CALCULATING MACHINE FOR FOURIER TRANSFORMS AND RELATED EXPRESSIONS

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TECHNICAL REPORT NO. 13
SEPTEMBER 3, 1946

RESEARCH LABORATORY OF ELECTRONICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

The research reported in this document was made possible through support extended the Massachusetts Institute of Technology, Research Laboratory of Electronics, jointly by the Army Signal Corps, the Navy Department (Office of Naval Research), and the Army Air Forces (Air Materiel Command), under the Signal Corps Contract No. W-36-039 sc-32037.

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By

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Abstract

This article describes a calculating machine for obtaining the Fourier transform, $\int f(x)\, e^{i[xy + \phi(x)]}dx$, or the sum of a Fourier series, $\sum a_n \sin(ny + b_n)$, as a continuous function of y. The method uses linear potentiometers for taking the product and the integral, while the complex exponential is generated by mechanical linkages. The machine may be adjusted to give direct readings of absolute value and phase as well as the real and imaginary parts. With minor modifications the same device also gives the convolution $\int_0^y f(x)g(x - t)dx$ as a function of t and y; the solution of simultaneous linear equations; and certain integrals containing a parameter, such as the Laplace transform. It is shown that the principles here used can be adapted to summation of complex power series or to computation of the general integral $\int f(x)\, e^{i[\phi(x) + k\rho(x)\cos(x + y)]}dx$, which often occurs in antenna work. The present report is concerned exclusively with design; the question of performance is considered in a report to be published by the Naval Research Laboratory, Washington, D. C.

Contents

Part I. Design of Machine for Specific Problem

1. Introduction. The expression

$$g(y) = \int f(x)\, e^{i[xy + \phi(x)]} dx \qquad (1)$$

frequently occurs in mathematical and engineering investigations. For various reasons however, the existing methods of evaluating it appear to be somewhat unsatisfactory in certain cases. The chief shortcoming of these methods is that they require a so-called point by point computation, in which one assigns a fixed numerical value to y, whereupon the expression becomes an ordinary integral not containing a parameter and may be computed on that basis. For each new value of y the entire operation is repeated. With such a procedure, which appears to be the only one hitherto available, one finds the cost of a complete curve to be prohibitively high. For example, a few computations of this sort carried out on the M.I.T. differential analyzer cost as much as $500 each, nor does it seem probable that the cost would be much below $50-100 even with quantity production. As far as principles are concerned, however, the evalution of such an integral on the differential analyzer presents no difficulty whatever; with y fixed, the exponential, product, and integral are readily generated, the curves f and ϕ being introduced in the usual manner. Thus, the high cost just noted is found chiefly because the computation must be repeated for each value of y. The same difficulty occurs in the use of punched card machines, which also lead to an estimated cost of not much less than $50 for each curve. The procedure here is to separate the integral into the four simpler integrals

$$\begin{Bmatrix} I_1 \\ I_2 \end{Bmatrix} = \int f \cos\phi \begin{Bmatrix} \sin xy \\ \cos xy \end{Bmatrix} dx$$
$$\begin{Bmatrix} I_3 \\ I_4 \end{Bmatrix} = \int f \sin\phi \begin{Bmatrix} \sin xy \\ \cos xy \end{Bmatrix} dx . \qquad (2)$$

These expressions suggest that one form the two products f cos ϕ, f sin ϕ and take a sine and cosine transform of each of these. From the four integrals thus obtained (each of which must be evaluated for a whole series of y's), the absolute value of the original integral is determined by the relation

$$|g(y)|^2 = (I_2 - I_3)^2 + (I_1 + I_4)^2 \qquad (3)$$

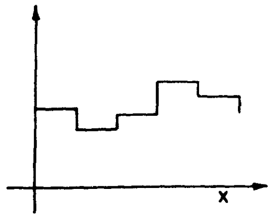
which is likewise to be computed as a function of y. Similar inconvenience is entailed by the rolling sphere method, and indeed by all the other methods that have been brought to our attention. In view of these difficulties it is believed that the device here described fills a definite need, even though the Fourier transform and allied expressions can be computed, in principle, by existing methods.

2. General Approach to the Problem. From the foregoing paragraph it is clear that much of the difficulty arises from the need for separate computation with each value of y. As one of the characteristics of the proposed machine, therefore, we shall require that this parameter be continuously variable. Such a requirement greatly restricts our choice of method, many of the design features being in fact completely determined. For example, since the value of the integral for any y depends on the functions f and ϕ over the

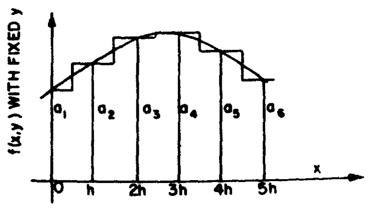
whole range of integration, it is clear that these entire curves must be introduced to the machine at the beginning of the calculation. In particular it will not suffice merely to trace the curves while computation progresses, as in the differential analyzer, or to introduce plotted points in rapid succession, as is done on punch-card machines. Now the introduction of a complete function in the manner here required presents considerable difficulty, in general, since its value must be specified at every point of the interval. On the other hand a step-function of sufficiently simple type leads to no inconvenience, and hence for such a function, at least, it appears that the design can perhaps be carried out as originally planned.

Suppose, then, that we have a device for evaluating integrals of the form $\int f(x,y)dx$ whenever the integrand, for each value of y, is of the form illustrated in Fig. 1a. Of course the curves ordinarily encountered are not of this type, but they may at least be approximated by such functions as shown in Fig. 1b., in which case the value computed, for any fixed y, is simply the trapezoidal-rule approximation to the desired integral. Instead of taking values at 0,h,2h, ... as in Fig. 1b, one could equally well take values at h/2, 3h/2, ... as shown in Fig. 1c. Such a computation is still in the form $\int f(x,y)dx$ with f(x,y) a step function of the proper type, and hence it too can be made by the hypothetical machine. But the arithmetic average of this value with the earlier one gives the result that would have been obtained with a basic interval of h/2 instead of h (Fig. 1d), and this is true for every value of y. Hence if we obtain a complete curve of $\int f(x,y)dx$ versus y by the procedure of Fig. 1b, and repeat as in Fig. 1c, then the arithmetic average of the two curves will give the curve corresponding to the finer subdivision of Fig. 1d. It is clear that this process, illustrated in Fig. 1e, could be indefinitely continued, and that the arithmetic mean of the k curves obtained by shifting a distance 0,h/k,2h/k, ... will give the single curve obtained for a subdivision h/k. Thus, if the machine be so designed that the integrand may be shifted sideways as here described, then the true integral may be approached as closely as desired even though at each step one obtains only an approximation. In the present machine there are 45 elements, that is, a single computation gives a trapezoidal-rule approximation based on 45 points of the curve. Repetition accordingly gives the result that would be obtained with 90 points, and so on. With 45 elements, a single computation is actually sufficient for many cases; the possibility of shifting, however, which is duly allowed for in the construction, shows that the restriction to step functions is not a serious disadvantage. Of course the functions considered must also be bounded, and equal to zero outside of some finite interval. Though objectionable, these latter conditions are quite difficult to avoid, and they appear to be required not only by the present machine, but by all others as well.

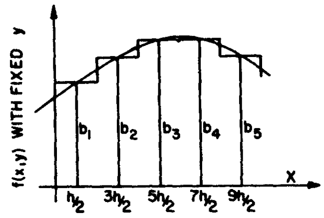
Before leaving this question of general procedure, it is of interest to compare the method here suggested with those used in similar conditions on the differential analyzer or on punch-card machines. In the former case one uses continuous variation of x but discrete values for y, so that one obtains a set of points on the desired curve rather than the whole curve; but each point so obtained is theoretically without approximation. With punch-card machines one uses discrete values for both x and y, obtaining a discrete set of values each of which entails an approximation similar to that described above. An operation



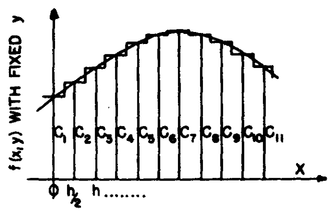
a) Step function with equal intervals.



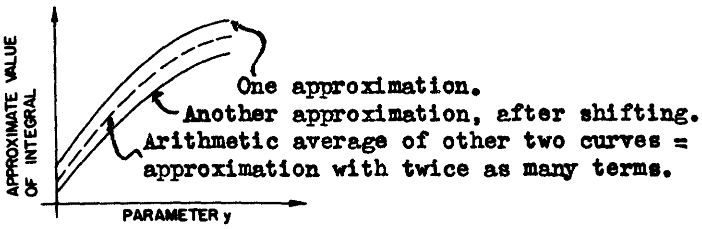
b) Approximation of an arbitrary curve by a function of type a). $\int \cong h \sum a_i$.



c) Same as b) with curve shifted. $\int \cong h \sum b_i$

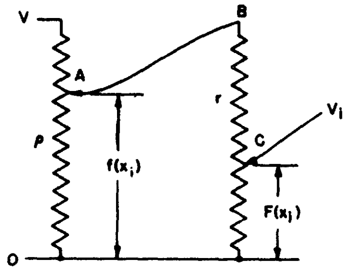


d) Result which would be obtained by adding approximation b) and c).

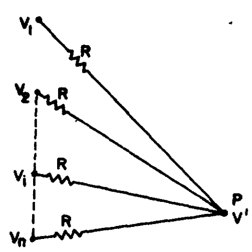


e) Application of same principle to integrals containing a parameter.

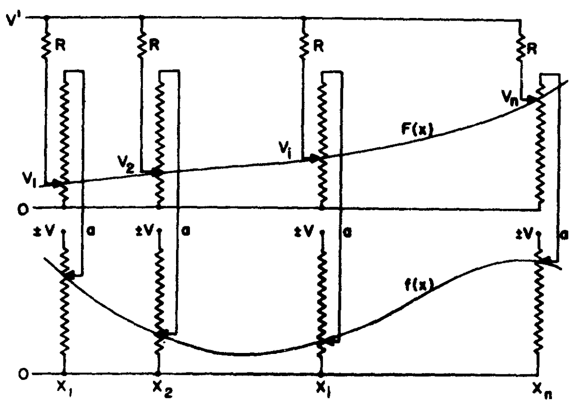
Figure 1. Approximation of an integral by a sum.



a) Circuit for giving a voltage equal to the product of two displacements.



b) Circuit for giving a voltage equal to the arithmetic average of any number of voltages.



c) Combination of a) and b) to obtain the sum of products at corresponding points on two different curves.

Figure 2. Basic circuit of machine

analogous to shifting cannot readily be carried out, but on the other hand the initial number of subdivisions can be made as large as we please by using a sufficient number of cards. In contrast to this method, where x and y assume a finite set of values, and in contrast also to the differential analyzer, where one uses discrete values for y and continuous variation of x, the proposed machine uses a discrete set of values x but continuous variation of y. Each point obtained is only approximate, but the entire curve may be found as a continuous function of y in accord with our initial requirements.

3. Basic Circuit. The real part of the desired integral (1) has the form

$$\int f(x) \cos(xy)\, dx \qquad (4)$$

whenever $\phi = 0$, and for fixed y this in turn is equivalent to the integral

$$\int f(x)\, F(x)\, dx. \qquad (5)$$

For evaluation of this latter expression, the procedure outlined above leads to three simple design problems, all of which are easily solved by a suitable d-c circuit. The first problem is to insert the values of f and F at equally spaced points on the curves. Because a large number of points must be used, it is clear that any device for this purpose should be convenient to construct and to operate; and the use of linear potentiometers is at once suggested (Fig. 2c). Equal increments of x are obtained automatically if the potentiometers are equally spaced, and hence it suffices merely to move each slider until it rests on the curve. It turns out that potentiometers wound on straight cards, which are clearly best suited to the present application, are less expensive than the circular type, and are also more accurate unless the latter are supplied with a correction cam. The accuracy is about 0.2 per cent of the full scale value. while the unit cost ranges from about $.50 to about $2.50 depending on the size. Further economy can be achieved by use of slide wires, although, on account of their low resistance, this procedure is not to be recommended. Instead of relying on the accuracy of the individual potentiometers one could of course make each setting by means of a bridge circuit, using an accurate decade box for comparison; but such a procedure introduces just the type of inconvenience which it is our object to avoid.

With the values of the two functions at x_i thus inserted into the machine, our next problem in the solution of (5) is to obtain the product at these points. A simple means of effecting this is shown in Fig. 2a, where the two potentiometers corresponding to f and F at $x = x_i$ are connected in tandem. Since the first potentiometer is linear, the voltage at point B, which is equal to that at A, is proportional to the displacement of the first slider; and this in turn is equal to $f(x_i)$. Similarly, the voltage at C is proportional to $F(x_i)$ and, moreover, it is clearly proportional to the voltage at B. Thus,

$$V_i = (\text{constant})\, f(x_i)\, F(x_i) \qquad (6)$$

as required. To complete the computation of (5), finally, we note that the procedure used here replaces the integral by a finite sum, so that we have to evaluate

$$\frac{1}{n}\sum_{1}^{n} f(x_i)\, F(x_i) \qquad (7)$$

where n is the number of elements (n = 45), and the voltages corresponding to $f(x_i)\ F(x_i)$ are available by the previous construction. This expression in turn is equal to the voltage V' in Fig. 2b, as we see by the relation

$$\sum_{1}^{n} \frac{V - V_i}{R} = 0 \qquad (8)$$

which follows from the fact that there can be no net current flow to the point P. The complete circuit[1] for approximate evaluation of (5) is shown in Fig. 2c, where input voltage on the potentiometer corresponding to x_i is to be +V if $f(x_i)\ F(x_i)$ is positive and -V if $f(x_i)\ F(x_i)$ is negative. We note that different choice of the resistance R would lead to different weightings; for example, the values R, R/2, R/4, R/2, ... would give an approximation based on Simpson's rule, and results with polynomials of higher degree could be similarly obtained. The above operation of shifting loses its merit with such a procedure, however, and there is additional disadvantage in that the error now depends on the higher derivatives. Also, non-uniform weighting leads to inconvenience when the machine is used for some of the calculations considered below; and for these reasons it was thought best to use equal values for the R's as described.

4. Sources of Error. The foregoing analysis contains a number of tacit assumptions which require additional investigation. In Fig. 2a, for example, the first potentiometer is loaded by the second, so that the voltage is not really proportional to distance as we have assumed above (see Fig. 3). Similar inaccuracy is introduced by use of a finite value for R, which affects both potentiometers to some extent; and the whole question is actually somewhat complicated, particularly when our purpose is optimum design rather than mere estimation of the error. In the first place, it is not desirable to use the entire potentiometer, even though the error is zero at each end, but instead one should so limit the range that the maximum positive and negative errors will be equal, if the most favorable proportionality constant is used. In other words the desired dependence of voltage on the distance S is taken as S (1 + tan θ), rather than as S alone, where θ (Fig. 3) and the range of S are so adjusted that the error c is equal to the common value of a and b. With this procedure, for which the author is indebted to H. Dowker, it is found that only about 7/9 of the potentiometer should be used or, which is practically the same thing, that a resistance $2\rho/7$ should be inserted at the high voltage end. The former method is followed in the present machine.

With the range determined, our next problem is to estimate the ratio of the potentiometer resistances r/ρ, R/r. Of course one could use a very high ratio (say 100:1)

1. This circuit, which was devised independently by the author in 1943, has also been used by others in problems leading to a sum of products. For solving linear equations it is described in the J. App. Phys. 17, 262 (1946), " A Computer for Solving Linear Simultaneous Equations", C. E. Berry, D. E. Wilcox, S. M. Rock, H. W. Washburn. The circuit has likewise been used for evaluating the Fourier coefficients of real functions, though the remainder of the method is different from that suggested here.
See: Rymer, T. B. and Butler, C. C., Phil. Mag., 35, 606 (1944).
Hagg, G. and Laurent, T. J. Sci. Instr., 23, No. 7 155, July 1946.

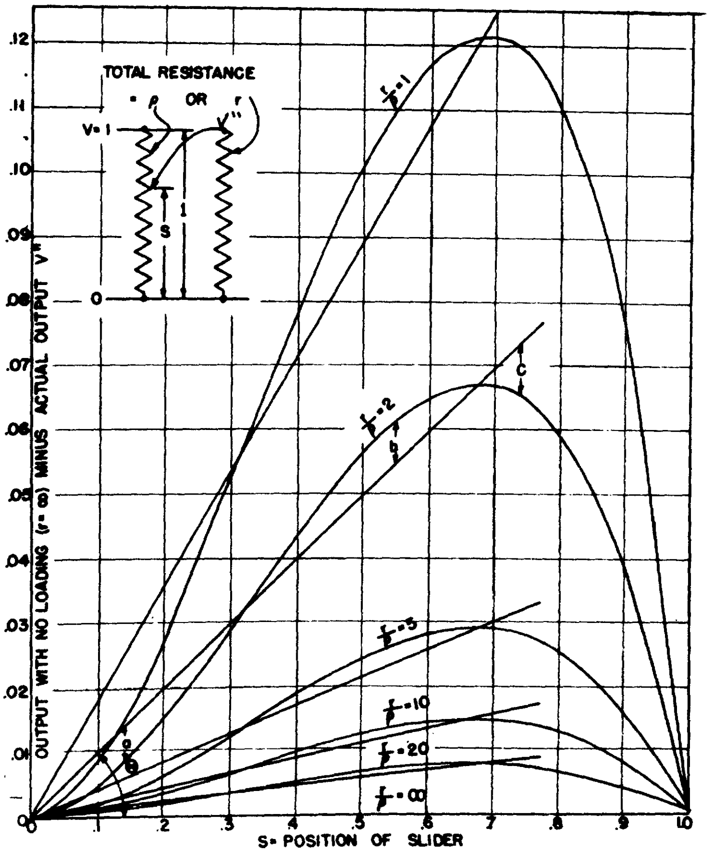


Figure 3. Voltage versus distance for a loaded potentiometer, and optimum linear approximation. In each case the straight line is drawn at such an angle θ that the maximum error **a** equals the maximum error **b**.

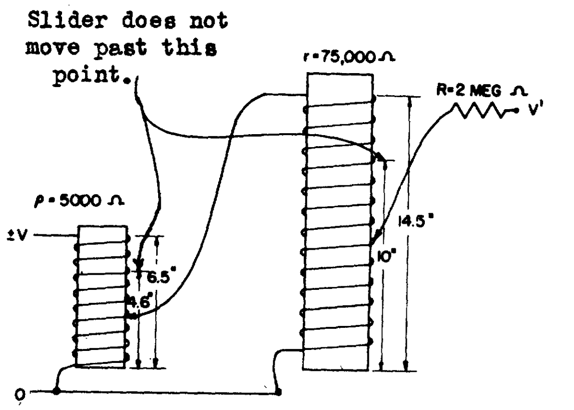


Figure 4. Dimensions and resistances used for an element of potentiometer assembly.

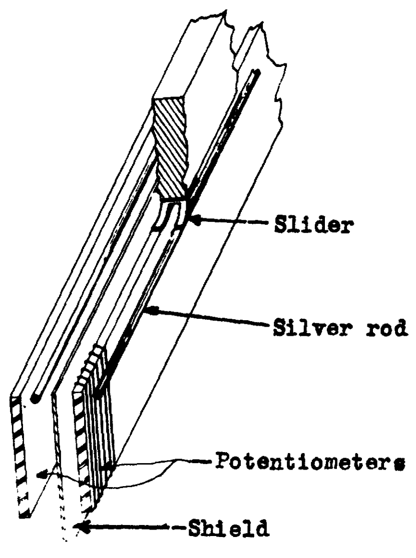


Figure 5. Detail of potentiometer, showing shield and methods of making connection to slider.

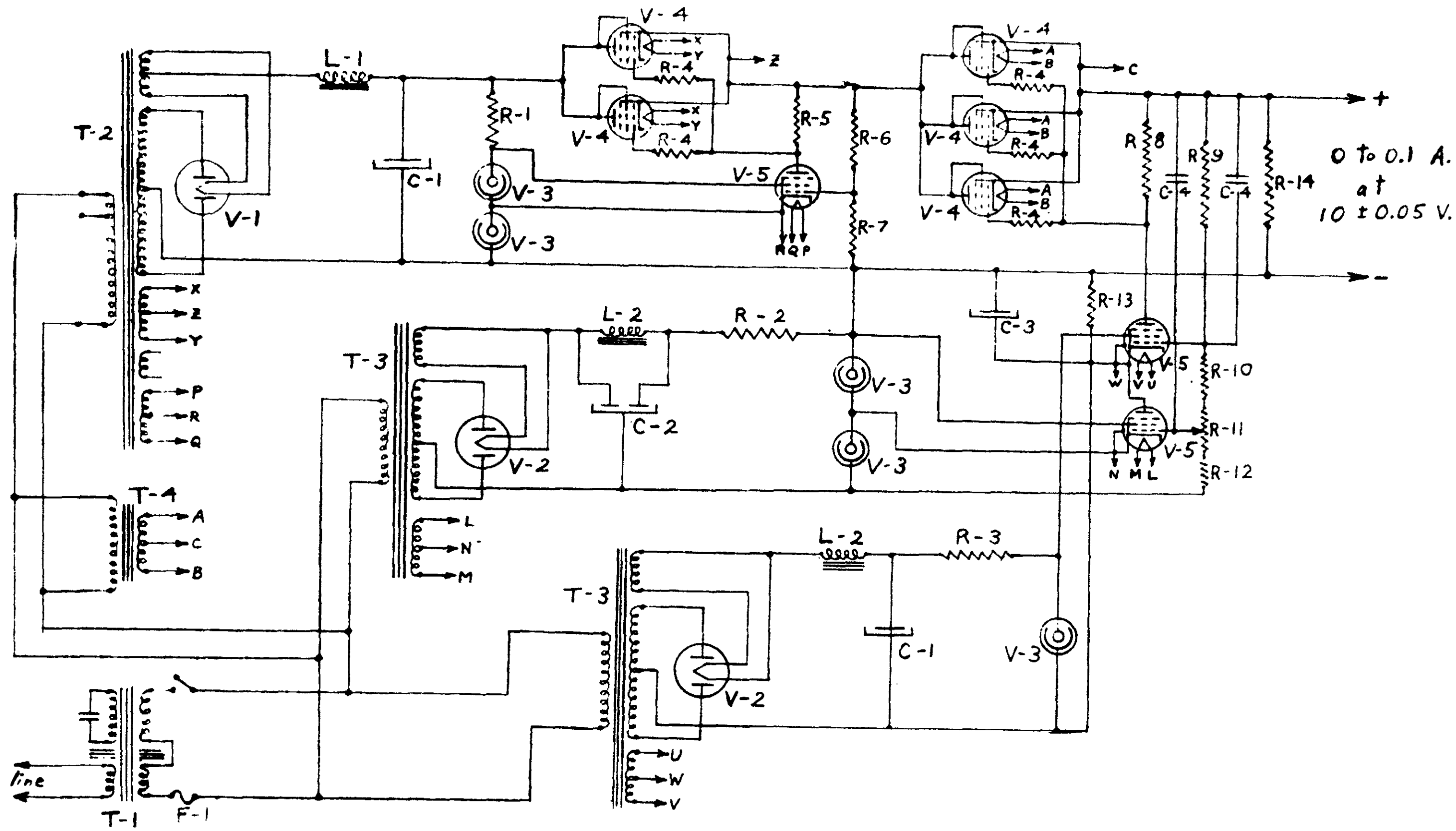
in each case, but such a procedure leads to a large output impedance and to consequent difficulty from noise. For this reason we require the minimum ratio consistent with the desired accuracy. The problem was investigated in considerable detail by H. Dowker, his procedure in estimating the error being to separate the various sources of error and to maximize each individually. The calculation was not defined to a single element (Fig. 2a), but took account of the circuit as a whole (Fig. 2c). The results finally obtained in this way show that, with the resistance ratios of Fig. 4, the error cannot exceed 1/4 per cent of the full scale reading plus 1/2 per cent of the actual output. It is worth noting, in this connection, that no error is introduced by the impedance of the voltmeter, provided this impedance remains constant [Eq. (8)].

After the ratios have been thus determined, it is a simple matter to obtain the actual values. Thus, we want the resistances to be low on account of noise, but high to obtain good resolution and to avoid overloading the generator. Considering only the question of resolution, we note that the functions are to be introduced manually, so that the maximum accuracy will hardly be better than $\pm$ 0.01 inch. If the potentiometer of lower resistance ρ is to have comparable accuracy, it is found that its resistance in the length used should not be much less than 5000 ohms. The resistances ρ, r, R, which should of course be rounded off to standard values, are thus completely determined (Fig. 4). The problem of loading may be dealt with in other ways ; if R is very large, for example, one could have $\rho = r$ and plot $f(x)$ on graph paper specially prepared to compensate the error. Alternatively, one could introduce a vacuum tube isolating stage at appropriate points in each element, a procedure suggested by O. A. Tysen. Both procedures are in many respects less simple than that actually used, however, though they would present certain advantages if the computation involved the product of more than two terms.

Besides these errors due to loading, there is additional error from stray fields and from variation in generator voltage. The former error can be eliminated by using direct current, though such a procedure complicates the problems of stabilization and detection. Instead of direct current, one could use alternating current at a frequency not a simple multiple of 60 cycles. In this way pick-up from power lines in the neighborhood of the apparatus is minimized. To reduce interaction between adjacent potentiometers, one may use shields as illustrated in Fig. 5 [1]. Both of these latter procedures are due to O. A. Tysen, who likewise designed and supervised the construction of the stabilized power supply illustrated diagrammatically in Fig. 6. The method of direct current is the one actually adopted.

The need for accurate stabilization of the generator is more urgent than at first appears to be the case. Thus, one might perhaps expect a fluctuation of 1% in generator voltage to introduce an error equal to 1% of the output; and it might be thought too that the error could be completely eliminated by use of a monitor. Actually, however, this desirable state of affairs does not prevail in practice. As we see by Eq.(4), the integrand is generally positive for some values of x, negative for others; and a small value of the output may accordingly arise through cancellation of voltages which are not themselves small. Without entering into great detail we note that the power supply must give the three

1. The method of making connection to the slider is taken from standard practice in the Potentiometer Group of the Radiation Laboratory, MIT.



F-1	2 A.	V-1	5U4-G	R-5	820K, ½ w.
T-1	Sola 30854	V-2	5Y3-G	R-6	33K, ½ w.
T-2	Stancor P-4004	V-3	VR-105	R-7	47K, ½ w.
T-3	Thordarson T-13R19	V-4	6Y6-G	R-8	1.5 M, ½ w.
L-1	T-67C49	V-5	6SH7	R-9	40K, 2 w.
L-2	T-13C27			R-10	15 K, 1 w.
C-1	10 μf. 450 v.	R-1	5 K, 5 w.	R-11	1 K, WW.
C-2	10-10 μf. 450 v	R-2	2 K, 2 w.	R-12	50 K, 2 w.
C-3	40 μf. 450 v	R-3	3.3 K, 2 w.	R-13	75 K, ½ w.
C-4	0.25 μf, 200 v	R-4	560 Ω, ½ w.	R-14	470 Ω, 1 w.

Antenna Computing Machine
Power Supply
(one unit of two)

Dwg. ARS-2021

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Figure 6. Wiring diagram for power supply (Courtesy of O.A. Tysen, N.R.L.)

voltages $-V$, 0, and $+V$. A given error, $|+V| - |-V|$, usually introduces an error of about the same magnitude in the output, and this in turn may represent a large percentage of the correct answer. Moreover, the condition $|+V| = |-V|$ must prevail even when the impedance on 0, $+V$ is different from that on 0, $-V$, since the impedance in either case may take values ranging approximately from ∞ to ρ/n in the course of the calculation. Hence the power supply must not only have an adjustment for setting $+V$ equal to $-V$, but it must be so stabilized that this equality will persist when the impedance changes as here described. Both conditions are adequately met by the circuit of Fig. 6, which also gives an output equal in absolute value to the optimum, if the optimum is taken as the largest voltage that will not lead to excessive heating of the first potentiometers. We remark in passing that the impedance change noted above, and hence the difficulty of stabilization, can be reduced by introduction of an extra resistance ρ with each potentiometer. The resistances would be connected across 0, $+V$ if the corresponding potentiometer is on 0, $-V$, but across 0, $-V$ if the potentiometer is on 0, $+V$. With such an arrangement the above limits ∞, ρ/n are replaced by values somewhat better than ρ/n, $\rho r/(n\rho + nr)$. In view of the success of the power supply, however, this added complication of the circuit, which also doubles the power requirement, was considered unnecessary. Incidentally, the question becomes irrelevant if any device in Fig. 8 is used.

Besides these errors due to electrical effects, further inaccuracy will be introduced by the mechanical tolerances. It turns out that this last is relatively unimportant with the proposed design; since the functions are introduced manually, an accuracy of about $\pm$ 0.01 inch is, as we observed above, about the best that can be expected. The mechanical tolerances, which should be somewhat smaller than this, are accordingly taken as 0.002 to 0.005 inch in typical cases. This order of precision is readily attained by routine methods. It is assumed, of course, that the error is not cumulative over the forty-five elements; and due care has been taken in the design to insure that this condition is actually satisfied.

Closely related to the question of mechanical tolerance is the question of potentiometer size, which is also determined by the 0.01 inch estimate when taken in conjunction with the percentage accuracy. Thus, one finds that commercial potentiometers are accurate within about 0.1 to 0.2 per cent of the full scale reading; and hence if this error is to be comparable in magnitude to the former figure, 0.01 inch, then the effective length should be five to ten inches. For reasons which will become apparent in what follows it is desirable that the first potentiometer be fairly short, although there is no especial restriction on the length of the second. The values finally determined, after revision to meet standard specifications, are shown in Fig. 4.

The sources of error in the proposed machine are, then, of three kinds. First are theoretical errors due to the use of a step-function approximation. These persist no matter how accurately the machine is constructed, but on the other hand they can be reduced as much as we please by repetition of the calculation. The second source of error is the fact that the response of a loaded potentiometer is not precisely linear. This error too

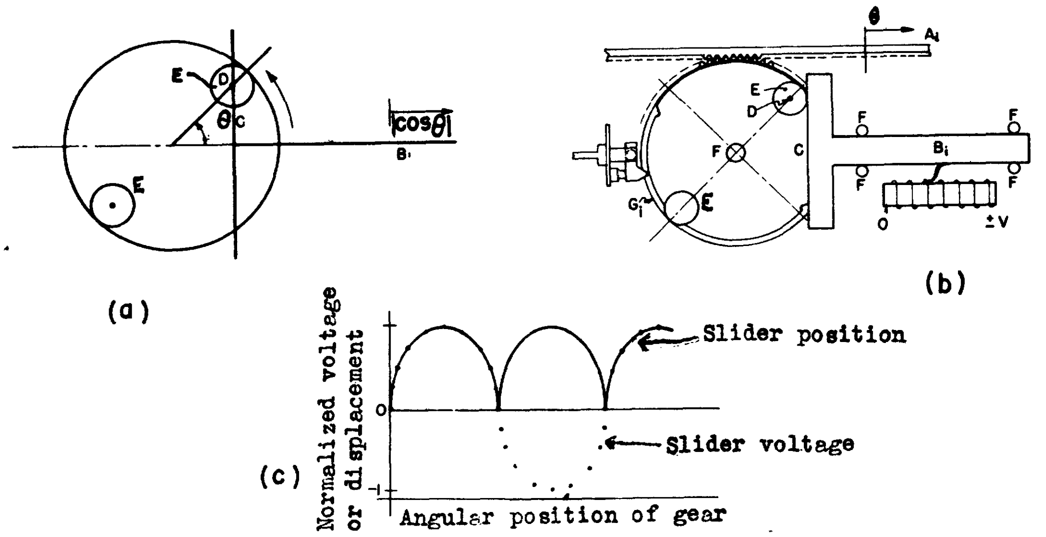


Figure 7. A method of generating a voltage proportional to cos θ. a) Simplified diagram illustrating principles. b) Details of construction. c) Comparison of slider voltage and slider displacement in b).

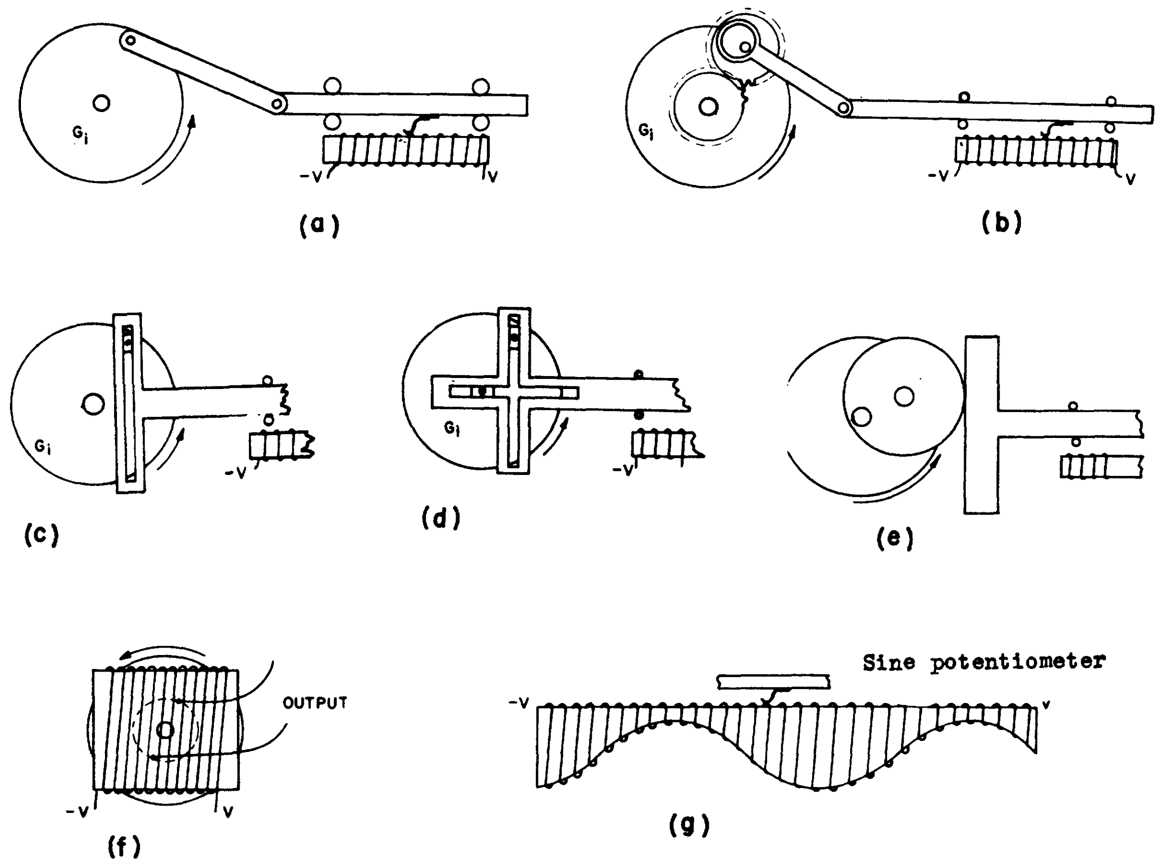


Figure 8. Alternative methods of generating a voltage proportional to cos θ.

can be reduced as much as desired by appropriate choice of the resistances, but on account of noise and other considerations, one finds in practice that a compromise must be reached. Closely allied to this error is that introduced by variation in the supply voltage, which, as we have seen, is of considerable importance, but can be sufficiently reduced with careful design of the power supply. The third source of error is inaccuracy of the mechanical parts, which of course cannot be manufactured with zero tolerances. The question of tolerance applies also to the potentiometers; even without loading, there would still be some departure from linearity. The foregoing discussion is intended to show that these errors, which are hardly avoidable with the proposed design, can be so reduced that the machine will still give results of practical utility.

In this connection we note that the machine takes an average of forty-five individual values. It is reasonable to suppose that the errors will be randomly distributed, with the result that the overall accuracy is frequently better than the above considerations at first appear to indicate. In summary, one may say that serious error should be noted only for small values of the output; every one of the above sources of inaccuracy is a magnitude effect rather than a percentage effect, and becomes of increasing (relative) importance as the output decreases. The machine tends to give results accurate to within a certain percentage of the full-scale output, rather than a certain percentage of the observed output.

5. Generation of cos θ. Up to this point we have discussed a circuit for approximate evaluation of the integral in Eq.(5). Turning now to the more complicated expression given in Eq.(4), we find the only new problem to be the generation of cos $x_1 y$ as a continuous function of y. If this problem is solved, then the previous construction, applied to $f(x) = f(x)$, $F(x) = \cos xy$, will give the desired integral (4). Assuming for the moment that $x_1 y$ has been obtained for each x_1, we see that it suffices merely to take the cosine. One method of doing so is illustrated in Fig. 7b, where the displacement of the T-shaped bar B_1 is proportional to cos θ, if θ is the displacement of the upper bar A from some suitably chosen origin. To see this, let us notice that the result is obviously true in the arrangement of Fig. 7a, where the edge C is constrained to pass through one of the points D rather than to rest on the disk E. But if the variation in Fig. 7a is sinusoidal, that in Fig. 7b must be also, since the displacements in the two figures differ only by the constant radius of the disk E.

This construction gives $|\cos \theta|$, while it is cos θ that is desired. Since we are really concerned with the voltage of the potentiometer slider, rather than with its displacement, the difficulty is easily avoided by the introduction of a reversing switch as shown in Fig. 7b. With this arrangement the voltage will follow the dotted line in Fig. 7c, while the position varies according to the solid line; that is, the slider voltage is proportional to cos θ even though its displacement is proportional to $|\cos \theta|$. The use of disks which are free to rotate, rather than fixed, is recommended to reduce friction and wear; it was suggested by H. Kylin of N.R.L., who also suggested the use of a cam machined directly into the gear surface, as shown in Fig. 7b, for operating the switch. The large gear diameter and fine pitch, which will be required later in connection with $\phi(x)$, have the further advantage of giving high accuracy without the use of close tolerances or precision-cut gears.

Before leaving this question of generating cos Θ, we observe that the problem is actually a simple one, for which a large number of constructions are readily obtained. In the course of designing the present machine, each of the constructions suggested in Fig. 8 was considered in some detail; and, with so many alternatives, it is natural to inquire whether the optimum construction is really given by Fig. 7. To consider the merits and shortcomings of each of these constructions would take us too far, and instead we list the advantages presented by the recommended arrangement. These advantages may also be of some interest in their own right, inasmuch as this construction is the one actually used.

a. Ease of assembly - Since we shall ultimately require forty-five units mounted side by side, it is quite important that this grouping can be carried out without undue difficulty. Such a requirement is met by the arrangement of Fig. 7, which is so designed as to be thin in the necessary direction. Also the whole set of forty-five elements is conveniently supported by the rods F in Fig. 7b, only the five rods shown being required in the completed machine. This property, that the individual units can be readily stacked, leads to material simplification in assembly.

b. Zero set - At the end of the calculation, the values of Θ for each x_i will generally be distributed more or less at random, and it is necessary to return them all to zero, or at any rate to some constant value, before a new computation can be carried out. This problem is readily solved by the method of Fig. 7, since the arrangement is in stable equilibrium when the two disks E both touch the T-shaped bar B_1. Hence for the zero set it suffices to push these bars as far as possible to the left in the figure, an operation that can be easily carried out for the forty-five elements simultaneously.

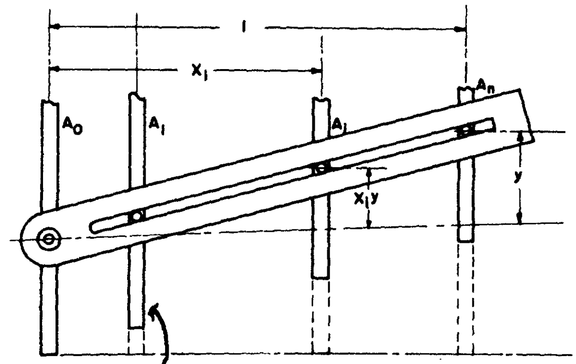
c. Potentiometer accuracy - As a third advantage of the proposed method we find that it doubles the effective length of the potentiometer. This property, which is sufficiently evident from the figure, leads to an increase of accuracy for given potentiometer length, or, which is nearly the same thing, it permits use of shorter potentiometers, and hence of smaller mechanical elements, for given accuracy.

d. Loading errors - In the course of investigating error due to the loading of one potentiometer by another, H. Dowker showed that the center of the first potentiometer should be grounded (i.e., connected to V = 0) for best results. So great is the difference between the two cases, grounded or ungrounded, that the latter leads to excessive error with any reasonable ratio of resistances, while the former can be used with success. With the devices shown in Fig. 8, the use of a grounded center requires insertion of a center tap on each potentiometer, an operation that presents considerable difficulty. In the system of Fig. 7, however, the desired condition is automatically achieved without the use of taps, for the role taken by the center of the potentiometer in Figs. 8a-e is taken by the left-hand end in Fig. 7b.

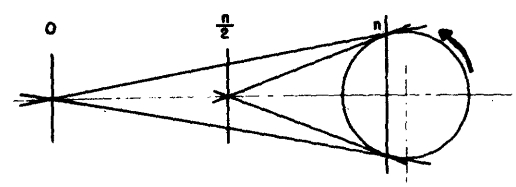
Although some of the devices of Fig. 8 offer the first advantage noted above, none appears to give the last three. It was believed, therefore, that the arrangement of Fig. 7 should be used, even though it requires an extra switch, and has the added disadvantage of reversing the slider motion at the point of maximum velocity. This last objection, in particular, is of no moment at low speeds, and because of the time required in setting up a calculation, the use of high speeds offers only slight advantage. Thus, it is permissible (in the author's opinion) that the time of calculation be comparable to the time required in setting up a problem; and with this condition the arrangement of Fig. 7, which permits a gear speed of about 20 rpm, is mechanically sound.

6. Generation of the Product. The only step in the computation of (4) which we have not yet considered is the determination of xy. Once this is solved, the preceding construction gives $\cos xy$; the circuit of Fig. 2a gives $f(x) \cos xy$; and the integral is obtained as suggested in Fig. 2c. To obtain xy, let us note that the length x is already available; it is simply the distance from some suitably chosen point to the corresponding potentiometer. Because of this rather fortuitous property, the problem in question turns out to be quite trivial. Thus, in the arrangement of Fig. 9a, the displacement of the rack corresponding to x_i is proportional to the fixed distance x_i in the figure, and also to the variable displacement y. Being proportional to each separately it is proportional to the product, as required.

At this point it is of interest to consider the range of variation in y. For Fourier transforms the values will generally be included between -20 and 20, while for Fourier series the corresponding limits may be from 0 to 100 or more if a complete period is to be obtained. In terms of the machine, this requires from 40 to 100 revolutions of the outer gears, that is, of the gears G_i, corresponding to x_o or to x_n. Since these gears are to be about 11 inches in diameter for high accuracy, the lever of Fig. 9 would have to be over a hundred feet long to function as described above. To circumvent this difficulty, one may proceed as suggested in Figs. 14,16. The lever is continuously moved back and forth, the racks being engaged during only half a complete cycle. In this way the restriction on range is entirely removed, while the desired dependence on xy is preserved. In general such a modification of the basic principle would be expected to introduce a cumulative error, since the racks will not exactly mesh when dropped on the gears at the beginning of successive cycles. If the dimensions be so adjusted, however, that the maximum swing of the lever moves each gear an integral number of teeth with respect to its neighbor, then no such difficulty arises; for the racks would now mesh exactly in the first case and hence in every subsequent case. Because the variation of position is always linear as we proceed from one gear to the next, such a condition, namely, displacement by an integral number of teeth in every case, is readily obtained. In addition it is desirable that the outer gear move a simple fraction of a revolution with each sweep of the lever, as this condition sometimes facilitates correlation of the value of y with the position of the hand crank. In



(a) Top view of the racks (bars A_i in Fig. 7)



(b)

Figure 9. Generation of xy. a) Simplified diagram illustrating principles. b) Adjustment of dimensions for satisfactory operation with pivot at center or end.

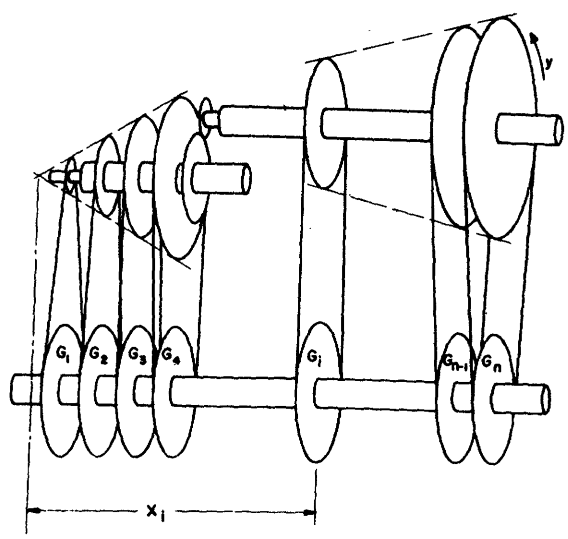


Figure 10. Alternative method of obtaining xy.

certain cases, moreover, we wish the integral between symmetric limits, $\int_{-a}^{a}$, rather than $\int_{0}^{a}$; with the former, Eq.(1) leads to a real result whenever f(x) is even and $\phi(x)$ is odd, as may be easily proved, while the latter is sometimes convenient for Fourier series. Of course any limits may be obtained by a transformation of the input and output functions, or, if one is content to use only part of the machine, by a simple translation of the input alone; but direct evaluation appears to be preferable. In the present machine, therefore, we have provided for pivoting the lever either at one end (Fig. 9) or in the center; the former arrangement gives $\int_{0}^{a}$ while the latter gives $\int_{-a}^{a}$. Other limits could be obtained by use of intermediate points.

The conditions previously mentioned, that adjacent gears be displaced an integral number of teeth and that one end should move a simple fraction of a revolution, must both prevail, then, whether the lever is pivoted at one end or in the center. It is not difficult to give necessary and sufficient conditions on the dimensions, number of elements, and number of gear teeth to insure that these requirements are all satisfied; in the present machine we use 45 elements with 538 teeth in each gear. other quantities being as shown in Fig. 15. One sweep of the lever moves the end gear through 1/6 revolution, so that adjacent gears are displaced exactly four teeth when the lever is pivoted at the center, two teeth when pivoted at one end. That the desired relations will prevail for the central pivot, if they are satisfied when the pivot is at one end, is insured by the geometrical construction of Fig. 9b. We remark in passing that the operation of shifting as described in Sec. 3 requires that the pivot be moved in small steps through a distance equal to the distance between elements. In the present machine a screw adjustment is provided for this purpose, the same mechanism being operable whether the pivot is at the center or at one end (see Figs. 14-16). In case we wish to equip the machine for automatic recording, it is convenient to have the recording drum rotate automatically with the independent variable y. Because of the intermittent operation of the lever, the drum cannot be connected to the crank, and instead is arranged to turn with the last gear G_n. To this end, a pinion is supplied in the completed machine.

For generation of xy it is worth noting that various other devices may be used, of which perhaps the simplest is suggested in Fig. 10. Adjacent sprockets differ by one tooth, additional gearing down being furnished at one end to avoid use of sprockets which are too small. Such an arrangement, which has been investigated in considerable detail, provides continuous rather than intermittent motion, and would be of interest if rapid calculations were contemplated. To this end it should be used with one of the devices of Fig. 8 rather than with that of Fig. 7. Besides the advantage of speed, however, the device in question has little to recommend it. In addition to the expense of making the sprockets, which do not appear to be stock items in the variety required, there is further difficulty due to the backlash and other inaccuracy introduced by the chains. Also the operation of shifting cannot be carried out by any obvious procedure, nor is the arrangement well adapted to the introduction of $\phi(x)$ or to the generation of absolute values. With regard to this last, in particular, the intermittent operation of the device in Fig. 9

will be found to be an advantage rather than otherwise, and the arrangement of Fig. 10 is accordingly dismissed forthwith.

7. Complex Functions. It has been assumed hitherto that $\phi(x)$, the phase associated with $f(x)$, is equal to zero. Turning now to the case in which this condition is not satisfied, we find that the real part of (1) takes the form

$$\int_{-a}^{a} f(x) \cos[xy + \phi(x)]dx \; . \tag{9}$$

Although this expression is apparently much more general than the one previously considered, Eq. (4), it may be computed without additional complication. Thus, instead of starting the calculation with the gears of Fig. 7 all at the zero position, we may equally well adjust their positions according to $\phi(x)$, the gear corresponding to x_0 being displaced initially through an angle $\phi(x_0)$, that for x_1 through an angle $\phi(x_1)$, and so on. It is clear that these initial conditions will persist, in the form of additive displacements, throughout the subsequent calculation. Where we formerly obtained xy we now have $xy + \phi(x)$; instead of $\cos xy$ we have $\cos[xy + \phi(x)]$; for $f(x) \cos xy$ the machine accordingly gives $f(x) \cos[xy + \phi(x)]$; and hence the final answer is the expression (9) rather than (4).

To introduce the function $\phi(x)$ in the manner here required, one may proceed as suggested in Fig. 11a. With the gears fixed at the zero position, the racks are released from the lever and set on the curve, each one being lifted so that it does not engage with the gear until the operation is actually completed. The racks are then returned to their original positions, the gears being engaged during the whole time. In practice the latter operation could be carried out for all elements simultaneously by use of a straight-edge. The machine is ready for use as soon as the racks are coupled to the lever, an operation for which a number of devices has been obtained; the procedure shown in Fig. 11b, which represents a substantial improvement over all the alternatives considered, is due to H. Kylin.

In the above process some slight error is to be anticipated in that the racks do not engage precisely with their gears when placed on the curve. This difficulty can be resolved by the device suggested in Fig. 11c, which is due to H. Dowker; in the present machine, however, such complications, which would give difficulty in taking absolute values, are avoided by large gears with small teeth. Thus, the error need never exceed half a tooth, which amounts to only 0.34° in the present case, and such accuracy is comparable to that assumed for the other components.

To obviate the excessive rack lengths entailed by use of large gears we modify the procedure outlined above. In the first place, the curve $\phi(x)$ may clearly be assumed to lie between $-\pi$ and π, since addition of $\pm 2k\pi$ to $\phi(x_1)$ has no significance. Addition of an odd multiple of π, moreover, merely changes the sign over the corresponding portion of the curve, and hence it could be compensated by means of reversing switches manually operated but otherwise similar to the cam-switches of Fig. 7. Switches of this kind are shown, with their connections, in Fig. 11d. For convenience they would be placed in a

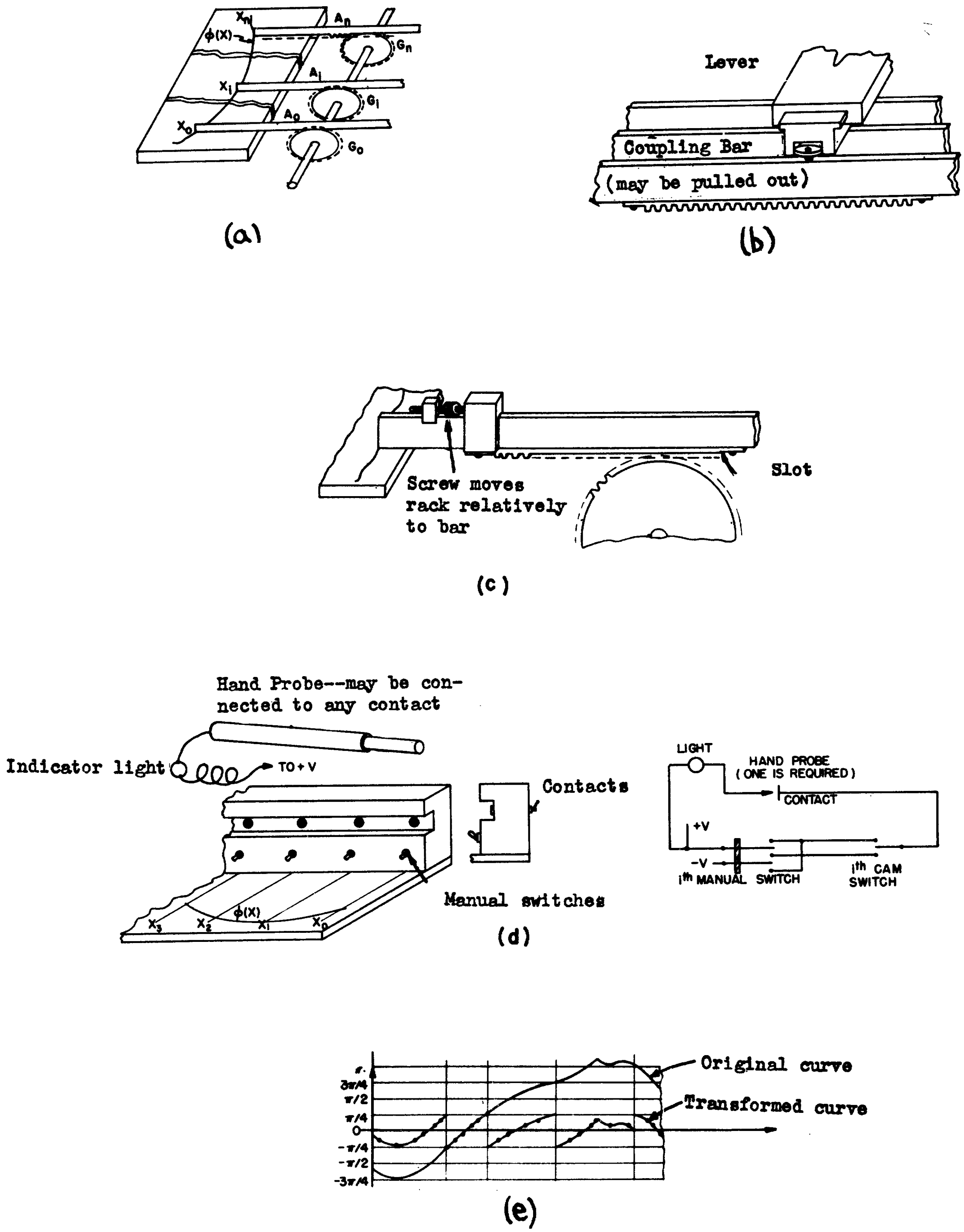


Figure 11. Method of inserting the phase $\phi(x)$ associated with $f(x)$. a) Simplified diagram of mechanism for introducing $\phi(x)$. b) Detail of method for engaging racks and lever. c) Method of obtaining precise meshing even when rack bars are placed exactly on the curve (optional). d) Switches for adding π to $\phi(x_i)$. e) Transformation of original curve to an equivalent one in the range $-\pi/4 \leq \phi(x) \leq \pi/4$.

horizontal line over the table on which $\phi(x)$ is to be plotted. The extra contacts in the figure are used to determine the position of the corresponding gear G_1, which admits an ambiguity of 1/2 revolution (see below). With proper adjustment of these switches, the curve may be assumed to lie between $-\pi/2$ and $\pi/2$. The corresponding travel for the racks, about 18", is still too great, and to achieve further reduction we introduce stops on each rack which restrict its motion to exactly 1/4 revolution of the corresponding gear. By pushing a rack through the extent of its travel we can conveniently add $\pm\ \pi/2$ to the function, and hence the curve may be assumed to be between $-\pi/4$ and $\pi/4$ (Fig. 11e).

The distance through which each rack must move now turns out to be about nine inches, and this is mechanically convenient. Similarly, the curve is plotted on a strip of about the same width, which is likewise a convenient size from the point of view of operation; and the accuracy is nevertheless equal to that obtained with direct representation when the curve covers a range of nearly three feet.

8. Absolute Value and Phase. When taken together, the above operations give the expression (9), which is the real part of the desired expression (1). The imaginary part is similarly obtained; it suffices in fact to repeat the previous calculation with $\phi(x)$ replaced by $\pi/2 + \phi(x)$ or, which is the same thing, with all gears displaced a quarter turn at the beginning of the new computation. Once the real and imaginary parts are known, the integral is completely determined; and since both are obtained as continuous functions of y, the problem facing us at the outset may be regarded as solved. There are many situations, nevertheless, in which we require the absolute value or phase rather than the real and imaginary parts, and such a requirement may well triple the time of computation, even though the additional operations are completely elementary. Thus, we must find both the real and imaginary parts, an operation which takes nearly twice as much time as finding either alone; and we must then compute the square root of the sum of the squares, or the inverse tangent of the ratios, for a whole set of values of y. The computation is much facilitated by graphical methods, of course. One obvious procedure is to plot the real part versus the imaginary part, whereupon the length of the radius vector gives the absolute value while its direction gives the phase. The time required would still be comparable to that consumed by the machine in finding the real part alone, however, and we thus obtain the time estimate given above.

It is desirable, then, that the machine give direct readings of absolute value and phase. To meet this requirement, let us observe that the absolute value of any complex number $r \exp(i\theta)$ may be obtained by the following somewhat devious procedure: First multiply by $\exp(iz)$ to get $r \exp[i(\theta + z)]$; then take the real part, $r \cos(\theta + z)$; and finally adjust z to make this real part a maximum. The maximum value so obtained is clearly equal to r, which is the absolute value of the original expression. Applying this principle to the problem at hand, we are led to consider

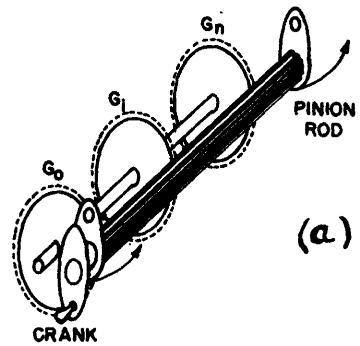
$$\int f(x)\cos[\phi(x) + xy + z]dx \qquad (10)$$

where z, like y, is an independent real variable. The maximum value of this expression, with respect to z, gives the absolute value of (1). In terms of the machine, the parameter z represents a constant addition to the angular position of the gears G_1, as we see by the above discussion of $\phi(x)$; and hence one may vary z, with y fixed, by rotating the whole set of 45 gears as a unit. If the output is presented on a meter, we simply set y to the desired value and adjust x for maximum deflection.

This method of taking the absolute value, which was suggested to the author by S. J. Mason, turns out to be particularly convenient when the other components are as described above. Thus, while the lever of Fig. 9a sweeps back and forth, the racks are automatically disengaged during each half cycle, and at the same time the brake is engaged to prevent rotation. If the brake be designed as shown in Fig. 12a, then the value of z is readily changed by simply turning the crank. In other words, during the return cycle of the lever in Fig. 9a, the machine is automatically prepared for determination of absolute values. Because the change of y introduced by a single sweep of the lever is very small, the restriction to values occurring during the return-sweep generally causes no inconvenience; on the actual machine, nevertheless, a switch is provided for engaging the brake and disengaging the racks at the mid-point of the cycle, so that y is almost unrestricted. For the few cases in which even these values do not suffice, one must use the real and imaginary parts.

Turning now to the question of phase, we find that it is determined, in principle, by the position of the crank at which the meter reading was maximum. This result, which is an immediate consequence of the elementary considerations noted above, cannot be used, however, for accurate computation. The difficulty is that the derivative of meter-deflection with respect to crank position is zero at the desired point, and hence the error is greatly magnified. Instead, we therefore adjust the crank for zero meter reading and subtract $\pi/2$ from the result thus obtained. If a logarithmic meter, or a linear meter of sufficiently high and variable sensitivity, be used, there is clearly no theoretical limit to the accuracy with which these settings can be made.

In the above operation it is the problem of meshing that prevents use of the method for arbitrary y, and this problem may well be considered in greater detail. When the brake of Fig. 12 is engaged, it is clear that every gear must line up with its neighbor, so that the successive teeth lie on a straight line. Not only must they correspond in this way with each other, but they must line up with the teeth of the pinion-rod forming the brake. For general positions of the lever, it is clear that neither condition is satisfied; and closer investigation shows that the proper positions occur only when the lever is at one of its stops or half way between them. That the gears will then line up, if due care is taken in the design, is insured by certain requirements stated above, which also show that the absolute position of the whole group will be invariant. Besides this condition on the gears we have a similar condition on the pinion rod, which must be rigidly clamped in the correct position, rather than free to rotate, while it is being engaged. Apart from the problem of meshing, incidentally, this clamping is also desirable whenever the pinion is used as a brake. Still a third requirement is that the racks must engage exactly, when the brake is released and the machine reset for adjustment of y. These conditions, and the further requirement that z be variable as above described, are met by the interlock shown



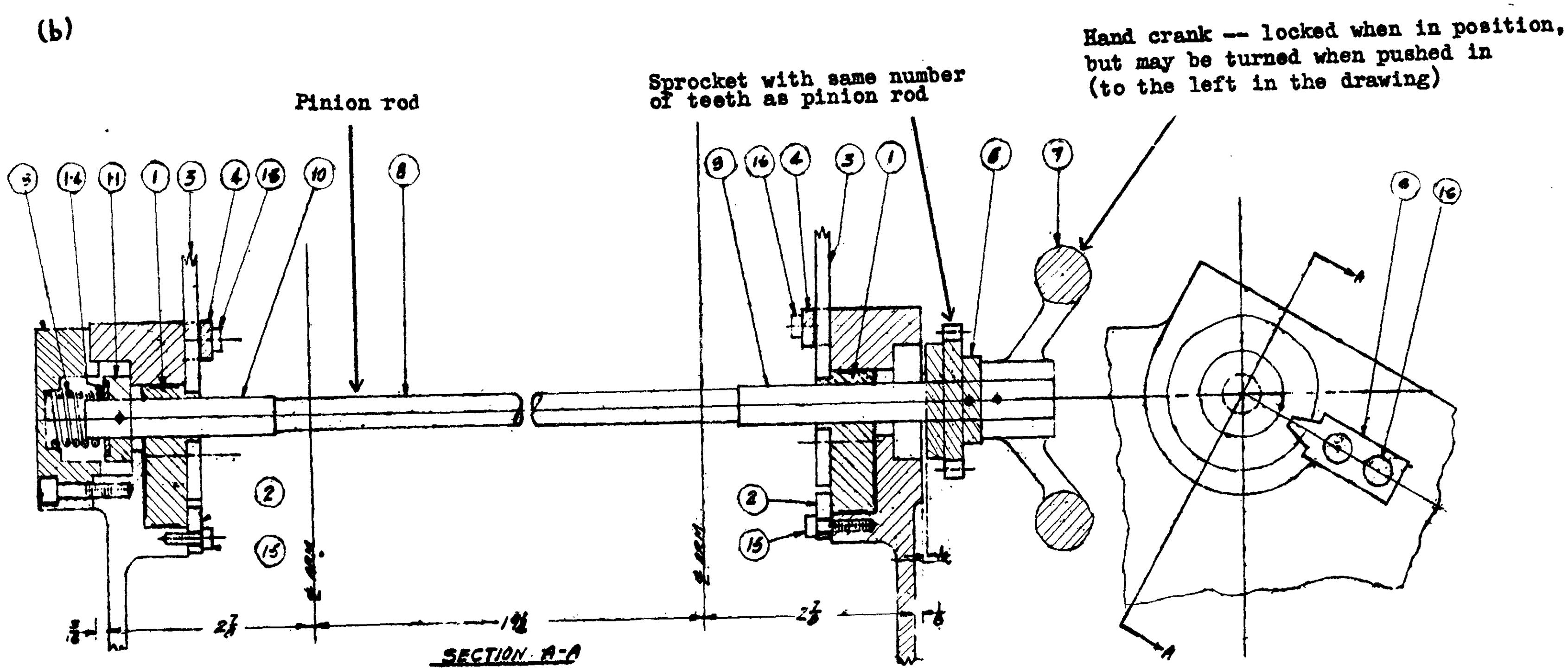


Figure 12. Method of finding absolute values and phase. (a) Use of brake for rotating gears as a unit, (b) detail of interlock.

in Fig. 12b, which was designed by H. Kylin and the author. The crank is free to turn only when it is pushed in, towards the machine; and after being engaged it cannot be released, unless it is lined up as required for proper meshing. The angular position of the pinion rod is fixed whenever it is being meshed or unmeshed; but this fixed position is itself adjustable to account for the change of origin introduced by shifting the lever sideways.

9. Output. The output of the machine is given in the form of a voltage, and may be evaluated in a number of ways. For example one may use a meter, as suggested in Fig. 17a, or a bridge circuit, as shown in Fig. 17b. Regardless of one's choice of method, the machine must be calibrated, for the operations hitherto described give a voltage proportional to the answer rather than the answer itself. Since the proportionality constant depends only on the machine, not on the problem, a simple method of calibration is to take $f(x) = 1$, $\phi(x) = y = 0$, or some other set of values leading to easily computed expressions, and to compare the observed with the predicted output. As far as principles are concerned, this is sufficient, since subsequent readings can all be normalized by division. For convenience, however, one may proceed as suggested in Fig. 17c. With simple functions in the machine, the potentiometer 1 is adjusted until the meter reading is equal to the predicted value; and thus in all later calculations the readings are normalized automatically. The calibration should be checked and the potentiometer adjusted from time to time if required. Besides this calibrating potentiometer 1, one may have a second one 2, linear or logarithmic, which is itself accurately calibrated. In this way the output can be readily multiplied by any desired factor, a possibility that is often convenient for purposes of plotting.

In many applications one requires the answer in the form of a curve. To avoid manual plotting, one may use a recording voltmeter of standard design, the drum being connected to G_n, the last of the gears G_i. For this purpose a pinion is provided on the completed machine (Fig. 15). In case greater accuracy is required than may be obtained with commercial recording meters, one may use the method of Fig. 17d, which has the further advantage that a logarithmic or other plot may be obtained as easily as a linear one by suitable choice of the potentiometer. The pen is kept off the paper by an electromagnet, which may either be operated by a foot switch or may be connected at the point normally occupied by the galvanometer. With this latter procedure the pen drops on the paper automatically when the desired point is reached, provided the adjustment of position is not made too rapidly.

All results hitherto obtained have been expressed in terms of the parameter y in Eq. (1). For antenna work it is often more convenient to express the result in terms of θ, where θ is given in terms of y by the relation

$$y = \frac{2\pi}{\lambda} \sin\theta$$

Such a change of variable can be made manually, if the curve is known as a function of y, but the devices for automatic recording are no longer applicable. To remedy this defect one may proceed as suggested in Fig. 17e. The device indicated is practicable, because one is

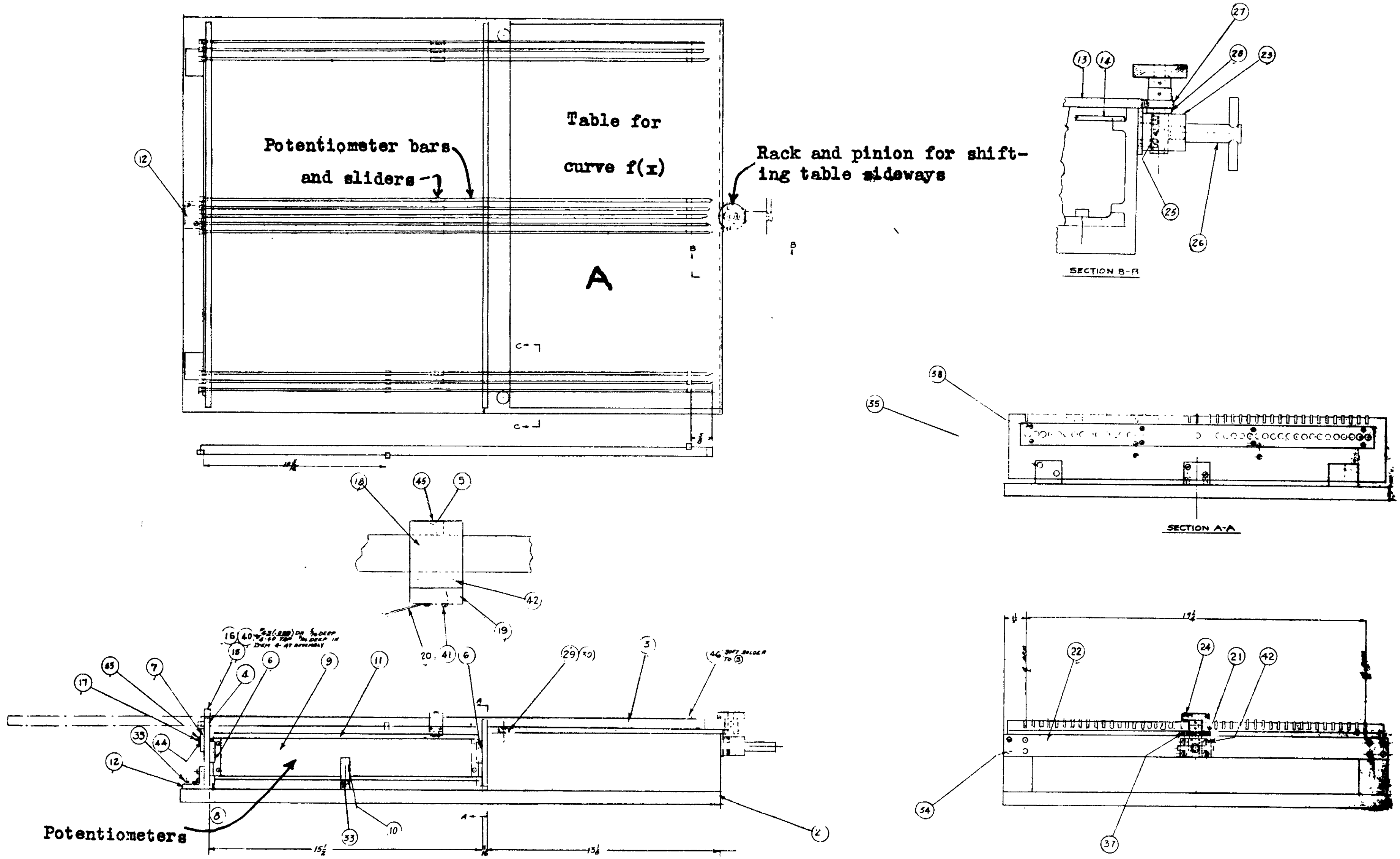


Figure 13. Potentiometer assembly for f(x)

interested only in the range $-\pi/2 \leq \theta \leq \pi/2$, and because the output gear G_n will generally make a large number of revolutions in the course of the calculation. In case this latter condition is not fulfilled, the vertical screw for adjusting the position of the T-bar should be replaced by a rack and pinion.

10. Assembly and Operation. A photograph of the completed machine is not available, but assembly drawings are given in Figs 13-16. There are two units, which are connected only by a cable and which may therefore be regarded, for purposes of fabrication, as completely independent. The first is a simple array of potentiometers for the function $f(x)$, and presents no serious design problems (see Fig. 13). The second unit is the linkage and potentiometer assembly for generation of $\cos[xy + \phi(x)]$; the individual components of this unit have been described in the foregoing pages, and it suffices here to indicate a few of the problems which apply to the assembly as a whole.

In the first place it is desirable that cumulative tolerances be avoided, a condition which is met by the indicated arrangements for supporting the T-bars, racks, and gears. Similarly, the individual components must be so mounted that they are accessible for repair, and this condition too is met by the suggested design. In particular, the potentiometers may be removed individually or, by unsoldering the connections, as a group; that is, the framework holding them is self-sufficient. This arrangement is especially to be recommended because it permits separate assembly of the potentiometers--an operation which is preferably to be done by electricians rather than by machinists. These and most of the other design features required in the assembly are due to H. Kylin.

The operation of the machine can perhaps be best illustrated by an example. Suppose, then, that we wish the absolute value of

$$\int_{-\frac{1}{2}}^{3} \sin x \, e^{i[xy + x^2]} dx$$

as a function of y. The first step is to obtain a curve of $f(x) = \sin x$ and $\phi(x) = x^2$ to the proper scale; on the present machine each graph would cover about 20 inches in x, 10 inches in y, the full scale preferably being used on account of the increased accuracy. The curves are then placed on the tables A and B of Figs. 13, 15, which, for convenience, may be removed from the machine. The potentiometer sliders in Fig. 13 and the racks in Fig. 15 are then placed on their respective curves, the switches of Fig. 11 being adjusted with due regard to the initial position of the gears and to the range of the curve $\phi(x) = x^2$. If the initial position of the gears was adjusted as suggested in Sec. 5, by simply moving the T-bars as far forward as possible, then there will be an ambiguity of 180°, as noted above; that is, a given cam-switch may be connected either to +V or to -V. To determine which connection prevails for a given gear we rotate the gears as a unit through a small distance, and then use the hand probe of Fig. 11d. If the light goes on when this probe is connected to the corresponding contact, then the cam-switch is connected to -V, and otherwise it is connected to +V. The connection may be reversed by means of the manual switches added for this purpose. In the present case we find that $\phi(x) = x^2$ will be less than π in the range $\frac{1}{2} \leq x \leq \sqrt{\pi}$, and hence in this range we want all cams

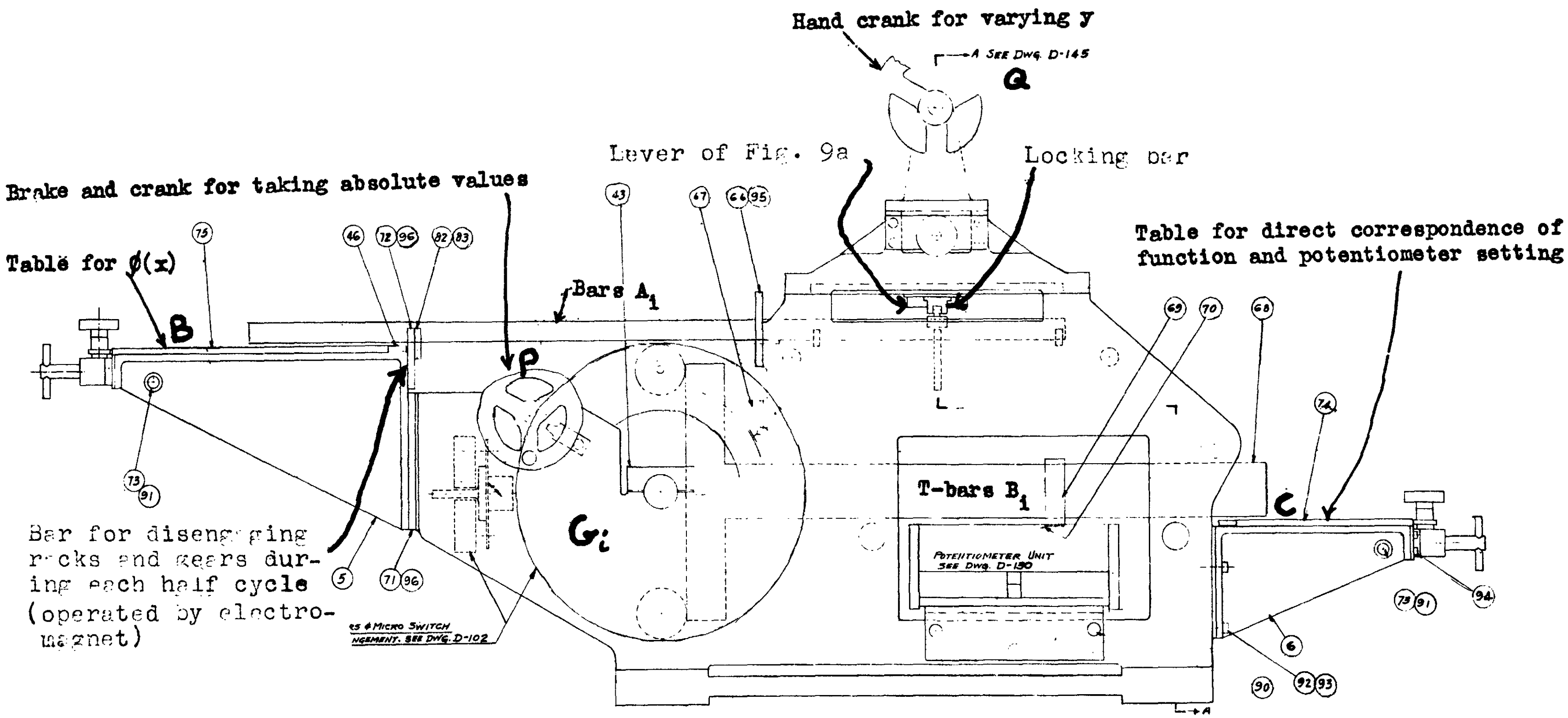


Figure 14. Unit for producing cos $[xy + \phi(x)]$ (side view)

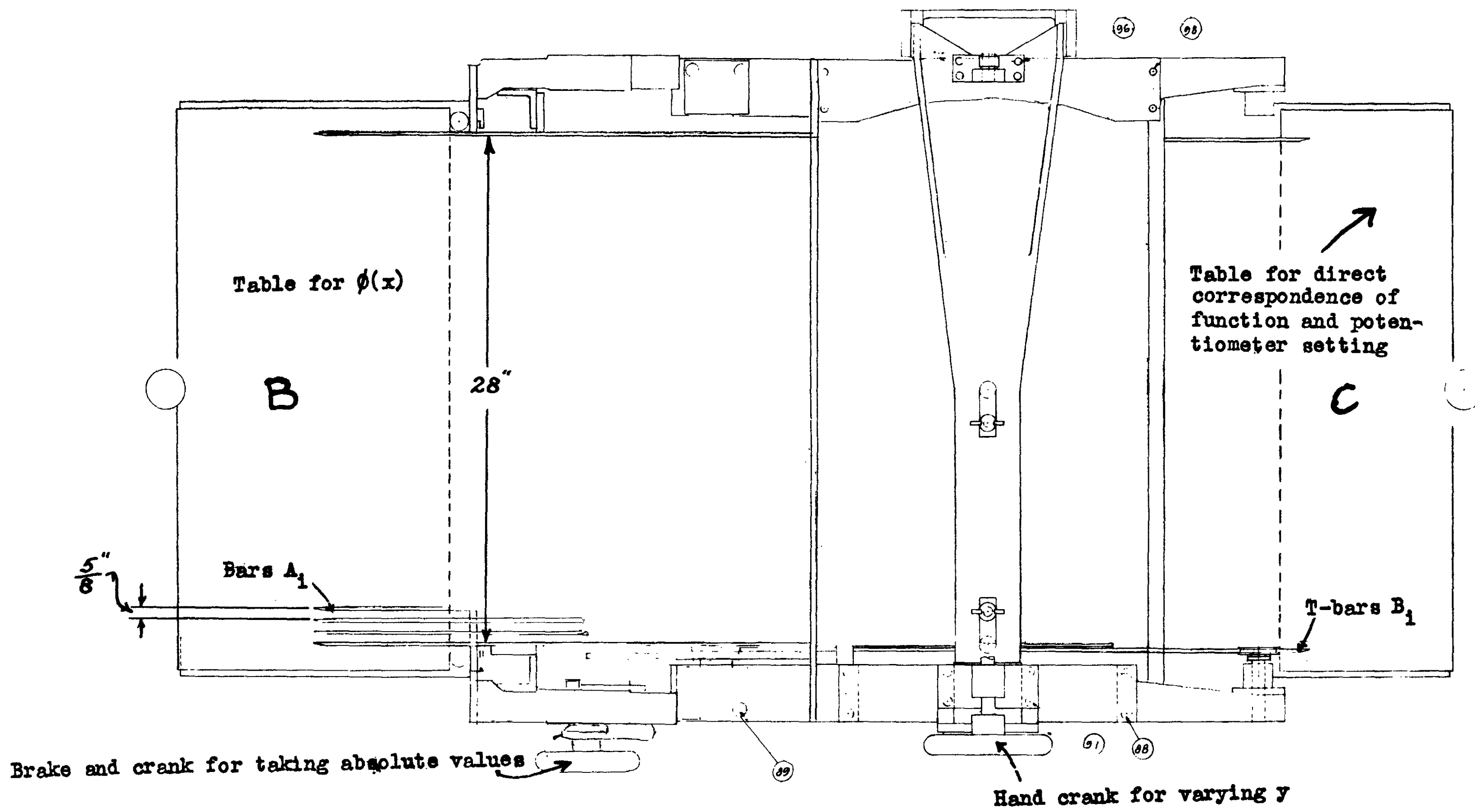


Figure 15. Top view of Fig. 14.

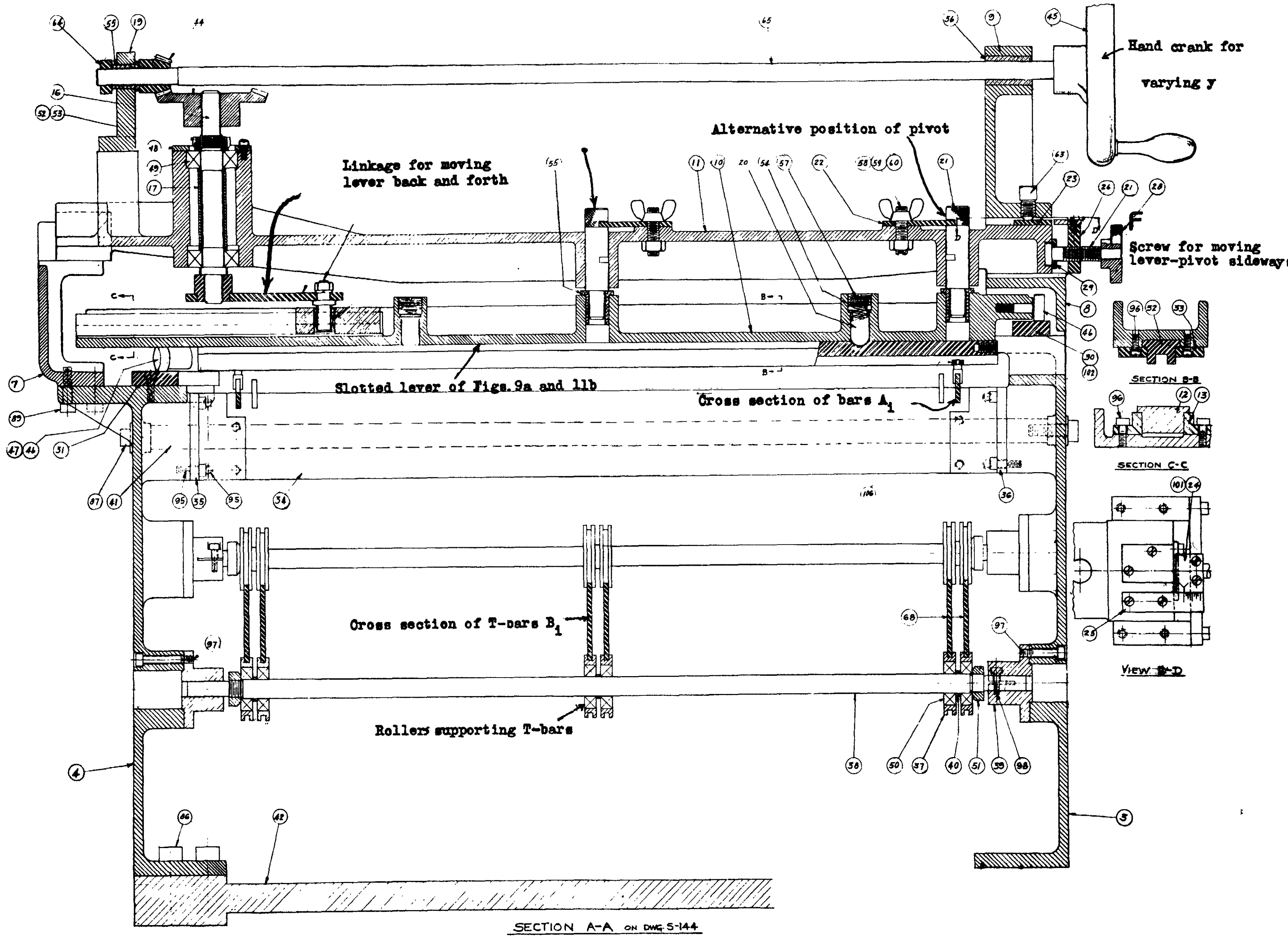


Figure 16. Sectional view of Fig. 14.

connected to +V. The manual switches of Fig. 11d must then be so adjusted that the light remains off when the probe is connected to each contact in turn. For $\sqrt{\pi} \leq x \leq \sqrt{2\pi}$, however, we have $\pi \leq \phi(x) \leq 2\pi$ and hence we must subtract π. For these values of x, we adjust the manual switches in such a way that the light is always on; and a similar process is used for the remainder of the interval $-\frac{1}{2} \leq x \leq 3$. The final adjustment can be checked by slowly drawing the hand probe through the slot containing the contacts.

In Figs. 14-16 the machine should have been horizontal when these operations were carried out, the brake should have been on, and the locking bar should have been withdrawn. After the racks have been placed on the curve, the brake is released, the racks are lined up by a straight-edge, the locking bar is inserted, and the machine is tipped to bring the T-bars against their cams. The machine is now ready for operation. We set the crank Q to a desired value of y, then turn the crank P to maximize the meter reading, and so on. It is worth noting that one generally requires an arbitrary set of values. In this case one may simply turn the crank Q through a single turn, take a reading, make another turn, take a reading, and so on. If fewer points are needed (i.e., if the function appears to be varying slowly), one may of course take readings every other turn, or less often, as desired. This procedure gives positive values of y only. For negative values of y a number of methods may be used, of which perhaps the simplest is to start at the smallest required value, rather than at zero as described above. This value is reached by turning the crank backwards from the zero position, the machine preferably being not yet tipped, so that there is no restriction on the speed with which the crank may be turned. Except for this difference in starting position, the procedure for negative y's is the same as above.

In case greater accuarcy is required one may carry out the operation of shifting. For sufficiently smooth functions f and ϕ, it is easily seen that difficulty is introduced by the exponential alone, and this too only when y is very large; hence it is generally sufficient to move the pivot (knob F in Fig. 16) without resetting the curves, and to repeat the calculation for large y. If f and ϕ are not smooth, the whole process may be repeated from the beginning, the only change being that the tables and the pivot are both shifted. We disengage the brake and the racks, when the calculation is finished, so that the T-bars can turn most of the gears to the equilibrium position; the remaining gears are treated by rotating the whole group through a quarter turn.

Part II. Other Problems Solved

11. Fourier Series. As the reader has perhaps noted already, it is the sum of a Fourier series, rather than a Fourier integral, which the machine actually computes. The most general expression obtained is the exponential series with complex coefficients,

$$\left.\begin{array}{l}\text{Real part}\\ \text{Imaginary part}\\ \text{Absolute value}\\ \text{Phase}\end{array}\right\} \text{ of } \sum_{0}^{44} a_n e^{ib_n + iny} \qquad (11)$$

The values a_n correspond to the points x_i used for f(x), while the b_n correspond to those for $\phi(x)$. To facilitate insertion of these constants the two tables are provided with a set of rulings. When the series contains less than 23 terms, it would often be advantageous to use every other potentiometer, rather than every one; if there were 15 terms or fewer, we should use every third one, and so on. Similarly, if the series contains more than 45 but less than 90 terms, we should put every other coefficient $a_n \exp(ib_n)$ into the machine, using all potentiometers, then repeat with the pivot shifted half the distance between elements, now using only the remaining terms $a_n \exp(ib_n)$. This latter operation is analogous to the shifting procedure noted above, and may be carried out as often as desired. The original sum gives 45 terms; one repeat gives 90, a second gives 135, and so on. It must be noted in this connection that the calculation of the Fourier transform was only approximate, and shifting was then used as a simple means of reducing the theoretical error. For Fourier series, on the other hand, there is no theoretical error of this type, and shifting is now used merely to permit summation of series with more than forty-five terms. Even when the range is thus extended, computation of Fourier series still entails no theoretical approximation.

For the special case in which $b_{2n} = 0$, $b_{2n+1} = \pi/2$, we find that the real part of (11) reduces to an ordinary Fourier series with real coefficients,

$$A_o + \sum_{1}^{22} [A_n \cos ny + B_n \sin ny] \qquad (12)$$

and similarly, the specializations $b_k = \pi/2$ or $b_k = 0$ give Fourier sine and cosine series:

$$\sum_{o}^{44} a_n \sin ny, \quad \sum_{o}^{44} a_n \cos ny \,. \qquad (13)$$

These expressions may all be computed, then, as continuous functions of y; and there is no theoretical error, nor any theoretical upper limit to the sum. Transferring the pivot to the center gives the sum between symmetric limits; corresponding expressions with cos, sin replaced by $|\cos|$, $|\sin|$ are given whenever the cam-switches in Fig. 7b are taken out of the circuit. A similar variation of the expression (1) also is thus obtained.

12. Convolution. In the circuit of Fig. 2c one may imagine that all the wires labeled a are cut, so that the lower potentiometers are connected to the upper ones only by the common ground. After this is done, suppose we reconnect the two left-hand wires only, then these two and also the next two, then the three left-hand pairs, and so on, until finally the original circuit is completely restored. A moment's reflection shows that this process is equivalent to varying the upper limit of the sum or integral computed by the machine; and hence every one of the integrals mentioned above may be replaced by $\int_a^w$ where w is a new parameter that may be varied at will. Similarly, the sums (11)-(13) are replaced by $\sum_1^k$ with k variable.

Instead of connecting the corresponding pairs in order as just described, one may connect the left-hand wire of the lower set to the right-hand wire of the upper set, then disconnect these and connect the two left-hand wires on the lower set to the two right-hand wires on the upper set, and so on. The process is readily visualized if we think of the lower group as sliding past the upper one. When f(x) represents the function

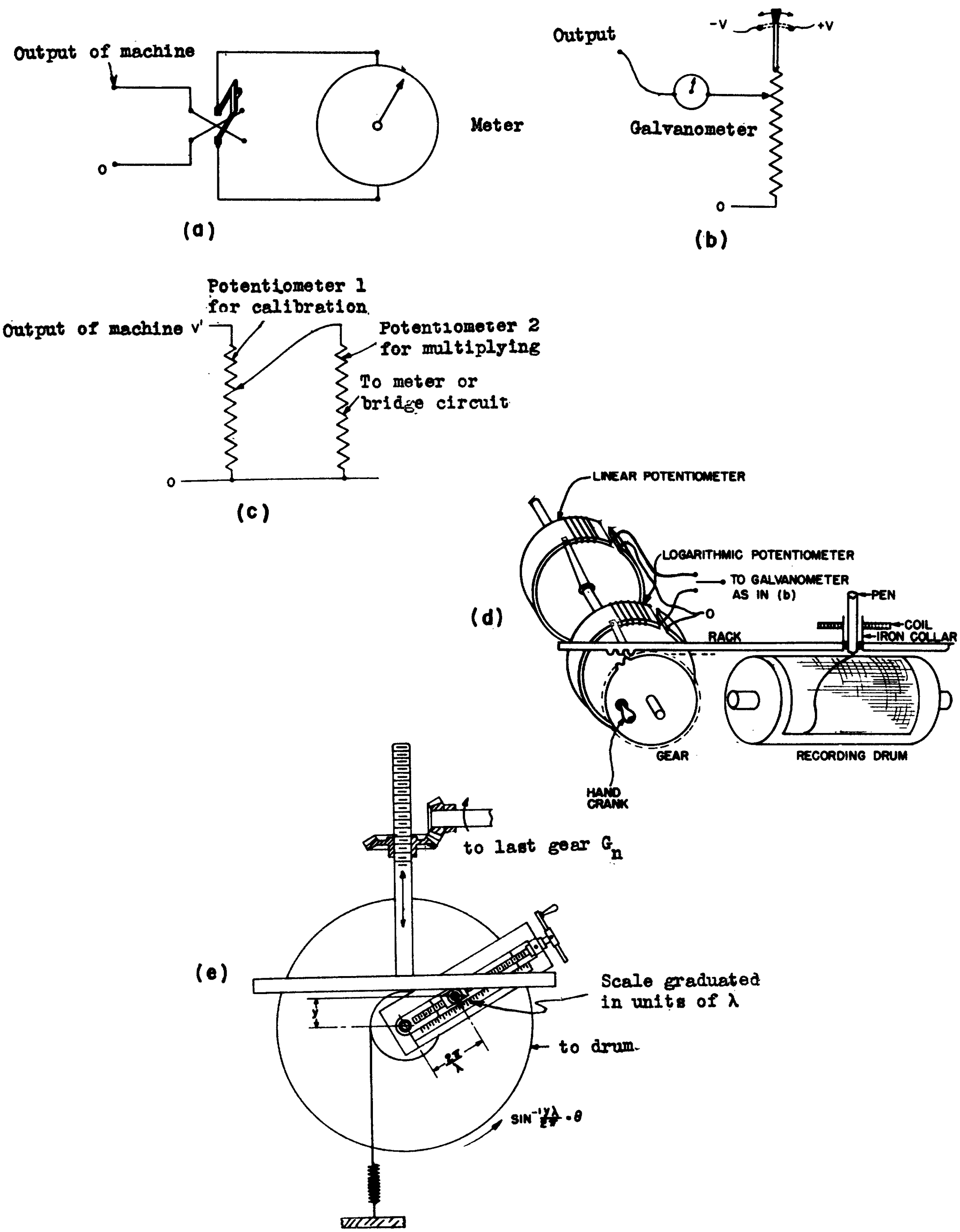


Figure 17. Problems connected with determination of output. a) Determination of output by means of a meter. The reversing switch is for negative values. b) Use of a bridge circuit for determining output. c) Circuit for calibration and multiplication. d) Arrangement for semi-automatic recording. e) An arangement for finding Θ from the equation $y = 2\pi/\lambda \sin \Theta$.

on the lower potentiometers, $F(x)$ that on the upper ones, then this second process gives

$$\int_a^b f(x)F(x - t)dx \qquad (14)$$

which is simply the convolution (Faltung) of the two functions f, F. In any of the integrals previously considered, then, the variable x in either one of the factors may be replaced by $x - t$, with t a new parameter that may be varied independently of the others. The most useful expression of this sort is that given in (14), rather than the corresponding form of (1); and hence the cams need not be used for such calculations. Their effect could be compensated for by performing a preliminary transformation on the function; that is, by putting $\cos^{-1} F(x)$ into the machine rather than $F(x)$, or which is the same thing, by plotting the original function F on suitably prepared graph paper. Such expedients are somewhat inconvenient, however, and the present machine is accordingly supplied with an extra table (see Figs. 14, 15). Functions placed on this table are inserted directly into the $F(x)$ potentiometers without intervention of the mechanism of Fig. 7b, and hence the integrals obtained are of the form (14) rather than (1).

The foregoing procedure gives the convolution (14) of two arbitrary functions as a function of t. Turning now to the question of mechanical design, we see that the arrangement suggested in Fig. 18 allows one to carry out the required operations, and is at the same time relatively simple to construct. The wires on the lower disk, which is fixed, go to the lower potentiometers in Fig. 2, while the wires on the upper disk, which is free to rotate through one revolution, are connected to the upper potentiometers. The disk in the center is also free to turn; it has a set of metal contacts, as shown, but requires no outside connections. The contacts are equally spaced, and the spacing, which must be the same on each disk, is such that slightly less than half of the circumference is required. For operation, we start with the three disks lined up. so that every contact on the lower disk is connected electrically to a contact on the upper disk. The complete circuit is then of the form shown in Fig. 2. To vary the upper limit w we turn the center disk, keeping the others fixed; to vary the parameter t we turn the top disk, keeping the lower ones fixed. It is clear that the two parameters t, w are completely independent: one may adjust the two lower disks to get an arbitrary value of w, then the top disk to vary t; or one may give an arbitrary value to t, and adjust the center disk to vary w.

Computation of (14) by the foregoing method makes no use of the linkage mechanism required for (1). In other words, the calculation could be effected by a machine having two units of the type shown in Fig. 13, plus one of the type shown in Fig. 18, and nothing else. The unit shown in Figs. 14-16 is really not required at all, and is used only because it happens to contain an array of potentiometers. In view of this simplicity, it is natural to inquire whether the expression (1) can perhaps be deduced from (14), with the result that the complicated unit of Figs. 14-16 (which accounts for by far the greater part of the cost of the machine) could be completely discarded. Such a transformation can indeed be carried out by suitable adjustment of the original functions and by choice of suitable curves for $F(x)$. Thus, in the next section we shall see that the expression

$$\int_a^b f(x)\ F(xt)\ dx \qquad (15)$$

can be obtained from (14); and this in turn is of the form (2). Calculation of the Fourier transform can therefore be made, in theory, by any machine which yields the expression (14). When we proceed from theory to practice, on the other hand, we find that such a procedure would be quite inferior to the standard methods now in use. Not only must one carry out all the preliminary calculations noted in connection with (2), but one must transform the original functions as described below. For large y the accuracy is poor; and, all things considered, the method appears so inconvenient that even manual computation (e.g., by means of vectors) would be preferable. Hence this procedure, which was contemplated before the unit of Figs. 14-16 was designed, is rejected. Though the parameter y is indeed (discontinuously) variable, the method is not in keeping with our basic aim, simplicity of operation.

13. Transformations of the Functions or Circuit. It has been assumed hitherto that the actual values of all functions and coefficients are put into the machine, without preliminary manipulation. Actually, however, the range of useful computations can be extended by use of suitable transformations, many of which are conveniently effected with non-linear graph paper. In the expression (14), for example, we may plot the two functions f and F on so-called semi-log paper, the logarithmic scale being used for the independent variable. Where we formerly obtained (14) we now obtain

$$\int \frac{f(x)}{x}\ g(xt')\ dx \qquad (16)$$

with $t' = e^t$ again an independent parameter. For the special case in which $g(x) = e^{-x}$ this expression reduces to the Laplace transform of $f(y)/y$,

$$\int \frac{f(x)}{x}\ e^{-xt'}\ dx\ , \qquad (17)$$

so that the transform of $f(x)$ can be obtained by carrying out the above operations on $xf(x)$.

Transformations of the type just described may be used in most devices for mechanical computation; they are not peculiar to the present machine, and are mentioned only on account of their great utility. There is another kind of transformation, however, which appears especially applicable to the particular machine here considered. Instead of using the circuit of Fig. 2c, one may equally well connect the potentiometers in other ways. Such a procedure would require a large but otherwise simple switch of conventional design, and presents no serious difficulties in either construction or operation. The expressions thus computed are of course different from those considered hitherto; and if the integrals contain a parameter (though not otherwise), the new class of functions is actually larger than the class obtained by using only the simple mathematical trans-

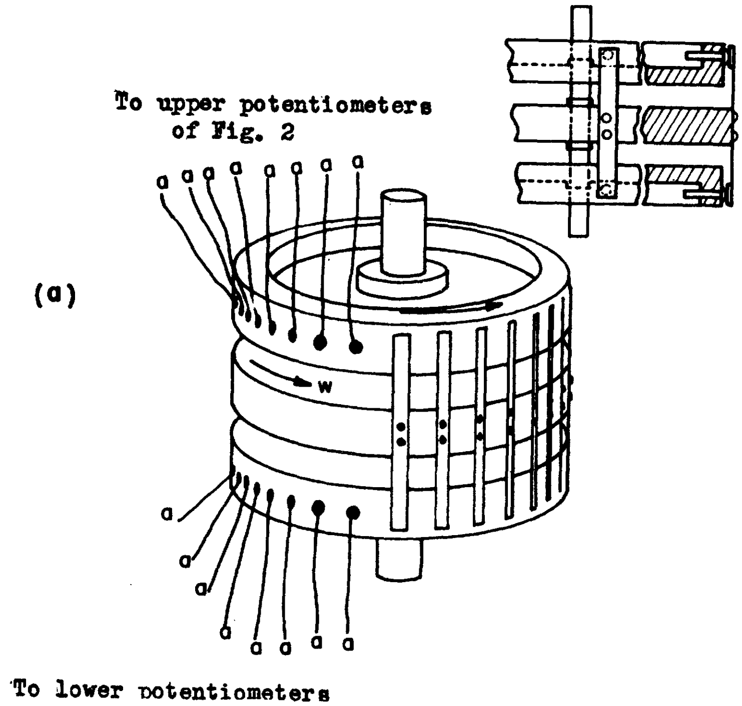


Figure 18. Switch for varying the upper limit in sums or integral, and for taking the convolution of two functions.

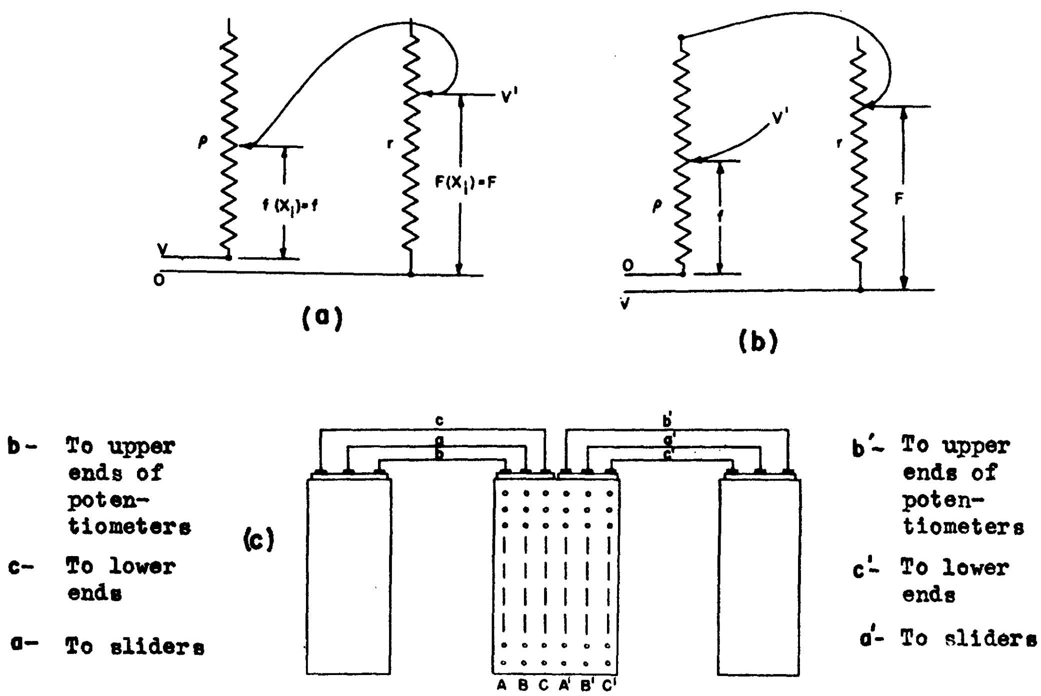


Figure 19. Examples of other connections possible with same potentiometers. a) Series connection. b) Another series connection. c) Method of making various connections.

formations mentioned above. Suppose, for example, that we revise the circuit as shown in Fig. 19a, connecting corresponding potentiometers in series rather than in tandem. It is easily seen that the voltage V' of that figure is given by

$$V' = \frac{rF}{rF + \rho f} V \tag{18}$$

so that, with choice of a suitable scale for f, one can evaluate integrals of the form

$$\int \frac{F(x)}{F(x) + af(x)} dx \tag{19}$$

where **a** is any constant. Similarly, if the potentiometers be connected as shown in Fig. 19b, then the voltage is

$$V' = \frac{F}{rf + \rho} V \tag{20}$$

and hence one obtains integrals of the form

$$\int \frac{F(x)}{f(x) + a} dx, \tag{21}$$

or of the form

$$\int \frac{F(x)}{g(x)} dx \tag{22}$$

by subtracting a suitable constant from the function g(x), that is, by displacing the graph paper through a constant distance ρ/r. Computation of (22) is subject to a natural restriction that f(x) be not too small; more precisely, the ratio of its maximum to its minimum value must never exceed $\rho/r = 1/15$.

These changes of circuit can be carried out by the device suggested in Fig. 19c. Each of the cables a, b, c, a', b', c' contains 45 separate conductors, one for each potentiometer. Similarly, the switchboard is supplied with 6 x 45 or 270 plugs, divided as shown into 45 rows. By connecting the plugs on the left to those on the right by cables, as in a manually-operated telephone switchboard, one can obtain every possible one-to-one connection of the potentiometers. This procedure allows the potentiometers to be separated into groups, different connections being permissible in different groups; and it would be useful for problems of the type considered below. In case we are concerned only with circuits hitherto described, however, in which the same connections are used for each pair of potentiometers, then the result can be obtained by connecting the cables a, b, c to a', b', c' directly, without use of the switchboard. For this reason it is desirable that the cables a, b, c, be supplied with male plugs while the others have female plugs. The original circuit of the machine, Fig. 2c, would then be obtained, for example, by connecting c and c' to ground (V = 0), while b is connected to V, a to b', and a' to the set of resistors R, which may be placed in a separate box and supplied with a plug for this purpose. If the input and output terminals of the switch (Fig. 18) are likewise connected to suitable plugs, then the switch may be inserted directly into the circuit at any desired point. For the convolution it would be inserted between a and b', the other connections being as described above.

There is one other modification of the circuit which is perhaps worth mentioning.

Since the potentiometer circuit generates the sum of products, one can obtain expressions of the form

$$\sum_{1}^{n} a_i b_i$$

by inserting the coefficients a_i on one set of potentiometers and the b_i on another set. As connected in Fig. 2, the circuit gives only a single expression of this form [Eq.(7)]; but if the units be divided internally as shown in Fig. 20a, then the number of sums is equal to the number of separate sets. By means of the switch A we can evaluate any of these sums independently, only one meter being required for the whole group. After thus subdividing the circuits by modifying the connections, one may unite corresponding potentiometers mechanically, in such a way that their sliders move in unison (see Fig. 20b). Alternatively, since only one of these potentiometers is required for any given position of the switch A, we may use a single group of six and switch this single group from one group of the upper potentiometers to the next, as switch A is changed. The need for the mechanical device of Fig. 20 is thus eliminated. This procedure, which was suggested to the author by J. C. Eaton of NRL, can be carried out by the switch of Fig. 18, if six extra contacts be added to the rotating disk. Whichever method is used, the expressions obtained are of the form

$$\sum_{i=1}^{p} a_{ij} u_i$$

with j taking the values 1, 2, ... as we proceed from one group to the next. By adjusting the first unknown u_1 to make the first sum equal to c_1, the second unknown u_2 to make the second sum equal to c_2, and so on, one can solve the system

$$\sum_{i=1}^{p} a_{ij} \bar{u}_i = c_j \quad (j = 1, 2, .. p).$$

The process is a well-known method of successive approximation, and converges rapidly whenever the coefficient matrix is suitably adjusted. Although found independently by the author in 1945, use of the circuit of Fig. 20a, or of one substantially equivalent to it, for solving equations in this way has been fully described elsewhere;[1] and hence, without entering into details, it suffices here merely to indicate the possibility of carrying out such calculations on the proposed machine. The number of equations solved is $[\sqrt{n}\,]$; with 45 potentiometers one could deal with six equations in six unknowns, while if the equations are adjusted to have at least four coefficients equal to zero, one can solve seven equations in seven unknowns.

It is evident that these and other forms of the circuit can be used in conjunction with the mathematical transformations just described, and that the two together can be combined with the mechanical units of Figs. 14 or 18 to give corresponding variations of all expressions hitherto considered. It should be noted too that the present discussion is representative only; of course one can make mathematical transformations different from the

1. Berry, E., et. al. loc. cit.

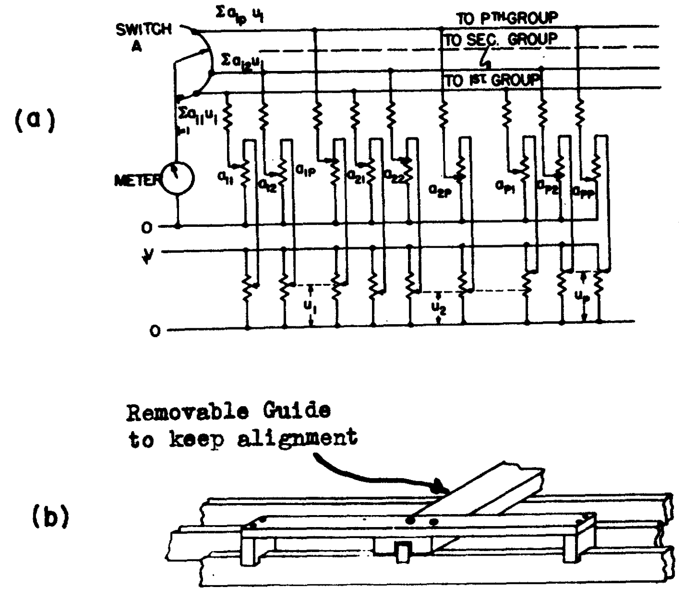


Figure 20. Arrangement of circuit for solving linear equations. a) Circuit. b) Method of moving potentiometer bars in unison.

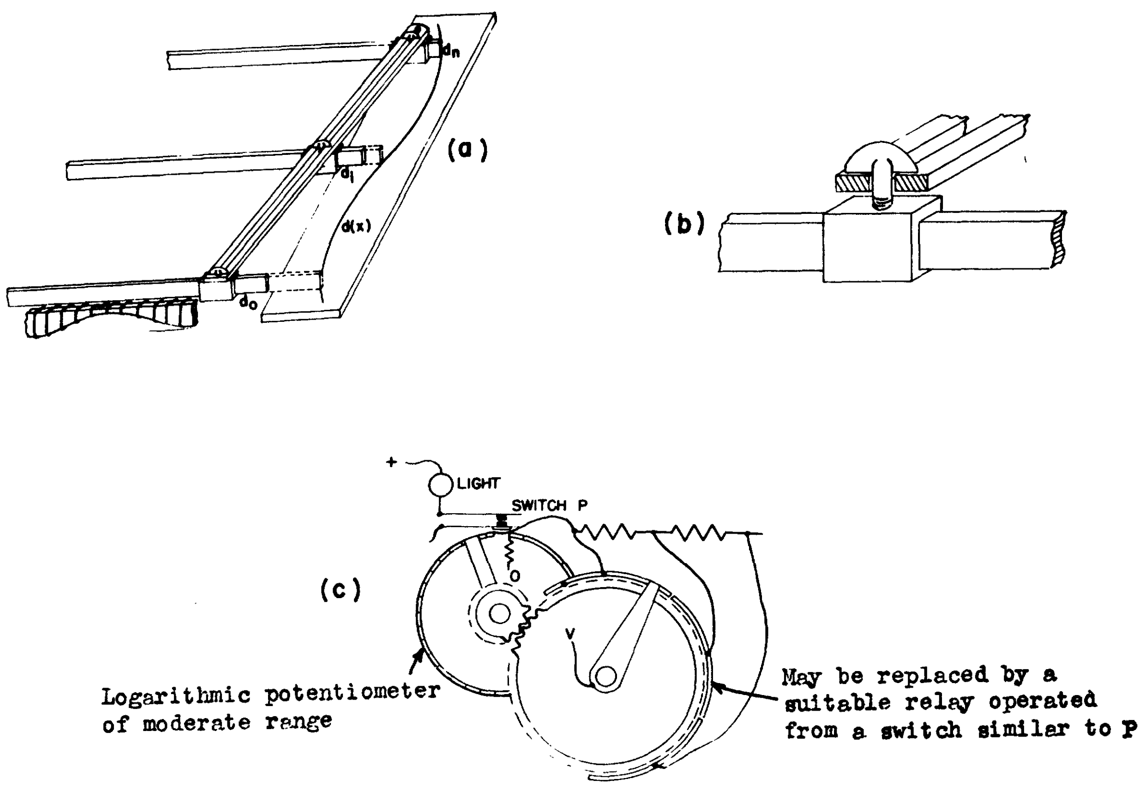


Figure 21. Use of non-linear potentiometers in connection with a simple linkage mechanism for introducing a parameter. a) General arrangement. b) Detail of device for locking bars to lever. c) A logarithmic potentiometer of large range.

substitution $f(x) \to f(e^x)$ considered above. Similarly, the possible forms of the circuit are not exhausted by Figs. 19, 20 even when there are only two sets of potentiometers; and if three or more sets are used, the variety is correspondingly increased. The device of Fig. 20c may be extended, in an obvious way, to give all possible connections for these more general circumstances; instead of six columns of plugs we should require nine for three sets of potentiometers, twelve for four, and so on.

14. Other Types of Potentiometers. The effect of the device in Fig. 7b is essentially to change a linear potentiometer to a sinusoidal one, that is, to a potentiometer in which the slider voltage varies as the sine of the displacement. This and linear variation form the only kinds hitherto considered, all the expressions noted above having been obtained either with linear potentiometers alone, or with linear and effectively sinusoidal ones in combination. When each potentiometer is adjusted manually, as for the insertion of $f(x)$, it is clear that nothing is gained by using a non-linear relation; for the same effect could always be obtained by plotting the function initially on suitable graph paper. When on the contrary the setting is determined mechanically, as for example in the unit of Fig. 7b, then substitution of non-linear for linear potentiometers can have effects not readily duplicated by any other procedure. This possibility, which is exemplified in the linkage mechanism described above, is perhaps worth considering in greater detail.

Suppose, then, that we use potentiometers for which the voltage varies according to the equation

$$V = P(d) \tag{23}$$

where d is the displacement of the slider. Such dependence is evidently obtained if the card upon which the wire is wound be shaped according to the equation $P'(d)$. With an initial displacement of d_i for the i-th potentiometer, the arrangement of Fig. 21 gives functions of the form

$$\sum_i P(iy + d_i), \tag{24}$$

when used in conjunction with the summing circuit previously described. The corresponding integral becomes

$$\int P[xy + d(x)]\ dx \tag{25}$$

if the number of elements is large, or if the calculation is repeated sufficiently often with a sideways shift of the functions. In this expression the function $P(x)$ is fixed for a given machine, and cannot be changed; but the curve $d(x)$ is an arbitrary function which need not be the same in different problems. Thus, $P(x)$ corresponds to cos x and $d(x)$ corresponds to $\phi(x)$ in the discussion given above. If the potentiometers are not identical the function P in (24) must be replaced by P_i. Similarly, if the potentiometers be used in conjunction with the linkage mechanism of Figs. 7, 9 rather than with that of Fig. 21a, then the functions $P(iy + d_i)$ are replaced by $P[\sin(iy + d_i)]$; this latter case is equivalent to a device using the simple mechanism of Fig. 21a and potentiometers with voltage versus distance given by $V = P(\sin d)$.

These general considerations can perhaps be illustrated by application to a specific example. A convenient expression for this purpose is the power series

$$\sum_0^n A_i\ z^i, \tag{26}$$

which also has a certain interest in its own right. At first confining our attention to the case in which both A_1 and z are real, we observe that a complete solution is given by the arrangement illustrated in Fig. 21, if the potentiometers are of the so-called logarithmic type,

$$P(d) = e^d,$$

and y is replaced by $\ln y$. The coefficients are introduced by taking $d_1 = \ln a_1$. To obtain logarithmic potentiometers of the required range one may proceed as suggested in Fig. 21c. An accurate logarithmic potentiometer is wound on a card which is later bent into a circle, the two ends of the winding being brought as close together as possible. The range may be any convenient value, which for definiteness we may assume to be 10:1. At each complete revolution of the potentiometer shaft the input voltage is to be reduced by a factor of ten, a requirement which may be met by use of a suitable set of fixed resistors and a simple switch. It is clear that the range thus attained may be as large as we please; a value of 50 or 100 db would doubtless be sufficient, however, for most applications. Of course the simple lever of Fig. 21a will no longer do for such an arrangement, and it is accordingly replaced by the mechanism of Fig. 16. The problem there encountered is identical with the present one, even as to details, and the solution previously obtained therefore applies without modification to the present case. We remark in passing that the potentiometer voltage will show an ambiguity of 10 db whenever the transition is made from one end of the card to the other. It is suggested, therefore, that each potentiometer be supplied with a switch which is on during this critical portion of its range. For all the potentiometers of the set, these switches are then connected in parallel, and the whole is connected in series with a light. No readings are to be taken when this light is on.

By the above procedure the sum of a real power series can be found as a continuous function of the variable; and the number of terms is restricted only by the number of (identical) potentiometers one wishes to use. Turning now to the less trivial case in which the coefficients and the variable $z = r\,e^{i\theta}$ may be complex, we find that this problem too depends for its solution only on principles already mentioned. Thus, the arrangement just described, Fig. 21a with $P(d) = e^d$, may be substituted for the f(x) unit of the foregoing discussion, Fig. 13. If the connections are made as before, we again obtain the sum of corresponding products, so that the machine now computes

$$\left.\begin{array}{l}\text{Real part}\\ \text{Imaginary part}\\ \text{Absolute value}\\ \text{Phase}\end{array}\right\} \text{ of } \sum_0^{44} \left[e^{(ny + a_n)}\right]\left[e^{i(ny' + b_n)}\right] \qquad (27)$$

instead of (1). By taking $a_n = \ln|A_n|$, $b_n = \underline{/A_n}$, $y = \ln|z|$, $y' = \underline{/z}$, we see that (27) reduces to (26), so that the sum of a power series has been obtained as a continuous function of the two parameters $|z| = r$, $\underline{/z} = \theta$. The logarithms need not be found in practice, incidentally, since the tables may be supplied with logarithmic rulings, or simply with a sheet of logarithmic paper, to permit direct insertion of $|A_n|$. Similarly, the scale giving the position of the lever for y would normally be logarithmic rather than linear, so that direct readings of $|z|$ would also be obtained. It is perhaps worth noting that the lever need not be pivoted at one end as shown in Fig. 21a, but can be pivoted instead at some intermediate point. The simple power series formerly obtained now becomes a Laurent series, with as many negative powers of z as there are potentiometers on the far side of the pivot.

15. Mechanical Additions. The modifications heretofore discussed have been of two kinds. The first kind, exemplified in the devices of Sec. 12, uses nothing but the original components, together with transformations of the input functions or of the circuit. The only extra items required are switches for carrying out the indicated operations. The second type of alteration is more extensive, since it requires construction of a whole new unit as, for example, the potentiometer unit described in Sec. 14. This second, more radical departure from our original design need not be confined to the potentiometers but can be extended to the linkage mechanism as well. In the course of antenna design, for example, one sometimes requires the secondary pattern produced by an illumination $f(x)\exp[i\phi(x)]$ on a cylindrical surface with polar equation $r = \rho(\theta)$. Such a calculation leads to an integral of the form

$$\int_a^b f(x) e^{i[\phi(x) + k\rho(x)\cos(x + y)]}\,dx \qquad (28)$$

for perpendicular polarization if one includes obliquity, and the same expression is obtained in the general case if obliquity is neglected. The only difference between this and the former expression (1), which is the analogous result for a plane aperture, is that we are now concerned with $\rho(x)\cos(x+y)$ where we previously had xy. Insertion of $f(x)$, $\phi(x)$, and generation of the exponential can be carried out, then, by the components previously described (Figs. 1-7); and it suffices to replace the lever of Fig. 9 by some system that will generate $\rho(x)\cos(x+y)$ as a continuous function of y. Such a device is represented in Fig. 22, where the guides for the flexible wires are placed along the curve $\rho(x)$ while the variable y is introduced by changing the indicated angle. For simplicity of fabrication one can group the adjacent wires as shown in Fig. 22c, the error introduced by non-parallelism being of the second order and completely negligible for most applications. Similarly, the rollers need not adjust themselves in such a way that the angle A is exactly equal to 90°; the relevant error is the error in length, rather than that in the angle. Use of a slotted metal sheet for holding the wire guides was suggested by H. Kylin.

Before the construction of Fig. 22 can be successfully operated, it must be coupled to the remainder of the machine; that is, one must arrange that the displacement of a given gear G_1 in Fig. 14 shall be proportional to the displacement of a corresponding wire in Fig. 22. In ordinary work the proportionality factor is rather large, so that a small motion of the wire must produce a large motion of the corresponding gear; and hence an amplification of torque will be required. We are thus led to the following problem:

a) The wires must be coupled to corresponding gears with amplification of torque.

b) The coupling must be reversible, so that the gear will turn backwards when the wire is slacked.

c) It must be possible to stack the succesive units conveniently.

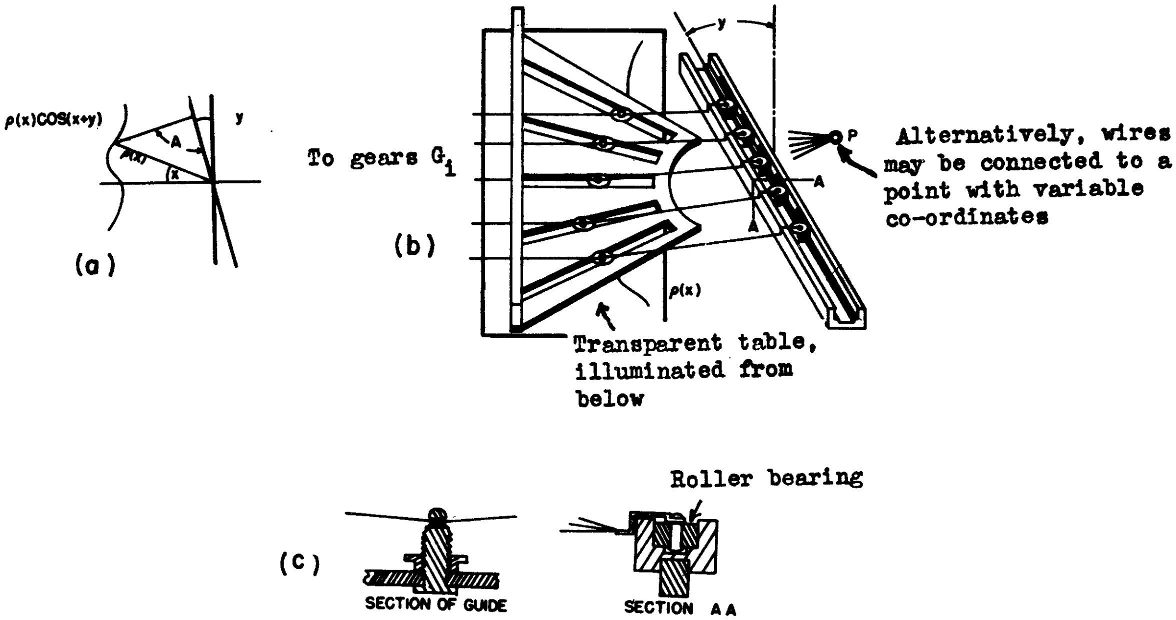


Figure 22. A method of obtaining displacements equal to $\rho(x_i)\cos(x_i+y)$ where $\rho(x)$ is an arbitrary curve. (a) Principle of operation, (b) actual arrangement, top view, (c) detail of guides and rollers.

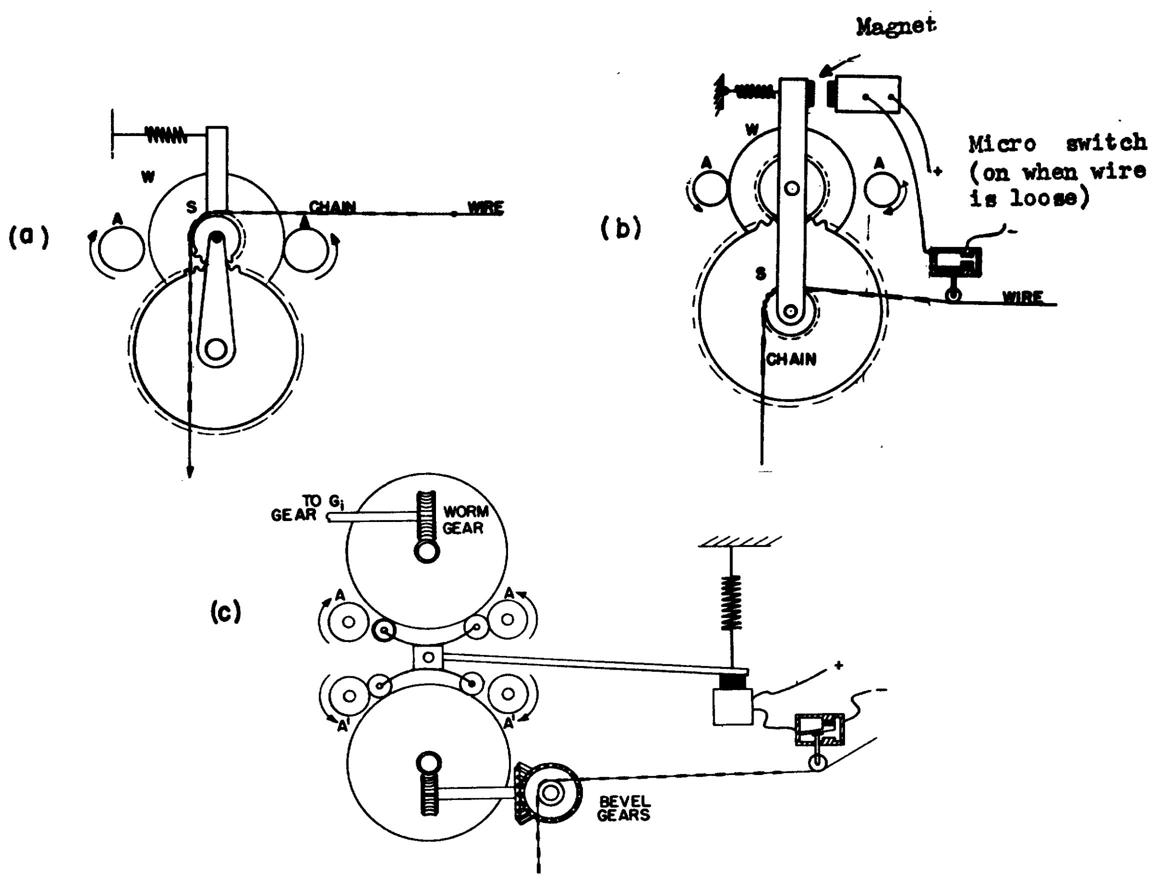


Figure 23. Methods of constraining a wheel to follow a wire in either direction with amplification of torque.

These requirements are met by the device illustrated in Fig. 23a. The shafts A are turned in the indicated directions by a motor, the torque being as large as is necessary for reliable operation. The two shafts shown and one motor serve for the entire assembly. It is readily seen that the gear will indeed follow the wire in either direction, as required, if the input tension is enough to couple the wheel W to the shaft A. The alternative arrangement of Fig. 23b avoids this difficulty, since the coupling is now effected by an electromagnet rather than by the tension of the wire. In either case the error cannot exceed the displacement required to actuate the mechanism once; it is not cumulative, and no error is introduced by slippage between the shafts and the wheel W.

With the foregoing device the parameter k is determined by the diameter of the sprocket S, and cannot be changed directly. This is not a serious disadvantage, since a change of k may be simulated by a change of scale in the curve $\rho(x)$. For taking the frequency dependence of an antenna pattern, however, one requires a set of curves for many values of k, and in this case a method of continuous or near-continuous variation would be desirable. Such a requirement is met, in principle, by the device shown in Fig. 23c. The ratio of wire motion to gear motion is proportional to the velocities of the two shafts A and A', and this ratio can be adjusted by change gears, only one gear box being required for the whole array. Unlike the simpler devices of Fig. 23a, b, however, this one leads to cumulative error. Thus, it is necessary that the time of contact be the same for the upper and lower rollers, and that the rates of slipping be equal. For quantitative investigation we observe that the problem is completely analogous to problem 3, page 147, of Uspensky's "Introduction to Mathematical Probability". Making use of the approximate results given on pages 153-154, one finds that the probability of a large error is very small whenever the initial errors are small, if positive and negative errors are equally likely. When this last condition is not satisfied, however, the cumulative error may become serious even in relatively short calculations, and hence the arrangement of Fig. 23c should be used only if each unit is accurately symmetrical. The present machine is so designed that any one of the three devices can be added at some future date if this should be found advisable.

It is worth noting that the wires in Fig. 22 can be connected to a point P rather than to the lever, as there indicated. In this case one obtains an expression which can sometimes be used for approximate evaluation of the field in the Fresnel region, although obliquity and inverse-distance attenuation are both neglected. The latter could be supplied by an extra bank of potentiometers, connected as described above for computation of (22), and arranged to have a slider motion proportional to the displacement of the corresponding wire in Fig. 23. It is doubtful, however, that such complications are economically justified. A similar construction can be used for calculating the field in the Fresnel region of a space curve, though the complications again are greater than the interest of the problem would appear to warrant.

Acknowledgment. The machine here described is being constructed by the Antenna Group of the Naval Research Laboratory under the direction of L. C. Van Atta. The author is responsible for the design only. The engineering and mechanical design is due to H. Kylin, now at the Naval Research Laboratory.

As far as principles are concerned, most of the devices here described were completed

in 1943; because of the pressure of war work, however, the subject was not pursued in detail until 1945, when it was reduced substantially to its present form. During this period (1943-1945) the author was working in the Antenna Group of the Radiation Laboratory, M.I.T., and he is glad of this opportunity to thank L. C. Van Atta, Group Leader of Group 54, and E. B. McMillan for the opportunity to carry out the research and development here described.