k$.

(4) $[f_{i_\alpha}]^{C2^{k+1}}\tau_{ki_1\ldots i_{m-1}} \in W\Omega^{fin}_{A/k}\left[[f_1],\ldots,[f_n],r_1,\ldots,r_n,\frac{1}{[f_{i_1\cdots\widehat{i_\alpha}\cdots i_{m-1}}]}\right]$ with degree $[f_{i_\alpha}]^{C2^{k+1}}\tau_{ki_1\ldots i_{m-1}} \leq C2^{k+1} + 24nC2^k$.

Then (2) implies that all the coefficients η of the polynomial representation (3) satisfy $\mathrm{ord}_p\,\eta \geq 2^k - 1$. Also (1) implies that $\partial(\sum_{k=0}^{\infty}\tau_k) = \sigma$. Using (2) and (3) we will show that $\sum_{k=0}^{\infty}\tau_k \in C^{m-1}$, i.e. is overconvergent.

Define elements $\sigma_{si_1,\ldots,i_m} \in W\Omega^d_{A_{f_{i_1}\cdots f_{i_m}}}$ for $n \geq 0$ by

$$\sigma_{si_1,\ldots,i_m} = \sum_{\alpha=0}^{2^{s+1}-1} M_\alpha^{i_1\ldots i_m}.$$

Then $\sigma_{si_1,\ldots,i_m} \equiv \sigma_{i_1,\ldots,i_m} \mod Fil^{2^{s+1}}$ and degree $\sigma_{si_1,\ldots,i_m} \leq C2^{2+1}$.

Define the cochain $\tau_0 \in C^{m-1}$ by

$$\tau_{0i_1\ldots i_{m-1}} = \sum_{i=1}^{n}\alpha_{i,2C}[f_i]^{2C}\sigma_{0i_1,\ldots,i_m i}.$$

Suppose we have constructed, for some integer $s > 0$, cochains $\tau_k \in C^{m-1}$ for $0 \leq k < s$ satisfying $(1)-(4)$. Then we construct τ_s as follows: Let $\gamma_{si_1\ldots i_m} = \sigma_{si_1,\ldots,i_m} - \partial(\sum_{k=0}^{s-1}\tau_k)_{i_1\ldots i_m}$. We see that $\gamma_{si_1\ldots i_m} \in Fil^{2^s-1}C^m$ is a cocycle modulo $Fil^{2^s+1}C^m$ and degree $\gamma_{si_1\ldots i_m} \leq 24nC2^{s-1}$.

Define

$$\tau_{si_1\ldots i_{m-1}} = \sum_{i=1}^{n} Q_{i,C2^{s+1}}[f_i]^{C2^{s+1}}\gamma_{si_1\ldots i_{m-1}i}.$$

Then $\sum_{k=0}^{s}\tau_k$ satisfies (1) by ([6], 1.2.4.). We have

$$\begin{aligned}[f_i]^{C2^{s+1}}\gamma_{si_1\ldots i_{m-1}i} &\in W^\dagger\Omega_{A_{f_{i_1}\cdots f_{i_{m-1}}}} \cap Fil^{2^s-1}W^\dagger\Omega_{f_{i_1}\cdots f_{i_m}/k} \\ &= Fil^{2^s-1}W^\dagger\Omega_{A_{f_{i_1}\cdots f_{i_{m-1}}}}\end{aligned}$$

and therefore $\tau_{si_1\ldots i_{m-1}}$ satisfies (2) (we have used (4) for $\tau_k, k < s$). Moreover,

$\tau_{si_1\dots i_{m-1}}$ has total degree bounded by

$$24nC2^{s-1} + 3nC2^{s+1} + C2^{s+1} \leq 24nC2^s$$

and τ_s satisfies (3). It is straightforward to show property (4) for τ_s. Therefore it remains to show that $\sum_{k=0}^{\infty} \tau_k$ is overconvergent. This will be derived from properties (2) and (3) as follows.

It follows from (3) that $\tau_{si_1\dots i_{m-1}}$ can be written as a finite sum $\tau_{si_1\dots i_{m-1}} = \sum_I r^I M_{s,I}$, where I runs through a finite set of multi-indices in $\mathbb{N}_0^n$, $r^I = r_1^{\lambda_1} \cdots r_n^{\lambda_n}$ for $I = (\lambda_1, \dots, \lambda_n)$ and $M_{s,I}$ is a finite sum of images of basic Witt differentials ω_s^t in variables $[T_1], \dots, [T_r], [Y_1], \dots, [Y_n], [Z]$ with

$$[T_j] \mapsto [t_j],\ [Y_j] \mapsto [f_j],\ [Z] \mapsto \prod_{j=1}^{m-1} \frac{1}{[f_{i_j}]}$$

with weights k_s^t satisfying

$$|I| + |k_s^t| \leq 24nC2^s = C'2^s$$

$(C' := 24nC)$ and

$$(*) \qquad \begin{aligned} \operatorname{ord}_p \omega_s^t \geq 2^s - 1 &= \frac{1}{C'}(C'2^s) - 1 \\ &\geq \frac{1}{C'}(|I| + |k_s^t|) - 1. \end{aligned}$$

For fixed I and varying s we get a sum

$$\sum_s r^I M_{s,I} = r^I \sum_s M_{s,I}.$$

Because of the condition (*), $\omega_I = \sum\limits_s M_{s,I}$ is overconvergent with radius of convergence $\varepsilon = \frac{1}{C'}$ and

$$\hat{\gamma}_{\frac{1}{C'}}(\omega_I) \geq \frac{1}{C'}|I| - 1.$$

Here $\hat{\gamma}_\varepsilon$ is the quotient norm of the canonical Gauss norm γ_ε on $W\Omega_{k[T_1,\dots,T_r,Y_1,\dots,Y_n,Z]/k}$.

We now look again at the definition of r_i. There exist liftings $\widetilde{\eta}, \widetilde{r}_i$ of η, r_i in $W^\dagger(S)$ and $\widetilde{a}_i$ of a_i in S where $\widetilde{\eta}$ is a finite sum of homogeneous elements such that

$$\widetilde{r}_i = (1 + {}^V\widetilde{\eta})^{-1}[\widetilde{a}_i].$$

For $\delta := \frac{1}{C'}$, there exists $\varepsilon > 0$, $\quad \frac{1}{C'} > \varepsilon$ such that

$$\breve{\gamma}_\varepsilon \left({}^V\widetilde{\eta}\right) \geq -\delta,$$

because we have a finite sum of homogeneous elements. By [3] Lemma 2.25,

$$\breve{\gamma}_\varepsilon(\widetilde{r}_i) \geq -\delta \text{ as well.}$$

Let $\widetilde{\omega}_I$ be a lifting of ω_I in $W^\dagger\Omega_{k[T_1,\ldots,T_r,Y_1,\ldots,Y_n,Z]/k}$ such that $\hat{\gamma}_\varepsilon(\omega_I) = \gamma_\varepsilon(\widetilde{\omega}_I)$. Then we obtain by Corollary 0.16 in [4],

$$\begin{aligned}
\hat{\gamma}_\varepsilon(r^I\omega_I) &\geq \gamma_\varepsilon(\widetilde{r}^I\widetilde{\omega}_I) \\
&\geq \gamma_\varepsilon(\widetilde{\omega}_I) + \breve{\gamma}_\varepsilon(\widetilde{r}^I) \\
&= \hat{\gamma}_\varepsilon(\omega_I) + \breve{\gamma}_\varepsilon(\widetilde{r}^I) \\
&\geq \hat{\gamma}_{\frac{1}{C'}}(\omega_I) + \breve{\gamma}_\varepsilon(\widetilde{r}^I) \\
&\geq \delta|I| - 1 + |I|(-\delta) = -1.
\end{aligned}$$

As this holds for all I, we see that $\sum_{s=0}^{\infty} \tau_{si_1\ldots i_{m-1}}$ is overconvergent with radius of convergence ε, and hence Proposition B.0.5 follows, and so does Proposition B.0.1. $\square$

Corollary B.0.6. *The complex $W^\dagger\Omega_{\mathrm{Spec}\,A/k}$, defined for each affine scheme as above, extends to a complex of Zariski sheaves $W^\dagger\Omega^d_{X/k}$ on any variety X/k.*

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