A q-analogue of Spanning Trees: Nilpotent Transformations over Finite Fields

by

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Abstract

The main result of this work is a q-analogue relationship between nilpotent transformations and spanning trees. For example, nilpotent endomorphisms on an n-dimensional vector space over $\mathbb{F}_q$ is a q-analogue of rooted spanning trees of the complete graph K_n. This relationship is based on two similar bijective proofs to calculate the number of spanning trees and nilpotent transformations, respectively.

We also discuss more details about this bijection in the cases of complete graphs, complete bipartite graphs, and cycles. It gives some refinements of the q-analogue relationship. As a corollary, we find the total number of nilpotent transformations with some restrictions on Jordan block sizes.

Thesis Supervisor: Richard P. Stanley
Title: Levinson Professor of Applied Mathematics

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Contents

Chapter 1

Introduction

The problem of enumerating the number of nilpotent matrices with certain restrictions over a finite field has attracted great attention in the literature.

In 1958, N.J. Fine, I.N. Herstein in [2], for the first time, found out that the number of nilpotent $n \times n$ matrices over the q-element finite field $\mathbb{F}_q$ is $q^{n(n-1)}$, by considering the decomposition according to Jordan canonical form. Later in 1961, M. Gerstenhaber gave another proof of it suggested by algebraic geometry in [3]. That was not the end of the story. After several years, in 1987, A. Kovacs considered the problem of when the product of k $n \times n$ matrices will be nilpotent, and gave a solution in [5] and [6]. There are more related results.

In general, people are interested in problems of the following form:

Problem. *Consider all nilpotent matrices over $\mathbb{F}_q$ of a fixed size with some entries set to be zero. How many are they?*

For instance, the problem of when the product of k $m \times m$ matrices, $A_1 A_2 \cdots A_k$, will be nilpotent is equivalent to consider when the following block matrix A is nilpotent:

$$A = \begin{pmatrix} 0 & A_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & A_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & A_{k-1} \\ A_k & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Let's start the discussion with some examples first.

Bad Example. Consider nilpotent 3×3 matrices of the following form:

$$A = \begin{pmatrix} * & 0 & * \\ 0 & * & * \\ * & * & 0 \end{pmatrix},$$

where $*$ denotes a entry from $\mathbb{F}_q$. One can easily compute the total number of nilpotent matrices over $\mathbb{F}_q$ of the above form is:

$$\begin{cases} 2q^3 - 2q^2 + 2q - 1, & \text{if } q \text{ is odd,} \\ q^3 + q^2 - 1, & \text{if } q \text{ is even.} \end{cases}$$

It is not a polynomial of q.

OK Example. Consider nilpotent $(n+1) \times (n+1)$ matrices of the following form:

$$A = \begin{pmatrix} * & \cdots & * & * \\ \vdots & \vdots & \vdots & \vdots \\ * & \cdots & * & * \\ * & \cdots & * & 0 \end{pmatrix},$$

where $*$ denotes a entry from $\mathbb{F}_q$. Define $\mathcal{M}_n$ to be the set of those matrices, i.e., all nilpotent $(n+1) \times (n+1)$ matrices $A = (a_{i,j})$ over $\mathbb{F}_q$ such that $a_{n+1,n+1} = 0$. We want to calculate the cardinality of $\mathcal{M}_n$.

Let $\mathcal{A}$ (resp. $\mathcal{B}$) be the subset of $\mathcal{M}_n$ such that the last row of any matrix in $\mathcal{A}$ (resp. in $\mathcal{B}$) is nonzero (resp. zero). We have $\mathcal{M}_n$ is the disjoint union of $\mathcal{A}$ and $\mathcal{B}$.

Let $\mathcal{C}$ (resp. $\mathcal{D}$) denote the set of all nilpotent $(n+1) \times (n+1)$ matrices $A = (a_{i,j})$ over $\mathbb{F}_q$ such that $(a_{n+1,1}, a_{n+1,2}, \ldots, a_{n+1,n})$ is not a zero vector (resp. is a zero vector). Hence, the set of all nilpotent $(n+1) \times (n+1)$ matrices over $\mathbb{F}_q$ is a disjoint union of $\mathcal{C}$ and $\mathcal{D}$. We have:

$$\#\mathcal{C} + \#\mathcal{D} = q^{n(n+1)},$$

where $\#S$ denotes the cardinality of set S. Since $A = (a_{i,j}) \in \mathcal{D}$ is nilpotent if and only if $a_{n+1,n+1} = 0$ and $(a_{i,j})_{i,j=1}^{n}$ is nilpotent, we have $\mathcal{B} = \mathcal{D}$ and:

$$\#\mathcal{B} = \#\mathcal{D} = q^{n(n-1)} \cdot q^n = q^{n^2} \quad \Rightarrow \quad \#\mathcal{C} = q^{n(n+1)} - q^{n^2}.$$

Define a map $\pi : \mathcal{A} \times \mathbb{F}_q^n \to \mathcal{C}$ by:

$$A \times v \mapsto \begin{pmatrix} I_n & v \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} I_n & -v \\ 0 & 1 \end{pmatrix}.$$

It is not hard to see that π is a q^{n-1} to 1 map. That implies:

$$\frac{\#\mathcal{A} \cdot q^n}{q^{n-1}} = \frac{\#\mathcal{C}}{1} \quad \Rightarrow \quad \#\mathcal{A} = q^{n(n+1)-1} - q^{n^2-1}.$$

Hence, the cardinality of $\mathcal{M}_n$ is $q^{n(n+1)-1} - q^{n^2-1} + q^{n^2}$, a polynomial in q.

Motivated by these two examples, we want to look at those "OK Examples". That is, we want to consider restrictions under which the number of nilpotent matrices over $\mathbb{F}_q$ is a polynomial in q.

Perfect Example. In M.C. Crabb's paper [1], he used a combinatorial method to calculate the number $N_q(n)$ of nilpotent endomorphisms on a n-dimensional vector space V_n over $\mathbb{F}_q$, i.e., nilpotent $n \times n$ matrices, and used an analogous method to count the number $N(n)$ of rooted spanning trees of complete graph K_n. From the result he found that the set of nilpotent transformations is a *"q-analogue"* of the set of rooted spanning trees, i.e., $N_q(n)$ is a "q-analogue" of $N(n)$:

$$N(n) = n^{n-1} = (\mathbf{n})^{n-1} \rightsquigarrow_q (\mathbf{q^n})^{n-1} = q^{n(n-1)} = N_q(n),$$

where $\mathbf{n}$ and $\mathbf{q^n}$ are the sizes of K_n and V_n, respectively.

Inspired by this, we want to focus on "Perfect Examples", instead of "OK Examples". That is, we want to consider restrictions under which not only the number of nilpotent transformations, equivalent to nilpotent matrices, over $\mathbb{F}_q$ is a polynomial in q, but also there exists a natural q-analogue relationship between spanning trees of some graph and those nilpotent transformations.

The "Perfect Example" is: given a digraph G with certain properties, we replace each vertex with a vector space, and consider the the nilpotent transformation that "maps along" the edges of G. These nilpotent transformations are the q-analogue of spanning trees of the *expanded digraph*, which can be get by replacing each vertex of G with given number of vertices and connecting them "corresponding to" edges of G. We calculate the total number these special nilpotent transformations and spanning trees in Theorem 2.4 and show the q-analogue relationship in Corollary 2.5.

Chapter 2

Definitions and Main theorems

2.1 Basic Definitions

We use the standard notations following Stanley [7]: let q be a fixed prime power and $\mathbb{F}_q$ denote the q element finite field. (All the vector spaces we talk about are over the field $\mathbb{F}_q$.) Let $\mathbb{N}$ and $\mathbb{P}$ denote the set of nonnegative integers and positive integers, respectively, and $[n] = \{1, 2, \ldots, n\}$, where $n \in \mathbb{P}$. For any finite set S, we let $\#S$ denote its cardinality. For any two finite set S and S', define $S \sqcup S'$ to be the disjoint union of S and S'. For a map $f : S_1 \to S_2$, where S_1 and S_2 are two sets, let $f(S)$ denote the image of $S \subset S_1$ under f in S_2, and if $S_1 = S_2$ let f^k be f composed with itself k times, for $k \in \mathbb{N}$.

Next let us recall some basic definitions about linear algebra and graph theory. We say a linear endomorphism f on vector space U is *nilpotent* if there exists some large enough $k \in \mathbb{P}$ such that $f^k = 0$.

About graph theory, we are mainly interested in *directed graphs* or *digraphs*. A directed graph or digraph G is a pair (V, E), where $V = [m]$ is a set of *vertices*, E is a set of (directed) *edges*, and the edge *from i to j*, i.e., with *initial vertex i* and *final vertex j*, where $i, j \in V$, is represented as $i \to j$. If $i = j$, then the edge is called a *loop*.[1] Let $\mathrm{Out}(i) = \{j \in V : i \to j \in E\}$. A *path* Γ in G *from i to j* is a sequence $i = i_0, i_1, \ldots, i_d = v$ such that $d \in \mathbb{P}, \{i_h : 0 \leq h \leq d\} \subset V$ and

[1] In fact, it means that the digraph can have loops but no multiple edges with the same orientation.

$\{i_h \to i_{h+1} : 0 \leq h < d\}$ are distinct edges of G that are not loops. We call it the *length* of Γ. An (oriented) *tree with root* i is a digraph T with i as one of its vertices, such that there is a unique path from any vertex j to i. An (oriented) *forest* F *with root set* I is a collection of disjoint trees with I as the collection of the roots of them. A *spanning* tree (resp. forest) of a digraph G consists of all the vertices and some edges of G, such that it forms a tree (resp. forest).

We will be interested in a special kind of digraph. In a digraph G, let $\operatorname{outdeg}(i) = \#\{j \in V : i \to j \in E\}$ (resp. $\operatorname{indeg}(i) = \#\{j \in V : j \to i \in E\}$) denote the *outdegree* (resp. *indegree*) of vertex i, and $\operatorname{outdeg}(G) = \max\{\operatorname{outdeg}(i) : i \in V\}$. The digraphs G we will consider are the ones such that $\operatorname{outdeg}(G) = 1$, called a *unidigraph*. Clearly, a tree or a forest satisfies this condition. In this case, we define $o(i)$ be the unique vertex, if exists, in $\operatorname{Out}(i)$, otherwise let $o(i) = 0$.

An example of a unidigraph G_0 is given in Figure 2-1. In Figure 2-2 we list all spanning forests of G_0, where I and II are the spanning trees.

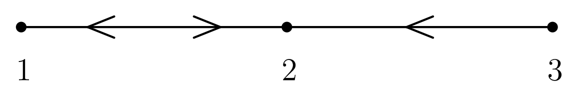

Figure 2-1: Digraph G_0.

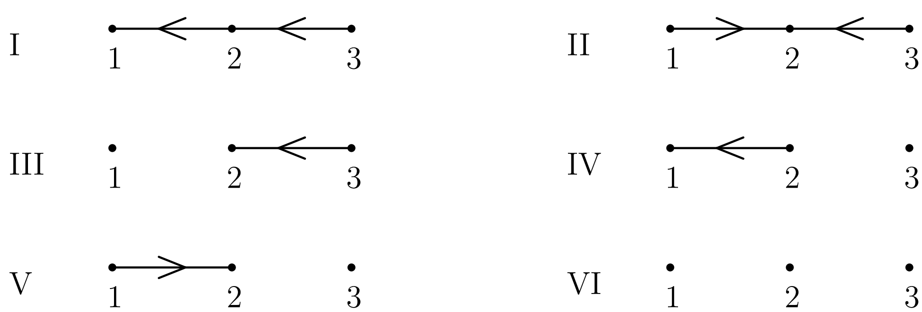

Figure 2-2: Forests of digraph G_0.

2.2 Digraphs and Vector Spaces

Definition 2.1. *Given a digraph* $G = (V, E)$*, the* expanded *digraph* $G_{\bar{n}} = (V_{\bar{n}}, E_{\bar{n}})$ of rank $\bar{n} = (n_1, n_2, \ldots, n_m)$ *is defined by the following conditions:*

1. $V_{\bar{n}} = \sqcup_{i=1}^{m} V_i$*, where each* $V_i = \{x_1^i, x_2^i, \ldots, x_{n_i}^i\}$ *is a vertex set of size* n_i *for* $i = 1, 2, \ldots, m$.

2. *For any* $x \in V_i, y \in V_j$*,* $x \to y \in E_{\bar{n}}$ *if and only if* $i \to j \in E$*, for any* $i, j \in V$*.*

For example, with $\bar{n} = (2, 2, 1)$, the expanded digraph $(G_0)_{\bar{n}}$ is given below:

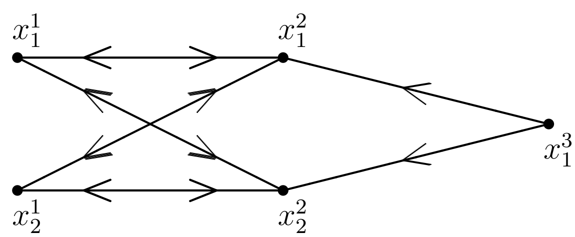

Figure 2-3: Expanded digraph $(G_0)_{\bar{n}}$.

Definition 2.2. *Given a digraph* $G = (V, E)$*, a* G-space of rank $\bar{n} = (n_1, n_2, \ldots, n_m)$ *is a vector space* $U = U_G(\bar{n}) = \oplus_{i=1}^{m} U_i$*, where each* U_i *is a vector space of dimension* n_i *for* $i = 1, 2, \ldots, m$*, and* m *is the number of vertices of* G*, i.e.,* $V = [m]$*.*

Definition 2.3. *Given a digraph* $G = (V, E)$ *and a* G*-space* $U = U_G(\bar{n})$*, a* G-space linear transformation $f = f_{G,U}$ *is an endomorphism of* U *satisfying:*

$$f|_{U_i} : U_i \to \bigoplus_{j \in \mathrm{Out}(i)} U_j \,,$$

where $\oplus_{j \in \varnothing} U_j$ *is the zero space.*[2]

Pick a basis $\{x_l^i : 1 \leq l \leq n_i\}$ for each U_i and let $\tilde{x}_l^i$ be the nature promotion of x_l^i from U_i to U, for $i = 1, 2, \ldots, m$. Let $n = |\bar{n}| = n_1 + n_2 + \cdots + n_m$. Then the

[2] The G-space together with the transformation is the same as a representation of the quiver G. But in the quiver case, the digraph is allowed to have multiple edges and each linear transformation along a edge is considered separately. Here we won't consider the multiple edges case and will treat all transformations as one on the direct sum of all subspaces.

transition matrix M_f of f under the basis $\{\tilde{x}_l^i : 1 \leq i \leq m, 1 \leq l \leq n_i\}$ is an $n \times n$ matrix that can be broken into blocks $M_f = (M_{i,j})_{i,j=1}^m$, where $M_{i,j}$ is an $n_i \times n_j$ matrix (see Figure 2-4). Moreover $M(i,j) = 0$ if $i \to j$ is not an edge in G.

$$
\begin{array}{cc}
 & \begin{array}{cccc} n_1 & n_2 & \cdots & n_m \end{array} \\
\begin{array}{c} n_1 \\ n_2 \\ \vdots \\ n_m \end{array} &
\begin{pmatrix}
M_{1,1} & M_{1,2} & \cdots & M_{1,m} \\
M_{2,1} & M_{2,2} & \cdots & M_{2,m} \\
\vdots & \vdots & \vdots & \vdots \\
M_{m,1} & M_{m,2} & \cdots & M_{m,m}
\end{pmatrix}
\end{array}
$$

Figure 2-4: Transition matrix.

If G is a unidigraph, the condition will be just:

$$
f|_{U_i} : U_i \longrightarrow U_{o(i)} ,
$$

where U_0 is the zero space. For example, if we consider G_0 from Figure 2-1, the G_0-space linear transformation f is required to map between spaces U_1 and U_2 and map from U_3 to U_2 (see Figure 2-5).

$$
\begin{tikzcd}
U_1 \arrow[r, shift left, "f|_{U_1}"'] & U_2 \arrow[l, shift left, "f|_{U_2}"'] & U_3 \arrow[l, "f|_{U_3}"']
\end{tikzcd}
$$

Figure 2-5: G_0-space linear transformation.

Let $\bar{n} = (2, 2, 1)$. Define a linear map f_0 on U by the transition matrix given in Figure 2-6. Since, except for $M_{1,2}$ and $M_{2,1}$, the other blocks are all zero, the map f_0 is a G_0-space linear transformation.

$$M_{f_0} = \left(\begin{array}{cc:cc:c} 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Figure 2-6: Transition matrix of f_0.

2.3 Main Theorem

Define $\mathrm{Tr}(G, \bar{n})$ and $\mathrm{Nil}(G, \bar{n})$ to be the set of spanning trees of the expanded digraph $G_{\bar{n}}$ and the set of nilpotent G-space linear transformations, respectively.

Theorem 2.4. *Given a digraph G, we have:*

1. *The number of spanning trees of the expanded digraph $G_{\bar{n}}$ is:*

$$\#\mathrm{Tr}(G, \bar{n}) = \prod_{i=1}^{m} \left(\sum_{j \in \mathrm{Out}(i)} n_j \right)^{n_i - 1} \cdot \sum_{T} \left(\prod_{i \neq I_T} n_{p_T(i)} \right). \tag{2.1}$$

In particular, when G is a unidigraph we have:

$$\#\mathrm{Tr}(G, \bar{n}) = \prod_{i=1}^{m} n_{o(i)}^{n_i - 1} \cdot \sum_{T} \left(\prod_{i \neq I_T} n_{o(i)} \right), \tag{2.2}$$

where $n_0 = 0$, and the two sums are taken over all spanning trees of G and I_T is the root of tree T, $p_T(i)$ is the parent vertex of i in T.

2. *When G is a unidigraph, the number of nilpotent G-space linear transformations is:*

$$\#\mathrm{Nil}(G, \bar{n}) = \prod_{i=1}^{m} (q^{n_{o(i)}})^{n_i - 1} \cdot \sum_{F} \left(\prod_{i \notin I_F} (q^{n_{o(i)}} - 1) \right), \tag{2.3}$$

where $n_0 = 0$, and the sum is taken over all spanning forests of G and I_F is the root set of the forest F.

This is a direct corollary of Lemma 3.8 and 3.9 from Chapter 3.

Corollary 2.5. *When G is a unidigraph, $\#\mathrm{Nil}(G,\bar{n})$ is a q-analogue of $\#\mathrm{Tr}(G,\bar{n})$, i.e., the set of nilpotent G-space linear transformations is a q-analogue of the set of spanning trees of expanded digraph $G_{\bar{n}}$.*

Proof. Recall from Chapter 1, the q-analogue we are considering is to replace a n-element set with a n-dimensional vector space over $\mathbb{F}_q$, and correspondingly replace n with q^n. With the formula given in Theorem 2.4, we have:

$$\begin{array}{ccc}
\#\mathrm{Tr}(G,\bar{n}) & & \#\mathrm{Nil}(G,\bar{n}) \\
\| & & \| \\
\prod_{i=1}^{m}(\mathbf{n_{o(i)}})^{n_i-1}\cdot\sum_F\left(\mathbf{0}^{\#I_F-1}\prod_{i\notin I_F}(\mathbf{n_{o(i)}}-\mathbf{0})\right) & \rightsquigarrow_q & \prod_{i=1}^{m}(\mathbf{q}^{\mathbf{n_{o(i)}}})^{n_i-1}\cdot\sum_F\left((\mathbf{q}^{\mathbf{0}})^{\#I_F-1}\prod_{i\notin I_F}(\mathbf{q}^{\mathbf{n_{o(i)}}}-\mathbf{q}^{\mathbf{0}})\right).
\end{array}$$

The summation in equation (2.2) is taken over all spanning trees of G. In the above diagram, to make the summation range over all spanning forests, we add an extra term $0^{\#I_F-1}$ and treat a spanning tree as a special spanning forest F with only one root. It is the same as considering the partial summation taken over all spanning trees in equation (2.3), i.e., considering only the "leading terms". □

Let us still take G_0 from Figure 2-1 as an example. If $\bar{n}=(2,2,1)$, from Theorem 2.4 together with the list of all spanning forest in Figure 2-2, we have:

$$\begin{aligned}
\#\mathrm{Tr}(G_0,\bar{n}) &= \left(n_2^{n_1-1}\cdot n_1^{n_2-1}\cdot n_2^{n_3-1}\right)\cdot\Big(\ \overset{\substack{\mathrm{I}\\ \downarrow}}{n_1\cdot n_2}\ +\ \overset{\substack{\mathrm{II}\\ \downarrow}}{n_2\cdot n_2}\ \Big)\\
&= n_1^{n_2-1}\cdot n_2^{n_1+n_3-2}\cdot n_2\cdot(n_1+n_2) = 32.
\end{aligned}$$

$$
\begin{aligned}
\#\mathrm{Nil}(G,\bar{n}) &= (q^{n_2(n_1-1)}\cdot q^{n_1(n_2-1)}\cdot q^{n_2(n_3-1)})\Big(\overset{\overset{\text{I}}{\downarrow}}{(q^{n_1}-1)(q^{n_2}-1)} + \overset{\overset{\text{II}}{\downarrow}}{(q^{n_2}-1)(q^{n_2}-1)} \\
&\qquad + \overset{\overset{\text{III}}{\downarrow}}{(q^{n_2}-1)} + \overset{\overset{\text{IV}}{\downarrow}}{(q^{n_1}-1)} \\
&\qquad + \overset{\overset{\text{V}}{\downarrow}}{(q^{n_2}-1)} + \overset{\overset{\text{VI}}{\downarrow}}{1} \Big) \\
&= q^{n_2\cdot(n_1-1)}\cdot q^{n_1\cdot(n_2-1)}\cdot q^{n_2\cdot(n_3-1)}\cdot q^{n_2}\cdot(q^{n_1}+q^{n_2}-1) \\
&= q^6(2q^2-1).
\end{aligned}
$$

Figure 2-7 is an example of the 16 spanning trees of $(G_0)_{\bar{n}}$ (see Figure 2-3). And f_0 with transition matrix in Figure 2-6 is one of the $q^5(2q^2-1)$ nilpotent G_0-space transformations.

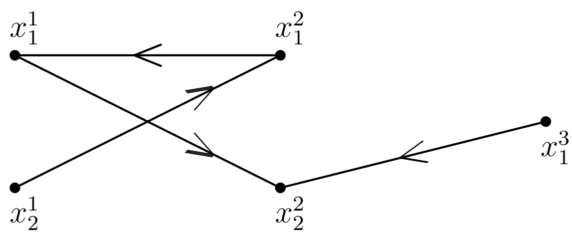

Figure 2-7: A spanning tree of expanded digraph $(G_0)_{\bar{n}}$.

Chapter 3

Proofs

To prove Theorem 2.4 from Section 2.3, we define two bijections from $\mathrm{Tr}(G, \bar{n})$ and $\mathrm{Nil}(G, \bar{n})$, respectively, to two sets whose cardinalities are easy to calculate.

Define $\mathrm{S}_{\mathrm{Tr}}(G, \bar{n}) = \{\bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)\}$ and $\mathrm{S}_{\mathrm{Nil}}(G, \bar{n}) = \{\bar{\beta} = (\beta_1, \beta_2, \ldots, \beta_m)\}$, such that $\alpha_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,n_i})$, $\beta_i = (b_{i,1}, b_{i,2}, \ldots, b_{i,n_i})$, and $a_{i,l} \in \{0\} \sqcup (\cup_{j \in \mathrm{Out}(i)} V_j)$, $b_{i,l} \in \oplus_{j \in \mathrm{Out}(i)} U_j$, for any $1 \leq i \leq m$, $1 \leq l \leq n_i$. They are called the set of *tree codes* and the set of *transformation codes*, respectively.

Definition 3.6. *A tree code $\bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ from $\mathrm{S}_{\mathrm{Tr}}(G, \bar{n})$ is* good *if there exists a spanning tree T of G with root $I = I_T$ such that:*

1. *$a_{i,l} = 0$ if and only if $i = I$ and $l = n_I$, for any $1 \leq i \leq m, 1 \leq l \leq n_i$.*

2. *For any $i \neq I$, we have $a_{i,n_i} \in V_{p_T(i)}$, where $p_T(i)$ is the parent vertex of i in T.*

Define $\mathrm{GS}_{\mathrm{Tr}}(G, \bar{n})$ to be the subset of $\mathrm{S}_{\mathrm{Tr}}(G, \bar{n})$ that contains only good tree codes.

Definition 3.7. *A transformation code $\bar{\beta} = (\beta_1, \beta_2, \ldots, \beta_m)$ from $\mathrm{S}_{\mathrm{Nil}}(G, \bar{n})$ is* good *if there exists a spanning forest F of G with root set $I = I_F$ such that:*

$$b_{i,n_i} = 0 \text{ if and only if } i \in I.$$

Define $\mathrm{GS}_{\mathrm{Nil}}(G, \bar{n})$ to be the subset of $\mathrm{S}_{\mathrm{Nil}}(G, \bar{n})$ that contains only good transformation code.

Take graph G_0 from Figure 2-1 as an example. Let $\bar{n} = (2, 2, 1)$. The expanded digraph $(G_0)_{\bar{n}}$ is given in Figure 2-3. Then $\bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, where:

$$\begin{cases} \alpha_1 & = (x_1^2, x_2^2), \\ \alpha_2 & = (x_1^1, 0), \\ \alpha_3 & = (x_2^2), \end{cases}$$

is one of the good tree codes corresponding to spanning tree II from Figure 2-2. And $\bar{\beta} = (\beta_1, \beta_2, \beta_3)$, where:

$$\begin{cases} \beta_1 & = (x_2^2, 0), \\ \beta_2 & = (x_1^1, -x_1^1 + x_2^1), \\ \beta_3 & = (0). \end{cases}$$

is one of the good transformation codes corresponding to forest IV from Figure 2-2.

Now we can define these two bijections.

Lemma 3.8. *There exists a bijection from* $\mathrm{Tr}(G, \bar{n})$ *to* $\mathrm{GS}_{\mathrm{Tr}}(G, \bar{n})$.

Lemma 3.9. *When G is a unidigraph, we have that $b_{i,l} \in U_{o(i)}$ for $i = 1, 2, \ldots, m$, $l = 1, 2, \ldots, n_i$. In this case, there exists a bijection from* $\mathrm{Nil}(G, \bar{n})$ *to* $\mathrm{GS}_{\mathrm{Nil}}(G, \bar{n})$.

3.1 Lemma 3.8

3.1.1 Proof

Proof. The proof is divided into 3 parts.

1. A Bijection from all spanning trees of $G_{\bar{n}}$ to all rate 1 nilpotent set maps on $V_{\bar{n}}$.

Given a spanning tree $T_{\bar{n}}$ of the expanded digraph $G_{\bar{n}} = (V_{\bar{n}}, E_{\bar{n}})$, we consider the set map $f_{T_{\bar{n}}}$ on $V_{\bar{n}} \sqcup \{0\}$ given by:

$$f_{T_{\bar{n}}}(x) = \begin{cases} p_{T_{\bar{n}}}(x), & \text{if } x \text{ is not the root of } T_{\bar{n}}, \\ 0, & \text{if } x \text{ is the root of } T_{\bar{n}}, \\ 0, & \text{if } x = 0, \end{cases}$$

where $p_{T_{\bar{n}}}(x)$ is the parent of x in $T_{\bar{n}}$.

Define a *nilpotent set map* f on the set S to be the map $f : S \sqcup \{0\} \to S \sqcup \{0\}$ such that $f(0) = 0$ and for large enough $k \in \mathbb{P}$ we have $f^k(S) = \{0\}$. And define the *rate* of f to be the number of elements $x \in S$ such that $f(x) = 0$. Hence, the map $T_{\bar{n}} \to f_{T_{\bar{n}}}$ gives a bijection from all spanning trees of $G_{\bar{n}}$ to all rate 1 nilpotent set maps on $V_{\bar{n}}$.

2. Define map $T_{\bar{n}} \to \bar{\alpha}^{T_{\bar{n}}}$ *from* $\mathrm{Tr}(G, \bar{n})$ *to* $\mathrm{GS}_{\mathrm{Tr}}(G, \bar{n})$.

For any $1 \leq i \leq m$, let $V_i^{(k)} = f^k_{T_{\bar{n}}}(V_{\bar{n}}) \cap V_i$ and r_i be the smallest integer such that $V_i^{(r_i)} = \varnothing$. Then we have:

$$V_i = V_i^{(0)} \supset V_i^{(1)} \supset \cdots \supset V_i^{(r_i-1)} \supsetneq V_i^{(r_i)} = \varnothing.$$

Recall that $V_i = \{x_1^i, x_2^i, \ldots, x_{n_i}^i\}$. Make the list $y_1^i, y_2^i, \ldots, y_{n_i}^i$ as follows: firstly the elements of $V_i^{(0)} - V_i^{(1)}$ (if any) in increasing order of the lower indices, secondly the elements of $V_i^{(1)} - V_i^{(2)}$ (if any) in increasing order of the lower indices, and so on, until finally the elements of $V_i^{(r_i-1)} - V_i^{(r_i)}$ in increasing order of the lower indices.

For any spanning tree $T_{\bar{n}}$ of $G_{\bar{n}}$ and the corresponding rate 1 nilpotent set map $f_{T_{\bar{n}}}$, we define $\bar{\alpha}^{T_{\bar{n}}} = (\alpha_1^{T_{\bar{n}}}, \alpha_2^{T_{\bar{n}}}, \ldots, \alpha_m^{T_{\bar{n}}})$ to be:

$$\alpha_i^{T_{\bar{n}}} = \left(a_{i,1}^{T_{\bar{n}}}, a_{i,2}^{T_{\bar{n}}}, \ldots, a_{i,n_i}^{T_{\bar{n}}}\right) = \left(f_{T_{\bar{n}}}(y_1^i), f_{T_{\bar{n}}}(y_2^i), \ldots, f_{T_{\bar{n}}}(y_{n_i}^i)\right).$$

In order to prove Lemma 3.8, it suffices to show that the map $T_{\bar{n}} \to \bar{\alpha}^{T_{\bar{n}}}$ is a bijection from $\mathrm{Tr}(G, \bar{n})$ to $\mathrm{GS}_{\mathrm{Tr}}(G, \bar{n})$.

Firstly, by the definition of $f_{T_{\bar{n}}}$, we have $a_{i,l}^{T_{\bar{n}}} \in (\cup_{j \in \mathrm{Out}(i)} V_j) \cup \{0\}$ for $i = 1, 2, \ldots, m$, $l = 1, 2, \ldots, n_i$. Hence $\bar{\alpha}^{T_{\bar{n}}} \in \mathrm{S}_{\mathrm{Tr}}(G, \bar{n})$.

Secondly, in order to show that $\bar{\alpha}^{T_{\bar{n}}} \in \mathrm{GS}_{\mathrm{Tr}}(G, \bar{n})$, we need the following claim.

(We will prove it latter.)

Claim 3.10. *$f_{T_{\bar{n}}}$ is a rate 1 nilpotent set map on $V_{\bar{n}}$ if and only the following conditions are satisfied:*

1. *For any $1 \le i \le m$, $1 \le l < n_i$, we have $a_{i,l}^{T_{\bar{n}}} \neq 0$.*

2. *There exists a unique $I \in [m]$ such that $a_{I,n_I}^{T_{\bar{n}}} = 0$.*

3. *For any $i \neq I$, there exists a sequence $i = i_0, i_1, \ldots, i_d = I$ such that $i_h \to i_{h+1}$ is an edge of G, and the last entry of $\alpha_{i_h}^{T_{\bar{n}}}$ satisfies that $a_{i_h,n_{i_h}}^{T_{\bar{n}}} \in V_{i_{h+1}}$, for any $0 \le h < d$.*

Define a spanning tree T of G to be with root I and the unique path from any vertex i to I is the one given in condition 3 for any $i \neq I$. It is not hard to see that the three conditions in Claim 3.10 is equivalent to the definition of $\mathrm{GS}_{\mathrm{Tr}}(G, \bar{n})$. Hence, the map $T_{\bar{n}} \to \bar{\alpha}^{T_{\bar{n}}}$ is a map from $\mathrm{Tr}(G, \bar{n})$ to $\mathrm{GS}_{\mathrm{Tr}}(G, \bar{n})$.
3. Define inverse map $\bar{\alpha} \to T_{\bar{n}}^{\bar{\alpha}}$ from $\mathrm{GS}_{\mathrm{Tr}}(G, \bar{n})$ to $\mathrm{Tr}(G, \bar{n})$.

To prove that it is a bijection, it suffices to find the inverse map $\bar{\alpha} \to T_{\bar{n}}^{\bar{\alpha}}$ from $\mathrm{GS}_{\mathrm{Tr}}(G, \bar{n})$ to $\mathrm{Tr}(G, \bar{n})$.

For any $\bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)\} \in \mathrm{GS}_{\mathrm{Tr}}(G, \bar{n})$ and $\alpha_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,n_i})$, we define a map $f^{\bar{\alpha}}$ on $V_{\bar{n}} \cup \{0\}$ as following:

1. $f^{\bar{\alpha}}(0) = 0$.

2. Define $V_{\bar{n}}^{(0)} = V_{\bar{n}}$ and $V_{\bar{n}}^{(1)} = \{a_{i,l} : 1 \le i \le m, 1 \le l \le n_i\} \cap V_{\bar{n}}$. Let $V_i^{(0)} = V_i$ and $V_i^{(1)} = V_{\bar{n}}^{(1)} \cap V_i$. Define $s_i^{(0)} = 1$ and $s_i^{(1)} = n_i - \#V_i^{(1)} + 1$.

3. Inductively, for $k = 1, 2, \ldots$, define $V_{\bar{n}}^{(k+1)} = \{a_{i,l} : 1 \le i \le m, s_i^{(k)} \le l \le n_i\} \cap V_{\bar{n}}$. Let $V_i^{(k+1)} = V_{\bar{n}}^{(k+1)} \cap V_i$. Define $s_i^{(k+1)} = n_i - \#V_i^{(k+1)} + 1$.

4. Stop when $s_i^{(k+1)} > n_i$ for all $1 \le i \le m$, i.e., when $V_{\bar{n}}^{(k+1)} = \varnothing$. Define r_i to be the smallest integer such that $V_i^{(r_i)} = \varnothing$.

5. For each $i = 1, 2, \ldots, m$, list the element of $V_i = \{x_1^i, x_2^i, \ldots, x_{n_i}^i\}$ as following: first list the $s_i^{(1)} - s_i^{(0)}$ elements from $V_i^{(0)} - V_i^{(1)}$ in increasing order of the

lower indices, then the $s_i^{(2)} - s_i^{(1)}$ elements from $V_i^{(1)} - V_i^{(2)}$ in increasing order of the lower indices, and so on, until finally the $s_i^{(r_i-1)} - s_i^{(r_i)}$ elements from $V_i^{(r_i-1)} - V_i^{(r_i)}$ in increasing order of the lower indices, as $y_1^i, y_2^i, \ldots, y_{n_i}^i$.

6. Define $f^{\bar{\alpha}}(y_l^i) = a_{i,l}$ for $i = 1, 2, \ldots, m, l = 1, 2, \ldots, n_i$.

By the definition of $\mathrm{GS}_{\mathrm{Tr}}(G, \bar{n})$, we know that $y_{n_I}^I$ is always an element of $V_{\bar{n}}^{(k)}$ as long as it is not an empty set, i.e., $s_I^{(k)} \le n_I$. Since $a_{I,n_I} = 0$, we have:

$$\begin{aligned}
V_{\bar{n}}^{(k+1)} &= \{a_{i,l} : 1 \le i \le m, s_i^{(k)} \le l \le n_i\} \cap V_{\bar{n}} \\
&= \Big(\{a_{i,l} : i \neq I, s_i^{(k)} \le l \le n_i\} \cup \{a_{I,l} : s_I^{(k)} \le l \le n_I - 1\} \Big) \cap V_{\bar{n}} \\
&\subset \{a_{i,l} : i \neq I, s_i^{(k)} \le l \le n_i\} \cup \{a_{I,l} : s_I^{(k)} \le l \le n_I - 1\}.
\end{aligned}$$

Hence, $\#V_{\bar{n}}^{(k+1)} \le \sum_{i=1}^m (n_i - s_i^{(k)} + 1) - 1 = \#V_{\bar{n}}^{(k)} - 1 < \#V_{\bar{n}}^{(k)}$. It is not hard to see that $V_{\bar{n}}^{(k+1)} \subset V_{\bar{n}}^{(k)}$ for any $k \in \mathbb{N}$. And because $V_{\bar{n}}^{(0)} = V_{\bar{n}}$ is a finite set, there exists a large enough $r \in \mathbb{P}$ such that $V_{\bar{n}}^{(r)} = \varnothing$. The inductive definition procedure in Step 3 will end as said in Step 4. So we showed that $f^{\bar{\alpha}}$ is a well-defined set map on $V_{\bar{n}} \cup \{0\}$.

From the definition procedure, we know that $(f^{\bar{\alpha}})^k(V_{\bar{n}}) \cap V_{\bar{n}} = V_{\bar{n}}^{(k)}$ for all $k \in \mathbb{N}$. Hence, there is a unique element x in $V_{\bar{n}}$ such that $f^{\bar{\alpha}}(x) = 0$. In fact, $x = y_{n_I}^I$. And $(f^{\bar{\alpha}})^r(V_{\bar{n}}) = V_{\bar{n}}^{(r)} \cup \{0\} = \{0\}$. That is, $f^{\bar{\alpha}}$ is a rate 1 nilpotent set map on $V_{\bar{n}}$, which bijectively gives a spanning tree $T_{\bar{n}}^{\bar{\alpha}}$ of $G_{\bar{n}}$.

One can easily check that the two maps $T_{\bar{n}} \to \bar{\alpha}^{T_{\bar{n}}}$ and $\bar{\alpha} \to T_{\bar{n}}^{\bar{\alpha}}$ are inverse map to each other. Hence, $T_{\bar{n}} \to \bar{\alpha}^{T_{\bar{n}}}$ is a bijection from $\mathrm{Tr}(G, \bar{n})$ to $\mathrm{GS}_{\mathrm{Tr}}(G, \bar{n})$. That proves Lemma 3.8. □

Proof of Claim 3.10. "$\Rightarrow$", given that $f_{T_{\bar{n}}}$ is a rate 1 nilpotent set map on $V_{\bar{n}}$, we need to show that the three conditions are satisfied.

Since $f_{T_{\bar{n}}}$ is a rate 1 nilpotent set map, we have $T_{\bar{n}}$ is a spanning tree of $G_{\bar{n}}$. $T_{\bar{n}}$ has a unique root $x \in V_{\bar{n}}$. Assume $x \in V_I$ for some $1 \le I \le m$.

Definition 3.11. *A* leaf *of an (oriented) tree T is a vertex with indegree* 0.

Definition 3.12. *For each vertex x of an (oriented) tree T, we say it is* in level k *if the longest path from any leaf to x is of length k. For instance, all the leaves form level* 0*.*

By the definition of $f_{T_{\bar{n}}}$, we have, for any $k \in \mathbb{N}$, $f_{T_{\bar{n}}}^k(V_{\bar{n}})$ is always 0 union the vertices of level k or higher. Thus, as the root of $T_{\bar{n}}$, x is the only element in $V_i^{(r_i-1)} - V_i^{(r_i)}$. Hence, $x = y_{n_I}^I$ and $a_{I,n_I}^{T_{\bar{n}}} = f_{T_{\bar{n}}}(x) = 0$. And for any $(i,l) \neq (I, n_I)$, since y_l^i is not the root, we have $a_{i,l}^{T_{\bar{n}}} = f_{T_{\bar{n}}}(y_l^i) \neq 0$. This gives condition 1 and 2.

For condition 3, for any $i \neq I$, let $i_0 = i$, for $h \in \mathbb{N}$ inductively define i_{h+1} to be the index such that the last entry of $\alpha_{i_h}^{T_{\bar{n}}}$ satisfies that $a_{i_h,n_{i_h}}^{T_{\bar{n}}} \in V_{i_{h+1}}$, and stop when $i_{h+1} = I$. There are two possible cases:

Case 1: The sequence stops at i_d. Thus, we got a sequence $i = i_0, i_1, \ldots, i_d = I$ satisfying that the last entry of $\alpha_{i_h}^{T_{\bar{n}}}$ satisfies that $a_{i_h,n_{i_h}}^{T_{\bar{n}}} \in V_{i_{h+1}}$, for any $0 \leq h < d$. And by the definition of $f_{T_{\bar{n}}}$ and $G_{\bar{n}}$, we have $i_h \to i_{h+1}$ is an edge of G. This gives the condition 3.

Case 2: The sequence repeats. Assume, without loss of generality, that $0 \leq d_1 < d_2$ and $i_{d_1} = i_{d_2}$. For any $k \in \mathbb{N}$:

$$\left.\begin{array}{l} f_{T_{\bar{n}}}(y_{n_{i_h}}^{i_h}) = a_{i_h,n_{i_h}}^{T_{\bar{n}}} \in V_{i_{h+1}} \\ y_{n_{i_h}}^{i_h} \in V_{i_h}^{(r_{i_h}-1)} \end{array}\right\} \quad \Rightarrow \quad a_{i_h,n_{i_h}}^{T_{\bar{n}}} \in f_{T_{\bar{n}}}^{r_{i_h}}(V_{\bar{n}}) \cap V_{i_{h+1}} = V_{i_{h+1}}^{(r_{i_h})} \neq \varnothing.$$

Thus, $r_{i_h} < r_{i_{h+1}}$. Hence, $r_{i_{d_1}} < r_{i_{d_1+1}} < \cdots < r_{i_{d_2-1}} < r_{i_{d_2}} = r_{i_{d_1}}$, a contradiction!

As a whole. we proved that $f_{T_{\bar{n}}}$ is a rate 1 nilpotent set map on $V_{\bar{n}}$ implies the three conditions.

"$\Leftarrow$", given the three conditions, we want to show that $f_{T_{\bar{n}}}$ is a rate 1 nilpotent set map on $V_{\bar{n}}$.

With condition 1 and 2, if $f_{T_{\bar{n}}}$ is nilpotent, it is of rate 1. Hence, it suffices to show that for some $r \in \mathbb{P}$:

$$V_{\bar{n}} = V_{\bar{n}}^{(0)} \supsetneq V_{\bar{n}}^{(1)} \supsetneq \cdots \supsetneq V_{\bar{n}}^{(r-1)} \supsetneq V_{\bar{n}}^{(r)} = \varnothing,$$

where $V_{\bar{n}}^{(k)} = f_{T_{\bar{n}}}^k(V_{\bar{n}}) \cap V_{\bar{n}} = \cup_{i=1}^m V_i^{(k)}$ for $k \in \mathbb{N}$.

It is not hard to see that:

$$V_{\bar{n}} = V_{\bar{n}}^{(0)} \supset V_{\bar{n}}^{(1)} \supset \cdots \supset V_{\bar{n}}^{(k)} \supset \cdots .$$

Since $V_{\bar{n}}$ is a finite set, it suffices to show that $\#V_{\bar{n}}^{(k)} > \#V_{\bar{n}}^{(k+1)}$ for any $k \in \mathbb{N}$ if $V_{\bar{n}}^{(k)} \neq \varnothing$.

By the definition of $V_{\bar{n}}^{(k)}$ and condition 3, we know that $y_{n_I}^I$ is always an element of $V_{\bar{n}}^{(k)}$ as long as it is not an empty set. Assume that $V_{\bar{n}}^{(k)} = \{z_1, z_2, \ldots, z_N\}$, where $z_1 = y_{n_I}^I$ and $N = \#V_{\bar{n}}^{(k)}$. Since $f_{T_{\bar{n}}}(y_{n_I}^I) = 0$, we have:

$$\begin{aligned} V_{\bar{n}}^{(k+1)} &= f_{T_{\bar{n}}}(V_{\bar{n}}^{(k)}) - \{0\} \\ &= \{f_{T_{\bar{n}}}(z_t) : 1 \leq t \leq N\} - \{0\} \\ &\subset \{f_{T_{\bar{n}}}(z_t) : 2 \leq t \leq N\}. \end{aligned}$$

Hence, $\#V_{\bar{n}}^{(k+1)} \leq N - 1 = \#V_{\bar{n}}^{(k)} - 1 < \#V_{\bar{n}}^{(k)}$. This proves that $f_{T_{\bar{n}}}$ is a rate 1 nilpotent set map on $V_{\bar{n}}$. $\square$

3.1.2 Example

Consider $G = G_0$ as given in Figure 2-1 and $\bar{n} = (2, 2, 1)$. In Figure 2-3 and Figure 2-7, we give the expanded digraph $(G_0)_{\bar{n}}$ and one of its spanning trees T_0. As discussed in the proof of Lemma 3.8, the spanning tree T_0 can be coded as a rate 1 nilpotent set map f_0 on $V_{\bar{n}} = \{x_1^1, x_2^1, x_1^2, x_2^2, x_1^3\}$, where:

$$\begin{cases} (f_0(x_1^1), f_0(x_2^1)) = (x_2^2, x_1^2), \\ (f_0(x_1^2), f_0(x_2^2)) = (x_1^1, 0), \\ (f_0(x_1^3)) = (x_2^2). \end{cases}$$

Hence, in terms of $V_i = V_i^{(0)} \supset V_i^{(1)} \supset \cdots \supset V_i^{(r_i-1)} \supsetneq V_i^{(r_i)} = \varnothing$, we have:

$$\begin{cases} V_1 = \{x_1^1, x_2^1\} \supset \{x_1^1\} \supset \{x_1^1\} \supset \varnothing, \\ V_2 = \{x_1^2, x_2^2\} \supset \{x_1^2, x_2^2\} \supset \{x_2^2\} \supset \{x_2^2\} \supset \varnothing, \\ V_3 = \{x_1^3\} \supset \varnothing. \end{cases}$$

This gives:

$$(y_1^1, y_2^1) = (x_2^1, x_1^1), \quad (y_1^2, y_2^2) = (x_1^2, x_2^2), \quad (y_1^3) = (x_1^3).$$

Hence, T_0 is bijectively mapped to $\bar{\alpha}^{T_0} = (\alpha_1^{T_0}, \alpha_2^{T_0}, \alpha_3^{T_0})$, where:

$$\begin{cases} \alpha_1^{T_0} & = (x_1^2, x_2^2), \\ \alpha_2^{T_0} & = (x_1^1, 0), \\ \alpha_3^{T_0} & = (x_2^2). \end{cases}$$

This $\bar{\alpha}^{T_0}$ satisfies the conditions in Lemma 3.8.

3.2 Lemma 3.9

3.2.1 "Adapt" a Basis

Before we prove Lemma 3.9, let us define the way to find a special basis.

For a n-dimensional vector space W with a given basis $\{x_1, x_2, \ldots, x_n\}$, we can uniquely *adapt* the basis to W' and get a new basis $\{y_1, y_2, \ldots, y_n\}$ satisfying that $\{y_{n-n'+1}, y_{n-n'+2}, \ldots, y_n\}$ generate the given n'-dimensional subspace $W' \subset W$ as follows:

1. For $s = 0, 1, \ldots, n$, let X_s be the $(n-s)$-dimensional subspace of W generated by $\{x_{s+1}, x_{s+2}, \ldots, x_n\}$. Define $S = \{s : W' \cap X_{s-1} \neq W' \cap X_s\}$.

2. For $s \in S$, define z_s to be the unique vector in $W' \cap X_{s-1}$ such that $z_s - x_s$ lies in the subspace of X_s spanned by the vectors $\{x_t : t > s, t \notin S\}$.

3. List the elements of $[n] - S$ and S in increasing order as $t_1 < t_2 < \cdots < t_{n-n'}$ and $s_1 < s_2 < \cdots < s'_n$.

4. Define $(y_1, y_2, \ldots, y_n) = (x_{t_1}, x_{t_2}, \ldots, x_{t_{n-n'}}, z_{s_1}, z_{s_2}, \ldots, z_{s_{n'}})$.

The above construction is a well-known method from the theory of Schubert cells in Grassmann varieties, see, for example, [4]. It can also be defined through reduced row echelon forms of matrices.

Definition 3.13. *A matrix is in* reduced row echelon form *if:*

1. *All nonzero rows are above any rows of all zeroes.*

2. *The leading coefficient (also called* pivot*) of each nonzero row is always strictly to the right of the leading coefficient of the row above it.*

3. *Every leading coefficient is 1 and the only nonzero entry in its column.*

Given the definition above, it is not hard to see that the construction we gave above is equivalent to the following one in terms of reduced row echelon form:

1. Pick a basis $\{z_1, z_2, \ldots, z_{n'}\}$ of W', write it as linear combinations of $\{x_1, x_2, \ldots, x_n\}$ as $(z_1, z_2, \ldots, z_{n'})^T = M(x_1, x_2, \ldots, x_n)^T$ where M is a $n' \times n$ matrix.

2. Consider the reduced row echelon form E of M. And let S be the set of column indices of all pivots. List $[n] - S$ in increasing order as $t_1 < t_2 < \cdots < t_{n-n'}$. Let $(z'_1, z'_2, \ldots, z'_{n'})^T = E(x_1, x_2, \ldots, x_n)^T$.

3. Define $(y_1, y_2, \ldots, y_n) = (x_{t_1}, x_{t_2}, \ldots, x_{t_{n-n'}}, z'_1, z'_2, \ldots, z'_{n'})$.

3.2.2 Proof

Proof. For Lemma 3.9, we assume that G is a unidigraph. The proof is similar to the one given in the previous section. It is divided into two parts.

1. Define a map $f \to \bar{\beta}^f$ from $\mathrm{Nil}(G, \bar{n})$ *to* $\mathrm{GS}_{\mathrm{Nil}}(G, \bar{n})$.

For any $1 \le i \le m$, let $U_i^{(k)} = f^k(U) \cap U_i$ and r_i be the smallest integer such that $U_i^{(r_i)} = \{0\}$. Then we have:

$$U_i = U_i^{(0)} \supset U_i^{(1)} \supset \cdots \supset U_i^{(r_i-1)} \supsetneq U_i^{(r_i)} = \{0\}.$$

Recall that, for $i = 1, 2, \dots, m$, U_i has the basis $\{x_1^i, x_2^i, \dots, x_{n_i}^i\}$. Take it and adapt it to $U_i^{(1)}$ to get another basis, then take the new basis and adapt it to $U_i^{(2)}$, and so on, until finally adapting to $U_i^{(r_i-1)}$ to get the basis $\{y_1^i, y_2^i, \dots, y_{n_i}^i\}$.

For any nilpotent G-space linear transformation f, we define $\bar{\beta}^f = (\beta_1^f, \beta_2^f, \dots, \beta_m^f)$ to be:

$$\beta_i^f = \left(b_{i,1}^f, b_{i,2}^f, \dots, b_{i,n_i}^f\right) = \left(f(y_1^i), f(y_2^i), \dots, f(y_{n_i}^i)\right).$$

In order to prove Lemma 3.9, it suffices to show that the map $f \to \bar{\beta}^f$ is a bijection from $\mathrm{Nil}(G, \bar{n})$ to $\mathrm{GS}_{\mathrm{Nil}}(G, \bar{n})$.

Firstly, by the definition of f, we have $b_{i,l}^f \in U_{o(i)}$ for $i = 1, 2, \dots, m$, $l = 1, 2, \dots, n_i$. Hence $\bar{\beta}^f \in \mathrm{S}_{\mathrm{Nil}}(G, \bar{n})$.

Secondly, in order to show that $\bar{\beta}^f \in \mathrm{GS}_{\mathrm{Nil}}(G, \bar{n})$, we need the following claim. (We will prove it latter.)

Claim 3.14. *f is a nilpotent G-space linear transformation if and only the following conditions are satisfied:*

1. *If $I = \{i \in [m] : b_{i,n_i}^f = 0\}$, then $I \neq \varnothing$.*
2. *For any $i \notin I$, there exists a sequence $i = i_0, i_1, \dots, i_d$ such that $o(i_h) = i_{h+1}$, for any $0 \le h < d$, and $i_d \in I$.*

Define a spanning forest F of G to have root set I and the unique path from any vertex i to a root is the one given in condition 3 for any $i \notin I$. It is not hard to see that the three conditions in Claim 3.14 are equivalent to the definition of $\mathrm{GS}_{\mathrm{Nil}}(G, \bar{n})$. Hence, the map $f \to \bar{\beta}^f$ is a map from $\mathrm{Nil}(G, \bar{n})$ to $\mathrm{GS}_{\mathrm{Nil}}(G, \bar{n})$.

2. Define the inverse map $\bar{\beta} \to f^{\bar{\beta}}$ from $\mathrm{GS}_{\mathrm{Nil}}(G, \bar{n})$ to $\mathrm{Nil}(G, \bar{n})$.

To prove that it is a bijection, it suffices to find the inverse map $\bar{\beta} \to f^{\bar{\beta}}$ from $\mathrm{GS}_{\mathrm{Nil}}(G, \bar{n})$ to $\mathrm{Nil}(G, \bar{n})$.

For any $\bar{\beta} = (\beta_1, \beta_2, \ldots, \beta_m)\} \in \mathrm{GS}_{\mathrm{Nil}}(G, \bar{n})$ and $\beta_i = (b_{i,1}, b_{i,2}, \ldots, b_{i,n_i})$, we define a G-space linear transformation $f^{\bar{\beta}}$ as follows:

1. Define $U^{(0)} = U$ and $U^{(1)} = \mathrm{span}_{\mathbb{F}_q}\{b_{i,l} : 1 \le i \le m, 1 \le l \le n_i\}$. Let $U_i^{(0)} = U_i$ and $U_i^{(1)} = U^{(1)} \cap U_i$. Define $s_i^{(0)} = 1$ and $s_i^{(1)} = n_i - \dim U_i^{(1)} + 1$.

2. Inductively, for $k = 1, 2, \ldots$, define $U^{(k+1)} = \mathrm{span}_{\mathbb{F}_q}\{b_{i,l} : 1 \le i \le m, s_i^{(k)} \le l \le n_i\}$. Let $U_i^{(k+1)} = U^{(k+1)} \cap U_i$. Define $s_i^{(k+1)} = n_i - \dim U_i^{(k+1)} + 1$.

3. Stop when $s_i^{(k+1)} > n_i$ for all $1 \le i \le m$, i.e., when $U^{(k+1)} = \{0\}$. Define r_i to be the smallest integer such that $U_i^{(r_i)} = \{0\}$.

4. For each $i = 1, 2, \ldots, m$, take the basis $\{x_1^i, x_2^i, \ldots, x_{n_i}^i\}$ and adapt it to $U_i^{(1)}$ to get another basis, then take the new basis and adapt it to $U_i^{(2)}$, and so on, until finally adapting to $U_i^{(r_i-1)}$ to get the basis $\{y_1^i, y_2^i, \ldots, y_{n_i}^i\}$.

5. Define $f^{\bar{\beta}}(y_l^i) = b_{i,l}$, for $i = 1, 2, \ldots, m, l = 1, 2, \ldots, n_i$, and linearly generate $f^{\bar{\beta}}$ to be a G-space linear transformation.

By the definition of $\mathrm{S}_{\mathrm{Nil}}(G, \bar{n})$ and given that G is a unidigraph, we know that at least one element of $\{y_{n_i}^i : i \in I\}$ is a vector from $U^{(k)}$ as long as it is not the zero space, i.e., there exists $i(0) \in I$ such that $s_{i(0)}^{(k)} \le n_{i(0)}$. Since $b_{i(0),n_{i(0)}} = 0$, we have:

$$\begin{aligned} U^{(k+1)} &= \mathrm{span}_{\mathbb{F}_q}\{b_{i,l} : 1 \le i \le m, s_i^{(k)} \le l \le n_i\} \\ &= \mathrm{span}_{\mathbb{F}_q}\{b_{i,l} : i \ne i(0), s_i^{(k)} \le l \le n_i\} \oplus \mathrm{span}_{\mathbb{F}_q}\{b_{i(0),l} : s_{i(0)}^{(k)} \le l \le n_{i(0)} - 1\}. \end{aligned}$$

Hence, $\dim U^{(k+1)} \le \sum_{i=1}^m (n_i - s_i^{(k)} + 1) - 1 = \dim U^{(k)} - 1 < \dim U^{(k)}$. It is not hard to see that $U^{(k+1)} \subset U^{(k)}$ for any $k \in \mathbb{N}$. And because $U^{(0)} = U$ is a finite-dimensional vector space, there exists a large enough $r \in \mathbb{P}$ such that $U^{(r)} = \{0\}$. Since G is unidigraph, we have $U^{(r)} = \oplus_{i=1}^m U_i^{(r)} = \cup_{i=1}^m U_i^{(r)}$. Hence, the inductive definition procedure in Step 3 will end as said in Step 4. So we showed that $f^{\bar{\beta}}$ is a well-defined G-space linear transformation.

From the definition procedure, we know that $(f^{\bar{\beta}})^k(U) = U^{(k)}$ for all $k \in \mathbb{N}$. Hence, $(f^{\bar{\beta}})^r(U) = U^{(r)} = \{0\}$. That is, $f^{\bar{\beta}}$ is a nilpotent G-space linear transformation.

One can easily check that the two maps $f \to \bar{\beta}^f$ and $\bar{\beta} \to f^{\bar{\beta}}$ are inverse maps to each other. Hence, $f \to \bar{\beta}^f$ is a bijection from $\mathrm{Nil}(G, \bar{n})$ to $\mathrm{GS}_{\mathrm{Nil}}(G, \bar{n})$. That proves Lemma 3.9. $\square$

Proof of Claim 3.14. "$\Rightarrow$", given that f is a nilpotent G-space linear transformation, we need to show that the two conditions are satisfied.

Let $r = \max\{r_i : 1 \leq i \leq m\}$ and $I' = \{i \in [m] : r_i = r\}$, so $I' \neq \varnothing$. Assume, for contradiction, that there exists $i(0) \in I'$ such that $b^f_{i(0),n_{i(0)}} \neq 0$. Thus:

$$\left.\begin{array}{l} 0 \neq b^f_{i(0),n_{i(0)}} \in U_{o(i(0))} \\ b^f_{i(0),n_{i(0)}} = f(y^{i(0)}_{n_{i(0)}}) \in f^{r_{i(0)}-1} \\ r_{i(0)} = r \end{array}\right\} \quad \Rightarrow \quad 0 \neq b^f_{i(0),n_{i(0)}} \in f^r \neq \{0\}.$$

Contradiction to the definition of r! Hence, $b^f_{i,n_i} = 0$ for all $i \in I'$, i.e., $I \supset I'$ and $I \neq \varnothing$. This gives condition 1.

For condition 2, for any $i \notin I$, let $i_0 = i$, and for $h \in \mathbb{N}$ inductively define $i_{h+1} = o(i_h)$, and stop when $i_{h+1} \in I$. There are two possible cases:

Case 1: The sequence stops at i_d. Thus, we get a a sequence $i = i_0, i_1, \ldots, i_d$ satisfying $o(i_h) = i_{h+1}$, for any $0 \leq h < d$, and $i_d \in I$. This gives the condition 2.

Case 2: The sequence repeats. Assume, without loss of generality, that $0 \leq d_1 < d_2$ and $i_{d_1} = i_{d_2}$. For any $k \in \mathbb{N}$:

$$\left.\begin{array}{l} f(y^{i_h}_{n_{i_h}}) = b^f_{i_h,n_{i_h}} \in U_{i_{h+1}} \\ y^{i_h}_{n_{i_h}} \in U^{(r_{i_h}-1)}_{i_h} \\ h \notin I \Rightarrow b^f_{i_h,n_{i_h}} \neq 0 \end{array}\right\} \quad \Rightarrow \quad 0 \neq b^f_{i_h,n_{i_h}} \in f^{r_{i_h}}(U) \cap U_{i_{h+1}} = U^{(r_{i_h})}_{i_{h+1}} \neq \varnothing.$$

Thus, $r_{i_h} < r_{i_{h+1}}$. Hence, $r_{i_{d_1}} < r_{i_{d_1+1}} < \cdots < r_{i_{d_2-1}} < r_{i_{d_2}} = r_{i_{d_1}}$, a contradiction!

As a whole. we proved that f being a nilpotent G-space linear transformation implies the two conditions.

"$\Leftarrow$", given the two conditions, we want to show that f is a nilpotent G-space linear transformation.

It suffices to show that for some $r \in \mathbb{P}$:

$$U = U^{(0)} \supsetneq U^{(1)} \supsetneq \cdots \supsetneq U^{(r-1)} \supsetneq U^{(r)} = \{0\},$$

where $U^{(k)} = f^k(U) = \oplus_{i=1}^m U_i^{(k)} = \cup_{i=1}^m U_i^{(k)}$ for $k \in \mathbb{N}$.

It is not hard to see that:

$$U = U^{(0)} \supset U^{(1)} \supset \cdots \supset U^{(k)} \supset \cdots .$$

Since U is a finite-dimensional vector space, it suffices to show that $\dim U^{(k)} > \dim U^{(k+1)}$ for any $k \in \mathbb{N}$ if $U^{(k)} \neq \{0\}$.

By the definition of $U^{(k)}$ and condition 2, we know that at least one element of $\{y^i_{n_i} : i \in I\}$ is a vector from $U^{(k)}$ as long as it is not the zero space, i.e., there exists $i(0) \in I$ such that $y^{i(0)}_{n_{i(0)}} \in U^{(k)}$. Assume that $U^{(k)} = \mathrm{span}_{\mathbb{F}_q}\{z_1, z_2, \ldots, z_N\}$, where $z_1 = y^{i(0)}_{n_{i(0)}}$ and $N = \dim U^{(k)}$. Since $f(y^{i(0)}_{n_{i(0)}}) = 0$, we have:

$$\begin{aligned} U^{(k+1)} &= f(U^{(k)}) \\ &= \mathrm{span}_{\mathbb{F}_q}\{f(z_t) : 1 \le t \le N\} \\ &= \mathrm{span}_{\mathbb{F}_q}\{f(z_t) : 2 \le t \le N\}. \end{aligned}$$

Hence, $\dim U^{(k+1)} \le N - 1 = \dim U^{(k)} - 1 < \dim U^{(k)}$. This proves that f is a nilpotent G-space linear transformation. □

3.2.3 Example

Consider $G = G_0$ as given in Figure 2-1 and $\bar{n} = (2, 2, 1)$. In Figure 2-6, we give the transition matrix of a nilpotent G-space linear transformation f_0. As discussed in the proof of Lemma 3.9, in terms of $U_i = U_i^{(0)} \supset U_i^{(1)} \supset \cdots \supset U_i^{(r_i-1)} \supsetneq U_i^{(r_i)} = \{0\}$, we have:

$$
\begin{cases}
U_1 = \operatorname{span}_{\mathbb{F}_q}\{x_1^1, x_2^1\} \supset \operatorname{span}_{\mathbb{F}_q}\{x_1^1, x_2^1\} \supset \operatorname{span}_{\mathbb{F}_q}\{-x_1^1 + x_2^1\} \\
\qquad \supset \operatorname{span}_{\mathbb{F}_q}\{-x_1^1 + x_2^1\} \supset \{0\}, \\
U_2 = \operatorname{span}_{\mathbb{F}_q}\{x_1^2, x_2^2\} \supset \operatorname{span}_{\mathbb{F}_q}\{x_2^2\} \supset \operatorname{span}_{\mathbb{F}_q}\{x_2^2\} \supset \{0\}, \\
U_3 = \operatorname{span}_{\mathbb{F}_q}\{x_1^3\} \supset \{0\}.
\end{cases}
$$

This gives:

$$
(y_1^1, y_2^1) = (x_2^1, x_1^1 - x_2^1), \quad (y_1^2, y_2^2) = (x_1^2, x_2^2), \quad (y_1^3) = (x_1^3).
$$

Hence, f_0 is bijectively mapped to $\bar{\beta}^{f_0} = (\beta_1^{f_0}, \beta_2^{f_0}, \beta_3^{f_0})$, where:

$$
\begin{cases}
\beta_1^{f_0} & = (x_2^2, 0), \\
\beta_2^{f_0} & = (x_1^1, -x_1^1 + x_2^1), \\
\beta_3^{f_0} & = (0).
\end{cases}
$$

This $\bar{\beta}^{T_0}$ satisfies the conditions in Lemma 3.9.

Chapter 4

Examples

In this chapter, we discuss several applications of the proofs of Lemma 3.8 and 3.9 to complete graphs, complete bipartite graphs, and cycles. We will show that we can decompose the set of nilpotent endomorphisms such that each subset is a q-analogue of a subset of spanning trees from a corresponding decomposition (see Corollary 4.20, 4.23, 4.29, 4.32, 4.38, and 4.40). In addition, as in Corollary 4.22 and 4.31, we can find the total number of nilpotent transformations with some restrictions on Jordan block sizes.

4.1 Complete Graphs

A *complete graph* K_n is a graph on the vertex set $[n]$ such that every pair of distinct vertices is connected by an edge. In digraph language, a complete graph on n vertices is equivalent to a digraph on vertex set $[n]$ such that every pair of distinct vertices is connected by two edges of opposite orientations. In terms of the expanded digraph, it is the same as the expanded digraph $H_{(n)}$, where H is the digraph with one vertex and a loop on this vertex (see Figure 4-1).

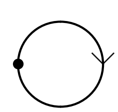

Figure 4-1: Digraph H.

4.1.1 Spanning Trees

A *rooted spanning tree* T of K_n is equivalent to a spanning tree $\overrightarrow{T}$ of $H_{(n)}$ if we orient all the edges of T towards the root. By Lemma 3.8, $\overrightarrow{T}$ is in bijection with $\bar{\alpha} = (\alpha) \in \mathrm{GS}_{\mathrm{Tr}}(H, (n))$, where $\alpha = \alpha_1 = (a_1, a_2, \ldots, a_n)$ and $a_i = a_i^1$ for $i = 1, 2, \ldots, n$. Since H has only one spanning tree that is a single vertex, we have the following result.

Proposition 4.15. *A rooted spanning tree T of K_n is in bijection with $\alpha = (a_1, a_2, \ldots, a_n)$ such that:*

1. $a_n = 0$.

2. $a_i \in [n]$ *for* $i = 1, 2, \ldots, n-1$.

Hence, the total number of rooted spanning trees of K_n is n^{n-1}.

The bijection is similar to the Prüfer code method.

In fact, the bijection gives us more than the above property. For example, it gives a bijective proof of Theorem 5.3.4 from [7].

Let's consider a rooted spanning tree T of K_n with $n-d$ *leaves*, i.e., leaves in $\overrightarrow{T}$ (see Definition 3.11).

Theorem 4.16. *A rooted spanning tree T of K_n with $n-d$ leaves is in bijection with $\alpha = (a_1, a_2, \ldots, a_n)$ such that:*

1. $a_n = 0$.

2. $\{a_1, a_2, \ldots, a_{n-1}\}$ *contains only d distinct numbers from $[n]$.*

Hence, the total number $N(n, d)$ of rooted spanning trees of K_n with $n-d$ leaves is:

$$\begin{aligned} N(n,d) &= \prod_{i=1}^{d}(n-i+1) \cdot \sum_{\lambda \subset d \times (n-1-d)} \lambda_1 \lambda_2 \cdots \lambda_{n-1-d} \\ &= \binom{n}{d} \cdot \sigma(d; n-1), \end{aligned} \tag{4.1}$$

where in the sum, λ ranges over all partitions with $n-1-d$ parts and largest part $\leq d$. $\sigma(s;t)$ is the number of ways to put s distinct numbers into t positions such that each number appears at least once, and:

$$\sigma(s;t) = \sum_{i=0}^{s}(-1)^i \binom{s}{i} (s-i)^t.$$

Proof. Since $\{a_1, a_2, \ldots, a_{n-1}\}$ contains all the nonleaf vertices, the bijection holds. Hence, $N(n,d)$ is also the total number of sequences α that satisfy conditions 1 and 2.

Given $\alpha = (a_1, a_2, \ldots, a_{n-1}, 0)$ that satisfies the condition that $\{a_1, a_2, \ldots, a_{n-1}\}$ contains only d distinct numbers, we run the following algorithm:

1. Let $j = 1, t = 1$ and set $A = \varnothing$.

2. Consider a_{n-j}. If $a_{n-j} \in A$, then do nothing; otherwise, let $i_t = j$, put a_{n-j} into A, and increase t by 1.

3. Increase j by 1. If $j \leq n-1$ and $t \leq d$, repeat step 2; otherwise, stop.

When finished, we get a special set of distinct numbers $A = \{a_{i_1}, a_{i_2}, \ldots, a_{i_d}\}$. Now write all a_i's in terms of numbers in A, we have:

$$(a_{n-1}, a_{n-2}, \ldots, a_1) = (a_{i_1}, a_{i_2}, \ldots, a_{i_d})P,$$

where P is a $d \times (n-1)$ matrix in reduced row echelon form (see Definition 3.13) that has a unique 1 in each column and zeros otherwise. In fact, the i_t-th column of P has a single 1 in the t-th row and zero otherwise, and columns between the i_t-th column and the i_{t+1}-th column has the 1 in rows $1, 2, \ldots, t$, for $0 \leq t \leq d$, where $i_0 = 0$ and $i_{d+1} = n$. For instance, if $n = 9, d = 4$ and $(i_1, i_2, i_3, i_4) = (2, 4, 5, 7)$, then P has the form:

$$\begin{pmatrix} 0 & 1 & * & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{pmatrix}, \tag{4.2}$$

where $*$'s in the same column denote a possible position for the unique 1.

Delete columns $i_1, i_2, \ldots, i_d$. We get a $d \times (n-1-d)$ matrix $\tilde{P}$. Let the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1-d})$ be as follows: λ_s is the number of $*$'s in the $(n-d-s)$-th column, for $s = 1, 2, \ldots, n-1-d$, i.e., the shape of λ is the same as the shape of all $*$'s in $\tilde{P}$ flipped horizontally. For instance, with the same example as above, we have $\lambda = (4, 3, 1, 0)$. Conversely, given any partition λ with $n-1-d$ parts and largest part $\leq d$, we can define $i_1, i_2, \ldots, i_d$ as follows: i_t is $n-1-(d-t)$ minus the number of parts $\geq t$ in λ, and there are $\lambda_1 \lambda_2 \cdots \lambda_{n-1-d}$ possible reduced row echelon matrices P with $i_1, i_2, \ldots, i_d$ having the same meaning as above.

Since $N(n,d)$ equals the number of possible $\alpha = (a_1, a_2, \ldots, a_{n-1}, 0)$'s, which is the same as the number of possible $(a_{i_1}, a_{i_2}, \ldots, a_{i_d})$'s times the number of possible P's, we have:

$$N(n,d) = (n(n-1)\cdots(n-d+1)) \cdot \sum_{\lambda \subset d \times (n-1-d)} \lambda_1 \lambda_2 \cdots \lambda_{n-1-d}.$$

Consider $\sigma(s;t)$. It is the number of ways to put s distinct numbers into t positions such that each number appears at least once. Assume we have numbers $m_1, m_2, \ldots, m_s$ and m_i appears ρ_i times, for $i = 1, 2, \ldots, s$. Then by the Principle of Inclusion-Exclusion, we have:

$$\sigma(s;t) = \sum_{\rho_i > 0} \binom{t}{\rho_1, \rho_2, \ldots, \rho_s} = \sum_{i=0}^{s} (-1)^i \binom{s}{i} (s-i)^t.$$

Since $N(n,d)$ is also equal to the number of ways to choose d distinct numbers from $[n]$ and put them in to $(a_1, a_2, \ldots, a_{n-1})$ such that each number appears at least once, we have:

$$N(n,d) = \binom{n}{d} \cdot \sigma(d; n-1).$$

This proves equation (4.1). $\square$

In a rooted spanning tree T of K_n, we say that a vertex is *in level* k if it is in level k in $\overrightarrow{T}$ (see Definition 3.12). The same idea in the above proof can be used to show the following theorem.

Theorem 4.17. *Consider a rooted spanning tree T of K_n with δ_k vertices in level k, where $\delta_0 \geq \delta_1 \geq \cdots \geq \delta_{r-1} > \delta_r = 0$, and $\sum_{k=0}^{r} \delta_k = n$. It is in bijection with $\alpha = (a_1, a_2, \ldots, a_n)$ such that $\{a_n, a_{n-1}, \ldots, a_{n-d_k+1}\}$ contains only d_{k+1} distinct numbers from $[n]$, where $d_k = \delta_r + \delta_{r-1} + \cdots + \delta_k$, for $k = 0, 1, \ldots, r$.*

Let $\bar{d} = (d_1, d_2, \ldots, d_r)$, the total number $N(n, \bar{d})$ of rooted spanning trees with above property is:

$$\begin{aligned} N(n,\bar{d}) &= \prod_{i=1}^{d_1} (n-i+1) \cdot \sum_{(\lambda^0, \lambda^1, \cdots, \lambda^{r-1})} \Pi(\lambda^0)\Pi(\lambda^1)\cdots\Pi(\lambda^{r-1}) \\ &= \binom{n}{\delta_0, \delta_1, \ldots, \delta_{r-1}} \cdot \prod_{k=0}^{r-1} \sigma(\delta_{k+1}, d_{k+2}; \delta_k), \end{aligned} \tag{4.3}$$

where $d_{r+1} = 0$, λ^k ranges over all partitions with $\delta_k - \delta_{k+1}$ parts and largest part $\leq d_{k+1}$ smallest part $\geq d_{k+2}$, and $\Pi(\lambda^k) = \lambda_1^k \lambda_2^k \cdots \lambda_{\delta_k - \delta_{k+1}}^k$, for $k = 0, 1, \ldots, r-1$. $\sigma(s_1, s_2; t)$ is the number of ways to put $s_1 + s_2$ distinct numbers into t positions such that each of the first s_1 numbers appears at least once, and:

$$\sigma(s_1, s_2; t) = \sum_{i=0}^{s_1} (-1)^i \binom{s_1}{i} (s_1 + s_2 - i)^t.$$

Proof. The bijection is implied by the proof of Lemma 3.8 in Section 3.1. Hence, $N(n, \bar{d})$ is also the total number of sequence α that satisfies the condition.

Given $\alpha = (a_1, a_2, \ldots, a_n)$, we run the following algorithm:

1. Let $j = 1, t = 1$ and set $A = \varnothing$.

2. Consider a_{n-j+1}. If $a_{n-j+1} \in A$, then do nothing; otherwise, let $i_t = j$, put a_{n-j+1} into A, and increase t by 1.

3. Increase j by 1. If $j \leq n$ and $t \leq d_1$, repeat step 2; otherwise, stop.

When finished, we get a special set of distinct numbers $A = \{a_{i_1}, a_{i_2}, \ldots, a_{i_{d_1}}\}$. Now writing all a_i's in terms of numbers in A, we have:

$$(a_n, a_{n-1}, \ldots, a_1) = (a_{i_1}, a_{i_2}, \ldots, a_{i_{d_1}})P,$$

where $P = (P_{i,j})_{i,j=1}^r$ is a $d_1 \times n$ block matrix in reduced row echelon form (see Definition 3.13) that has a unique 1 in each column and zeros otherwise. And each $P_{i,j}$ is a $\delta_{r+1-i} \times \delta_{r-j}$ matrix that satisfies:

1. If $i > j$, $P_{i,j} = 0$.

2. If $i = j$, $P_{i,j}$ is a reduced row echelon matrix.

3. If $i < j$, $P_{i,j}$ is a matrix with a column equal to 0 if it corresponds to a pivot in $P_{j,j}$, and arbitrary otherwise.

That is, the matrix P has the form:

$$\begin{pmatrix} P_{1,1} & * & \cdots & * \\ 0 & P_{2,2} & \cdots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & P_{r,r} \end{pmatrix},$$

where the $*$ in column j denotes a matrix that has all zeros in a column if $P_{j,j}$ has a pivot in the same column, and other entries are possible positions for the unique 1 in that column of P.

Using the same technique as in the proof of Theorem 4.16, we can bijectively get a partition $\hat{\lambda}^k$ from each $P_{r+1-k,r+1-k}$ such that $\hat{\lambda}^k$ has $\delta_k - \delta_{k+1}$ parts and the largest part is $\leq \delta_{k+1}$, for $k = 0, 1, \ldots, r-1$. Let λ^k be the partition after adding d_{k+2} to all parts of μ^k. Then the total number of reduced row echelon matrices P corresponding to partitions $(\lambda^0, \lambda^1, \ldots, \lambda^{r-1})$ is:

$$\begin{aligned}&\prod_{k=0}^{r-1}\left((\hat{\lambda}_1^k + d_{k+2})(\hat{\lambda}_2^k + d_{k+2}) \cdots (\hat{\lambda}_{\delta_k-\delta_{k+1}}^k + d_{k+2})\right)\\ =&\prod_{k=0}^{r-1}\left(\lambda_1^k\lambda_2^k \cdots \lambda_{\delta_k-\delta_{k+1}}^k\right) = \prod_{k=0}^{r-1}\Pi(\lambda^k).\end{aligned}$$

Similar to the proof of Theorem 4.16, we have:

$$N(n, \bar{d}) = \prod_{i=1}^{d_1}(n-i+1) \cdot \sum_{(\lambda^0,\lambda^1,\cdots,\lambda^{r-1})} \Pi(\lambda^0)\Pi(\lambda^1)\cdots\Pi(\lambda^{r-1}).$$

Consider $\sigma(s_1, s_2; t)$. It is the number of ways to put $s_1 + s_2$ distinct numbers into t positions such that each of the first s_1 numbers appears at least once. Assume we have numbers $m_1, m_2, \ldots, m_{s_1}, m'_1, m'_2, \ldots, m'_{s_2}$ and m_i appears ρ_i times, for $i = 1, 2, \ldots, s_2$, and m'_i appears ρ'_i times, for $i = 1, 2, \ldots, s_2$. Then using the Principle of Inclusion-Exclusion, we have:

$$\begin{aligned}\sigma(s_1, s_2; t) &= \sum_{\rho_i>0,\rho'_i\geq 0}\binom{t}{\rho_1, \rho_2, \ldots, \rho_{s_1}, \rho'_1, \rho'_2, \ldots, \rho'_{s_2}}\\ &= \sum_{i=0}^{s_1}(-1)^i\binom{s_1}{i}(s_1+s_2-i)^t.\end{aligned}$$

Now $N(n, \bar{d})$ is also obtained as follows: first choose δ_r distinct numbers from $[n]$ and put them into $\{a_n, a_{n-1}, \ldots, a_{n-d_{r-1}+1}\}$ such that each number appears at least once; then for $k = r-2, r-3, \ldots, 0$, choose δ_{k+1} distinct numbers from the remaining $n - d_{k+2}$ numbers, together with the d_{k+2} chosen numbers, and put them into $\{a_{n-d_{k+1}}, a_{n-d_{k+1}-1}, \ldots, a_{n-d_k+1}\}$ such that each of the δ_{k+1} numbers appears at least once. Hence, we have:

$$N(n,\bar{d}) = \left(\binom{n}{\delta_r}\sigma(\delta_r;d_{r-1})\right)\cdot\prod_{k=0}^{r-2}\left(\binom{n-d_{k+2}}{\delta_{k+1}}\sigma(\delta_{k+1},d_{k+2};\delta_k)\right)$$
$$= \binom{n}{\delta_0,\delta_1,\ldots,\delta_{r-1}}\cdot\prod_{k=0}^{r-1}\sigma(\delta_{k+1},d_{k+2};\delta_k).$$

This proves equation (4.3). $\square$

4.1.2 Nilpotent Transformations

Now we want to consider a nilpotent H-space linear transformation. With $\bar{n}=(n)$, it is the same as a nilpotent endomorphism $f:U\to U$, where U is an n-dimensional vector space over $\mathbb{F}_q$.

By Lemma 3.9, f is in bijection with $\bar{\beta}=(\beta)\in \mathrm{GS}_{\mathrm{Nil}}(U,(n))$, where $\beta=\beta_1=(b_1,b_2,\ldots,b_n)$ and $b_i=b_i^1$ for $i=1,2,\ldots,n$. Since H has only one spanning forest that is a single vertex, we have the following result.

Proposition 4.18. *A nilpotent endomorphism f on n-dimensional vector space U is in bijection with $\beta=(b_1,b_2,\ldots,b_n)$ such that:*

1. $b_n=0$.

2. $b_i\in U$ *for* $i=1,2,\ldots,n-1$.

Hence, the total number of nilpotent endomorphisms on U is $q^{n(n-1)}$.

This bijection was also given in [1].

In fact, the bijection gives us more than the above property. We can also enumerate the number of nilpotent endomorphisms of fixed rank.

Theorem 4.19. [1] *A nilpotent endomorphism f on n-dimensional vector space U of rank d is in bijection with $\beta=(b_1,b_2,\ldots,b_n)$ such that:*

[1]This result was also given in Remark 3.1 of [1].

1. $b_n = 0$.

2. $\{b_1, b_2, \ldots, b_{n-1}\}$ *spans a* d*-dimensional subspace of* U.

Hence, the total number $N_q(n; d)$ *of nilpotent endomorphisms on* U *of rank* d *is:*

$$\begin{aligned} N_q(n;d) &= \prod_{i=1}^{d}(q^n - q^{i-1}) \cdot \sum_{\lambda \subset d\times(n-1-d)} q^{\lambda_1} q^{\lambda_2} \cdots q^{\lambda_{n-1-d}} \\ &= \prod_{i=1}^{d}(q^n - q^{i-1}) \cdot \begin{bmatrix} n-1 \\ d \end{bmatrix}_q, \end{aligned} \tag{4.4}$$

where in the sum, λ *ranges over all partitions with* $n-1-d$ *parts and largest part* $\leq d$*, and*

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{(q^m-1)(q^m-q)\cdots(q^m-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})}.$$

Proof. Since $f(U) = \mathrm{span}_{\mathbb{F}_q}\{b_1, b_2, \ldots, b_{n-1}\}$, the bijection holds. Hence, $N_q(n;d)$ is also the total number of sequences β that satisfy conditions 1 and 2.

Given $\beta = (b_1, b_2, \ldots, b_{n-1}, 0)$ satisfying the condition that $\{b_1, b_2, \ldots, b_{n-1}\}$ spans a d-dimensional subspace, we run the following algorithm:

1. Let $j = 1, t = 1$ and set $B = \varnothing$.

2. Consider b_{n-j}. If $b_{n-j} \in \mathrm{span}_{\mathbb{F}_q} B$, then do nothing; otherwise, let $i_t = j$, put b_{n-j} into B, and increase t by 1.

3. Increase j by 1. If $j \leq n-1$ and $t \leq d$, repeat step 2; otherwise, stop.

When finished, we get a special set of independent vectors $B = \{b_{i_1}, b_{i_2}, \ldots, b_{i_d}\}$. Now writing all b_i's as linear combinations of vectors in B, we have:

$$(b_{n-1}, b_{n-2}, \ldots, b_1) = (b_{i_1}, b_{i_2}, \ldots, b_{i_d})E,$$

where E is a $d \times (n-1)$ matrix in reduced row echelon form (see Definition 3.13). In fact, the i_t-th column of E has a single 1 in the t-th row and zero otherwise, and the columns between the i_t-th column and the i_{t+1}-th column has zeros in rows $t+1, t+2, \ldots, d$, for $0 \le t \le d$, where $i_0 = 0$ and $i_{d+1} = n+1$. For instance, if $n = 9, d = 4$ and $(i_1, i_2, i_3, i_4) = (2, 4, 5, 7)$, then E has the form as in equation (4.2), where $*$ denotes a entry from $\mathbb{F}_q$.

Deleting columns $i_1, i_2, \ldots, i_d$, we get a $d \times (n-1-d)$ matrix $\tilde{E}$. Let the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1-d})$ be as follows: λ_s is the number of $*$'s in the $(n-d-s)$-th column, for $s = 1, 2, \ldots, n-1-d$, i.e., the shape of λ is the same as the shape of all $*$'s in $\tilde{E}$ flipped horizontally. Conversely, given any partition λ with $n-1-d$ parts and largest part $\le d$, we can define $i_1, i_2, \ldots, i_d$ by the condition that i_t is $n-1-(d-t)$ minus the number of parts $\ge t$ in λ, and there are $q^{\lambda_1} q^{\lambda_2} \cdots q^{\lambda_{n-1-d}}$ possible reduced row echelon matrices E with $i_1, i_2, \ldots, i_d$ having the same meaning as above.

Since $N_q(n; d)$ equals to the number of possible $\beta = (b_1, b_2, \ldots, b_{n-1}, 0)$'s, which is the same as the number of possible $(b_{i_1}, b_{i_2}, \ldots, b_{i_d})$'s times the number of possible E's, we have:

$$N_q(n; d) = (q^n - 1)(q^n - q) \cdots (q^n - q^{d-1}) \cdot \sum_{\lambda \subset d \times (n-1-d)} q^{\lambda_1} q^{\lambda_2} \cdots q^{\lambda_{n-1-d}}.$$

By Proposition 1.3.19 from [7], we know that the sum in the equation is equal to the total number of d-dimensional subspaces of a $(n-1)$-dimensional space. This proves equation (4.4). □

Compare equation (4.1) and (4.4), we can see that $N_q(n; d)$ is a q-analogue of $N(n, d)$.

Corollary 4.20. *The set of nilpotent endomorphisms on n-dimensional vector space of rank d is a q-analogue of the set of rooted spanning trees of complete graph K_n with $n-d$ leaves.*

The same idea in the proof of Theorem 4.19 can be used to prove the following theorem.

Theorem 4.21. [2] *Consider a nilpotent endomorphism f on n-dimensional vector space U satisfying that* $\dim f^k(U) = d_k$ *for* $k = 0, 1, \ldots, r$, *where* $n = d_0 > d_1 > \cdots > d_r = 0$. *It is in bijection with* $\beta = (b_1, b_2, \ldots, b_n)$ *such that* $\{b_n, b_{n-1}, \ldots, b_{n-d_k+1}\}$ *spans a* d_{k+1}*-dimensional subspace of* U, *for* $k = 0, 1, \ldots, r-1$.

Let $\bar{d} = (d_1, d_2, \ldots, d_r)$. *The total number* $N_q(n; \bar{d})$ *of nilpotent endomorphisms with above property is:*

$$\begin{aligned} N_q(n;\bar{d}) &= \prod_{i=1}^{d_1}(q^n - q^{i-1}) \cdot \sum_{(\lambda^0,\lambda^1,\cdots,\lambda^{r-1})} q^{|\lambda^0|}q^{|\lambda^1|}\cdots q^{|\lambda^{r-1}|} \\ &= \prod_{i=1}^{d_1}(q^n - q^{i-1}) \cdot \prod_{k=0}^{r-1}\left(q^{(\delta_k-\delta_{k+1})d_{k+2}} \cdot \begin{bmatrix} \delta_k \\ \delta_{k+1} \end{bmatrix}_q \right), \end{aligned} \tag{4.5}$$

where $d_{r+1} = 0$, $\delta_k = d_k - d_{k+1}$ *for* $k = 0, 1, \ldots, r$, λ^k *ranges over all partitions with* $\delta_k - \delta_{k+1}$ *parts and largest part* $\leq d_{k+1}$ *smallest part* $\geq d_{k+2}$, *and* $|\lambda^k| = \lambda^k_1 + \lambda^k_2 + \cdots + \lambda^k_{\delta_k-\delta_{k+1}}$, *for* $k = 0, 1, \ldots, r-1$.

Proof. The bijection is an easy corollary of Lemma 3.9 that is proved in Section 3.2. Hence $N_q(n; \bar{d})$ is also the total number of sequence β that satisfies the condition.

Given $\beta = (b_1, b_2, \ldots, b_n)$, we run the following algorithm:

1. Let $j = 1, t = 1$ and set $B = \varnothing$.

2. Consider b_{n-j+1}. If $b_{n-j+1} \in \operatorname{span}_{\mathbb{F}_q} B$, then do nothing; otherwise, let $i_t = j$, put b_{n-j+1} into B, and increase t by 1.

3. Increase j by 1. If $j \leq n$ and $t \leq d_1$, repeat step 2; otherwise, stop.

When finished, we get a special set of independent vectors $B = \{b_{i_1}, b_{i_2}, \ldots, b_{i_{d_1}}\}$. Now write all b_i's as linear combinations of vectors in B, we have:

$$(b_n, b_{n-1}, \ldots, b_1) = (b_{i_1}, b_{i_2}, \ldots, b_{i_{d_1}})E,$$

[2] This result was mentioned also in Remark 3.2 of [1], but was stated incorrectly.

where $E = (E_{i,j})_{i,j=1}^{r}$ is a $d_1 \times n$ block matrix in reduced row echelon form (see Definition 3.13). Each $E_{i,j}$ is a $\delta_{r+1-i} \times \delta_{r-j}$ matrix that satisfies:

1. If $i > j$, $E_{i,j} = 0$.
2. If $i = j$, $E_{i,j}$ is a reduced row echelon matrix.
3. If $i < j$, $E_{i,j}$ is a matrix with a column equal to 0 if it corresponds to a pivot in $E_{j,j}$, and arbitrary otherwise.

That is, matrix E has the form:

$$\begin{pmatrix} E_{1,1} & * & \cdots & * \\ 0 & E_{2,2} & \cdots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & E_{r,r} \end{pmatrix},$$

where the $*$ in column j denotes a matrix that has all zeros in a column if $E_{j,j}$ has a pivot in the same column, and other entries are arbitrary elements in $\mathbb{F}_q$.

Using the same technique as in the proof of Theorem 4.19, we can bijectively get a partition $\hat{\lambda}^k$ from each $E_{r+1-k,r+1-k}$ such that $\hat{\lambda}^k$ has $\delta_k - \delta_{k+1}$ parts and the largest part is $\leq \delta_{k+1}$, for $k = 0, 1, \ldots, r-1$. Let λ^k be the partition after adding d_{k+2} to all parts of μ^k. Then the total number of reduced row echelon matrices E corresponding to partitions $(\lambda^0, \lambda^1, \cdots, \lambda^{r-1})$ is:

$$\left(q^{|\hat{\lambda}^0|} q^{|\hat{\lambda}^1|} \cdots q^{|\hat{\lambda}^{r-1}|}\right) \cdot \prod_{i<j} q^{(\delta_{r-j}-\delta_{r+1-j})\delta_{r+1-i}} = \prod_{k=0}^{r-1} q^{|\hat{\lambda}^k|} \cdot \prod_{k=0}^{r-1} q^{(\delta_k - \delta_{k+1})d_{k+2}} = \prod_{k=0}^{r-1} q^{|\lambda^k|}.$$

Similarly to the proof of Theorem 4.19, we have:

$$
\begin{aligned}
N_q(n;\bar{d}) &= \prod_{i=1}^{d_1}(q^n - q^{i-1}) \cdot \sum_{(\lambda^0,\lambda^1,\cdots,\lambda^{r-1})} q^{|\lambda^0|}q^{|\lambda^1|}\cdots q^{|\lambda^{r-1}|} \\
&= \prod_{i=1}^{d_1}(q^n - q^{i-1}) \cdot \prod_{k=0}^{r-1}\left(\sum_{\hat{\lambda}^k \subset \delta_{k+1}\times(\delta_k-\delta_{k+1})} q^{|\hat{\lambda}^k|}\right) \cdot \prod_{k=0}^{r-1} q^{(\delta_k-\delta_{k+1})d_{k+2}} \\
&= \prod_{i=1}^{d_1}(q^n - q^{i-1}) \cdot \prod_{k=0}^{r-1}\begin{bmatrix}\delta_k \\ \delta_{k+1}\end{bmatrix}_q \cdot \prod_{k=0}^{r-1} q^{(\delta_k-\delta_{k+1})d_{k+2}} \\
&= \prod_{i=1}^{d_1}(q^n - q^{i-1}) \cdot \prod_{k=0}^{r-1}\left(q^{(\delta_k-\delta_{k+1})d_{k+2}} \cdot \begin{bmatrix}\delta_k \\ \delta_{k+1}\end{bmatrix}_q\right).
\end{aligned}
$$

This proves equation (4.5). □

Corollary 4.22. *The total number of nilpotent endomorphisms on an n-dimensional vector space U with Jordan block sizes equal to the parts of the partition $\nu \vdash n$ is $N_q(n,\nu')$, where ν' is the conjugate partition of ν.*

Proof. For a nilpotent endomorphism f, given its sizes of all Jordan blocks, the dimensions of $f^k(U)$ is the same as the number of parts of ν that are $\geq k$, i.e., $\dim f^k(U) = \nu'_k$, for $k = 0, 1, 2, \ldots$. The proof follows from Theorem 4.21. □

Comparing equations (4.3) and (4.5), we can see that $N_q(n;\bar{d})$ is a q-analogue of $N(n,\bar{d})$.

Corollary 4.23. *The set of nilpotent endomorphisms on an n-dimensional vector space U satisfying* $\dim f^k(U) = d_k$ *for $k = 0, 1, \ldots, r$ is a q-analogue of the set of rooted spanning trees of the complete graph K_n with δ_k vertices in level k, where $\delta_k = d_k - d_{k+1}$ for $k = 0, 1, \ldots, r$.*

4.2 Complete Bipartite Graphs

A *complete bipartite graph* $K_{n,m}$ is a graph on the vertex set $V_1 \sqcup V_2$ with $V_1 = [n], V_2 = [m]$, and each vertex of V_1 is connected by an edge to each vertex of V_2. In digraph language, the complete bipartite graph $K_{n,m}$ is equivalent to a digraph on vertex set $\overrightarrow{V_1} \sqcup \overrightarrow{V_2}$ and there are two edges of opposite orientations between each vertex of V_1 and each vertex of V_2. In terms of the expanded digraph, it is also the same as the expanded digraph $D_{(n,m)}$, where D is the digraph with two vertices and two edges between them (see Figure 4-2).

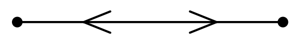

Figure 4-2: Digraph D.

Assume that in this section, $\star$ ranges over $\{1, 2\}$, and $\diamond$ denotes $\{1, 2\} - \{\star\}$.

4.2.1 Spanning Trees

A rooted spanning tree T of $K_{n,m}$ is equivalent to a spanning tree $\overrightarrow{T}$ of $D_{(n,m)}$ if we orient all the edges of T towards the root. By Lemma 3.8, $\overrightarrow{T}$ is in bijection with $\bar{\alpha} = (\alpha_1, \alpha_2) \in \mathrm{GS}_{\mathrm{Tr}}(D, (n, m))$, where $\alpha_1 = (a^1_1, a^1_2, \ldots, a^1_n), \alpha_2 = (a^2_1, a^2_2, \ldots, a^2_m)$. Since D has two spanning trees, we have the following result.

Proposition 4.24. *A rooted spanning tree T of $K_{n,m}$ is in bijection with:*

$$\begin{cases} \alpha_1 = (a^1_1, a^1_2, \ldots, a^1_n), \\ \alpha_2 = (a^2_1, a^2_2, \ldots, a^2_m), \end{cases}$$

such that:

1. *$a^1_n \in [m] \sqcup \{0\}, a^2_m \in [n] \sqcup \{0\}$. And one of a^1_n and a^2_m is zero.*

2. *$a^1_i \in [m], a^2_j \in [n]$ for $i = 1, 2, \ldots, n-1, j = 1, 2, \ldots, m-1$.*

Hence, the total number of rooted spanning trees of $K_{n,m}$ is $(n+m)n^{m-1}m^{n-1}$.

In fact, the bijection gives us more than the above property. For example, it gives a bijective proof of Exercise 5.30 from [8].

Let's consider a rooted spanning tree T of $K_{n,m}$ with $n-d^1$ leaves in V_1 and $m-d^2$ leaves in V_2.

Theorem 4.25. *A rooted spanning tree T of $K_{n,m}$ with $n-d^1$ leaves in V_1 and $m-d^2$ leaves in V_2 is in bijection with:*

$$\begin{cases} \alpha_1 = (a_1^1, a_2^1, \ldots, a_n^1), \\ \alpha_2 = (a_1^2, a_2^2, \ldots, a_m^2), \end{cases}$$

such that:

1. *Exactly one of a_n^1 and a_m^2 is zero.*

2. *If $a_n^1 = 0$, then $\{a_1^1, a_2^1, \ldots, a_{n-1}^1\}$ contains only d^2 distinct numbers from $[m]$, and $\{a_1^2, a_2^2, \ldots, a_m^2\}$ contains only d^1 distinct numbers from $[n]$.*

3. *If $a_m^2 = 0$, then $\{a_1^1, a_2^1, \ldots, a_n^1\}$ contains only d^2 distinct numbers from $[m]$, and $\{a_1^2, a_2^2, \ldots, a_{m-1}^2\}$ contains only d^1 distinct numbers from $[n]$.*

Hence, the total number $NM(n,m;d^1,d^2)$ of rooted spanning trees of $K_{n,m}$ with $n-d^1$ leaves in V_1 and $m-d^2$ leaves in V_2 is:

$$\begin{aligned} & NM(n,m;d^1,d^2) \\ = & \prod_{i=1}^{d^1}(n-i+1) \cdot \prod_{i=1}^{d^2}(m-i+1) \\ & \cdot \Big(\sum_{\lambda \subset d^2 \times (n-1-d^2)} \lambda_1\lambda_2\cdots\lambda_{n-1-d^2} \sum_{\mu \subset d^1 \times (m-d^1)} \mu_1\mu_2\cdots\mu_{m-d^1} \\ & + \sum_{\lambda \subset d^2 \times (n-d^2)} \lambda_1\lambda_2\cdots\lambda_{n-d^2} \sum_{\mu \subset d^1 \times (m-1-d^1)} \mu_1\mu_2\cdots\mu_{m-1-d^1} \Big) \\ = & \binom{n}{d^1} \cdot \binom{m}{d^2} \cdot \left(\sigma(d^2;n-1)\cdot\sigma(d^1;m) + \sigma(d^2;n)\cdot\sigma(d^1;m-1)\right). \end{aligned} \tag{4.6}$$

Proof. Since $\{a_1^\star, a_2^\star, \ldots, a_n^\star\}$ contains all the nonleaf vertices of $V_\diamond$, the bijection holds. Hence, $NM(n,m;d^1,d^2)$ is also the total number of pairs (α_1, α_2) that satisfy conditions 1, 2 and 3.

Assume, without loss of generality, that $a_n^1 = 0$.

Given $\alpha_1 = (a_1^1, a_2^1, \ldots, a_{n-1}^1, 0), \alpha_2 = (a_1^2, a_2^2, \ldots, a_m^2)$, we run the following two algorithms:

1. Algorithm 1:
   (a) Let $i = 1, s = 1$ and set $A^1 = \varnothing$.
   (b) Consider a_{n-i}^1. If $a_{n-i}^1 \in A^1$, then do nothing; otherwise, let $i_s = i$, put a_{n-i}^1 into A^1, and increase s by 1.
   (c) Increase i by 1. If $i \leq n-1$ and $s \leq d^2$, repeat step b; otherwise, stop.
2. Algorithm 2:
   (a) Let $j = 1, t = 1$ and set $A^2 = \varnothing$.
   (b) Consider a_{m+1-j}^2. If $a_{m+1-j}^2 \in A^2$, then do nothing; otherwise, let $j_t = j$, put a_{m+1-j}^2 into A^2, and increase t by 1.
   (c) Increase j by 1. If $j \leq m$ and $t \leq d^1$, repeat step b; otherwise, stop.

When finished, we get two special sets of distinct numbers $A^1 = \{a_{i_1}^1, a_{i_2}^1, \ldots, a_{i_{d^2}}^1\}$, $A^2 = \{a_{j_1}^2, a_{j_2}^2, \ldots, a_{j_{d^1}}^2\}$. Now writing all a_i^1's and a_j^2's in terms of numbers in A^1 and A^2 respectively, we have:

$$\begin{cases} (a_{n-1}^1, a_{n-2}^1, \ldots, a_1^1) = (a_{i_1}^1, a_{i_2}^1, \ldots, a_{i_{d^2}}^1)P^1, \\ (a_m^2, a_{m-1}^2, \ldots, a_1^2) = (a_{j_1}^2, a_{j_2}^2, \ldots, a_{j_{d^1}}^2)P^2, \end{cases}$$

where P^1 and P^2 are $d^2 \times (n-1)$ and $d^1 \times m$ matrices, respectively, in reduced row echelon form (see Definition 3.13) that have a unique 1 in each column and zeros otherwise.

Using the same technique as in the proof of Theorem 4.16, we can bijectively get partitions λ and μ from P^1 and P^2 such that they have $n-1-d^2$ and $m-d^1$ parts and the largest parts are $\leq d^2$ and $\leq d^1$, respectively. Then the total number of reduced row echelon matrices P^1 and P^2 corresponding to partitions λ and μ is $(\lambda_1\lambda_2\lambda_{n-1-d^2})(\mu_1\mu_2\mu_{m-d^1})$.

By symmetry, this shows the first part of equation (4.6).

Since $NM(n,m;d^1,d^2)$ is also equal to the number of ways to choose d^2 distinct numbers from $[m]$ and d^1 distinct numbers from $[n]$, and put them in to $(a_1^1,a_2^1,\dots,a_{n-1}^1)$ and $(a_1^2,a_2^2,\dots,a_m^2)$ (or $(a_1^1,a_2^1,\dots,a_n^1)$ and $(a_1^2,a_2^2,\dots,a_{m-1}^2)$), respectively, such that each number appears at least once, we have:

$$NM(n,m;d^1,d^2) = \binom{n}{d^1}\cdot\binom{m}{d^2}\cdot\left(\sigma(d^2;n-1)\cdot\sigma(d^1;m)+\sigma(d^2;n)\cdot\sigma(d^1;m-1)\right).$$

This proves the rest of equation (4.6). □

The same idea in the above proof can be used to show the following theorem.

Theorem 4.26. *Consider a rooted spanning tree T of $K_{n,m}$ with $\delta_k^\star$ vertices from $V_\star$ in level k, where $\delta_k^\star \geq \delta_{k+1}^\diamond$, $\delta_r^\star = 0$, $\delta_{r-1}^1+\delta_{r-1}^2>0$, and $\sum_{k=0}^r \delta_k^1 = n$, $\sum_{k=0}^r \delta_k^2 = m$. It is in bijection with:*

$$\begin{cases} \alpha_1 = (a_1^1,a_2^1,\dots,a_n^1), \\ \alpha_2 = (a_1^2,a_2^2,\dots,a_m^2), \end{cases}$$

such that $\{a_n^1,a_{n-1}^1,\dots,a_{n-d_k^1+1}^1\}$ contains only d_{k+1}^2 distinct numbers from $[m]$, and $\{a_m^2,a_{m-1}^2,\dots,a_{m-d_k^2+1}^2\}$ contains only d_{k+1}^1 distinct numbers from $[n]$, where $d_k^\star = \delta_r^\star+\delta_{r-1}^\star+\cdots+\delta_k^\star$, for $k=0,1,\dots,r$.

Letting $\bar{d}^\star = (d_1^\star,d_2^\star,\dots,d_r^\star)$, the total number $NM(n,m;\bar{d}^1,\bar{d}^2)$ of rooted spanning trees with the above property is:

$$
\begin{aligned}
&NM(n,m;\bar{d}^1,\bar{d}^2)\\
=&\prod_{i=1}^{d_1^1}(n-i+1)\cdot\prod_{i=1}^{d_1^2}(m-i+1)\\
&\cdot\sum_{(\lambda^0,\lambda^1,\ldots,\lambda^{r-1})}\Pi(\lambda^0)\Pi(\lambda^1)\cdots\Pi(\lambda^{r-1})\cdot\sum_{(\mu^0,\mu^1,\ldots,\mu^{r-1})}\Pi(\mu^0)\Pi(\mu^1)\cdots\Pi(\mu^{r-1}) \qquad (4.7)\\
=&\binom{n}{\delta_0^1,\delta_1^1,\ldots,\delta_{r-1}^1}\cdot\binom{m}{\delta_0^2,\delta_1^2,\ldots,\delta_{r-1}^2}\cdot\prod_{k=0}^{r-1}\prod_{\star}\sigma(\delta_{k+1}^{\diamond},d_{k+2}^{\diamond};\delta_k^{\star}),
\end{aligned}
$$

where $d_{r+1}^{\star}=0$, λ^k ranges over all partitions with $\delta_k^1-\delta_{k+1}^2$ parts and largest part $\leq d_{k+1}^2$ smallest part $\geq d_{k+2}^2$, and μ^k ranges over all partitions with $\delta_k^2-\delta_{k+1}^1$ parts and largest part $\leq d_{k+1}^1$ smallest part $\geq d_{k+2}^1$.

Proof. The bijection is implied by the proof of Lemma 3.8 in Section 3.1. Hence, $NM(n,m;\bar{d}^1,\bar{d}^2)$ is also the total number of pairs (α_1,α_2) that satisfy the condition.

Given $\alpha_1=(a_1^1,a_2^1,\ldots,a_n^1),\alpha_2=(a_1^2,a_2^2,\ldots,a_m^2)$, we run the following two algorithms:

1. Algorithm 1:

   (a) Let $i=1,s=1$ and set $A^1=\varnothing$.

   (b) Consider a_{n+1-i}^1. If $a_{n+1-i}^1\in A^1$, then do nothing; otherwise, let $i_s=i$, put a_{n+1-i}^1 into A^1, and increase s by 1.

   (c) Increase i by 1. If $i\leq n$ and $s\leq d_1^2$, repeat step (b); otherwise, stop.

2. Algorithm 2:

   (a) Let $j=1,t=1$ and set $A^2=\varnothing$.

   (b) Consider a_{m+1-j}^2. If $a_{m+1-j}^2\in A^2$, then do nothing; otherwise, let $j_t=j$, put a_{m+1-j}^2 into A^2, and increase t by 1.

   (c) Increase j by 1. If $j\leq m$ and $t\leq d_1^1$, repeat step (b); otherwise, stop.

When finished, we get two special sets of distinct numbers $A^1 = \{a^1_{i_1}, a^1_{i_2}, \ldots, a^1_{i_{d^2_1}}\}$, $A^2 = \{a^2_{j_1}, a^2_{j_2}, \ldots, a^2_{j_{d^1_1}}\}$. Now writing all a^1_i's and a^2_j's in terms of numbers in A^1 and A^2 respectively, we have:

$$\begin{cases} (a^1_n, a^1_{n-1}, \ldots, a^1_1) = (a^1_{i_1}, a^1_{i_2}, \ldots, a^1_{i_{d^2_1}})P^1, \\ (a^2_m, a^2_{m-1}, \ldots, a^2_1) = (a^2_{j_1}, a^2_{j_2}, \ldots, a^2_{j_{d^1_1}})P^2, \end{cases}$$

where $P^1 = (P^1_{i,j})^r_{i,j=1}$ and $P^2 = (P^2_{i,j})^r_{i,j=1}$ are $d^2_1 \times n$ and $d^1_1 \times m$ block matrices, respectively, in reduced row echelon form (see Definition 3.13) that have a unique 1 in each column and zeros otherwise. And each $P^\star_{i,j}$ is a $\delta^\diamond_{r+1-i} \times \delta^\star_{r-j}$ matrix that satisfies:

1. If $i > j$, $P^\star_{i,j} = 0$.
2. If $i = j$, $P^\star_{i,j}$ is a reduced row echelon matrix.
3. If $i < j$, $P^\star_{i,j}$ is a matrix with a column equal to 0 if it corresponds to a pivot in $P^\star_{j,j}$, and arbitrary otherwise.

Using the same technique as in the proof of Theorem 4.16, we can bijectively get a partition $\hat{\lambda}^k$ from each $P^1_{r+1-k,r+1-k}$ such that $\hat{\lambda}^k$ has $\delta^1_k - \delta^2_{k+1}$ parts and the largest part is $\leq \delta^2_{k+1}$, and also a partition $\hat{\mu}^k$ from each $P^2_{r+1-k,r+1-k}$ such that $\hat{\mu}^k$ has $\delta^2_k - \delta^1_{k+1}$ parts and the largest part is $\leq \delta^1_{k+1}$, for $k = 0, 1, \ldots, r-1$. Let λ^k be the partition after adding d^2_{k+2} to all parts of $\hat{\lambda}^k$, and μ^k be the partition after adding d^1_{k+2} to all parts of $\hat{\mu}^k$. Then the total number of reduced row echelon matrices P^1 and P^2 corresponding to partitions $(\lambda^0, \lambda^1, \cdots, \lambda^{r-1})$ and $(\mu^0, \mu^1, \cdots, \mu^{r-1})$ is:

$$\prod_{k=0}^{r-1} \left(\lambda^k_1 \lambda^k_2 \cdots \lambda^k_{\delta^1_k - \delta^2_{k+1}}\right) \left(\mu^k_1 \mu^k_2 \cdots \mu^k_{\delta^2_k - \delta^1_{k+1}}\right) = \prod_{k=0}^{r-1} \Pi(\lambda^k)\Pi(\mu^k).$$

This shows the first part of equation (4.7).

Since $NM(n, m; \bar{d}^1, \bar{d}^2)$ is also equal to the number of ways to first choose δ^2_r (resp. δ^1_r) distinct numbers from V_2 (resp. V_1) and put them into $\{a^1_n, a^1_{n-1}, \ldots, a^1_{n-d^1_{r-1}+1}\}$ (resp. $\{a^2_m, a^2_{m-1}, \ldots, a^2_{m-d^2_{r-1}+1}\}$) such that each number appears at least once; then for $k = r-2, r-3, \ldots, 0$, choose δ^2_{k+1} (resp. δ^1_{k+1}) distinct numbers from the rest $m - d^2_{k+2}$ (resp. $n - d^1_{k+2}$) numbers from V_2 (resp. V_1), together with the d^2_{k+2} (resp. d^1_{k+2}) chosen numbers, put them into $\{a^1_{n-d^1_{k+1}}\}, a^1_{n-d^1_{k+1}-1}\}, \ldots, a^1_{n-d^1_k+1}\}$ (resp. $\{a^2_{m-d^2_{k+1}}\}, a^2_{m-d^2_{k+1}-1}\}, \ldots, a^2_{m-d^2_k+1}\}$) such that each of the δ^2_{k+1} (resp. δ^1_{k+1}) numbers appears at least once. Hence we have:

$$\begin{aligned}
&NM(n,m;\bar{d}^1,\bar{d}^2)\\
=&\left(\binom{n}{\delta^1_r}\sigma(\delta^1_r;d^2_{r-1})\right)\cdot\left(\binom{m}{\delta^2_r}\sigma(\delta^2_r;d^1_{r-1})\right)\\
&\cdot\prod_{k=0}^{r-2}\left(\binom{n-d^1_{k+2}}{\delta^1_{k+1}}\sigma(\delta^1_{k+1},d^1_{k+2};\delta^2_k)\right)\cdot\prod_{k=0}^{r-2}\left(\binom{m-d^2_{k+2}}{\delta^2_{k+1}}\sigma(\delta^2_{k+1},d^2_{k+2};\delta^1_k)\right)\\
=&\binom{n}{\delta^1_0,\delta^1_1,\ldots,\delta^1_{r-1}}\cdot\binom{m}{\delta^2_0,\delta^2_1,\ldots,\delta^2_{r-1}}\cdot\prod_{k=0}^{r-1}\prod_{\star}\sigma(\delta^\diamond_{k+1},d^\diamond_{k+2};\delta^\star_k).
\end{aligned}$$

This proves the rest of Equation 4.7. $\square$

4.2.2 Nilpotent Transformations

Now we want to consider the nilpotent D-space linear transformation. With $\bar{n} = (n,m)$, it is the same as a nilpotent endomorphism $f : U_1 \oplus U_2 \to U_1 \oplus U_2$ satisfying $f_\star := f|_{U_\star} : U_\star \to U_\diamond$, where U_1 (resp. U_2) is a n-dimensional (resp. m-dimensional) vector space over $\mathbb{F}_q$.

By Lemma 3.9, f is in bijection with $\bar{\beta} = (\beta_1,\beta_2) \in \mathrm{GS}_{\mathrm{Nil}}(D,(n,m))$, where $\beta_1 = (b^1_1, b^1_2, \ldots, b^1_n), \beta_2 = (b^2_1, b^2_2, \ldots, b^2_m)$. Since D has three spanning forests, we have the following result.

Proposition 4.27. *A nilpotent endomorphism f on $U_1 \oplus U_2$ satisfying $f_\star : U_\star \to U_\diamond$ is in bijection with:*

$$\begin{cases}\beta_1 = (b^1_1, b^1_2, \ldots, b^1_n),\\ \beta_2 = (b^2_1, b^2_2, \ldots, b^2_m),\end{cases}$$

such that:

1. *At least one of b_n^1 and b_m^2 is zero.*

2. $b_i^1 \in U_2, b_j^2 \in U_1$ *for* $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m.$

Hence, the total number of nilpotent endomorphisms on $U_1 \oplus U_2$ with the above property is $(q^n + q^m - 1)q^{n(m-1)}q^{m(n-1)}$.

In fact, the bijection gives us more than the above property. We can also enumerate the number of nilpotent transformations f of fixed ranks for f_1 and f_2.

Theorem 4.28. *Consider a nilpotent endomorphism f on $U_1 \oplus U_2$ that satisfies $f_\star : U_\star \to U_\diamond$ and f_{star} is of rank $d^\diamond$. It is in bijection with:*

$$\begin{cases} \beta_1 = (b_1^1, b_2^1, \ldots, b_n^1), \\ \beta_2 = (b_1^2, b_2^2, \ldots, b_m^2), \end{cases}$$

such that:

1. *At least one of b_n^1 and b_m^2 is zero.*

2. $\{b_1^1, b_2^1, \ldots, b_n^1\}$ *spans a d^2-dimensional subspace of U_2, and* $\{b_1^2, b_2^2, \ldots, b_n^2\}$ *spans a d^1-dimensional subspace of U_1.*

Hence, the total number $NM_q(n, m; d^1, d^2)$ of nilpotent endomorphisms on $U_1 \oplus U_2$ with the above property is:

$$\begin{aligned} &NM(n, m; d^1, d^2) \\ =&\prod_{i=1}^{d^1}(q^n - q^{i-1}) \cdot \prod_{i=1}^{d^2}(q^m - q^{i-1}) \cdot \Big(\sum_{\lambda \subset d^2 \times (n-1-d^2)} q^{|\lambda|} \sum_{\substack{(1^{m-d^1}) \subset \mu \\ \subset d^1 \times (m-d^1)}} q^{|\mu|} \\ &+ \sum_{\substack{(1^{n-d^2}) \subset \lambda \\ \subset d^2 \times (n-d^2)}} q^{|\lambda|} \sum_{\mu \subset d^1 \times (m-1-d^1)} q^{|\mu|} + \sum_{\lambda \subset d^2 \times (n-1-d^2)} q^{|\lambda|} \sum_{\mu \subset d^1 \times (m-1-d^1)} q^{|\mu|} \Big) \\ =&\prod_{i=1}^{d^1}(q^n - q^{i-1}) \cdot \prod_{i=1}^{d^2}(q^m - q^{i-1}) \cdot \Big(q^{m-d^1} \begin{bmatrix} n-1 \\ d^2 \end{bmatrix}_q \cdot \begin{bmatrix} m-1 \\ d^1-1 \end{bmatrix}_q \\ &+ q^{n-d^2} \begin{bmatrix} n-1 \\ d^2-1 \end{bmatrix}_q \cdot \begin{bmatrix} m-1 \\ d^1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ d^2 \end{bmatrix}_q \cdot \begin{bmatrix} m-1 \\ d^1 \end{bmatrix}_q \Big). \end{aligned} \tag{4.8}$$

Proof. Since $f_1(U_1) = f(U_1 \oplus U_2) \cap U_2 = \mathrm{span}_{\mathbb{F}_q}\{b_1^1, b_2^1, \dots, b_n^1\}, f_2(U_2) = f(U_1 \oplus U_2) \cap U_1 = \mathrm{span}_{\mathbb{F}_q}\{b_1^2, b_2^2, \dots, b_m^2\}$, the bijection holds. Hence, $NM_q(n, m; d^1, d^2)$ is also the total number of pairs (β_1, β_2) that satisfy the condition.

Assume, without loss of generality, that $b_n^1 = 0, b_m^2 \neq 0$.

Given $\beta_1 = (b_1^1, b_2^1, \dots, b_{n-1}^1, 0), \beta_2 = (b_1^2, b_2^2, \dots, b_m^2)$, we run the following two algorithms:

1. Algorithm 1:

    (a) Let $i = 1, s = 1$ and set $B^1 = \varnothing$.

    (b) Consider b_{n-i}^1. If $b_{n-i}^1 \in \mathrm{span}_{\mathbb{F}_q} B^1$, then do nothing; otherwise, let $i_s = i$, put b_{n-i}^1 into B^1, and increase s by 1.

    (c) Increase i by 1. If $i \leq n-1$ and $s \leq d^2$, repeat step (b); otherwise, stop.

2. Algorithm 2:

    (a) Let $j = 1, t = 1$ and set $B^2 = \varnothing$.

    (b) Consider b_{m+1-j}^2. If $b_{m+1-j}^2 \in \mathrm{span}_{\mathbb{F}_q} B^2$, then do nothing; otherwise, let $j_t = j$, put b_{m+1-j}^2 into B^2, and increase t by 1.

    (c) Increase j by 1. If $j \leq m$ and $t \leq d^1$, repeat step (b); otherwise, stop.

When finished, we get two special sets of independent vectors $B^1 = \{b_{i_1}^1, b_{i_2}^1, \dots, b_{i_{d^2}}^1\}, B^2 = \{b_{j_1}^2, b_{j_2}^2, \dots, b_{j_{d^1}}^2\}$, where $j_1 = 1$. Now writing all b_i^1's and b_j^2's in terms of numbers in B^1 and B^2 respectively, we have:

$$\begin{cases} (b_{n-1}^1, b_{n-2}^1, \dots, b_1^1) = (b_{i_1}^1, b_{i_2}^1, \dots, b_{i_{d^2}}^1)E^1, \\ (b_m^2, b_{m-1}^2, \dots, b_1^2) = (b_{j_1}^2, b_{j_2}^2, \dots, b_{j_{d^1}}^2)E^2, \end{cases}$$

where E^1 and E^2 are $d^2 \times (n-1)$ and $d^1 \times m$ matrices, respectively, in reduced row echelon form (see Definition 3.13). In fact, the i_s-th (resp. j_t-th) column of E^1 (resp. E^2) has a single 1 in the s-th (resp. t-th) row and zero otherwise, and columns between the i_s-th (resp. j_t-th) column and the i_{s+1}-th (resp. j_{t+1}-th) column has zeros in rows $s+1, s+2, \ldots, d^2$ (resp. $t+1, t+2, \ldots, d^1$), for $0 \le s \le d^2, 0 \le t \le d^1$, where $i_0 = j_0 = 0$ and $i_{d^2+1} = n, j_{d^1+1} = m+1$.

Use the same technique as in the proof of Theorem 4.16, we can bijectively get partitions λ and μ from E^1 and E^2 such that they have $n-1-d^2$ and $m-d^1$ parts and the largest parts are $\le d^2$ and $\le d^1$, respectively. And since $j_1 = 1$, we have $(1^{m-d^1}) \subset \mu$. Then the total number of reduced row echelon matrices E^1 and E^2 corresponding to the partitions λ and μ is $q^{|\lambda|} \cdot q^{|\mu|}$.

By symmetry, this shows the first part of equation (4.8).

The rest of equation (4.8) follows directly from Proposition 1.3.19 in [7]. $\square$

Comparing equations (4.6) and (4.8), we can see that $NM_q(n,m;d^1,d^2)$ is a q-analogue of $NM(n,m;d^1,d^2)$. (Use the same trick as in proof of part 3 of Theorem 2.4.)

Corollary 4.29. *The set of nilpotent endomorphisms f on $U_1 \oplus U_2$ satisfying $f_\star : U_\star \to U_\diamond$ and f_{star} is of rank $d^\diamond$ is a q-analogue of the set of rooted spanning trees of $K_{n,m}$ with $n-d^1$ leaves in V_1 and $m-d^2$ leaves in V_2.*

The same idea in the proof of Theorem 4.28 can be used to prove the following theorem.

Theorem 4.30. *Consider a nilpotent endomorphism f on $U_1 \oplus U_2$ that satisfies $f_\star : U_\star \to U_\diamond$ and $\dim f^k(U_1 \oplus U_2) \cap U_\star = d_k^\star$, where $n = d_0^1 \ge d_1^1 \ge \cdots \ge d_r^1 = 0, m = d_0^2 \ge d_1^2 \ge \cdots \ge d_r^2 = 0$, and $d_{r-1}^1 + d_{r-1}^2 > 0$. It is in bijection with:*

$$\begin{cases} \beta_1 = (b_1^1, b_2^1, \ldots, b_n^1), \\ \beta_2 = (b_1^2, b_2^2, \ldots, b_m^2), \end{cases}$$

such that $\{b_n^1, b_{n-1}^1, \ldots, b_{n-d_k^1+1}^1\}$ spans a d_{k+1}^2-dimensional subspace of U_2, and $\{b_m^2, b_{m-1}^2, \ldots, b_{m-d_k^2+1}^2\}$ spans a d_{k+1}^1-dimensional subspace of U_1, for $k = 0, 1, \ldots, r-1$.

Letting $\bar{d}^\star = (d_1^\star, d_2^\star, \ldots, d_r^\star)$, the total number $NM_q(n,m;\bar{d}^1,\bar{d}^2)$ of nilpotent endomorphisms with the above property is:

$$
\begin{aligned}
&NM_q(n,m;\bar{d}^1,\bar{d}^2)\\
=&\prod_{i=1}^{d_1^1}(q^n-q^{i-1})\cdot\prod_{i=1}^{d_1^2}(q^m-q^{i-1})\\
&\cdot\sum_{(\lambda^0,\lambda^1,\cdots,\lambda^{r-1})} q^{|\lambda^0|}q^{|\lambda^1|}\cdots q^{|\lambda^{r-1}|}\cdot\sum_{(\mu^0,\mu^1,\cdots,\mu^{r-1})} q^{|\mu^0|}q^{|\mu^1|}\cdots q^{|\mu^{r-1}|} \qquad (4.9)\\
=&\prod_{i=1}^{d_1^1}(q^n-q^{i-1})\cdot\prod_{i=1}^{d_1^2}(q^m-q^{i-1})\cdot\prod_{k=0}^{r-1}\prod_{\star}\left(q^{(\delta_k^\star-\delta_{k+1}^\diamond)d_{k+2}^\diamond}\cdot\begin{bmatrix}\delta_k^\star\\ \delta_{k+1}^\diamond\end{bmatrix}_q\right),
\end{aligned}
$$

where $d_{r+1}^\star=0$, $\delta_k^\star=d_k^\star-d_{k+1}^\star$ *for* $k=0,1,\ldots,r$, λ^k *ranges over all partitions with* $\delta_k^1-\delta_{k+1}^2$ *parts and largest part* $\le d_{k+1}^2$ *and smallest part* $\ge d_{k+2}^2$*, and* μ^k *ranges over all partitions with* $\delta_k^2-\delta_{k+1}^1$ *parts and largest part* $\le d_{k+1}^1$ *and smallest part* $\ge d_{k+2}^1$*.*

Proof. The bijection is a easy corollary of Lemma 3.9 that is proved in Section 3.2. Hence, $NM_q(n,m;\bar{d}^1,\bar{d}^2)$ is also the total number of pairs (β_1,β_2) that satisfy the condition.

Given $\beta_1=(b_1^1,b_2^1,\ldots,b_n^1),\beta_2=(b_1^2,b_2^2,\ldots,b_m^2)$, we run the following two algorithms:

1. Algorithm 1:

   (a) Let $i=1,s=1$ and set $B^1=\varnothing$.

   (b) Consider b_{n+1-i}^1. If $b_{n+1-i}^1\in\mathrm{span}_{\mathbb{F}_q}B^1$, then do nothing; otherwise, let $i_s=i$, put b_{n+1-i}^1 into B^1, and increase s by 1.

   (c) Increase i by 1. If $i\le n$ and $s\le d_1^2$, repeat step (b); otherwise, stop.

2. Algorithm 2:

   (a) Let $j=1,t=1$ and set $B^2=\varnothing$.

   (b) Consider b_{m+1-j}^2. If $b_{m+1-j}^2\in\mathrm{span}_{\mathbb{F}_q}B^2$, then do nothing; otherwise, let $j_t=j$, put b_{m+1-j}^2 into B^2, and increase t by 1.

(c) Increase j by 1. If $j \leq m$ and $t \leq d_1^1$, repeat step (b); otherwise, stop.

When finished, we get two special sets of independent vectors $B^1 = \{b^1_{i_1}, b^1_{i_2}, \ldots, b^1_{i_{d_1^2}}\}$, $B^2 = \{b^2_{j_1}, b^2_{j_2}, \ldots, b^2_{j_{d_1^1}}\}$. Now writing all b_i^1's and b_j^2's in terms of numbers in B^1 and B^2 respectively, we have:

$$\begin{cases} (b^1_n, b^1_{n-1}, \ldots, b^1_1) = (b^1_{i_1}, b^1_{i_2}, \ldots, b^1_{i_{d_1^2}})E^1, \\ (b^2_m, b^2_{m-1}, \ldots, b^2_1) = (b^2_{j_1}, b^2_{j_2}, \ldots, b^2_{j_{d_1^1}})E^2, \end{cases}$$

where $E^1 = (E^1_{i,j})^r_{i,j=1}$ and $E^2 = (E^2_{i,j})^r_{i,j=1}$ are $d_1^2 \times n$ and $d_1^1 \times n$ matrices, respectively, in reduced row echelon form (see Definition 3.13). And each $E^\star_{i,j}$ is a $\delta^\diamond_{r+1-i} \times \delta^\star_{r-j}$ matrix that satisfies:

1. If $i > j$, $E^\star_{i,j} = 0$.
2. If $i = j$, $E^\star_{i,j}$ is a reduced row echelon matrix.
3. If $i < j$, $E^\star_{i,j}$ is a matrix with a column equal to 0 if it corresponds to a pivot in $E^\star_{j,j}$, and arbitrary otherwise.

Using the same technique as in the proof of Theorem 4.16, we can bijectively get a partition $\hat{\lambda}^k$ from each $E^1_{r+1-k,r+1-k}$ such that $\hat{\lambda}^k$ has $\delta^1_k - \delta^2_{k+1}$ parts and the largest part is $\leq \delta^2_{k+1}$, and also a partition $\hat{\mu}^k$ from each $E^2_{r+1-k,r+1-k}$ such that $\hat{\mu}^k$ has $\delta^2_k - \delta^1_{k+1}$ parts and the largest part is $\leq \delta^1_{k+1}$, for $k = 0, 1, \ldots, r-1$. Let λ^k be the partition after adding d^2_{k+2} to all parts of $\hat{\lambda}^k$, and μ^k be the partition after adding d^1_{k+2} to all parts of $\hat{\mu}^k$. Then the total number of reduced row echelon matrices E^1 and E^2 corresponding to partitions $(\lambda^0, \lambda^1, \cdots, \lambda^{r-1})$ and $(\mu^0, \mu^1, \cdots, \mu^{r-1})$ is:

$$\left(q^{|\lambda^0|}q^{|\lambda^1|}\cdots q^{|\lambda^{r-1}|}\right) \cdot \left(q^{|\mu^0|}q^{|\mu^1|}\cdots q^{|\mu^{r-1}|}\right).$$

This shows the first part of equation (4.9).

Since $|\lambda^k| = |\hat{\lambda}^k| + (\delta^1_k - \delta^2_{k+1})d^2_{k+2}$ and $|\mu^k| = |\hat{\mu}^k| + (\delta^2_k - \delta^1_{k+1})d^1_{k+2}$, the rest of equation (4.9) follows directly from Proposition 1.3.19 in [7]. $\square$

Corollary 4.31. *Consider a nilpotent endomorphism f on $U_1 \oplus U_2$ with $f_\star : U_\star \to U_\diamond$. If $f_2 f_1$ has Jordan block sizes equal to the parts of the partition $\nu^1 \vdash n$, and $f_1 f_2$ has Jordan block sizes equal to the parts of the partition $\nu^2 \vdash n$, then the total number of possible f's is a sum of $NM_q(n, m; \bar{d}^1, \bar{d}^2)$ over all possible $\{\delta^\star_{2k} \in \mathbb{N} : k \in \mathbb{N}\}$ satisfying:*

$$\delta^\diamond_{2k} + \delta^\star_{2k} \geq (\nu^\star)'_k - (\nu^\star)'_{k+1} \geq \delta^\diamond_{2k} + \delta^\star_{2k+2},$$

where:

$$\begin{cases} d^\star_{2k} = (\nu^\star)'_k, \\ d^\star_{2k+1} = (\nu^\star)'_k - \delta^\star_{2k}, \\ \delta^\star_{2k+1} = (\nu^\star)'_k - (\nu^\star)'_{k+1} - \delta^\star_{2k}. \end{cases}$$

Proof. For a nilpotent endomorphism $f_\diamond f_\star$, given the sizes of its Jordan blocks, the dimensions of $(f_\diamond f_\star)^k(U_\star)$ is the same as the number of parts of $\nu^\star$ that are $\geq k$, i.e., $\dim (f_\diamond f_\star)^k(U_\star) = (\nu^\star)'_k$, for $k = 0, 1, 2, \ldots$.

Since $(f_\diamond f_\star)^k(U_\star) = f^{2k}(U_1 \oplus U_2) \cap U_\star$, the proof follows from Theorem 4.30. $\square$

Comparing equations (4.7) and (4.9), we can see that $NM_q(n, m; \bar{d}^1, \bar{d}^2)$ is a q-analogue of $NM(n, m; \bar{d}^1, \bar{d}^2)$.

Corollary 4.32. *The set of nilpotent endomorphisms f on $U_1 \oplus U_2$ that satisfy $f_\star : U_\star \to U_\diamond$ and $\dim f^k(U_1 \oplus U_2) \cap U_\star = d^\star_k$ is a q-analogue of the set of rooted spanning trees of $K_{n,m}$ with $\delta^\star_k$ vertices from $V_\star$ in level k, where $\delta^\star_k = d^\star_k - d^\star_{k+1}$ for $k = 0, 1, \ldots, r$.*

4.3 Cycles

An *m-cycle* C_m is a digraph on the vertex set $[m]$, whose edges are $i \to i + 1$, for $i = 1, 2, \ldots, m$, where $m + 1$ is the same vertex as 1 (see Figure 4-3).

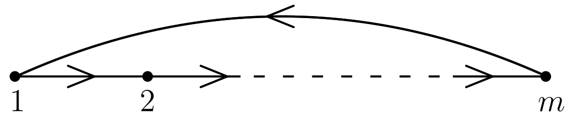

Figure 4-3: Digraph C_m.

We want to consider spanning trees of the expanded digraph $(C_m)_{\bar{n}} = (V_{\bar{n}}, E_{\bar{n}})$ and the nilpotent C_m-space linear transformations.

4.3.1 Spanning Trees

By Lemma 3.8, a spanning tree $T_{\bar{n}}$ of the expanded digraph $(C_m)_{\bar{n}}$ is in bijection with $\bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathrm{GS}_{\mathrm{Tr}}(C_m, \bar{n})$, where $\alpha_i = (a_1^i, a_2^i, \ldots, a_{n_i}^i)$, for $i = 1, 2, \ldots, m$. Since C_m has m spanning trees, we have the following result.

Proposition 4.33. *A spanning tree $T_{\bar{n}}$ of the expanded digraph $(C_m)_{\bar{n}}$ is in bijection with $\bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathrm{GS}_{\mathrm{Tr}}(C_m, \bar{n})$, such that there exists a unique $I \in [m]$ satisfying:*

1. *$a_{n_I}^I = 0$, and $a_l^I \in V_{I+1}$ for $l = 1, 2, \ldots, n_I - 1$.*

2. *$a_l^i \in V_{i+1}$ for $i \neq I, l = 1, 2, \ldots, n_i$.*

Hence, the total number of spanning trees of $(C_m)_{\bar{n}}$ is

$$\prod_{i=1}^{m} n_{i+1}^{n_i} \cdot \sum_{i=1}^{m} \frac{1}{n_i}\,.$$

Using the same technique as in the complete bipartite graph case in Section 4.2, we can get the following theorems.

Theorem 4.34. *A spanning tree $T_{\bar{n}}$ of the expanded digraph $(C_m)_{\bar{n}}$ with $n_i - d^i$ leaves in V_i, for $i = 1, 2, \ldots, m$, is in bijection with $\bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathrm{GS}_{\mathrm{Tr}}(C_m, \bar{n})$, such that there exists a unique $I \in [m]$ satisfying:*

1. *$a_{n_I}^I = 0$.*

2. $\{a_1^I, a_2^I, \ldots, a_{n_I-1}^I\}$ *contains only* d^{I+1} *distinct numbers from* V_{I+1}*, and* $\{a_1^i, a_2^i, \ldots, a_{n_i}^i\}$ *contains only* d^{i+1} *distinct numbers from* V_{i+1} *for any* $i \neq I$.

Hence, the total number $CN(\bar{n}; d^1, d^2, \ldots, d^m)$ *of spanning trees of* $(C_m)_{\bar{n}}$ *with* $n_i - d^i$ *leaves in* V_i *is:*

$$\begin{aligned} &CN(\bar{n}; d^1, d^2, \ldots, d^m) \\ =&\prod_{i=1}^{m}\left(\prod_{j=1}^{d^i}(n_i - j + 1)\right) \cdot \sum_{I=1}^{m}\prod_{i=1}^{m}\left(\sum_{\lambda^i \subset d^{i+1}\times(n_i - d^{i+1} - \varepsilon_{i,I})} \lambda_1^i \lambda_2^i \cdots \lambda_{n_i - d^{i+1} - \varepsilon_{i,I}}^i\right) \\ =&\prod_{i=1}^{m}\binom{n_i}{d^i} \cdot \sum_{I=1}^{m}\prod_{i=1}^{m}\sigma(d^{i+1}; n - \varepsilon_{i,I}), \end{aligned} \tag{4.10}$$

where:

$$\varepsilon_{i,I} = \begin{cases} 1, & \textit{if } i = I, \\ 0, & \textit{if } i \neq I. \end{cases}$$

Theorem 4.35. *Consider a spanning tree* $T_{\bar{n}}$ *of the expanded digraph* $(C_m)_{\bar{n}}$ *with* δ_k^i *vertices from* V_i *in level* k*, where* $\delta_k^i \geq \delta_{k+1}^{i+1}$*,* $\delta_r^i = 0$*,* $\sum_{i=1}^m \delta_{r-1}^i > 0$*, and* $\sum_{k=0}^r \delta_k^i = n_i$*, for* $i = 1, 2, \ldots, m$*. It is in bijection with* $\bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathrm{GS}_{\mathrm{Tr}}(C_m, \bar{n})$*, such that* $\{a_{n_i}^i, a_{n_i-1}^i, \ldots, a_{n_i - d_k^i + 1}^i\}$ *contains only* d_{k+1}^{i+1} *distinct numbers from* V_{i+1}*, where* $d_k^i = \delta_r^i + \delta_{r-1}^i + \cdots + \delta_k^i$*, for* $i = 1, 2, \ldots, m$ *and* $k = 0, 1, \ldots, r$.

Set $\bar{d}^i = (d_1^i, d_2^i, \ldots, d_r^i)$*. Then the total number* $CN(\bar{n}; \bar{d}^1, \bar{d}^2, \ldots, \bar{d}^m)$ *of spanning trees with the above property is:*

$$\begin{aligned} &CN(\bar{n}; \bar{d}^1, \bar{d}^2, \ldots, \bar{d}^m) \\ =&\prod_{i=1}^{m}\left(\prod_{j=1}^{d_1^i}(n_i - j + 1)\right) \cdot \prod_{i=1}^{m}\left(\sum_{(\lambda_0^i, \lambda_1^i, \cdots, \lambda_{r-1}^i)} \Pi(\lambda_0^i)\Pi(\lambda_1^i)\cdots\Pi(\lambda_{r-1}^i)\right) \\ =&\prod_{i=1}^{m}\binom{n_i}{\delta_0^i, \delta_1^i, \ldots, \delta_{r-1}^i,} \cdot \prod_{i=1}^{m}\prod_{k=0}^{r-1}\sigma(\delta_{k+1}^{i+1}, d_{k+2}^{i+1}; \delta_k^i), \end{aligned} \tag{4.11}$$

where $d_{r+1}^i = 0$*,* λ_k^i *ranges over all partitions with* $\delta_k^i - \delta_{k+1}^{i+1}$ *parts and largest part* $\leq d_{k+1}^{i+1}$ *and smallest part* $\geq d_{k+2}^{i+1}$*, for* $i = 1, 2, \ldots, m$ *and* $k = 0, 1, \ldots, r-1$.

4.3.2 Nilpotent Transformations

By Lemma 3.9, a nilpotent C_m-space linear transformation f is in bijection with $\bar{\beta} = (\beta_1, \beta_2, \ldots, \beta_m) \in \mathrm{GS}_{\mathrm{Nil}}(C_m, \bar{n})$, where $\beta_i = (b_1^i, b_2^i, \ldots, b_{n_i}^i)$, for $i = 1, 2, \ldots, m$. Since C_m has $2^m - 1$ spanning forests, we have the following result.

Proposition 4.36. *A nilpotent C_m-space linear transformation f is in bijection with $\bar{\beta} = (\beta_1, \beta_2, \ldots, \beta_m) \in \mathrm{GS}_{\mathrm{Nil}}(C_m, \bar{n})$, such that:*

1. *At least one of $\{b_{n_1}^1, b_{n_2}^2, \ldots, b_{n_m}^m\}$ is zero.*

2. *$b_l^i \in U_{i+1}$ for $i = 1, 2, \ldots, m$ and $l = 1, 2, \ldots, n_i$.*

Hence, the total number of nilpotent C_m-space linear transformations is:

$$\prod_{i=1}^{m} q^{n_{i+1}(n_i-1)} \cdot \left(\prod_{i=1}^{m} q^{n_i} - \prod_{i=1}^{m} (q^{n_i} - 1) \right).$$

This implies the result in [6].

Using the same technique as in the complete bipartite graph case in Section 4.2, we can get the following theorems.

Theorem 4.37. *Consider a nilpotent C_m-space linear transformation f that satisfies $\dim f(U_i) = d^{i+1}$ for $i = 1, 2, \ldots, m$. It is in bijection with $\bar{\beta} = (\beta_1, \beta_2, \ldots, \beta_m) \in \mathrm{GS}_{\mathrm{Nil}}(C_m, \bar{n})$, such that:*

1. *At least one of $\{b_{n_1}^1, b_{n_2}^2, \ldots, b_{n_m}^m\}$ is zero.*

2. *$\{b_1^i, b_2^i, \ldots, b_{n_i}^i\}$ spans a d^{i+1}-dimensional subspace of U_{i+1} for $i = 1, 2, \ldots, m$.*

Hence, the total number $CN_q(\bar{n}; d^1, d^2, \ldots, d^m)$ of nilpotent C_m-space linear transformations with the above property is:

$$
\begin{aligned}
&CN_q(\bar{n}; d^1, d^2, \ldots, d^m)\\
=&\prod_{i=1}^{m}\prod_{j=1}^{d^i}(q^{n_i}-q^{j-1})\\
&\cdot \sum_{\varnothing\neq I\subset[m]}\prod_{i\in I}\left(\sum_{\lambda^i\subset d^{i+1}\times(n_i-d^{i+1}-1)} q^{|\lambda^i|}\right)\cdot\prod_{i\notin I}\left(\sum_{\substack{(1^{n_i-d^{i+1}})\subset\lambda^i\\ \subset d^{i+1}\times(n_i-d^{i+1})}} q^{|\lambda^i|}\right) \qquad (4.12)\\
=&\prod_{i=1}^{m}\prod_{j=1}^{d^i}(q^{n_i}-q^{j-1})\cdot\sum_{\varnothing\neq I\subset[m]}\prod_{i\in I}\begin{bmatrix} n_i-1\\ d^{i+1}\end{bmatrix}_q\cdot\prod_{i\notin I}\left(q^{n_i-d^{i+1}}\begin{bmatrix} n_i-1\\ d^{i+1}-1\end{bmatrix}_q\right).
\end{aligned}
$$

Corollary 4.38. *The set of nilpotent C_m-space linear transformations f that satisfy $\dim f(U_i)=d^{i+1}$, for $i=1,2,\ldots,m$, is a q-analogue of the set of spanning trees of expanded digraph $(C_m)_{\bar{n}}$ with n_i-d^i leaves in V_i, for $i=1,2,\ldots,m$.*

Theorem 4.39. *Consider a nilpotent C_m-space linear transformation f that satisfies $\dim f^k(U)\cap U_i = d^i_k$, where $n_i=d^i_0\geq d^i_1\geq\cdots\geq d^i_r=0$, and $\sum_{i=1}^m d^i_{r-1}>0$, for $i=1,2,\ldots,m$. It is in bijection with $\bar{\beta}=(\beta_1,\beta_2,\ldots,\beta_m)\in \mathrm{GS}_{\mathrm{Nil}}(C_m,\bar{n})$, such that $\{b^i_{n_i}, b^i_{n_i-1},\ldots,b^i_{n_i-d^i_k+1}\}$ spans a d^{i+1}_{k+1}-dimensional subspace of U_{i+1}, for $i=1,2,\ldots,m, k=0,1,\ldots,r-1$.*

Set $\bar{d}^i=(d^i_1,d^i_2,\ldots,d^i_r)$. Then the total number $CN_q(\bar{n};\bar{d}^1,\bar{d}^2,\ldots,\bar{d}^m)$ of nilpotent C_m-space linear transformations with above property is:

$$
\begin{aligned}
&CN_q(\bar{n};\bar{d}^1,\bar{d}^2,\ldots,\bar{d}^m)\\
=&\prod_{i=1}^{m}\left(\prod_{j=1}^{d^i_1}(q^{n_i}-q^{j-1})\right)\cdot\prod_{i=1}^{m}\left(\sum_{(\lambda^i_0,\lambda^i_1,\cdots,\lambda^i_{r-1})} q^{|\lambda^i_0|}q^{|\lambda^i_1|}\cdots q^{|\lambda^i_{r-1}|}\right) \qquad (4.13)\\
=&\prod_{i=1}^{m}\left(\prod_{j=1}^{d^i_1}(q^{n_i}-q^{j-1})\right)\cdot\prod_{i=1}^{m}\prod_{k=0}^{r-1}\left(q^{(\delta^i_k-\delta^{i+1}_{k+1})d^{i+1}_{k+2}}\cdot\begin{bmatrix}\delta^i_k\\ \delta^{i+1}_{k+1}\end{bmatrix}_q\right),
\end{aligned}
$$

where $d^i_{r+1}=0$, $\delta^i_k=d^i_k-d^i_{k+1}$, λ^i_k ranges over all partitions with $\delta^i_k-\delta^i_{k+1}$ parts and largest part $\leq d^{i+1}_{k+1}$ and smallest part $\geq d^{i+1}_{k+2}$, for $i=1,2,\cdots,m$ and $k=0,1,\ldots,r$.

Corollary 4.40. *The set of nilpotent C_m-space linear transformations f that satisfy $\dim f^k(U) \cap U_i = d^i_k$, for $i = 1, 2, \ldots, m$ and $k = 0, 1, \ldots, r-1$, is a q-analogue of the set of spanning trees of the expanded digraph $(C_m)_{\bar{n}}$ with δ^i_k vertices from V_i in level k, for $i = 1, 2, \ldots, m$ and $k = 0, 1, \ldots, r-1$.*

Bibliography

[1] M.C. Crabb. Counting nilpotent endomorphisms. *Finite Fields and Their Applications*, 12 (2006), 151-154.

[2] N.J. Fine, I.N. Herstein. The probability that a matrix be nilpotent. *Illinois Journal of Mathematics*, 2 (1958), 499-504.

[3] M. Gerstenhaber. On the number of nilpotent matrices with coefficients in a finite field. *Illinois Journal of Mathematics*, 5 (1961), 330-333.

[4] P. Griffiths, J. Harris. *Principles of Algebraic Geometry*, Wiley, New York, 1978.

[5] A. Kovacs. On the probability that the product of k $n \times n$ matrices over a finite field will be zero. *Journal of Combinatorial Theory, Series A*, 45 (1987), No. 2, 290-299.

[6] A. Kovacs. Some enumeration problems for matrices over a finite field. *Linear Algebra Applications*, 94 (1987), 223–236.

[7] R. Stanley. *Enumerative Combinatorics*, vol. 1. Wadsworth and Brooks/Cole, Pacific Grove, CA, 1986; second printing, Cambridge University Press, Cambridge, 1996.

[8] R. Stanley. *Enumerative Combinatorics*, vol. 2. Cambridge University Press, Cambridge, 1999.