# THEORY OF THE STRUCTURE OF FERROMAGNETIC DOMAINS

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# Theory of the Structure of Ferromagnetic Domains in Films and Small Particles

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The theory is developed of the domain structure of ferromagnetic bodies whose smallest dimension is comparable with the thickness of the Weiss domains as found in crystals of ordinary size. Calculations of the domain boundary, magnetic, and anisotropy energies of various domain configurations are given for thin films, small particles, and long needles of ferromagnetic material. For sufficiently small dimensions the optimum structure consists of a single domain magnetized to saturation in one direction. This result implies unusual magnetic characteristics, such as have in fact been reported by a number of experimenters. The critical dimensions for transition from a configuration with domain structure to a saturated configuration are estimated $\sim 3 \times 10^{-5}$ cm in films and $\sim 2 \times 10^{-6}$ cm in particles or grains. These estimates are based on typical values of the relevant material constants, and may be increased or decreased by a factor of ten for other values of the constants.

THE present paper discusses the theory of the domain structure of ferromagnetic bodies whose smallest dimension is comparable with the thickness of the Weiss domains as found in crystals of ordinary size. The problem does not appear to have been considered previously. It will be shown that the normal macroscopic magnetization characteristics of ferromagnetic materials may be expected to alter radically as one or more dimensions of the test specimen are reduced below a critical length which is of the order of $10^{-5}$ to $10^{-6}$ cm. A number of experimental results in the literature are in agreement with the theory put forward here.

In the absence of an applied magnetic field it is a matter of common observation that the demagnetized state is the stable state in large ferromagnetic crystals. In the demagnetized state the domains are oriented so that the magnetic flux circuit lies almost entirely within the specimen. Internal flux closure implies a high degree of ordering in the domain configurations —this is contrary to the usual statement that in the demagnetized condition the domains are oriented at random. Geometrical configurations of domains in large crystals were first discussed theoretically in an important and often neglected paper by Landau and Lifshitz[1]; further contributions have been made by Kennard,[2] Lifshitz,[3] and Néel.[4]

As the dimensions of the specimen are diminished, the relative contributions of the various energy terms to the total energy are changed, and surface energies become more important than volume energies. The surface of the boundary (Bloch wall) between domains is a surface energy, while the energy in the magnetic field (magnetostatic self-energy) is a volume energy. When very small dimensions are reached, *there will be a point at which it is favorable energetically to do away with the domain boundaries, so that the whole specimen becomes one domain and acts as a permanent magnet.* This was first predicted by Frenkel and Dorfman.[5] Experimental evidence of permanent magnetization in small ferromagnetic particles was discovered by Elmore,[6] who worked with magnetite particles with a diameter of the order of $2 \times 10^{-6}$ cm.

The conditions in which the lowest energy state of a specimen represents a structure with a single domain are expected to differ somewhat in films, needles, and powders. In a film one dimension is small; in a needle or wire two dimensions are small, while in a powder particle of the material all three dimensions are small.

* Fellow of the John Simon Guggenheim Memorial Foundation.
[1] L. Landau and E. Lifshitz, Physik. Zeits. Sowjetunion 8, 153 (1935).
[2] E. H. Kennard, Phys. Rev. 55, 312 (1939).

[3] E. Lifshitz, J. Phys. U.S.S.R. 8, 337 (1944).
[4] L. Néel, J. de Physique [8] 5, 241, 265 (1944).
[5] J. Frenkel and J. Dorfman, Nature 126, 274 (1930); in this note the surface energy of the boundary between domains is overestimated by a factor of the order of 50, so that a critical length $L_c$ for particles $\sim 10^{-4}$ cm is obtained, instead of $\sim 2 \times 10^{-6}$ cm which we find. A correct method for treating the boundary surface energy was given first by F. Bloch, Zeits. f. Physik 74, 295 (1932).
[6] W. C. Elmore, Phys. Rev. 54, 1092 (1938); 60, 593 (1941).

## ENERGY RELATIONS

The energy relationships in ferromagnetic single crystals have been discussed by Landau and Lifshitz,[1,3] and Brown.[7] The results of these papers are used as a starting point for the present treatment. We shall assume that no applied magnetic field is acting, and that the specimens are each composed of an ideal crystal lattice. The effect of lattice distortions at the surface of the specimen is not considered. We shall make other simplifications which are pointed out below. These simplifications appear to be justified in a first approach to the subject and can be eliminated in favor of a more detailed treatment when adequate experimental data become available on the crystal orientation and mechanical structure of the specimens.

The terms in the free energy of the specimen which are relevant to the present problem may be represented as the sum

$$F = F_w + F_m + F_a, \tag{1}$$

where

$F_w$ = surface energy of the boundary surfaces between domains;
$F_m$ = magnetic field energy of the configuration;
$F_a$ = anisotropy energy of spin orientation.

The free energy is a minimum for a stable configuration* in the absence of external forces. The essential theoretical problem is to determine the balance between the three components of the free energy which will minimize the sum $F$. The determination of the configuration of domains which will minimize the free energy is a problem in the calculus of variations for which an exact solution has been found only in a few special cases. In other cases of interest we assume a solution which appears plausible on physical grounds and which contains one or more arbitrary parameters. The free energy is then minimized with respect to the parameters.

## Wall Energy $F_w$

In discussing the surface energy of the domain boundaries it is convenient to introduce the surface energy per unit area denoted by $\sigma_w$.

Naturally $\sigma_w$ will be a function of the crystallographic direction of the boundary plane and of the total change in spin direction across the boundary. Calculations by Lifshitz and others suggest values of $\sigma_w$ of the order of 1 to 5 ergs/cm². In the present paper the arbitrary value $\sigma_w = 3$ ergs/cm² will be used for the sake of concreteness in all cases. We may, therefore, write

$$F_w = \sigma_w S, \tag{2}$$

where $S$ is the total area of the domain boundaries within the body.

No account is taken of the special situation at boundary junctions, where there is a small additional energy caused by the finite limiting radius of curvature of the Bloch walls. This situation is probably responsible for the existence of narrow lines of free poles on the surface of magnetic specimens as revealed by colloid powder patterns.

## Magnetic Energy $F_m$

The magnetic field energy $F_m$ represents the work necessary to assemble the distribution of effective magnetic poles within and on the surface of the specimen, starting from a configuration with **H** everywhere zero. This energy is given by

$$F_m = -\tfrac{1}{2}\int (\mathbf{H} \cdot \mathbf{M}) dV, \tag{3}$$

where **M** is the magnetization or magnetic moment per unit volume and **H** is the magnetic field arising from the magnetization. The integral is to be carried out over the volume of the specimen. It should be emphasized that this relation supposes that there is no external applied magnetic field. A careful discussion of Eq. (3) has been given by Jeans and also by Guggenheim.[8]

The special case of a uniformly magnetized ellipsoid is of particular importance. Here a demagnetizing factor $N$ may be defined such that in the absence of an applied field the internal field is given by $H = -NM$, giving

$$F_m = NM^2 V/2, \tag{4}$$

[7] W. F. Brown, Jr., Phys. Rev. **58**, 736 (1940).

* It is supposed that the stable configuration may be attained by carrying the specimen through a suitable "demagnetization" process, as by cooling in field-free space from above the Curie point.

[8] J. H. Jeans *Electricity and Magnetism* (Cambridge University Press, Cambridge, 1933), fifth edition, p. 398; E. A. Guggenheim, Proc. Roy. Soc. **A155**, 49 (1936). Guggenheim takes B = 0 as his standard zero energy state; this convention is somewhat clumsy and for our purposes may be replaced by **H** = 0, so that the standard state is represented by a flux-closure configuration.
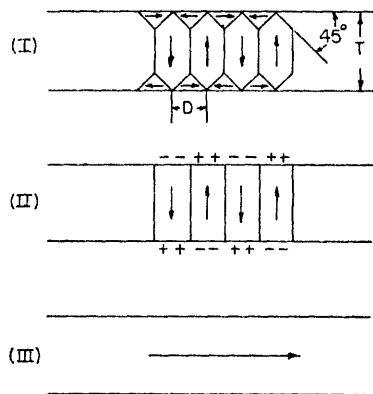
FIG. 1. Domain configurations in films.

where $V$ is the volume. Numerical values of the demagnetizing factors for the general ellipsoid have been tabulated by Osborn.[9]

In other cases of interest the magnetic energy is computed using

$$F_m = (1/8\pi) \int H^2 dV, \tag{5}$$

where now the integral is extended over all space. For parallel coplanar strips of width $D$ with magnetic pole densities alternately $+M$ and $-M$ per unit area the magnetic energy density per unit area $\sigma_m$ is given by

$$\sigma_m = 0.85 M^2 D. \tag{6}$$

This formula will be used later and is derived in Appendix A.

It is possible to devise domain configurations such that there are no magnetic poles within or on the surface of the specimen. An obvious example would be ring-shaped "domains" in a toroidal specimen. It is also possible to have sudden changes in spin direction between adjacent domains without giving rise to "free" magnetic poles. To have this. requires only the continuity of the normal component of the magnetization across the boundary.

### Anisotropy Energy $F_a$

The anisotropy energy arises from the existence of preferred axes of easiest magnetization in ferromagnetic crystals. The interactions responsible for anisotropy have been treated by Van Vleck,[10] Brooks,[11] and others. It will suffice

for the present purpose to suppose that there is an anisotropy energy density $\rho_a$ associated with domains oriented in directions away from an axis of easy magnetization, so that

$$F_a = \rho_a V_a, \tag{7}$$

where $V_a$ is the total volume of the domains in the specimen *not* oriented near an easy direction. Values[12] of $\rho_a$ are of the order of $10^5$ and $10^6$ ergs/cm$^3$. For the sake of concreteness we shall use the arbitrary value $\rho_a = 5 \times 10^5$ ergs/cm$^3$. Following Landau and Lifshitz we shall usually assume that the crystal is *uniaxial*, as in cobalt or in iron under tension. In general, as Lifshitz has shown, this assumption does not change the nature of the results, thanks to contributions from the magnetostrictive energy of the domains of closure. This energy we do not consider explicitly.

In treating thin films we suppose that the preferred axis is normal to the plane of the film: this would lead to erroneous results if there existed one or more other preferred axes in the plane of the film. In the case of iron this would occur if the [100] direction were normal to the film. In actual fact this orientation does not occur.[13]

### APPLICATION TO PARTICULAR CONFIGURATIONS

#### Films

We shall consider the energy of various configurations of the film of thickness $T$ shown in Fig. 1. The surface of the film is assumed to be perpendicular to the preferred direction of magnetization.

*Case I.* Figure 1-I shows a cross section of a structure which provides flux closure within the film. The domain width is $D$. The area of the Bloch walls per unit area of film is $2\sqrt{2} + (T-D)/D$, so that

$$F_w = \sigma_w [2\sqrt{2} + (T-D)/D], \tag{8}$$

where $\sigma_w$ is the surface energy density of the Bloch wall.

The triangular prisms contribute an anisotropy

[9] J. A. Osborn, Phys. Rev. **67**, 351 (1945).
[10] J. H. Van Vleck, Phys. Rev. **52**, 1178 (1937).
[11] H. Brooks, Phys. Rev. **58**, 909 (1940).

[12] R. Becker and W. Döring, *Ferromagnetismus* (Edwards Brothers, Ann Arbor, 1943).
[13] See, for example, O. Beeck, A. Wheeler, and A. E. Smith, Phys. Rev. **55**, 601A (1939); O. Beeck, Rev. Mod. Phys. **17**, 61 (1945).
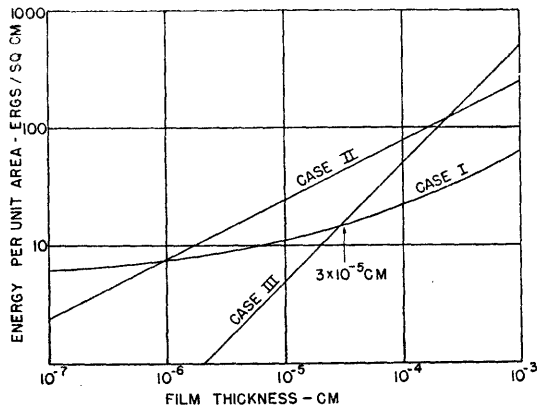
Fig. 2. Energy of various domain configurations in films.

energy given by the product of the anisotropy energy density $\rho_a$ and the volume of the prisms, which is $D/2$ per unit area of film, so that

$$F_a = \rho_a D/2. \tag{9}$$

The magnetic energy $F_m$ is zero for this configuration.

The total energy per unit area is then

$$F = \sigma_w[2\sqrt{2} + (T-D)/D] + \rho_a D/2. \tag{10}$$

This is a minimum with respect to the domain width $D$ when

$$D = (2\sigma_w T/\rho_a)^{\frac{1}{2}}, \tag{11}$$

so that

$$F_{\min} = (2\sqrt{2} - 1)\sigma_w + (2\sigma_w T\rho_a)^{\frac{1}{2}}. \tag{12}$$

*Case II.* Figure 1-II shows a structure for which the flux circuit is open. The magnetic energy per unit area is given by Eq. (6) with $M$ set equal to $M_s$, the saturation magnetization:

$$F_m = 1.7 M_s^2 D, \tag{13}$$

when it is remembered that the film has two surfaces. It is assumed here that $T > D$. The wall energy is

$$F_w = \sigma_w T/D. \tag{14}$$

The anisotropy energy $F_a$ is zero.

The total energy is a minimum for

$$D = (\sigma_w T/1.7 M_s^2)^{\frac{1}{2}}, \tag{15}$$

giving

$$F_{\min} = 2(1.7\sigma_w T)^{\frac{1}{2}} M_s. \tag{16}$$

*Case III.* In Fig. 1-III the film is uniformly magnetized in its own plane. If the film is very long in comparison with the thickness the

demagnetization factor and hence the magnetic field energy is small. The principal contribution to the energy comes from the anisotropy, supposing always that the plane of the film does not contain a direction of easy magnetization. We have

$$F = \rho_a T. \tag{17}$$

Numerical values of the energy for cases I, II, and III as a function of film thickness are plotted in Fig. 2, taking $\sigma_w = 3$ ergs/cm$^2$, $\rho_a = 5 \times 10^5$ ergs/cm$^3$ and $M_s = 1700$ gauss. It is seen that for films greater than $3 \times 10^{-5}$ cm in thickness, case I has the lowest energy of the three cases considered. This is not surprising, as case I is the structure evolved by Landau and Lifshitz in their original study of large uniaxial crystals. It is interesting to note that an experimental demonstration of the existence of "domains of closure" has been given by Shoenberg and Wilson[14] by means of magnetic torque measurements on a Si-Fe crystal.

For thinner films case III has the lowest energy. If the demagnetization factor parallel to the surface is not negligible, because of the finite extent of the film, then the case I structure will persist to smaller dimensions than calculated above. Case II occurs only with very much larger anisotropy.*
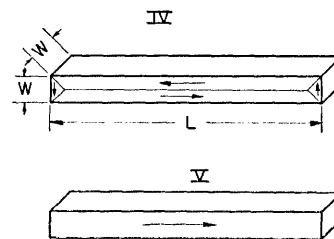


Fig. 3. Domain configurations in needles.

[14] D. Shoenberg and A. J. C. Wilson, Nature **157**, 548 (1946).

* In cobalt the anisotropy is unusually pronounced: $\rho_a = 5.1 \times 10^6$ ergs/cm$^3$, according to R. Gans and E. Czerlinsky, Ann. d. Physik [5] **16**, 625 (1933). The saturation magnetization at room temperature is $M_s = 1400$ gauss, according to K. Honda and H. Masumoto, Sci. Rep. Tôhoku Univ. **20**, 323 (1931). In the limit of large crystals we have for case I the value $F(I) = 3.2(\sigma_w T)^{\frac{1}{2}} \times 10^3$ ergs, and for case II the value $F(II) = 3.6(\sigma_w T)^{\frac{1}{2}} \times 10^3$ ergs, so that there is not much difference energetically between the two configurations. If the constants are varied slightly so that $F(I) = F(II)$, it may be shown that the most favorable configuration is somewhat intermediate between these two cases, with part of the unit flux circuit completed internally and part giving rise to a free pole distri-

At the critical thickness of $3\times10^{-5}$ cm the domain width for case I is $2\times10^{-5}$ cm.

## Needles and Wires

The general relationship between the energies of structures I, II, and III may be expected to hold also in long wires, provided that small adjustments are made in numerical factors.

The relations for needles are of somewhat greater interest. For the sake of example the anisotropy energy is omitted.

*Case IV.* The needle (Fig. 3-IV) of dimensions $L\times W\times W$ consists of two long domains with the flux circuit closed internally by triangular caps. The energy is largely in the wall separating the long domains, and is given approximately by

$$F=\sigma_w LW. \qquad (18)$$

*Case V.* The needle (Fig. 3-V) consists of one domain magnetized along the length of the needle. The energy is entirely magnetic and is given by

$$F=NM_s^2 LW^2/2, \qquad (19)$$

following Eq. (4).

The critical dimension below which case V is the probable state is given by $W_c=2\sigma_w/NM_s^2$. Values of $W_c$ and the associated $L_c$ are given in Table I, which is calculated for $\sigma_w=3$ ergs/cm$^2$, $M_s=1700$ gauss.

The values of the demagnetizing factor were taken to be equal to the values for a prolate spheroid with the same axial ratio.

## Particles

*Case VI.* Figure 4-VI represents a cube of side $L$. The particle consists of one domain. The

TABLE I.

| $L/W$ | $N$ | $W_c$ (cm) | $L_c$ (cm) |
|---|---|---|---|
| 5 | 0.702 | $2.8\times10^{-6}$ | $1.4\times10^{-5}$ |
| 10 | 0.255 | $7.9\times10^{-6}$ | $7.9\times10^{-5}$ |
| 15 | 0.135 | $1.5\times10^{-5}$ | $2.2\times10^{-4}$ |
| 20 | 0.085 | $2.4\times10^{-5}$ | $4.7\times10^{-4}$ |

bution. The domains of partial flux closure occupy the fraction $P$ of the surface, where $\sin(\pi P/2)\approx 4P/\pi$; the root is $P\leqq0.7$. L. H. Germer, Phys. Rev. **62**, 295A (1942) has studied by electron beams the magnetic fields at the hexagonal surface of a "demagnetized" cobalt crystal, and finds indication of a non-closure or type II configuration. The possibility of non-closure configurations was apparently overlooked by Landau and Lifshitz. I wish to thank Dr. Germer for a discussion of the results on which his published abstract is based.
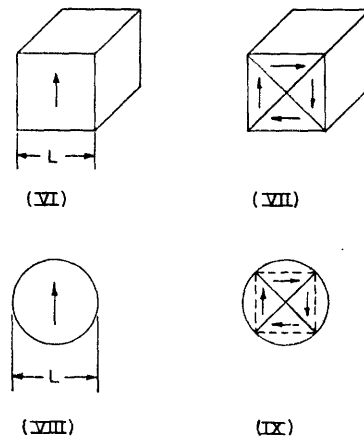


FIG. 4. Domain configurations in small particles.

energy is entirely magnetic and is given by

$$F\cong(2\pi/3)M_s^2 L^3, \qquad (20)$$

where the effective demagnetization factor has been taken as $4\pi/3$, the value for a sphere.

*Case VII.* Figure 4-VII shows a cube with domains arranged to give internal flux closure. The magnetic energy is zero. The wall energy is

$$F_w=\sigma_w 2\sqrt{2}L^2. \qquad (21)$$

The anisotropy energy is approximately

$$F_a=\rho_a L^3/2 \qquad (22)$$

and is relatively small.

The critical diameter below which case VI becomes the probable configuration is given by

$$L_c=\frac{2\sqrt{2}\sigma_w}{(2\pi/3)M_s^2-\rho_a/2}\cong1.5\times10^{-6}\text{ cm}, \qquad (23)$$

using the numerical values previously assumed.

The relations for the small sphere pictured in Figs. 4-VIII and 4-IX are similar to the relations for cubes. The energy in case VIII is magnetic and equal to $L^3/9$. In case IX most of the energy is in the walls, and $F_w=\pi\sigma_w L^2/2$. The flux closure in case IX is not complete. Fields of octopole symmetry arise from the volume between the surface of the sphere and the boundary of an inscribed cube. The critical dimension for a sphere is close to that for a cube.

For particles just larger than the critical dimension the domain structure shown in Figs. 4-VII and 4-IX should not be interpreted too

literally, since here the width of a Bloch wall may be of the same order of magnitude as the domain dimensions. In this case there will not be a sharply defined boundary between discrete domains, but there will be rather a gradual rotation of spin direction within the particle.

## EXPERIMENTAL RESULTS

The theory developed above suggests that films $\sim 10^{-5}$ cm thick and small particles $\sim 10^{-6}$ cm across should exhibit quite different magnetic properties from bulk specimens of ferromagnetic materials. The evidences we look for include:

(A) *Permanent saturation magnetization.* Specimens consisting of a single domain should behave as if magnetized to saturation in one direction. It is supposed that ordinary remanent effects have been distinguished by carrying the material through a demagnetization process, as by cooling from above the Curie point in field-free space.

(B) *High coercive force.* If no domain boundaries are formed the only magnetization changes in a specimen occur through spin rotation. Spin rotation is opposed by the anisotropy forces, which are usually much greater than the local forces opposing movement of a domain boundary. With decreasing particle size it may therefore be expected that the coercive force will increase and approach the anisotropy force in value.

(C) *Low initial permeability.* If there are no domain boundaries the magnetization changes in weak fields must come from reversible spin rotations. The associated permeabilities are quite low.

Relevant experimental results are reviewed briefly below. The results as a whole are in very good agreement with theory.

(1) Elmore[6] has obtained positive evidence that small particles behave as permanent magnets. He studied the susceptibility of colloidal suspensions of iron oxides. On the assumption[15] that the particles behave as the molecules of a classical paramagnetic gas he is able to fit the experimental susceptibility curve. Magnetite ($Fe_3O_4$) particles of mean diameter $1.7 \times 10^{-6}$ cm show an average permanent magnetization equal to 44 percent of the saturation value. Siderac

($\gamma$-$Fe_2O_3$) particles of mean diameter $2.1 \times 10^{-6}$ show 38 percent of the saturation value. The discrepancy between the observed and the saturation values has been attributed to the presence of non-ferromagnetic oxides of iron in the colloid. Heaps[16] has also detected a tendency of particles to agglomerate in groups which were only partly magnetic. Consideration shows that group formation is likely to occur when the magnetic bond energy between particles is large in comparison with $kT$. This is satisfied in the conditions of Elmore's experiments.

(2) Montgomery,[15] Heaps,[16] Benedikt,[17] and Mueller and Shamos[18] have made measurements by various methods using particles larger than those used by Elmore. The particles averaged between $10^{-3}$ and $10^{-5}$ cm. The interpretation of some of the results is complicated. In general there is an indication of permanent magnetism, but less than in Elmore's measurements. The theory predicts a critical dimension $\sim 2 \times 10^{-6}$ cm for particles, so that the measurements fit into the expected pattern when allowance is made for dispersion in particle size and for the unavoidable presence of some remanent magnetization in the larger particles.

(3) In the older literature[19-22] a considerable amount of work is reported on the properties of iron and nickel films deposited under the influence of weak magnetic fields in the plane of the film. The effects observed, such as quadrilateral hysteresis loops, are generally compatible with the concept that thin films consist of a single domain. It would be valuable to repeat some of the early work using electron diffraction methods[23] to give some control of the quality of the film deposited.

(4) High values of the coercive force ($\sim 300$ gauss) in small particles and thin films have been measured by Sappa,[24] Gottschalk,[25] Procopiu,[26]

[15] C. G. Montgomery, Phys. Rev. **38**, 1782 (1931); **39**, 162 (1931).

[16] C. W. Heaps, Phys. Rev. **57**, 528 (1940).
[17] E. T. Benedikt, J. App. Phys. **13**, 105 (1942).
[18] H. Mueller and M. Shamos, Phys. Rev. **61**, 631 (1942).
[19] C. Maurain, J. de phys. et rad. [3] **10**, 123 (1901); [4] **1**, 90, 151 (1902).
[20] W. Leick, Wied. Ann. **58**, 691 (1896).
[21] K. Schild, Ann. d. Physik [4] **25**, 586 (1908).
[22] W. Kaufmann and W. Meyer, Physik. Zeits. **12**, 513 (1911).
[23] L. Germer, Phys. Rev. **55**, 605 (1939); **56**, 58 (1939).
[24] O. Sappa, Ricerca Scient. **8**, 413 (1937).
[25] V. H. Gottschalk, Physics **6**, 127 (1935).
[26] S. Procopiu, J. de phys. et rad. [7] **5**, 199 (1934).

...and Maurain.[19] The result that small grains of a material show high values of the coercive force has been applied to the selection of materials for permanent magnets.

(5) Low values of the initial permeability have been reported for thin films by Florescu[27] and for small particles by Gottschalk and Wartman,[28] and Chevalier and Mathieu.[29] Florescu finds $\mu = 3.5$ for an iron film $8 \times 10^{-6}$ cm thick at a frequency near 100 kc/sec. An iron wire $5 \times 10^{-3}$ cm in diameter at the same frequency gave $\mu = 69.2$. Gottschalk and Wartman find $\mu = 4.62$ for Ural Mountain magnetite of grain size $180 \times 10^{-4}$ cm, and $\mu = 2.53$ for grain size $3.5 \times 10^{-4}$ cm.

## CONCLUDING REMARKS

The study of the magnetic behavior of thin films and small particles appears to offer considerable promise of elucidating some of the problems of domain theory. It would be of particular interest to study magnetization processes in films using a number of different techniques, including magnetometer measurements, electron diffraction studies of the surface structure, and magneto-optic and colloid pattern studies of the domain structure. In this way we might hope to establish quantitative relationships between domain displacements and magnetization in configurations which may be amenable to theoretical treatment. It must be emphasized, however, that the film technique will only be useful if a reasonably homogeneous film can be prepared, and one which is tolerably free from internal stresses.

[27] N. A. Florescu, Comptes rendus (Paris) **208**, 1717 (1939).

[28] V. H. Gottschalk and F. S. Wartman, U. S. Bureau of Mines Report of Investigations **3268**, 67 (1935).

[29] R. Chevalier and S. Mathieu, Comptes rendus (Paris) **204**, 854 (1937).

Measurements of the transmission of microwaves through thin films are of particular interest in the study[30] of the dispersion of ferromagnetism at high frequencies.

I am indebted to Dr. R. M. Bozorth of the Bell Telephone Laboratories for discussion on the subject of this paper.

## APPENDIX A

### Magnetic Field Energy of Coplanar Strips of Alternate Sign

Let the plane of the strips be the $(x, y)$ plane with the $y$ axis parallel to the axis of the strips; the width of a single strip is $D$, and the pole strength per unit area of a strip is $\pm M$. The vertical or $z$ component of the magnetic field directly above the plane of the strips is given by the Fourier expansion of a square-wave of amplitude $2\pi M$. The appropriate solution of the Laplace equation is

$$H_z = 2\pi M[(4/\pi) \sin kx e^{-kz} + \text{terms in odd multiples of } k],$$

where $k = \pi/D$. For the present the harmonic terms in the expansion will be neglected. To satisfy div $\mathbf{H} = 0$ we must have

$$H_x = -2\pi M[(4/\pi) \cos kx e^{-kz}].$$

The energy per unit area above the strips is given, using Eq. (5), by $(1/8\pi) \int_0^\infty H^2 dz$, so that the surface energy density $\sigma_m$ is given by

$$\sigma_m = (1/4\pi)(2\pi M)^2 (4/\pi)^2 \int_0^\infty e^{-2kz} dz,$$

when both faces of the plane of the strips are taken into account. This gives $\sigma_m = (8/\pi^2) M^2 D$, which is the contribution from the first order term alone. The complete expression including the effect of the harmonic terms is obtained by multiplying by $\Sigma n^{-3}$, where the sum is over odd integers and is approximately equal to 1.05. We have finally

$$\sigma_m \cong 0.85 M^2 D,$$

as given in Eq. (6), and in agreement with Néel.[4]

The treatment of double layers of charged strips proceeds similarly, but is somewhat more complicated.

[30] C. Kittel, Phys. Rev. **70**, 281 (1946). An interesting dispersion calculation by G. Heller is referred to by M. J. O. Strutt and K. S. Knol, Physica **7**, 635 (1940).