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by

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Abstract

A continuous function, such as an intelligence wave, can be simulated by various types of step approximations. In this paper a simple type of such a step approximation of a sinusoidal wave is subjected to a spectrum analysis. The results show to what extent the approximated function differs from the original function, and how this difference can be reduced.

ANALYSIS OF STEP APPROXIMATION TO A CONTINUOUS FUNCTION

It is occasionally convenient or necessary in practice to know the result of replacing a given continuous function, such as the modulating wave of a transmitter, by a "staircase" approximation. An example of a function thus approximated is shown in Fig. 1. Such a function occurs in certain pulse-amplitude modulation

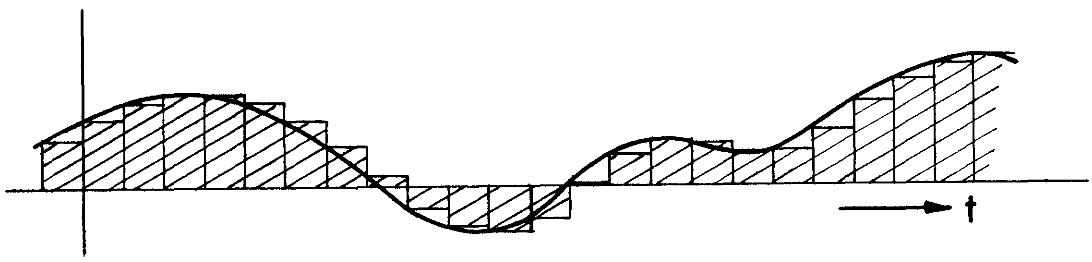


Figure 1. Step-approximated continuous function.

decoders and may be useful in fundamental analyses of modulation processes.

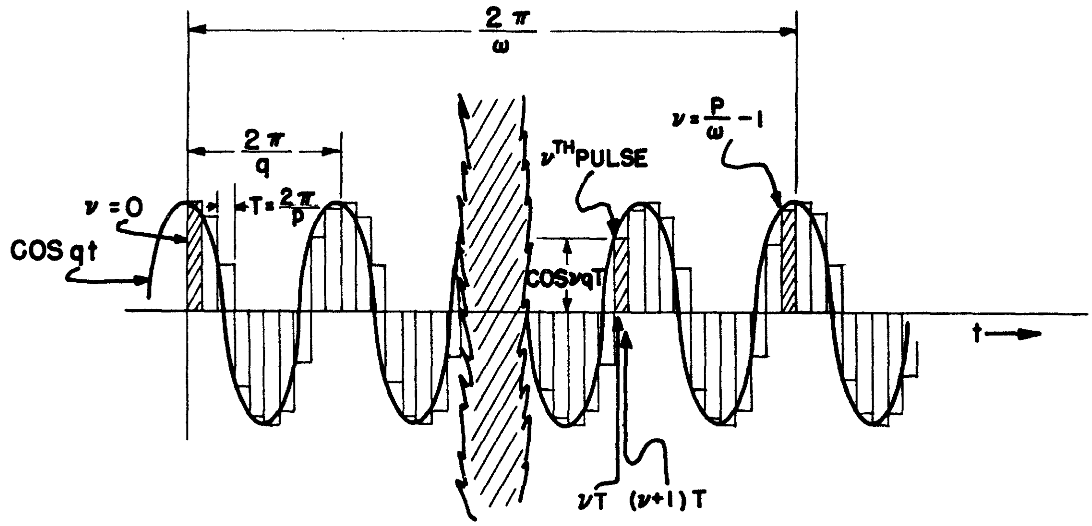
This problem is perhaps the most basic of the many which arise whenever a function is sampled at discrete intervals. An understanding of this simple case is of considerable importance in the analysis of more complex problems along similar lines.

The following analysis is aimed at determining the components of such a wave and their behavior as a function of signal-to-sample frequency ratio. The step-approximated function consists of a series of square pulses, separated by zero time intervals. In the case to be analyzed the height of each pulse is equal to the signal value at the instant at which that pulse begins. There are, of course, various other possibilities. The pulses are of equal width, which we shall designate by T, corresponding to a fundamental radian frequency p. The continuous function, i.e., the signal, will be represented by a cosine wave of radian frequency q. The use of a cosine wave of fixed phase rather than variable phase is shown not to detract from the generality of the desired results (see Appendix).

The step-approximated wave is not generally periodic in the audio frequency q, since q is not in general a factor of the sampling frequency p. There is, however, a frequency ω which is the highest common factor of p and q, and the wave is periodic in this frequency[1]. While this statement is not necessarily true mathematically speaking, since p and q may differ by an irrational or transcendental number, it is always true to a degree of precision greater than any degree of precision that can be specified.

1. This method of reasoning is used in Appendix III of Fredendall, Schlesinger, and Schroeder, "Transmission of Sound on the Picture Carrier", Proc. I.R.E., 34, pp. 49-61, February 1946.

A Fourier analysis will be carried out over the period $\frac{2\pi}{\omega}$, the fundamental repetition period. There are a total of $\frac{p}{\omega}$ pulses in this period. It is a simple matter to find the Fourier series for the νth pulse, say, since it is of the same height every cycle. This height is equal to $\cos(2\pi \frac{q}{p} \nu)$ or $\cos \nu qT$, if $\nu = 0$ and $\nu = \frac{p}{\omega} - 1$ are the index numbers, respectively, of the first and last pulses in the integration period. We shall therefore carry out a Fourier analysis of the νth pulse, having a width T, a height $\cos \nu qT$, an angular repetition frequency ω, and the result will then be summed over all integral values of ν extending from zero to $\frac{p}{\omega} - 1$, or $\frac{2\pi}{\omega T} - 1$. Figure 2 shows a summary of the ideas presented in the preceding paragraphs.



Signal frequency	$= q$
Sampling frequency	$= p$
Sampling period $= \frac{2\pi}{p}$	$= T$
Frequency of wave	$= \omega$
Pulse index number	$= \nu$
Number of pulses per cycle	$= \frac{p}{\omega} = \frac{2\pi}{\omega T}$
Height of νth pulse	$= \cos \nu qT$

Figure 2. Step-approximated cosine wave and notation used.

If the exponential form of Fourier analysis is used, the complex Fourier coefficient for the νth pulse chain is given by

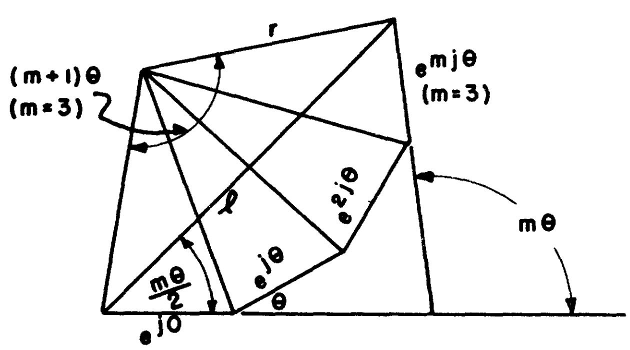
$$a_n = \frac{\omega}{2\pi} \int_{\nu T}^{(\nu+1)T} \left[\cos \nu qT\right] e^{-jn\omega t} dt \tag{1}$$

$$\begin{aligned}
a_n &= \frac{1}{2\pi n} \left[\cos \nu qT\right] \left[e^{-jn\omega(\nu+1)T} - e^{-jn\omega\nu T}\right] \\
&= \frac{1}{2\pi n} \left[\cos \nu qT\right] \left[\sin n\omega(\nu+1)T - \sin n\omega\nu T + j\cos n\omega(\nu+1)T - j\cos n\omega\nu T\right] \\
&= \frac{1}{2\pi n} \left[\cos \nu qT\right] \Big[\sin(n\omega T\nu)\cos(n\omega T) + \cos(n\omega T\nu)\sin(n\omega T) - \sin(n\omega T\nu) \\
&\qquad + j\cos(n\omega T\nu)\cos(n\omega T) - j\sin(n\omega T\nu)\sin(n\omega T) - j\cos(n\omega T\nu)\Big] \\
&= \frac{1}{4\pi n}\Big\{ \left[\cos n\omega T\right]\left[\sin(n\omega T+qT)\nu + \sin(n\omega T - qT)\nu\right] \\
&\qquad + \left[\sin n\omega T\right]\left[\cos(n\omega T+qT)\nu + \cos(n\omega T - qT)\nu\right] \\
&\qquad - \left[\sin(n\omega T + qT)\nu + \sin(n\omega T - qT)\nu\right] \\
&\qquad + j\left[\cos n\omega T\right]\left[\cos(n\omega T + qT)\nu + \cos(n\omega T - qT)\nu\right] \\
&\qquad - j\left[\sin n\omega T\right]\left[\sin(n\omega T + qT)\nu + \sin(n\omega T - qT)\nu\right] \\
&\qquad - j\left[\cos(n\omega T + qT)\nu + \cos(n\omega T - qT)\nu\right]\Big\}.
\end{aligned} \tag{2}$$

Next this expression must be summed over ν. For this purpose we make use of the following relations, which are derived by trigonometry from Fig. 3, as outlined in the three steps indicated there.

$$\sum_{\nu=0}^{m} \cos \nu\theta = \cos\frac{m\theta}{2} \sin (m+1)\frac{\theta}{2} \operatorname{cosec}\frac{\theta}{2}$$

$$\sum_{\nu=0}^{m} \sin \nu\theta = \sin\frac{m\theta}{2} \sin (m+1)\frac{\theta}{2} \operatorname{cosec}\frac{\theta}{2}$$



1) $r = \cos\frac{\theta}{2}/\sin\theta$

2) $\ell = r\sin(m+1)\theta/\cos(m+1)\frac{\theta}{2}$

3) (a) $\sum_{\nu=0}^{m} \cos\nu\theta = \ell\cos\frac{m\theta}{2}$

(b) $\sum_{\nu=0}^{m} \sin\nu\theta = \ell\sin\frac{m\theta}{2}$

Figure 3. Diagram for the summation of cos $\nu\theta$ and sin $\nu\theta$ from $\nu = 0$ to $\nu = m$.

We shall now form the complex Fourier coefficient $\alpha_n = \sum_{\nu=0}^{\nu=\frac{2\pi}{\omega T}-1} a_n$ for the entire wave. To do this, the summation sign is put in front of Equation (2) and the above identities substituted, letting $\theta = n\omega T \mp qT$, and $m = \frac{2\pi}{\omega T} - 1$.

$$\alpha_n = \frac{1}{4\pi m}\begin{bmatrix}\cos n\omega T - 1 \\ - j\sin n\omega T\end{bmatrix}\begin{bmatrix}\sin\frac{1}{2}(\frac{2\pi}{\omega T})(n\omega T + qT)\sin\frac{1}{2}(\frac{2\pi}{\omega T} - 1)(n\omega T + qT)\operatorname{cosec}\frac{1}{2}(n\omega T + qT) \\ +\sin\frac{1}{2}(\frac{2\pi}{\omega T})(n\omega T - qT)\sin\frac{1}{2}(\frac{2\pi}{\omega T} - 1)(n\omega T - qT)\operatorname{cosec}\frac{1}{2}(n\omega T - qT)\end{bmatrix}$$
$$+ \frac{1}{4\pi m}\begin{bmatrix}\sin n\omega T \\ +j\cos n\omega T \\ - j\end{bmatrix}\begin{bmatrix}\sin\frac{1}{2}(\frac{2\pi}{\omega T})(n\omega T + qT)\cos\frac{1}{2}(\frac{2\pi}{\omega T} - 1)(n\omega T + qT)\operatorname{cosec}\frac{1}{2}(n\omega T + qT) \\ +\sin\frac{1}{2}(\frac{2\pi}{\omega T})(n\omega T - qT)\cos\frac{1}{2}(\frac{2\pi}{\omega T} - 1)(n\omega T - qT)\operatorname{cosec}\frac{1}{2}(n\omega T - qT)\end{bmatrix}. \tag{3}$$

The factor $\sin\frac{1}{2}(\frac{2\pi}{\omega T})(n\omega T \mp qT)$, which is present in each of the four members of Equation (3), can be rewritten as $\sin\pi(n \mp \frac{q}{\omega})$. But $\frac{q}{\omega}$ is an integer by definition of ω, so that the quantity is always zero. Hence the only way in which a non-zero solution is possible is to have $\operatorname{cosec}(n\omega T \mp qT) = \infty$. Therefore we must have

$$\begin{aligned} n\omega T \mp qT &= 2\pi N, \quad N \text{ an integer} \\ \frac{2\pi n\omega}{p} \mp \frac{2\pi q}{p} &= 2\pi N \\ n\omega &= Np \pm q. \end{aligned} \tag{4}$$

Substitution of (4) in (3) would result in an indeterminate expression, which can be evaluated by L'Hospital's rule. The indeterminate part of the first half of (3) becomes

$$\lim_{x\to n} \frac{\sin\pi\left[x + \frac{q}{\omega}\right]\sin\pi\left[x + \frac{q}{\omega} - \overbrace{x\frac{\omega}{p} \pm \frac{q}{p}}^{N}\right]}{\sin\pi\left[\underbrace{x\frac{\omega}{p} \mp \frac{q}{p}}_{N}\right]} = \frac{\pi\left[1 - \frac{\omega}{q}\right]\sin(k_1\pi)\cos(k_2\pi) + \pi\cos(k_1\pi)\sin(k_2\pi)}{\frac{\pi\omega}{p}\cos(k_3\pi)} = 0,$$

where k_1 and k_2 are integers. For the second part of Equation (3)

$$\lim_{x\to n} \frac{\sin\pi\left[x + \frac{q}{\omega}\right]\cos\pi\left[x + \frac{q}{\omega} - \overbrace{x\frac{\omega}{q} \pm \frac{q}{p}}^{N}\right]}{\sin\pi\left[\underbrace{x\frac{\omega}{q} \mp \frac{q}{p}}_{N}\right]} = \frac{\pi\left[1 - \frac{\omega}{q}\right]\sin(k_1\pi)\cos(k_2\pi) + \pi\cos\pi(n + \frac{q}{\omega})\cos\pi(n + \frac{q}{\omega} - N)}{\frac{\pi\omega}{q}\cos\pi N}$$

$$= \frac{(-1)^N}{(-1)^N} = \frac{\pi}{\pi\frac{\omega}{p}} = \frac{p}{\omega} = \frac{np}{n\omega} = \frac{np}{Np \pm q} = \frac{n}{N \pm q}. \tag{5}$$

The first half of Equation (3) is therefore zero, and the equation reduces to

$$\alpha_n = \frac{1}{4\pi}\left[\sin n\omega T + j\cos n\omega T - j\right]\frac{1}{N \pm \frac{q}{p}} \tag{6}$$

Substituting $\quad n\omega T = 2\pi N \pm \frac{2\pi q}{p}$,

one obtains

$$\alpha_{(Np \pm q)} = \frac{1}{4\pi(N \pm \frac{q}{p})}\left[\pm\sin(\frac{2\pi q}{p}) + j\cos(\frac{2\pi q}{p}) - j\right]. \tag{7}$$

If the function is expressed in the form

$$f(t) = \sum \left[A_n \cos n\omega t + B_n \sin n\omega t \right]$$

then

$$A_n = 2\,\mathrm{Re}(\alpha_n)$$

and

$$B_n = -2\,\mathrm{Im}(\alpha_n).$$

We therefore have for our final result[2]

$$f(t) = \sum_{N=0}^{\infty} \frac{1}{2\pi(N \pm \frac{q}{p})} \left\{ \left[1 - \cos(\tfrac{2\pi q}{p})\right] \sin(Np \pm q)t \pm \left[\sin(\tfrac{2\pi q}{p})\right] \cos(Np \pm q)t \right\}. \quad (8)$$

The step-approximated wave is seen to contain components of the intelligence frequency, as well as all harmonics of the sampling frequency plus and minus the intelligence frequency. The simplest way of checking Equation (8) is to let the signal-to-sampling frequency ratio approach zero. In that case the expression reduces to cos qt, as it should. For $\frac{q}{p} < 0.05$, we have, with approximately one per cent accuracy, for the intelligence component

$$f(t) = \left[1 - \frac{1}{6}(\frac{2\pi q}{p})^2\right] \cos qt + \left[\frac{\pi q}{p}\right] \sin qt. \quad (9)$$

Equation (9) shows that, while the amplitude stays practically constant, the phase is retarded as $\frac{q}{p}$ increases. For small values of $\frac{q}{p}$, the resulting angle of lag is approximately $\tan^{-1}(\frac{\pi q}{p})$.

Before interpreting the general results given by Eq. (8), it is of interest to examine certain degenerate cases. These will provide a further check on Eq. (8) and will help clarify its use in special cases. In these special cases, the results are not independent of the signal-wave phase (see Appendix).

Let us consider, first, the case of $\frac{q}{p} = \frac{1}{2}$, which is of special interest, since $\frac{1}{2}$ is the highest practical frequency ratio. The corresponding time function is shown in Fig. 4. It is an ordinary square wave of fundamental angular frequency q, and its Fourier series is

$$\sum_{n=1,3,5,\ldots}^{\infty} \frac{4}{\pi n} \sin nqt\,. \quad (10)$$

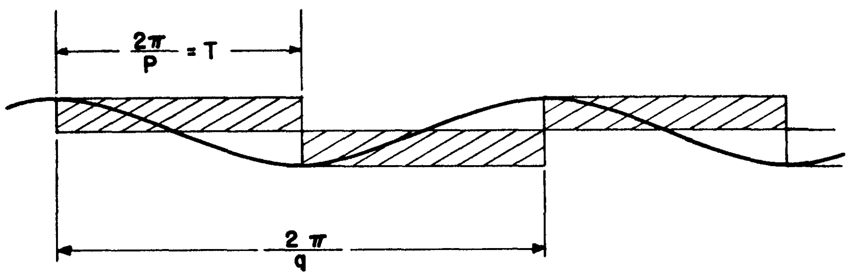


Figure 4. Degenerate case of step-approximated cosine wave. Signal-to-sampling frequency ratio $\frac{q}{p} = \frac{1}{2}$.

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2. By making the phase relationship between signal and sample variable, a more general result is obtained. This is given in the Appendix, where it is shown that Eq. (8) is nevertheless sufficiently general for present purposes since the magnitudes of the individual components are independent of this phase relationship.

By substitution of the condition $\frac{q}{p} = \frac{1}{2}$ into Equation (8), we obtain for the various components

$$A_q = \frac{2}{\pi}\sin qt, \quad A_{p+q} = \frac{2}{3\pi}\sin 3qt, \quad A_{2p+q} = \frac{2}{5\pi}\sin 5qt, \cdots$$

$$A_{p-q} = \frac{2}{\pi}\sin qt, \quad A_{2p-q} = \frac{2}{3\pi}\sin 3qt, \quad A_{2p-q} = \frac{2}{5\pi}\sin 5qt, \cdots$$

so that

$$f(t) = \frac{4}{\pi}\sin qt + \frac{4}{3\pi}\sin 3qt + \frac{4}{5\pi}\sin 5qt + \ldots,$$ which is identical with (10).

An example of $\frac{q}{p} > 1$ is given by the degenerate case $\frac{q}{p} = \frac{3}{2}$, for which we have a square wave as in Fig. 4, except with three times as large a period. The fundamental component is then of frequency $\frac{q}{3}$, its amplitude $\frac{4}{\pi}$ being given by $A_{p-q} + A_{p-2q}$. The third harmonic of the square wave, $\frac{4}{3\pi}\sin qt$, is given by $A_q + A_{3p-q}$. Recalling that this third harmonic is the intelligence, one readily sees that this case is of academic interest only. The two cases examined have the common property that two intermodulation components of equal amplitude always coincide in frequency to form a single harmonic.

Another special case of interest is that for which $\frac{q}{p} = \frac{1}{4}$. The corresponding function is shown in Fig. 5. In this case, as in the case shown in Fig. 4,

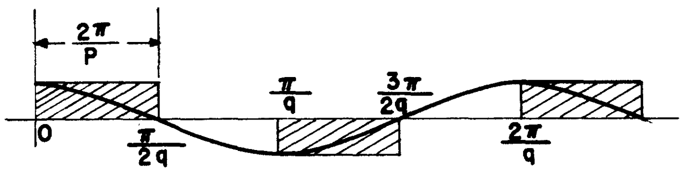


Figure 5. Degenerate case of step-approximated cosine wave. Signal-to-sampling frequency ratio $\frac{q}{p} = \frac{1}{4}$.

the fundamental square-wave component represents the intelligence wave, lagging it by 45 degrees rather than 90 degrees as it did when $\frac{q}{p}$ was twice as large. Either direct Fourier analysis of the function of Fig. 5 or substitution of $\frac{q}{p} = \frac{1}{4}$ in Eq. (8) yield

$$f(t) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{2}{n\pi}\left[(-1)^{\frac{n-1}{2}} \cos nqt + \sin nqt\right]. \tag{11}$$

The fundamental (signal) component is given by A_q, the third harmonic by A_{p-q}, the fifth by A_{p+q}, the seventh by A_{2p-q}, and so on.

Having checked a few special cases, we proceed to evaluate the results as given by Eq. (8) on a general basis. The major components present in the step-approximated wave have been plotted in Fig. 6 as a function of $\frac{q}{p}$, the signal-to-sampling frequency ratio. For the larger values of $\frac{q}{p}$, the component of frequency p-q is seen to give strong distortion which reaches 100 per cent when $\frac{q}{p}$ is one-half. In order to prevent this distorting component from falling within the audio band, one

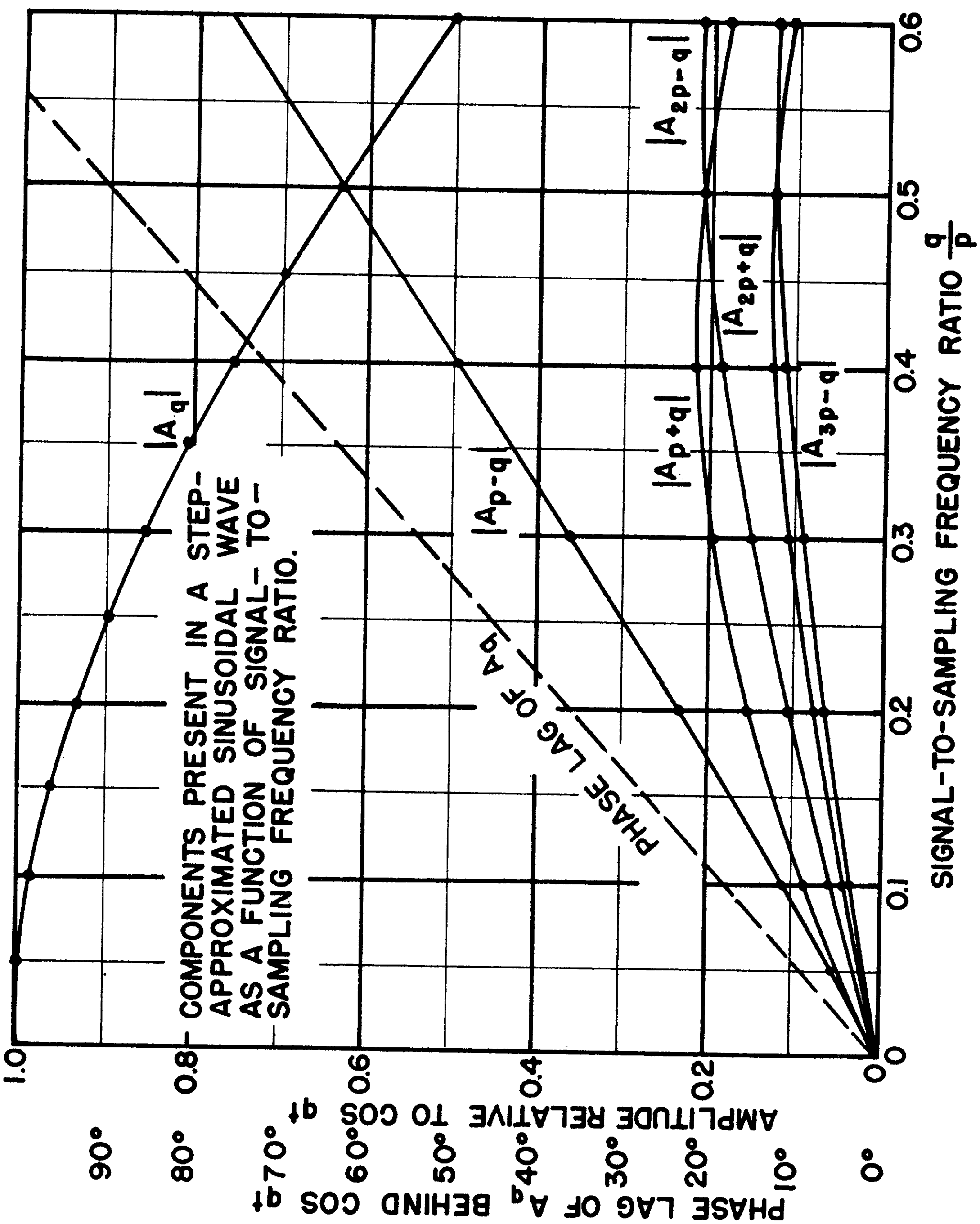
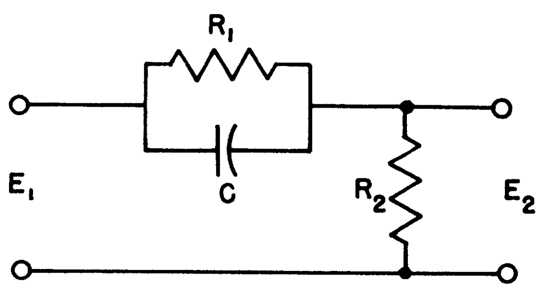

COMPONENTS PRESENT IN A STEP-APPROXIMATED SINUSOIDAL WAVE AS A FUNCTION OF SIGNAL- TO-SAMPLING FREQUENCY RATIO.
$|A_q|$
$|A_{p-q}|$
$|A_{p+q}|$
$|A_{2p-q}|$
$|A_{2p+q}|$
$|A_{3p-q}|$
PHASE LAG OF A_q
SIGNAL-TO-SAMPLING FREQUENCY RATIO $\frac{q}{p}$
AMPLITUDE RELATIVE TO COS qt
PHASE LAG OF A_q BEHIND COS qt
0
0.1
0.2
0.3
0.4
0.5
0.6
1.0
0.8
0.6
0.4
0.2
0
90°
80°
70°
60°
50°
40°
30°
20°
10°
0°

Figure 6

should make $p-q_{max} > q_{max}$, or $p > 2q_{max}$. The sampling frequency should exceed twice the highest intelligence frequency, and a low-pass filter with about 40 db attenuation at the frequency $p-q_{max}$ should precede the output in a practical case. Thus all other $A_{Np \pm q}$ $(N \neq 0)$ components are also lost, since they all exceed A_{p-q} in frequency.

It remains merely to consider the intelligence component A_q, which we should like to have equal to cosqt. Figure 6 shows, however, that the magnitude decreases from one to $\frac{2}{\pi}$ and the phase angle of lag increases linearly from zero to 90 degrees as $\frac{q}{p}$ increases from zero to one-half. Linear phase shift implies constant time delay and the only defect of the approximated function is therefore the drop in amplitude at high intelligence frequencies.

In practice it may be desirable to correct the amplitude variation. A simple network which will accomplish this is shown in Fig. 7. While it reduces the amplitude drop of over 30 per cent to a maximum amplitude deviation (from one) of 5 per cent, it unfortunately has a non-linear phase characteristic (see Fig. 8). The values of α



$$\tau = R_1 C = 2.38$$

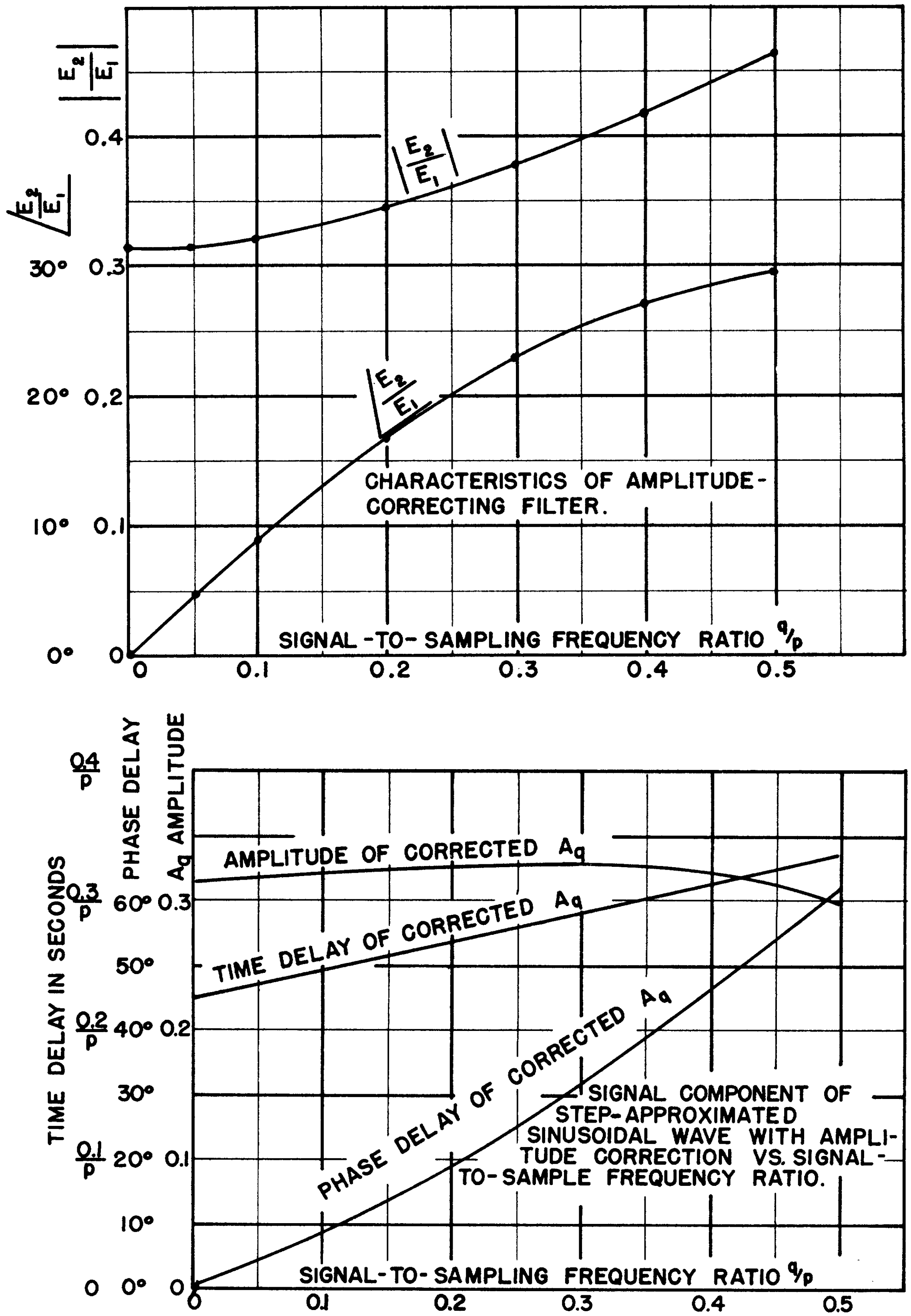
$$\alpha = \frac{R_2}{R_1 + R_2} = 0.316$$

$$\frac{E_2}{E_1} = \frac{\alpha(1 + j\frac{q}{p}\tau)}{1 + j\frac{q}{p}\alpha\tau} = \frac{0.316(1 + j2.38\frac{q}{p})}{1 + j0.752\frac{q}{p}}$$

Figure 7. Amplitude-correcting network.

and τ have been chosen to give the most nearly constant resultant amplitude response over the range of $\frac{q}{p}$ from zero to one-half. The transfer characteristics of the network as well as the resulting behavior of the intelligence component are shown in Fig. 8. When the filter is used, the time delay, instead of being constant, increases by a total of about $\frac{0.1}{p}$ seconds as $\frac{q}{p}$ increases from zero to one-half.

Thus, in cases where phase is not a primary factor (e.g., in audio work), the intelligence can be restored virtually to its original value, as long as $\frac{q_{max}}{p} < \frac{1}{2}$. If exact linearity of phase shift is essential, however, an amplitude drop of over 3 db must be accepted at the top intelligence frequencies, unless a more suitable network can be designed.



CHARACTERISTICS OF AMPLITUDE-CORRECTING FILTER.
SIGNAL-TO-SAMPLING FREQUENCY RATIO q/p
AMPLITUDE OF CORRECTED A_q
TIME DELAY OF CORRECTED A_q
PHASE DELAY OF CORRECTED A_q
SIGNAL COMPONENT OF STEP-APPROXIMATED SINUSOIDAL WAVE WITH AMPLITUDE CORRECTION VS. SIGNAL-TO-SAMPLE FREQUENCY RATIO.
TIME DELAY IN SECONDS
PHASE DELAY
A_q AMPLITUDE

Figure 8

Appendix

The analysis presented in the preceding text is limited inasmuch as the signal wave is a fixed cosine wave, so that sampling (at time $t = 0$) always begins at the peak of the wave. A question therefore arises as to how much generality is lost by this simplification.

In general, the sampling frequency is not an integral multiple of the signal frequency, and it would seem in that case that the relative phase at zero time between signal and sampling is immaterial, since it is continuously changing and passing through all possible positions.

In the special cases where the sampling frequency is a multiple of the signal frequency, the situation is quite different. The phase, amplitude, and even the type of the resultant wave may depend on the phase of the signal wave at which sampling begins. This fact can readily be checked by redrawing Fig. 4 or 5 with sine rather than cosine waves.

The result as given by Eq. (8) is therefore obviously limited to a cosine wave in the above-mentioned special cases. These cases, however, are not of interest here; firstly, they occupy a vanishingly small amount of frequency spectrum, and secondly, the resulting waves are purely periodic and can easily be analyzed by ordinary Fourier analysis.

It must be shown now, how Eq. (8) gives the desired results in the general case, in spite of the fact that it is not generally applicable in certain cases. If the entire mathematical analysis is repeated using $\cos(qt + \varphi)$ instead of $\cos qt$ as the intelligence wave, and with a sampling period still beginning at zero time as before, the following result is obtained in place of Eq. (8):

$$f(t) = \sum_{N=0}^{\infty} \frac{1}{2\pi(N \pm \frac{q}{p})} \left[\left\{ \left[1 - \cos\frac{2\pi q}{p}\right] \cos\varphi - \left[\sin\frac{2\pi q}{p}\right] \sin\varphi \right\} \sin(Np \pm q)t + \left\{ \left[\pm \sin\frac{2\pi q}{p}\right] \cos\varphi \pm \left[1 - \cos\frac{2\pi q}{p}\right] \sin\varphi \right\} \cos(Np \pm q)t \right]$$

This new result can be written as

$$f(t) = B(\varphi) \sin n\omega t + A(\varphi) \cos n\omega t$$

as contrasted to the original less general result

$$f(t) = C \cos n\omega t \ .$$

Now, it can readily be seen by carrying out the appropriate algebraic process that $\sqrt{A^2(\varphi) + B^2(\varphi)} = C$, independently of φ, so that the amplitude $|A_{Np \pm q}|$ of a given component of frequency $Np \pm q$ is totally independent of the phase angle φ. This proves that Eq. (8) is perfectly general for the general case where p is not an integral multiple of q, since in that case no two components $A_{Np \pm q}$ are alike in frequency and their relative phases are therefore immaterial.

In the special cases where p is an integral multiple of q, however, the relative phases of various components suddenly become of extreme importance. The frequencies of all components are then harmonics of a common fundamental, and several components $A_{Np \pm q}$ may even be of identical frequency; for example, for $q/p = 1/2$, A_q and A_{p-q} are components having identical frequencies. The amplitude of each of the components is still independent of the phase angle φ, but their relative phases are not. Hence the phase angle φ determines the resultant sum of the various components. It should be emphasized, however, that Eq. (8) and the plot of Fig. 6 give the amplitudes of the individual components correctly and independently of φ, even in the degenerate cases, but these amplitudes alone provide insufficient information in these cases.