Appendix A

The Outer Integral

Throughout this appendix, $(X, \mathscr{B}, p)$ is a probability space. Unless otherwise specified, f, g, and h are functions from X to $[-\infty, \infty]$.

Definition A.1 If $f \geq 0$, the *outer integral* of f with respect to p is defined by

$$\int^* f\, dp = \inf\left\{\int g\, dp \middle| f \leq g,\ g \text{ is } \mathscr{B}\text{-measurable}\right\}. \tag{1}$$

If f is arbitrary, define

$$\int^* f\, dp = \int^* f^+\, dp - \int^* f^-\, dp, \tag{2}$$

where

$$f^+(x) = \max\{0, f(x)\}, \qquad f^-(x) = \max\{0, -f(x)\},$$

and we set $\infty - \infty = \infty$.

Lemma A.1 If $f \geq 0$, then there exists a $\mathscr{B}$-measurable g with $g \geq f$, such that

$$\int^* f\, dp = \int g\, dp. \tag{3}$$

Proof Choose $g_n \geq f$, g_n $\mathscr{B}$-measurable, so that

$$\int g_n \, dp \downarrow \int^* f \, dp.$$

We assume without loss of generality that $g_1 \geq g_2 \geq \cdots$. Let $g = \lim_{n\to\infty} g_n$. Then $g \geq f$, g is $\mathscr{B}$-measurable, and (3) holds. Q.E.D.

Lemma A.2 If $f \geq 0$ and $h \geq 0$, then

$$\int^*(f + h)\, dp \leq \int^* f \, dp + \int^* h \, dp. \tag{4}$$

If either f or h is $\mathscr{B}$-measurable, then equality holds in (4).

Proof Suppose $g_1 \geq f$, $g_2 \geq f$, g_1 and g_2 are $\mathscr{B}$-measurable, and $\int^* f \, dp = \int g_1 \, dp$, $\int^* h \, dp = \int g_2 \, dp$. Then $g_1 + g_2 \geq f + h$ and (4) follows from (1).

Suppose h is $\mathscr{B}$-measurable and $\int h \, dp < \infty$. [If $\int h \, dp = \infty$, equality is easily seen to hold in (4).] Suppose $f + h \leq g$, where g is $\mathscr{B}$-measurable and

$$\int^*(f + h)\, dp = \int g \, dp.$$

Then $f \leq g - h$ and $g - h$ is $\mathscr{B}$-measurable, so

$$\int^* f \, dp \leq \int g \, dp - \int h \, dp,$$

which implies

$$\int^* f \, dp + \int h \, dp \leq \int^*(f + h)\, dp.$$

Therefore equality holds in (4). Q.E.D.

We provide an example to show that strict inequality can occur in (4), even if $f + h$ is $\mathscr{B}$-measurable. For this and subsequent examples we will need the following observation: For any $E \subset X$,

$$\int^* \chi_E \, dp = p^*(E), \tag{5}$$

where $p^*(E)$ is p-outer measure defined by

$$p^*(E) = \inf\{p(B) \mid E \subset B, B \in \mathscr{B}\}$$

and χ_E is the indicator function of E defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

To verify (5), note that if $\chi_E \leq g$ and g is $\mathscr{B}$-measurable, then $\{x|g(x) \geq 1\}$ is a $\mathscr{B}$-measurable set containing E and consequently

$$\int g\,dp \geq p^*(E).$$

Definition A.1 implies

$$\int^* \chi_E\,dp \geq p^*(E). \tag{6}$$

On the other hand, if $\{B_n\}$ is a sequence of $\mathscr{B}$-measurable sets with $E \subset B_n$ and $p(B_n) \downarrow p^*(E)$, then $p(\bigcap_{n=1}^\infty B_n) = p^*(E)$. By construction, $\chi_{\bigcap_{n=1}^\infty B_n} \geq \chi_E$. But $\chi_{\bigcap_{n=1}^\infty B_n}$ is $\mathscr{B}$-measurable, and

$$\int \chi_{\bigcap_{n=1}^\infty B_n}\,dp = p^*(E).$$

The reverse of inequality (6) follows. Note that the preceding argument shows that for any set E, there exists a set $B \in \mathscr{B}$ such that $E \subset B$ and $p(B) = p^*(E)$.

EXAMPLE 1 Let $X = [0, 1]$, let $\mathscr{B}$ be the Borel σ-algebra, and let p be Lebesgue measure restricted to $\mathscr{B}$. Let $E \subset X$ be a set for which $p^*(E) = p^*(X - E) = 1$ (see [H1, Section 16, Theorem E]). Then

$$\int (\chi_E + \chi_{X-E})\,dp = \int 1\,dp = 1,$$

$$\int^* \chi_E\,dp + \int^* \chi_{X-E}\,dp = 2,$$

and strict inequality holds in (4).

Lemma A.2 cannot be extended to (possibly negative) bounded functions, even if h is $\mathscr{B}$-measurable, as the following example demonstrates.

EXAMPLE 2 Let $(X, \mathscr{B}, p)$ and E be as before. Let $f = \chi_E - \chi_{X-E}$, $h = 1$. Then

$$\int^* (f + h)\,dp = \int^* 2\chi_E\,dp = 2,$$

$$\int^* f\,dp + \int h\,dp = \int^* \chi_E\,dp - \int^* \chi_{X-E}\,dp + 1 = 1.$$

Lemma A.3

(a) If $f \leq g$, then $\int^* f\,dp \leq \int^* g\,dp$.

(b) If $\varepsilon > 0$ and $f \leq g \leq f + \varepsilon$, then

$$\int^* f\,dp \leq \int^* g\,dp \leq \int^* f\,dp + 2\varepsilon. \tag{7}$$

(c) If $\int^* f^+ \, dp < \infty$ or $\int^* f^- \, dp < \infty$, then

$$\int^* (-f) \, dp = -\int^* f \, dp. \tag{8}$$

(d) If $A, B \in \mathscr{B}$ are disjoint, then for any f

$$\int^* \chi_{A \cup B} f \, dp = \int^* \chi_A f \, dp + \int^* \chi_B f \, dp. \tag{9}$$

(e) If $E \subset X$ satisfies $p^*(E) = 0$, then for any f

$$\int^* f \, dp = \int^* \chi_{X-E} f \, dp.$$

(f) If $p^*(\{x | f(x) = \infty\}) > 0$, then for every g, $\int^*(g + f) \, dp = \infty$.

(g) If $p^*(\{x | f(x) = -\infty\}) > 0$, then for every g either $\int^*(g + f) \, dp = \infty$ or $\int^*(g + f) \, dp = -\infty$.

Proof (a) If $f \leq g$, then $f^+ \leq g^+$ and $f^- \geq g^-$. By (1),

$$\int^* f^+ \, dp \leq \int^* g^+ \, dp, \qquad \int^* f^- \, dp \geq \int^* g^- \, dp.$$

The result follows from (2).

(b) In light of (a), it remains only to show that

$$\int^* (f + \varepsilon) \, dp \leq \int^* f \, dp + 2\varepsilon. \tag{10}$$

For $g_1 \geq f^+$, g_1 $\mathscr{B}$-measurable, and

$$\int^* f^+ \, dp = \int g_1 \, dp,$$

we have

$$(f + \varepsilon)^+ \leq g_1 + \varepsilon$$

so

$$\int^* (f + \varepsilon)^+ \, dp \leq \int g_1 \, dp + \varepsilon = \int^* f^+ \, dp + \varepsilon. \tag{11}$$

For $g_2 \geq (f + \varepsilon)^-$, g_2 $\mathscr{B}$-measurable, and

$$\int^* (f + \varepsilon)^- \, dp = \int g_2 \, dp,$$

we have

$$g_2 + \varepsilon \geq (f + \varepsilon)^- + \varepsilon = \max\{f^- - \varepsilon, 0\} + \varepsilon \geq f^-,$$

so

$$\varepsilon + \int^* (f + \varepsilon)^- \, dp = \varepsilon + \int g_2 \, dp = \int (g_2 + \varepsilon) \, dp \geq \int^* f^- \, dp. \tag{12}$$

Combine (11) and (12) to conclude (10).

(c) We have

$$\begin{aligned}\int^*(-f)\,dp &= \int^*(-f)^+\,dp - \int^*(-f)^-\,dp\\ &= \int^* f^-\,dp - \int^* f^+\,dp = -\left[\int^* f^+\,dp - \int^* f^-\,dp\right]\\ &= -\int^* f\,dp,\end{aligned}$$

where the assumption that $\int^* f^+\,dp < \infty$ or $\int^* f^-\,dp < \infty$ is necessary for the next to last equality.

(d) Suppose $f \geq 0$. Let g be a $\mathscr{B}$-measurable function with $g \geq \chi_{A\cup B} f$ and

$$\int^* \chi_{A\cup B} f\,dp = \int g\,dp.$$

Then $\chi_A g \geq \chi_A f$, $\chi_B g \geq \chi_B f$, so

$$\begin{aligned}\int^* \chi_{A\cup B} f\,dp &= \int \chi_A g\,dp + \int \chi_B g\,dp\\ &\geq \int^* \chi_A f\,dp + \int^* \chi_B f\,dp.\end{aligned} \tag{13}$$

Now suppose $g_1 \geq \chi_A f$, $g_2 \geq \chi_B f$ are $\mathscr{B}$-measurable and

$$\int g_1\,dp = \int^* \chi_A f\,dp, \qquad \int g_2\,dp = \int^* \chi_B f\,dp.$$

Then $g_1 + g_2 \geq \chi_{A\cup B} f$, so

$$\begin{aligned}\int^* \chi_A f\,dp + \int^* \chi_B f\,dp &= \int (g_1 + g_2)\,dp\\ &\geq \int^* \chi_{A\cup B} f\,dp.\end{aligned} \tag{14}$$

Combine (13) and (14) to conclude (9) for $f \geq 0$. The extension to arbitrary f is straightforward.

(e) Suppose $f \geq 0$. Choose $B \in \mathscr{B}$ with $p(B) = p^*(E) = 0$, $B \supset E$. By (d),

$$\int^* f\,dp = \int^* \chi_{X-B} f\,dp \leq \int^* \chi_{X-E} f\,dp \leq \int^* f\,dp.$$

Hence $\int^* f\,dp = \int^* \chi_{X-E} f\,dp$. The extension to arbitrary f is straightforward.

(f) We have $(g + f)^+(x) = \infty$ if $f(x) = \infty$, so that

$$p^*(\{x \mid (g + f)^+(x) = \infty\}) > 0.$$

Hence $\int^*(g + f)^+\,dp = \infty$, and it follows that $\int^*(g + f)\,dp = \infty$.

(g) Consider the sets $E = \{x \mid f(x) = -\infty\}$ and $E_g = \{x \mid f(x) = -\infty,$ $g(x) < \infty\}$. If $p^*(E_g) = 0$, then

$$p^*(E - E_g) = p^*(E - E_g) + p^*(E_g) \geq p^*(E) > 0.$$

Since we have $f(x) + g(x) = \infty$ for $x \in E - E_g$, it follows from (f) that $\int^*(g + f)\,dp = \infty$. If $p^*(E_g) > 0$, then $p^*(\{x \mid (g + f)^-(x) = \infty\}) \geq p^*(E_g) > 0$ and hence, by (f), $\int^*(g + f)^-\,dp = \infty$. Hence, if $\int^*(g + f)^+\,dp = \infty$, then $\int^*(g + f)\,dp = \infty$, while if $\int^*(g + f)^+\,dp < \infty$, then $\int^*(g + f)\,dp = -\infty$. Q.E.D.

The bound given in (7) is the sharpest possible. To see this, let f be as defined in Example 2, $g = f + 1$, and $\varepsilon = 1$. Despite these pathologies of outer integration, there is a monotone convergence theorem, which we now prove.

Proposition A.1 If $\{f_n\}$ is a sequence of nonnegative functions and $f_n \uparrow f$, then

$$\int^* f_n\,dp \uparrow \int^* f\,dp. \tag{15}$$

If $\{f_n\}$ is a sequence of nonpositive functions and $f_n \downarrow f$, then

$$\int^* f_n\,dp \downarrow \int^* f\,dp.$$

Proof We prove the first statement of the theorem. The second follows from the first and Lemma A.3(c). Assume $f_n \geq 0$ and $f_n \uparrow f$. Let $\{g_n\}$ be a sequence of $\mathscr{B}$-measurable functions such that $g_n \geq f_n$ and

$$\int^* f_n\,dp = \int g_n\,dp. \tag{16}$$

If, for some n, $\int g_n\,dp = \int^* f_n\,dp = \infty$, then (15) is assured. If not, then for every n,

$$\int g_n\,dp < \infty. \tag{17}$$

Suppose (17) holds for every n and for some n,

$$p(\{x \mid g_n(x) > g_{n+1}(x)\}) > 0.$$

Then since $g_{n+1} \geq f_{n+1} \geq f_n$, we have that $\bar{g}$ defined by

$$\bar{g}(x) = \begin{cases} g_n(x) & \text{if } \ g_n(x) \leq g_{n+1}(x), \\ g_{n+1}(x) & \text{if } \ g_n(x) > g_{n+1}(x), \end{cases}$$

satisfies $g_n \geq \bar{g} \geq f_n$ everywhere and $\bar{g} < g_n$ on a set of positive measure. This contradicts (16). We may therefore assume without loss of generality that (17) holds and $g_1 \leq g_2 \cdots$. Let $g = \lim_{n\to\infty} g_n$. Then $g \geq f$ and

$$\lim_{n\to\infty} \int^* f_n\,dp = \lim_{n\to\infty} \int g_n\,dp = \int g\,dp \geq \int^* f\,dp.$$

But $f_n \leq f$ for every n, so the reverse inequality holds as well. Q.E.D.

One might hope that if $\{f_n\}$ is a sequence of functions which are bounded below and $f_n \uparrow f$, then (15) remains valid. This is not the case, as the following example shows.

EXAMPLE 3 Let $X = [0, 1)$, $\mathscr{B}$ be the Borel σ-algebra, and p be Lebesgue measure restricted to $\mathscr{B}$. Define an equivalence relation $\sim$ on X by

$$x \sim y \Leftrightarrow x - y \text{ is rational.}$$

Let F_0 be constructed by choosing one representative from each equivalence class. Let $Q = \{q_0, q_1, \ldots\}$ be an enumeration of the rationals in $[0, 1)$ with $q_0 = 0$ and define

$$F_k = \{x + q_k [\text{mod } 1] \mid x \in F_0\} = F_0 + q_k [\text{mod } 1] \qquad k = 0,1, \ldots .$$

Then $F_0, F_1, \ldots$ is a sequence of disjoint sets with

$$\bigcup_{k=0}^{\infty} F_k = [0, 1). \tag{18}$$

If for some $n < \infty$, we have $p^*(\bigcup_{k=n}^{\infty} F_k) < 1$, then $E = \bigcup_{k=0}^{n-1} F_k$ contains a $\mathscr{B}$-measurable set with measure $\delta > 0$. For $k = 1, \ldots, n-1$, let $q_k = r_k/s_k$, where r_k and s_k are integers and r_k/s_k is reduced to lowest terms. Let $\{p_1, p_2, \ldots\}$ be a sequence of prime numbers such that

$$\max_{1 \le k \le n-1} s_k < p_1 < p_2 < \cdots$$

Then the sets E, $E + p_1^{-1} [\text{mod } 1]$, $E + p_2^{-1} [\text{mod } 1], \ldots$ are disjoint, and by the translation invariance of p, each contains a $\mathscr{B}$-measurable set with measure $\delta > 0$. It follows that $[0, 1)$ must contain a $\mathscr{B}$-measurable set of infinite measure. This contradiction implies

$$p^*\left(\bigcup_{k=n}^{\infty} F_k\right) = 1 \tag{19}$$

for every n. Define

$$f_n = -\chi_{\bigcup_{k=n}^{\infty} F_k}, \qquad n = 0,1, \ldots .$$

Then $f_n \uparrow 0$, but (5) and (19) imply that for every n

$$\int^* f_n \, dp = -1.$$

By a change of sign in Example 3, we see that the second part of Theorem A.1 cannot be extended to functions which are bounded above unless additional conditions are imposed. We impose such conditions in order to prove a corollary.

Corollary A.1.1 Let $\{\varepsilon_n\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Let $\{f_n\}$ be a sequence with

$$\lim_{n\to\infty} f_n = f, \tag{20}$$

$$f \le f_n, \qquad n = 1, 2, \ldots, \tag{21}$$

$$f_n(x) \le f(x) + \varepsilon_n \qquad \text{if} \quad f(x) > -\infty, \tag{22}$$

$$f_n(x) \le f_{n-1}(x) + \varepsilon_n \qquad \text{if} \quad f(x) = -\infty, \quad n = 2, 3, \ldots, \tag{23}$$

$$\int^* f_1 \, dp < \infty. \tag{24}$$

Then

$$\lim_{n\to\infty} \int^* f_n \, dp = \int^* f \, dp. \tag{25}$$

Proof From (20) we have $\lim_{n\to\infty} f_n^+ = f^+$ and $\lim_{n\to\infty} f_n^- = f^-$. Now $\inf_{k\ge n} f_k^- \le f_n^- \le f^-$ and $\inf_{k\ge n} f_k^- \uparrow f^-$ as $n \to \infty$. By Proposition A.1,

$$\int^* f^- \, dp = \lim_{n\to\infty} \int^* \inf_{k\ge n} f_k^- \, dp \le \lim_{n\to\infty} \int^* f_n^- \, dp \le \int^* f^- \, dp,$$

so

$$\lim_{n\to\infty} \int^* f_n^- \, dp = \int^* f^- \, dp. \tag{26}$$

Let $A = \{x | f(x) = -\infty\}$. If $p^*(A) = 0$, then (21), (22), (24), and Lemmas A.3(b) and (e) imply

$$\int^* f^+ \, dp \le \int^* f_n^+ \, dp \le 2\varepsilon_n + \int^* f^+ \, dp < \infty,$$

so

$$\lim_{n\to\infty} \int^* f_n^+ \, dp = \int^* f^+ \, dp < \infty. \tag{27}$$

Combine (26) and (27) to conclude (25). If $p^*(A) > 0$, then $\int^* f^- \, dp = -\infty$ and (26) will imply (25) provided that

$$\int^* f^+ \, dp < \infty \tag{28}$$

and

$$\limsup_{n\to\infty} \int^* f_n^+ \, dp < \infty. \tag{29}$$

Conditions (21) and (24) imply (28). Conditions (21)–(23) imply for every $x \in X$

$$f_n(x) \le f_{n-1}(x) + \varepsilon_n, \qquad n = 2, 3, \ldots,$$

so

$$\int^* f_n^+ \, dp \le 2\varepsilon_n + \int^* f_{n-1}^+ \, dp$$

and

$$\int^* f_n^+ \, dp \le 2 \sum_{k=2}^{n} \varepsilon_k + \int^* f_1^+ \, dp.$$

The finiteness of $\sum_{k=2}^{\infty} \varepsilon_k$ and (24) imply (29). Q.E.D.

Appendix B

Additional Measurability Properties of Borel Spaces

This appendix supplements Section 7.6. The notation and terminology used here is the same as in that section and, in most cases, is defined in Section 7.1.

B.1 Proof of Proposition 7.35(e)

Our first task is to give a proof of Proposition 7.35(e). To do this, we introduce the space $N^* = \{1, 2, \ldots\} \cup \{\infty\}$ with the topology induced by the metric

$$d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|,$$

where we define $1/\infty = 0$. Let $\mathcal{N}^* = N^*N^* \cdots$ with the product topology. The space $\mathcal{N}$ of sequences of positive integers is a topological subspace of $\mathcal{N}^*$. The space $\mathcal{N}^*$ is compact by Tychonoff's theorem, while $\mathcal{N}$ is not. If $(X, \mathscr{P})$ and $(Y, \mathscr{Q})$ are paved spaces, we denote by $\mathscr{P}\mathscr{Q}$ the paving of XY:

$$\mathscr{P}\mathscr{Q} = \{PQ \mid P \in \mathscr{P}, Q \in \mathscr{Q}\}. \tag{1}$$

Proposition B.1 Let $(X, \mathscr{P})$ be a paved space and $\mathscr{K}$ the collection of compact subsets of $\mathcal{N}^*$. Then the projection on X of a set in $\mathscr{S}(\mathscr{P}\mathscr{K})$ is in $\mathscr{S}(\mathscr{P})$. Conversely, every set in $\mathscr{S}(\mathscr{P})$ is the projection on X of some set in $[(\mathscr{P}\mathscr{K})_\sigma]_\delta$.

Proof Let S be a Suslin scheme for $\mathscr{P}\mathscr{K}$. Then for every $s \in \Sigma$, $S(s)$ has the form $S(s) = S_1(s)S_2(s)$, where $S_1(s) \in \mathscr{P}$ and $S_2(s) \in \mathscr{K}$. Now

$$\begin{aligned} N(S) &= \bigcup_{z \in \mathscr{N}} \bigcap_{s<z} S(s) \\ &= \bigcup_{z \in \mathscr{N}} \bigcap_{s<z} [S_1(s)S_2(s)] \\ &= \bigcup_{z \in \mathscr{N}} \left\{ \left[\bigcap_{s<z} S_1(s) \right] \left[\bigcap_{s<z} S_2(s) \right] \right\}, \end{aligned}$$

so

$$\text{proj}_X[N(S)] = \bigcup_{z \in A} \bigcap_{s<z} S_1(s),$$

where

$$A = \left\{ z \in \mathscr{N} \,\middle|\, \bigcap_{s<z} S_2(s) \neq \varnothing \right\}.$$

Since each $S_2(s)$ is compact, we have

$$A = \left\{ (\zeta_1, \zeta_2, \ldots) \in \mathscr{N} \,\middle|\, \bigcap_{k=1}^{n} S_2(\zeta_1, \zeta_2, \ldots, \zeta_k) \neq \varnothing \;\forall n \right\}.$$

Define a Suslin scheme R for $\mathscr{P}$ by

$$R(\zeta_1, \ldots, \zeta_n) = \begin{cases} S_1(\zeta_1, \ldots, \zeta_n) & \text{if } \bigcap_{k=1}^{n} S_2(\zeta_1, \ldots, \zeta_k) \neq \varnothing, \\ \varnothing & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \text{proj}_X[N(S)] &= \bigcup_{z \in A} \bigcap_{s<z} S_1(s) \\ &= \bigcup_{z \in \mathscr{N}} \bigcap_{s<z} R(s) = N(R), \end{aligned}$$

so $\text{proj}_X[N(S)] \in \mathscr{S}(\mathscr{P})$.

For the second part of the proposition, suppose S is a Suslin scheme for $\mathscr{P}$. Define a Suslin scheme R for $\mathscr{K}$ by

$$R(\sigma_1, \ldots, \sigma_n) = \{(\zeta_1, \zeta_2, \ldots) \in \mathscr{N}^* | \zeta_1 = \sigma_1, \ldots, \zeta_n = \sigma_n\}.$$

For fixed $z_0 \in \mathscr{N}$, we have $\bigcap_{s<z_0} R(s) = \{z_0\}$, so

$$\begin{aligned} \bigcap_{s<z_0} [S(s)R(s)] &= \left[\bigcap_{s<z_0} S(s) \right] \left[\bigcap_{s<z_0} R(s) \right] \\ &= \left\{ (x, z_0) \,\middle|\, x \in \bigcap_{s<z_0} S(s) \right\}. \end{aligned} \tag{2}$$

Therefore,

$$\begin{aligned} N(S) &= \bigcup_{z\in\mathcal{N}} \bigcap_{s<z} S(s) \\ &= \bigcup_{z\in\mathcal{N}} \operatorname{proj}_X \left\{ \bigcap_{s<z} [S(s)R(s)] \right\} \\ &= \operatorname{proj}_X \left\{ \bigcup_{z\in\mathcal{N}} \bigcap_{s<z} [S(s)R(s)] \right\}, \end{aligned}$$

and it remains only to show that

$$\bigcup_{z\in\mathcal{N}} \bigcap_{s<z} [S(s)R(s)] \in [(\mathcal{P}\mathcal{K})_\sigma]_\delta. \tag{3}$$

If we can show that

$$\bigcup_{z\in\mathcal{N}} \bigcap_{s<z} [S(s)R(s)] = \bigcap_{k=1}^{\infty} \bigcup_{s\in\Sigma_k} [S(s)R(s)], \tag{4}$$

where Σ_k is the set of elements in Σ having k components, then (3) will follow. Let $x\in X$ and $z_0 = (\zeta_1^0, \zeta_2^0, \ldots)\in\mathcal{N}^*$ be given. Suppose

$$(x, z_0) \in \bigcap_{z\in\mathcal{N}} \bigcap_{s<z} [S(s)R(s)].$$

We see from (2) that $z_0\in\mathcal{N}$ and $(x, z_0)\in\bigcap_{s<z_0}[S(s)R(s)]$, so for every $k \geq 1$, $(x, z_0)\in S(\zeta_1^0, \ldots, \zeta_k^0)R(\zeta_1^0, \ldots, \zeta_k^0)$. This implies $(x, z_0)\in\bigcap_{k=1}^{\infty}\bigcup_{s\in\Sigma_k}[S(s)R(s)]$, and

$$\bigcup_{z\in\mathcal{N}} \bigcap_{s<z} [S(s)R(s)] \subset \bigcap_{k=1}^{\infty} \bigcup_{s\in\Sigma_k} [S(s)R(s)]. \tag{5}$$

On the other hand, if $(x, z_0)\in\bigcap_{k=1}^{\infty}\bigcup_{s\in\Sigma_k}[S(s)R(s)]$, then for each $k \geq 1$, $(x, z_0)\in\bigcup_{s\in\Sigma_k}[S(s)R(s)]$. This can happen only if $z_0\in\mathcal{N}$ and $(x, z_0)\in S(\zeta_1^0, \ldots, \zeta_k^0)R(\zeta_1^0, \ldots, \zeta_k^0)$. Therefore,

$$\begin{aligned} (x, z_0) &\in \bigcap_{k=1}^{\infty} [S(\zeta_1^0, \ldots, \zeta_k^0)R(\zeta_1^0, \ldots, \zeta_k^0)] \\ &= \bigcap_{s<z_0} [S(s)R(s)] \\ &\subset \bigcup_{z\in\mathcal{N}} \bigcap_{s<z} [S(s)R(s)], \end{aligned}$$

which proves the reverse of set containment (5). Equality (4) follows.

Q.E.D.

If $(X,\mathscr{P})$ is a paved space, Y is another space, and $Q \subset Y$, we define a paving of XY by

$$\mathscr{P}Q = \{PQ | P \in \mathscr{P}\}.$$

Lemma B.1 Let $(X,\mathscr{P})$ and $(Y,\mathscr{Q})$ be paved spaces. Then:

(a) $\mathscr{S}(\mathscr{P})Q = \mathscr{S}(\mathscr{P}Q)$ for every $Q \subset Y$;
(b) $\mathscr{S}(\mathscr{P})\mathscr{Q} \subset \mathscr{S}(\mathscr{P}\mathscr{Q})$.

Proof Part (a) is trivial and part (b) follows from (a). Q.E.D.

We are now in a position to prove part (e) of Proposition 7.35.

Proposition B.2 Let $(X,\mathscr{P})$ be a paved space. Then $\mathscr{S}(\mathscr{P}) = \mathscr{S}[\mathscr{S}(\mathscr{P})]$.

Proof In light of Proposition 7.35(d), we need only prove

$$\mathscr{S}(\mathscr{P}) \supset \mathscr{S}[\mathscr{S}(\mathscr{P})]. \tag{6}$$

Let $\mathscr{N}^*$ and $\mathscr{K}$ be as in Proposition B.1. If $A \in \mathscr{S}[\mathscr{S}(\mathscr{P})]$, then by the second part of Proposition B.1, $A = \mathrm{proj}_X(B)$ for some set $B \in ([\mathscr{S}(\mathscr{P})\mathscr{K}]_\sigma)_\delta$. By Lemma B.1(b) and Proposition 7.35(b) and (c), we have

$$B \in ([\mathscr{S}(\mathscr{P})\mathscr{K}]_\sigma)_\delta \subset ([\mathscr{S}(\mathscr{P}\mathscr{K})]_\sigma)_\delta = \mathscr{S}(\mathscr{P}\mathscr{K}).$$

The first part of Proposition B.1 implies that $A = \mathrm{proj}_X(B) \in \mathscr{S}(\mathscr{P})$ and (6) follows. Q.E.D.

B.2 Proof of Proposition 7.16

In Proposition 7.16 we stated that Borel spaces X and Y are Borel-isomorphic if and only if they have the same cardinality. A related result is that every uncountable Borel space is Borel-isomorphic to every other uncountable Borel space. We used the latter fact in Proposition 7.27 to assume without loss of generality that the Borel spaces under consideration were actually copies of $(0,1]$, we used it in Proposition 7.39 to transfer a statement about $\mathscr{N}$ to a statement about any uncountable Borel space, and we will use it again in Proposition B.7 to allow our treatment of the limit σ-algebra to center on the space $\mathscr{N}$. The proofs of Proposition 7.16 and Corollary 7.16.1 depend on the following lemma, which is an immediate consequence of Propositions 7.36 and 7.37. The reader may wish to verify that these propositions depend only on Propositions 7.35, B.1, and B.2, so no circularity is present in the arguments.

Lemma B.2 Let X be a nonempty Borel space. There is a continuous function f from $\mathscr{N}$ *onto* X.

Define $\mathscr{M}$ to be the set of infinite sequences of zeroes and ones. We can regard $\mathscr{M}$ as the countable product of copies of $\{0,1\}$ and endow it with the product topology, where $\{0,1\}$ has the discrete topology. By Tychonoff's theorem, $\mathscr{M}$ is compact with this topology. It is also metrizable as a complete separable space.

Our proof of Proposition 7.16 consists of three parts. We show first that every uncountable Borel space contains a Borel subset homeomorphic to $\mathscr{M}$, we show second that every uncountable Borel space is isomorphic to a Borel subset of $\mathscr{M}$, and we show finally that these first two facts imply that every uncountable Borel space is isomorphic to $\mathscr{M}$.

Lemma B.3 Let X be an uncountable Borel space. There exists a compact set $K \subset X$ such that $\mathscr{M}$ and K are homeomorphic.

Proof Let $f: \mathscr{N} \to X$ be the continuous, onto function of Lemma B.2. For each $x \in X$, choose an element $z_x \in \mathscr{N}$ such that $x = f(z_x)$. Let $S = \{z_x | x \in X\}$, so that f is a one-to-one function from S onto X. For $z \in S$, if possible choose an open neighborhood $T(z)$ of z such that $S \cap T(z)$ is countable. Let R be the set of all $z \in S$ for which such a $T(z)$ can be found. Since separable metrizable spaces have the Lindelöf property, there exists a countable subset R' of R such that $\bigcup_{z \in R} T(z) = \bigcup_{z \in R'} T(z)$, so

$$R \subset S \cap \left[\bigcup_{z \in R} T(z) \right] = \bigcup_{z \in R'} [S \cap T(z)],$$

and R is countable. Since S is uncountable, $S - R$ must be infinite. Furthermore, if $z \in S - R$, then every open neighborhood of z contains infinitely many points of $S - R$.

Let d be a metric on $\mathscr{N}$ consistent with its topology for which $(\mathscr{N}, d)$ is complete. For $\bar{z} \in \mathscr{N}$, the closed sphere of radius r centered at $\bar{z}$ is the set $\{z \in \mathscr{N} | d(z, \bar{z}) \le r\}$. The interior of this sphere, denoted $\operatorname{Int}\{z \in \mathscr{N} | d(z, \bar{z}) \le r\}$, is the set $\{z \in \mathscr{N} | d(z, \bar{z}) < r\}$. Let $z(0)$ and $z(1)$ be distinct points in $S - R$. Then $f[z(0)] \ne f[z(1)]$, so there exist disjoint open neighborhoods U and V of $f[z(0)]$ and $f[z(1)]$ respectively. Let $S(0)$ and $S(1)$ be disjoint closed spheres of radius no greater than one centered at $z(0)$ and $z(1)$ and contained in $f^{-1}(U)$ and $f^{-1}(V)$ respectively. We have that $f[S(0)]$ and $f[S(1)]$ are disjoint. Note also that for every $z \in (S - R) \cap \operatorname{Int} S(0)$, every open neighborhood of z contains infinitely many points of $(S - R) \cap \operatorname{Int} S(0)$, and the same is true of $S(1)$. By the same procedure we can choose distinct points $z(0,0)$ and $z(0,1)$ in $(S - R) \cap \operatorname{Int} S(0)$ and distinct points $z(1,0)$ and $z(1,1)$ in $(S - R) \cap \operatorname{Int} S(1)$, and we can also choose disjoint closed spheres $S(0,0)$, $S(0,1)$, $S(1,0)$ and $S(1,1)$ of radius no greater than $\frac{1}{2}$ centered at $z(0,0)$, $z(0,1)$, $z(1,0)$ and $z(1,1)$, respectively, so that $f[S(0,0)]$, $f[S(0,1)]$, $f[S(1,0)]$ and $f[S(1,1)]$ are all disjoint. We can choose these spheres so that $S(0,0)$ and

$S(0,1)$ are contained in $S(0)$, while $S(1,0)$ and $S(1,1)$ are contained in $S(1)$. At the kth step of this process, we choose a collection of disjoint closed spheres $S(\mu_1, \ldots, \mu_k)$ of radius no greater than $1/k$ centered at distinct points $z(\mu_1, \ldots, \mu_k)$ in $S - R$, where each μ_j is either zero or one. Furthermore, we can choose the spheres so that for each $(\mu_1, \ldots, \mu_{k-1})$

(i) $f[S(\mu_1, \ldots, \mu_{k-1}, 0)] \cap f[S(\mu_1, \ldots, \mu_{k-1}, 1)] = \varnothing$,

(ii) $S(\mu_1, \ldots, \mu_{k-1}, \mu_k) \subset S(\mu_1, \ldots, \mu_{k-1}), \qquad \mu_k = 0, 1.$

For fixed $m = (\mu_1, \mu_2, \ldots) \in \mathscr{M}$, the sets $\{S(\mu_1, \ldots, \mu_k)\}$ form a decreasing sequence of closed sets with radius converging to zero, so $\{z(\mu_1, \ldots, \mu_k)\}$ is Cauchy and thus has a limit $\varphi(m) \in \bigcap_{k=1}^{\infty} S(\mu_1, \ldots, \mu_k)$.

We show that $\varphi: \mathscr{M} \to \mathscr{N}$ is a homeomorphism. If $(\mu_1, \mu_2, \ldots)$ and $(v_1, v_2, \ldots)$ are distinct elements of $\mathscr{M}$, then for some integer k, we have $\mu_k \neq v_k$. Since $\varphi(\mu_1, \mu_2, \ldots) \in S(\mu_1, \ldots, \mu_k)$, $\varphi(v_1, v_2, \ldots) \in S(v_1, \ldots, v_k)$, and $S(\mu_1, \ldots, \mu_k)$ is disjoint from $S(v_1, \ldots, v_k)$, we see that $\varphi(\mu_1, \mu_2, \ldots) \neq \varphi(v_1, v_2, \ldots)$, so φ is one-to-one. To show φ is continuous, let $\{m_n\}$ be a sequence converging to $m \in \mathscr{M}$. Choose $\varepsilon > 0$ and let k be a positive integer such that $2/k < \varepsilon$. There exists an $\bar{n}$ such that whenever $n \geq \bar{n}$, the elements m_n and $m = (\mu_1, \mu_2, \ldots)$ agree in the first k components, so both $\varphi(m_n)$ and $\varphi(m)$ are in $S(\mu_1, \ldots, \mu_k)$. This implies $d(\varphi(m_n), \varphi(m)) \leq 2/k < \varepsilon$, so φ is continuous. To show that φ^{-1} is continuous, it suffices to show that $\varphi(F)$ is closed in $\varphi(\mathscr{M})$ whenever F is closed in $\mathscr{M}$. This follows from the fact that $\mathscr{M}$ is compact and φ is continuous. Define $\mathscr{N}_1 \subset \mathscr{N}$ to be the compact homeomorphic image of $\mathscr{M}$ under φ.

We now show that $f: \mathscr{N}_1 \to X$ is a homeomorphism. To see that f is one-to-one, choose distinct points z and $\hat{z}$ in $\mathscr{N}_1$. Then there exist distinct points $m = (\mu_1, \mu_2, \ldots)$ and $\hat{m} = (\hat{\mu}_1, \hat{\mu}_2, \ldots)$ in $\mathscr{M}$ such that $z = \varphi(m)$ and $\hat{z} = \varphi(\hat{m})$. For some k, we have $\mu_k \neq \hat{\mu}_k$, so by (i), $f[S(\mu_0, \ldots, \mu_k)] \cap f[S(\hat{\mu}_0, \ldots, \hat{\mu}_k)] = \varnothing$. Since $z \in S(\mu_0, \ldots, \mu_k)$ and $\hat{z} \in S(\hat{\mu}_0, \ldots, \hat{\mu}_k)$, we see that $f(z) \neq f(\hat{z})$, so f is one-to-one. Just as in the case of φ, the continuity of f^{-1} follows from the fact that f is continuous and has a compact domain.

The set $K = f(\mathscr{N}_1)$ is a compact subset of X homeomorphic to $\mathscr{M}$. Q.E.D.

Lemma B.4 Let X be an uncountable Borel space. There exists a Borel subset L of $\mathscr{M}$ such that X and L are Borel-isomorphic.

Proof By definition, X is homeomorphic to a Borel subset B of a complete separable metric space Y. By Urysohn's and Alexandroff's theorems (Propositions 7.2 and 7.3), Y is homeomorphic to a G_δ-subset of the Hilbert cube $\mathscr{H}$, so B and hence X are homeomorphic to a Borel subset of $\mathscr{H}$. It suffices then to show that $\mathscr{H}$ is Borel-isomorphic to a Borel subset of $\mathscr{M}$.

The idea of the proof is this. Each element in $\mathscr{H}$ is a sequence of real numbers in $[0,1]$. Each of these numbers has a binary expansion, and by mixing all these expansions, we obtain an element in $\mathscr{M}$. Let us first define $\psi:[0,1] \to \mathscr{M}$ which maps a real number into a sequence of zeroes and ones which is its binary expansion. It is easier to define ψ^{-1}, which we define on $\mathscr{M}_1 \cup \{(0,0,0,\ldots)\}$, where

$$\mathscr{M}_1 = \{(\mu_1, \mu_2, \ldots) \in \mathscr{M} | \mu_k = 1 \text{ for infinitely many } k\}.$$

It is given by

$$\psi^{-1}(\mu_1, \mu_2, \ldots) = \sum_{k=1}^{\infty} \mu_k/2^k,$$

and it is easily verified that ψ^{-1} is one-to-one, continuous, and maps onto $[0,1]$. Since $\mathscr{M} - \mathscr{M}_1$ is countable, the domain of ψ^{-1} is a Borel subset of $\mathscr{M}$, and Proposition 7.15 tells us that ψ is a Borel isomorphism. Since we have not proved Proposition 7.15, we show directly that ψ is Borel-measurable. Consider the collection of sets

$$R(k) = \{(\mu_1, \mu_2, \ldots) \in \mathscr{M} | \mu_k = 0\}, \qquad k = 1, 2, \ldots,$$
$$\tilde{R}(k) = \{(\mu_1, \mu_2, \ldots) \in \mathscr{M} | \mu_k = 1\}, \qquad k = 1, 2, \ldots.$$

These sets form a subbase for the topology of $\mathscr{M}$, so by the remark following Definition 7.6, we need only prove that $\psi^{-1}[R(k)]$ and $\psi^{-1}[\tilde{R}(k)]$ are Borel-measurable to conclude that ψ is. Since one of these sets is the complement of the other, we may restrict attention to $\psi^{-1}[R(k)]$. Remembering that the domain of ψ^{-1} is $\mathscr{M}_1 \cup \{0,0,0,\ldots)\}$, we have

$$\psi^{-1}[R(k)] = \left\{ \sum_{j=1}^{\infty} \frac{\mu_j}{2^j} \,\middle|\, (\mu_1, \mu_2, \ldots) \in \mathscr{M}_1, \quad \mu_k = 0 \right\} \cup \{0\},$$

and

$$\left\{ \sum_{j=1}^{\infty} \frac{\mu_j}{2^j} \,\middle|\, (\mu_1, \mu_2, \ldots) \in \mathscr{M}_1, \quad \mu_k = 0 \right\} = \bigcup_{(\mu_1, \ldots, \mu_{k-1})} \left\{ x + \sum_{j=1}^{k-1} \frac{\mu_j}{2^j} \,\middle|\, 0 < x \le \frac{1}{2^k} \right\},$$

which is a finite union of Borel sets.

The proof that $\mathscr{M}\mathscr{M}\cdots$ and $\mathscr{M}$ are homeomorphic is essentially the same one given in Lemma 7.25, and we do not repeat it here. Let θ mapping $\mathscr{M}\mathscr{M}\cdots$ onto $\mathscr{M}$ be a homeomorphism and define $\varphi:\mathscr{H} \to \mathscr{M}$ by

$$\varphi(x_1, x_2, \ldots) = \theta[\psi(x_1), \psi(x_2), \ldots].$$

Then φ is the required Borel-isomorphism. Q.E.D.

Lemma B.5 If K_1 and L are Borel subsets of $\mathscr{M}$, $K_1 \subset L$, and K_1 is Borel-isomorphic to $\mathscr{M}$, then L is Borel-isomorphic to $\mathscr{M}$.

Proof For Borel subsets A and B of $\mathscr{M}$, we write $A \approx B$ to indicate that A and B are Borel-isomorphic. Note that $A \approx B$ and $B \approx C$ implies

$A \approx C$. Also, if $A_1, A_2, \ldots$ is a sequence of disjoint Borel sets, if $B_1, B_2, \ldots$ is another such sequence, and if $A_i \approx B_i$ for every i, then $\bigcup_{i=1}^{\infty} A_i \approx \bigcup_{i=1}^{\infty} B_i$. We note finally that if $A = A_1 \cup A_2$ and $A \approx B$, then $B = B_1 \cup B_2$, where $A_1 \approx B_1$ and $A_2 \approx B_2$. If A_1 and A_2 are disjoint, then B_1 and B_2 can be taken to be disjoint.

Under the hypotheses of the lemma, let $D_1 = \mathscr{M} - K_1$. Since $\mathscr{M}_1 \approx K_1$ and $\mathscr{M} = K_1 \cup D_1$, there exist disjoint Borel sets K_2 and D_2 such that $K_1 = K_2 \cup D_2$, $K_1 \approx K_2$ and $D_1 \approx D_2$. Since $K_1 \approx K_2$ and $K_1 = K_2 \cup D_2$, there exist disjoint Borel sets K_3 and D_3 such that $K_2 = K_3 \cup D_3$, $K_2 \approx K_3$, and $D_2 \approx D_3$. Continuing in this manner, at the nth step we construct disjoint Borel sets K_n and D_n such that $K_{n-1} = K_n \cup D_n$, $K_{n-1} \approx K_n$, and $D_{n-1} \approx D_n$. Let $K_\infty = \bigcap_{n=1}^{\infty} K_n$. Then $\mathscr{M} = K_\infty \cup [\bigcup_{n=1}^{\infty} D_n]$, and all the sets on the right side of this equation are disjoint.

Let $A_1 = \mathscr{M} - L$ and $B_1 = L - K_1$. Then A_1 and B_1 are disjoint and $D_1 = A_1 \cup B_1$. For each n, $D_1 \approx D_n$, so $D_n = A_n \cup B_n$, where A_n and B_n are disjoint Borel sets and $A_1 \approx A_n$, $B_1 \approx B_n$. In particular, $A_n \approx A_{n+1}$ for $n = 1, 2, \ldots$, and we have

$$\begin{aligned}
\mathscr{M} &= K_\infty \cup \left[\bigcup_{n=1}^{\infty} D_n\right] = K_\infty \cup \left[\bigcup_{n=1}^{\infty} A_n\right] \cup \left[\bigcup_{n=1}^{\infty} B_n\right] \\
&\approx K_\infty \cup \left[\bigcup_{n=2}^{\infty} A_n\right] \cup \left[\bigcup_{n=1}^{\infty} B_n\right] \\
&= \left\{K_\infty \cup \left[\bigcup_{n=1}^{\infty} D_n\right]\right\} - A_1 = \mathscr{M} - A_1 = L. \qquad \text{Q.E.D.}
\end{aligned}$$

We can now prove Proposition 7.16, and the proof clearly shows that Corollary 7.16.1 is also true.

Proposition B.3 Let X and Y be Borel spaces. Then X and Y are isomorphic if and only if they have the same cardinality.

Proof If X and Y are isomorphic, then clearly they must have the same cardinality. If X and Y both have the same finite or countably infinite cardinality, then their Borel σ-algebras are their power sets and any one-to-one onto mapping from one to the other is a Borel-isomorphism.

If X is uncountable, then by Lemma B.4 there exists a Borel isomorphism $\varphi: X \to \mathscr{M}$ such that $L = \varphi(X)$ is a Borel subset of M. By Lemma B.3, X contains a compact set K which is homeomorphic to $\mathscr{M}$, so $\varphi(K)$ is Borel-isomorphic to $\mathscr{M}$ and $\varphi(K) \subset L$. Set $K_1 = \varphi(K)$ and use Lemma B.5 to conclude that L and $\mathscr{M}$ are isomorphic. It follows that X and $\mathscr{M}$ are isomorphic. If Y is uncountable, the same argument shows that Y and $\mathscr{M}$ are isomorphic, so X and Y are isomorphic. Q.E.D.

B.3 An Analytic Set Which Is Not Borel-Measurable

Suslin schemes can be used to generate a strictly increasing sequence of σ-algebras on any given uncountable Borel space X. The first σ-algebra in this sequence is the Borel σ-algebra $\mathscr{B}_X$ and the second is the analytic σ-algebra $\mathscr{A}_X$, and, as a result of the following discussion, we will see that $\mathscr{A}_X$ is strictly larger than $\mathscr{B}_X$. The proof of this depends on a contradiction involving universal functions, which we now introduce.

Let $\mathscr{M}_1$ be the set of sequences of zeroes and ones for which one occurs infinitely many times. If the nonzero components of $m \in \mathscr{M}_1$ are in positions $m_1, m_2, \ldots$, then we can think of m as a mapping from $\mathscr{N}$ to $\mathscr{N}$ defined by

$$m(\zeta_1, \zeta_2, \ldots) = (\zeta_{m_1}, \zeta_{m_2}, \ldots).$$

Definition B.1 Let $\mathscr{P}$ be a paving of $\mathscr{N}$. A *universal function* L for $\mathscr{P}$ is a mapping from $\mathscr{N}$ onto $\mathscr{P}$. If $\mathscr{Q}$ is another paving of $\mathscr{N}$ and

$$\{z \in \mathscr{N} | z \in L[m(z)]\} \in \mathscr{Q} \qquad \forall m \in \mathscr{M}_1, \tag{7}$$

we say L is *consistent with* $\mathscr{Q}$.

Proposition B.4 Let $\mathscr{G}$ be the collection of open subsets of $\mathscr{N}$. There exists a universal function for $\mathscr{G}$ consistent with $\mathscr{G}$.

Proof The space $\mathscr{N}$ is separable, so its topology has a countable base $\{G(1), G(2), \ldots\}$, where the empty set is included among these basic open sets. Define $L: \mathscr{N} \to \mathscr{G}$ by

$$L(\zeta_1, \zeta_2, \ldots) = \bigcup_{n=1}^{\infty} G(\zeta_n).$$

It is clear that L is a universal function for $\mathscr{G}$. Now choose $m \in \mathscr{M}_1$ and suppose the nonzero components of m are in positions $m_1, m_2, \ldots$. Choose $z_0 = (\zeta_1^0, \zeta_2^0, \ldots)$ in the set

$$\{z \in \mathscr{N} | z \in L[m(z)]\} = \left\{(\zeta_1, \zeta_2, \ldots) \in \mathscr{N} \,\middle|\, (\zeta_1, \zeta_2, \ldots) \in \bigcup_{k=1}^{\infty} G(\zeta_{m_k})\right\}.$$

Then for some $\bar{k}$, we have $z_0 \in G(\zeta_{m_{\bar{k}}}^0)$. Let

$$U_{\bar{k}}(z_0) = \{(\zeta_1, \zeta_2, \ldots) \in \mathscr{N} | \zeta_{m_{\bar{k}}} = \zeta_{m_{\bar{k}}}^0\}.$$

Then $G(\zeta_{m_{\bar{k}}}^0) \subset L[m(z)]$ for every $z \in U_{\bar{k}}(z_0)$, so $z \in L[m(z)]$ for every $z \in U_{\bar{k}}(z_0) \cap G(\zeta_{m_{\bar{k}}}^0)$. Therefore $U_{\bar{k}}(z_0) \cap G(\zeta_{m_{\bar{k}}}^0)$ is an open neighborhood of z_0 contained in $\{z \in \mathscr{N} | z \in L[m(z)]\}$, so this set is open. Q.E.D.

Given a paved space and a universal function for the paving which satisfies a condition like (7), it is possible to construct similar universal functions for larger pavings. We show first how this is done when the given paving is extended by the use of Suslin schemes.

Proposition B.5 Let $\mathscr{P}$ be a paving for $\mathscr{N}$ and suppose that there exists a universal function for $\mathscr{P}$ consistent with $\mathscr{S}(\mathscr{P})$. Then there exists a universal function for $\mathscr{S}(\mathscr{P})$ consistent with $\mathscr{S}(\mathscr{P})$.

Proof Fix a partition $\{P_s | s \in \Sigma\}$ of the positive integers into countably many countable sets, and define for each $s \in \Sigma$ a corresponding $m_s = (\mu_1(s), \mu_2(s), \ldots) \in \mathscr{M}_1$ by

$$\mu_k(s) = \begin{cases} 1 & \text{if } k \in P_s, \\ 0 & \text{if } k \notin P_s, \end{cases} \tag{8}$$

Let L be a universal function for $\mathscr{P}$ consistent with $\mathscr{S}(\mathscr{P})$. Define $K: \mathscr{N} \to \mathscr{S}(\mathscr{P})$ by

$$K(z_0) = \bigcup_{z \in \mathscr{N}} \bigcap_{s < z} L[m_s(z_0)]. \tag{9}$$

To show that K is onto, we must show that given any Suslin scheme S for $\mathscr{P}$, there exists $z_0 \in \mathscr{N}$ such that

$$S(s) = L[m_s(z_0)] \qquad \forall s \in \Sigma. \tag{10}$$

If $S: \Sigma \to \mathscr{P}$ is given and $s \in \Sigma$, then $S(s) \in \mathscr{P}$. Since L is a universal function for $\mathscr{P}$, there exists $z_s \in \mathscr{N}$ for which $S(s) = L(z_s)$. If z_0 is chosen so that $m_s(z_0) = z_s$ for every $s \in \Sigma$, then (10) is satisfied, and such a choice of z_0 is possible because $m_s(z_0)$ depends only on the components of z_0 with indices in P_s. Therefore K is a universal function for $\mathscr{S}(\mathscr{P})$.

If $m, n \in \mathscr{M}_1$, then there is an element in $\mathscr{M}_1$, which we denote by mn, such that $(mn)(z) = m[n(z)]$ for every $z \in \mathscr{N}$. In fact, if the nonzero elements of m are $(m_1, m_2, \ldots)$ and the nonzero elements of n are $(n_1, n_2, \ldots)$, then the nonzero elements of mn are $(n_{m_1}, n_{m_2}, \ldots)$. Now suppose $m \in \mathscr{M}_1$. We have

$$\begin{aligned} \{z_0 \in \mathscr{N} | z_0 \in K[m(z_0)]\} &= \left\{ z_0 \in \mathscr{N} \middle| z_0 \in \bigcup_{z \in \mathscr{N}} \bigcap_{s < z} L[(m_s m)(z_0)] \right\} \\ &= \bigcup_{z \in \mathscr{N}} \bigcap_{s < z} \{z_0 \in \mathscr{N} | z_0 \in L[(m_s m)(z_0)]\}, \end{aligned}$$

which, since L is consistent with $\mathscr{S}(\mathscr{P})$, is the nucleus of a Suslin scheme for $\mathscr{S}(\mathscr{P})$. It follows from Proposition B.2 that K is consistent with $\mathscr{S}(\mathscr{P})$.

Q.E.D.

Corollary B.5.1 There is a universal function for $\mathscr{S}(\mathscr{F}_{\mathscr{N}})$ consistent with $\mathscr{S}(\mathscr{F}_{\mathscr{N}})$.

Proof Let $\mathscr{G}$ be the collection of open subsets of $\mathscr{N}$. By Propositions B.4 and B.5, there is a universal function for $\mathscr{S}(\mathscr{G})$ consistent with $\mathscr{S}(\mathscr{G})$, and it remains only to show that $\mathscr{S}(\mathscr{G}) = \mathscr{S}(\mathscr{F}_{\mathscr{N}})$. Since $\mathscr{G} \subset \mathscr{B}_{\mathscr{N}}$, it follows from Proposition 7.36 that $\mathscr{S}(\mathscr{G}) \subset \mathscr{S}(\mathscr{F}_{\mathscr{N}})$. Since every closed subset of $\mathscr{N}$ is a G_δ-set and, by Proposition 7.35, $\mathscr{G}_\delta \subset \mathscr{S}(\mathscr{G})_\delta = \mathscr{S}(\mathscr{G})$, we see that $\mathscr{F}_{\mathscr{N}} \subset \mathscr{S}(\mathscr{G})$. Proposition B.2 implies that $\mathscr{S}(\mathscr{F}_{\mathscr{N}}) \subset \mathscr{S}[\mathscr{S}(\mathscr{G})] = \mathscr{S}(\mathscr{G})$. Q.E.D.

Corollary B.5.2 Let L be a universal function for $\mathscr{S}(\mathscr{F}_{\mathscr{N}})$ consistent with $\mathscr{S}(\mathscr{F}_{\mathscr{N}})$. The set

$$A_0 = \{z \in \mathscr{N} \mid z \in L(z)\} \tag{11}$$

is analytic but not Borel-measurable, and $\mathscr{N} - A_0$ is not analytic.

Proof The set A_0 is analytic because L is consistent with $\mathscr{S}(\mathscr{F}_{\mathscr{N}})$. We have

$$\mathscr{N} - A_0 = \{z \in \mathscr{N} \mid z \notin L(z)\}, \tag{12}$$

and if this set is analytic, then there exists $z_0 \in \mathscr{N}$ such that

$$\mathscr{N} - A_0 = L(z_0).$$

If $z_0 \in A_0$, then $z_0 \notin L(z_0)$, and (11) is contradicted. If $z_0 \in \mathscr{N} - A_0$, then $z_0 \in L(z_0)$ and (12) is contradicted. Therefore $\mathscr{N} - A_0$ is not analytic, thus not Borel-measurable, so A_0 is also not Borel-measurable. Q.E.D.

Proposition B.6 Let X be an uncountable Borel space. There exists an analytic subset A of X such that A is not Borel-measurable and $X - A$ is not analytic.

Proof Let $\varphi: \mathscr{N} \to X$ be a Borel isomorphism from $\mathscr{N}$ onto X (Corollary 7.16.1), and let $A_0 \subset \mathscr{N}$ be as in Corollary B.5.2. Then $A = \varphi(A_0)$ is analytic, but since $\mathscr{N} - A_0 = \varphi^{-1}(X - A)$ is not analytic, neither is $X - A$. It follows that A is not Borel-measurable. Q.E.D.

B.4 The Limit σ-algebra

We construct a collection of σ-algebras indexed by the countable ordinals, and at the end of this process we arrive at the limit σ-algebra, denoted by $\mathscr{L}_X$. The proofs of many of the properties of $\mathscr{L}_X$, and indeed the definition of $\mathscr{L}_X$, proceed by transfinite induction. We also make frequent use of the fact that if $\{\alpha_n\}$ is a sequence of countable ordinals, then there exists a countable ordinal $\bar{\alpha}$ such that $\alpha_n < \bar{\alpha}$ for every n. In keeping with standard convention, we denote by Ω the first uncountable ordinal.

Definition B.2 Let X be a Borel space and $\mathscr{G}_X$ the collection of open subsets of X. For each countable ordinal α, we define

$$\mathscr{L}_X^0 = \sigma(\mathscr{G}_X), \tag{13}$$

$$\mathscr{L}_X^\alpha = \sigma\left[\mathscr{S}\left(\bigcup_{\beta<\alpha} \mathscr{L}_X^\beta\right)\right]. \tag{14}$$

The *limit σ-algebra* is

$$\mathscr{L}_X = \bigcup_{\alpha<\Omega} \mathscr{L}_X^\alpha. \tag{15}$$

We prove later (Proposition B.10) that $\mathscr{L}_X$ is in fact a σ-algebra. Note that $\mathscr{L}_X^0 = \mathscr{B}_X$ and $\mathscr{L}_X^1 = \mathscr{A}_X$. When X is countable, $\mathscr{B}_X = \mathscr{L}_X^\alpha$ for every $\alpha < \Omega$. If X is uncountable, there is no loss of generality in assuming $X = \mathscr{N}$ when dealing with the σ-algebras $\mathscr{L}_X^\alpha$ and $\mathscr{L}_X$. This is the subject of the next proposition.

Proposition B.7 Let X be an uncountable Borel space and let $\varphi:\mathscr{N} \to X$ be a Borel isomorphism from $\mathscr{N}$ onto X. (Such an isomorphism exists by Corollary 7.16.1.) Then for every $\alpha < \Omega$,

$$\varphi(\mathscr{L}_{\mathscr{N}}^\alpha) = \mathscr{L}_X^\alpha, \qquad \mathscr{L}_{\mathscr{N}}^\alpha = \varphi^{-1}(\mathscr{L}_X^\alpha), \tag{16}$$

and

$$\varphi(\mathscr{L}_{\mathscr{N}}) = \mathscr{L}_X, \qquad \mathscr{L}_{\mathscr{N}} = \varphi^{-1}(\mathscr{L}_X). \tag{17}$$

Proof We prove (16) by transfinite induction. For $\alpha = 0$, (16) clearly holds. If (16) holds for all $\beta < \alpha$, where $\alpha < \Omega$, then we have

$$\varphi\left(\bigcup_{\beta<\alpha} \mathscr{L}_{\mathscr{N}}^\beta\right) = \bigcup_{\beta<\alpha} \mathscr{L}_X^\beta, \qquad \bigcup_{\beta<\alpha} \mathscr{L}_{\mathscr{N}}^\beta = \varphi^{-1}\left(\bigcup_{\beta<\alpha} \mathscr{L}_X^\beta\right).$$

Let S be a Suslin scheme for $\bigcup_{\beta<\alpha} \mathscr{L}_{\mathscr{N}}^\beta$. Then

$$\varphi[N(S)] = N(\varphi \circ S),$$

where

$$(\varphi \circ S)(s) = \varphi[S(s)] \qquad \forall s \in \Sigma.$$

Since $\varphi \circ S$ is a Suslin scheme for $\bigcup_{\beta<\alpha} \mathscr{L}_X^\beta$, we see that

$$\varphi\left[\mathscr{S}\left(\bigcup_{\beta<\alpha} \mathscr{L}_{\mathscr{N}}^\beta\right)\right] \subset \mathscr{S}\left(\bigcup_{\beta<\alpha} \mathscr{L}_X^\beta\right). \tag{18}$$

On the other hand, if R is a Suslin scheme for $\bigcup_{\beta<\alpha} \mathscr{L}_X^\beta$, then

$$N(R) = \varphi[N(\varphi^{-1} \circ R)],$$

where

$$(\varphi^{-1} \circ R)(s) = \varphi^{-1}[R(s)] \qquad \forall s \in \Sigma.$$

This shows that $N(R) \in \varphi[\mathscr{S}(\bigcup_{\beta<\alpha} \mathscr{L}_{\mathscr{N}}^{\beta})]$, which proves the reverse of set containment (18). Therefore,

$$\varphi\left[\mathscr{S}\left(\bigcup_{\beta<\alpha} \mathscr{L}_{\mathscr{N}}^{\beta}\right)\right] = \mathscr{S}\left(\bigcup_{\beta<\alpha} \mathscr{L}_{X}^{\beta}\right). \tag{19}$$

Since φ is one-to-one, we also have

$$\mathscr{S}\left(\bigcup_{\beta<\alpha} \mathscr{L}_{\mathscr{N}}^{\beta}\right) = \varphi^{-1}\left[\mathscr{S}\left(\bigcup_{\beta<\alpha} \mathscr{L}_{X}^{\beta}\right)\right]. \tag{20}$$

Now by (19), $\varphi(\mathscr{L}_{\mathscr{N}}^{\alpha})$ is a σ-algebra containing $\mathscr{S}(\bigcup_{\beta<\alpha} \mathscr{L}_{X}^{\beta})$, so

$$\varphi(\mathscr{L}_{\mathscr{N}}^{\alpha}) \supset \mathscr{L}_{X}^{\alpha}. \tag{21}$$

By (20), $\varphi^{-1}(\mathscr{L}_{X}^{\alpha})$ is a σ-algebra containing $\mathscr{S}(\bigcup_{\beta<\alpha} \mathscr{L}_{\mathscr{N}}^{\beta})$, so

$$\mathscr{L}_{\mathscr{N}}^{\alpha} \subset \varphi^{-1}(\mathscr{L}_{X}^{\alpha}). \tag{22}$$

Since φ is one-to-one, (21) implies

$$\mathscr{L}_{\mathscr{N}}^{\alpha} \supset \varphi^{-1}(\mathscr{L}_{X}^{\alpha}) \tag{23}$$

and (22) implies

$$\varphi(\mathscr{L}_{\mathscr{N}}^{\alpha}) \subset \mathscr{L}_{X}^{\alpha}. \tag{24}$$

Relations (21)–(24) imply (16). Relation (17) follows from (15) and (16).

Q.E.D.

We have already seen that in an uncountable Borel space X, $\mathscr{L}_{X}^{0}$ is properly contained in $\mathscr{L}_{X}^{1}$ (Proposition B.6). We would like to show more generally that if $\beta < \alpha < \Omega$, then $\mathscr{L}_{X}^{\beta}$ is properly contained in $\mathscr{L}_{X}^{\alpha}$. Our method for doing this is to generalize Corollary B.5.1 and then generalize Corollary B.5.2. The following lemmas are a step in this direction. If $\mathscr{P}$ is a paving for a space X, we denote by $\bar{\mathscr{P}}$ the paving

$$\bar{\mathscr{P}} = \mathscr{P} \cup \{X - P \mid P \in \mathscr{P}\}. \tag{25}$$

Lemma B.6 Let $\mathscr{P}$ be a paving for $\mathscr{N}$ which contains the open subsets of $\mathscr{N}$, and suppose there exists a universal function for $\mathscr{P}$ consistent with $\mathscr{P}$. Then there exists a universal function for $\bar{\mathscr{P}}$ consistent with $\sigma(\mathscr{P})$.

Proof Let L be a universal function for $\mathscr{P}$ consistent with $\mathscr{P}$. Define $K: \mathscr{N} \to \bar{\mathscr{P}}$ by

$$K(\zeta_1, \zeta_2, \ldots) = \begin{cases} L(\zeta_2, \zeta_3, \zeta_4, \ldots) & \text{if } \zeta_1 \text{ is odd,} \\ \mathscr{N} - L(\zeta_2, \zeta_3, \zeta_4, \ldots) & \text{if } \zeta_1 \text{ is even.} \end{cases}$$

It is clear that K is a universal function for $\bar{\mathscr{P}}$. As in the proof of Proposition B.4, choose $m \in \mathscr{M}_1$ and suppose that the nonzero components of m are in positions $m_1, m_2, \ldots$. Then

$$\begin{aligned}
&\{z \in \mathscr{N} | z \in K[m(z)]\} \\
&\quad = \{(\zeta_1, \zeta_2, \ldots) | \zeta_{m_1} \text{ is odd and } (\zeta_1, \zeta_2, \ldots) \in L(\zeta_{m_2}, \zeta_{m_3}, \ldots)\} \\
&\qquad \cup \{(\zeta_1, \zeta_2, \ldots) | \zeta_{m_1} \text{ is even and } (\zeta_1, \zeta_2, \ldots) \notin L(\zeta_{m_2}, \zeta_{m_3}, \ldots)\} \\
&\quad = \left(\left[\bigcup_{k=1}^{\infty} \{(\zeta_1, \zeta_2, \ldots) | \zeta_{m_1} = 2k - 1\} \right] \right. \\
&\qquad \left. \cap \{(\zeta_1, \zeta_2, \ldots) | (\zeta_1, \zeta_2, \ldots) \in L(\zeta_{m_2}, \zeta_{m_3}, \ldots)\} \right) \\
&\qquad \cup \left(\left[\bigcup_{k=1}^{\infty} \{(\zeta_1, \zeta_2, \ldots) | \zeta_{m_1} = 2k\} \right] \right. \\
&\qquad \left. \cap \{(\zeta_1, \zeta_2, \ldots) | (\zeta_1, \zeta_2, \ldots) \notin L(\zeta_{m_2}, \zeta_{m_3}, \ldots)\} \right). \qquad (26)
\end{aligned}$$

Since L is consistent with $\mathscr{P}$ and $\mathscr{P}$ contains every open set, we have that every set in (26) is in $\sigma(\mathscr{P})$. It follows that K is consistent with $\sigma(\mathscr{P})$. Q.E.D.

Lemma B.7 Let α be a countable ordinal. For each $\beta < \alpha$, let $\mathscr{P}_\beta$ be a paving for $\mathscr{N}$ which contains the collection $\mathscr{G}$ of open sets, and assume that there exists a universal function L_β for $\mathscr{P}_\beta$ consistent with $\mathscr{P}_\beta$. Then there exists a universal function for $\bigcup_{\beta<\alpha} \mathscr{P}_\beta$ consistent with $\mathscr{S}(\bigcup_{\beta<\alpha} \mathscr{P}_\beta)$.

Proof The set of ordinals $\{\beta | \beta < \alpha\}$ is countable whenever $\alpha < \Omega$, so there exists a partition $\{P(\beta) | \beta < \alpha\}$ of the positive integers such that $P(\beta)$ is nonempty for each $\beta < \alpha$. Define a universal function for $\bigcup_{\beta<\alpha} P(\beta)$ by

$$L(\zeta_1, \zeta_2, \ldots) = L_\beta(\zeta_2, \zeta_3, \ldots) \qquad \text{if} \quad \zeta_1 \in P(\beta).$$

Let $m \in \mathscr{M}_1$ have nonzero components $m_1, m_2, \ldots$. Then

$$\begin{aligned}
&\{z \in \mathscr{N} | z \in L[m(z)]\} \\
&\quad = \bigcup_{\beta<\alpha} \{(\zeta_1, \zeta_2, \ldots) | \zeta_{m_1} \in P(\beta) \text{ and } (\zeta_1, \zeta_2, \ldots) \in L_\beta(\zeta_{m_2}, \zeta_{m_3}, \ldots)\} \\
&\quad = \bigcup_{\beta<\alpha} [\{(\zeta_1, \zeta_2, \ldots) | \zeta_{m_1} \in P(\beta)\} \cap \{(\zeta_1, \zeta_2, \ldots) | (\zeta_1, \zeta_2, \ldots) \in L_\beta(\zeta_{m_2}, \zeta_{m_3}, \ldots)\}],
\end{aligned}$$

and this set is in $\mathscr{S}(\bigcup_{\beta<\alpha} \mathscr{P}_\beta)$ by Proposition 7.35(b), (c), and the fact that each L_β is consistent with $\mathscr{P}_\beta$. Q.E.D.

Proposition B.8 For each $\alpha < \Omega$, there is a universal function for $\mathscr{S}(\mathscr{L}_{\mathscr{N}}^{\alpha})$ consistent with $\mathscr{S}(\mathscr{L}_{\mathscr{N}}^{\alpha})$.

Proof For simplicity of notation, we suppress the subscript $\mathscr{N}$. The proof is by transfinite induction. When $\alpha = 0$, the result follows from Corollary B.5.1. Assume now that the result holds for every $\beta < \alpha$, where $\alpha < \Omega$. We prove it for α.

By Lemma B.7 and the induction assumption, there is a universal function for $\bigcup_{\beta<\alpha} \mathscr{S}(\mathscr{L}^\beta)$ consistent with $\mathscr{S}[\bigcup_{\beta<\alpha} \mathscr{S}(\mathscr{L}^\beta)]$. Now

$$\bigcup_{\beta<\alpha} \mathscr{L}^\beta \subset \bigcup_{\beta<\alpha} \mathscr{S}(\mathscr{L}^\beta) \subset \mathscr{S}\left(\bigcup_{\beta<\alpha} \mathscr{L}^\beta\right), \tag{27}$$

and applying $\mathscr{S}$ to both sides of (27) and using Proposition B.2, we obtain

$$\mathscr{S}\left(\bigcup_{\beta<\alpha} \mathscr{L}^\beta\right) = \mathscr{S}\left[\bigcup_{\beta<\alpha} \mathscr{S}(\mathscr{L}^\beta)\right]. \tag{28}$$

From Proposition B.5 and (28) we have the existence of a universal function for $\mathscr{S}(\bigcup_{\beta<\alpha} \mathscr{L}^\beta)$ consistent with $\mathscr{S}(\bigcup_{\beta<\alpha} \mathscr{L}^\beta)$, and Lemma B.6 implies existence of a universal function for $\overline{\mathscr{S}(\bigcup_{\beta<\alpha} \mathscr{L}^\beta)}$ consistent with $\mathscr{L}^\alpha$. From Corollary 7.35.1 we have

$$\mathscr{L}^\alpha = \sigma\left[\overline{\mathscr{S}\left(\bigcup_{\beta<\alpha} \mathscr{L}^\beta\right)}\right] \subset \mathscr{S}\left[\overline{\mathscr{S}\left(\bigcup_{\beta<\alpha} \mathscr{L}^\beta\right)}\right], \tag{29}$$

so we have a universal function for $\overline{\mathscr{S}(\bigcup_{\beta<\alpha} \mathscr{L}^\beta)}$ consistent with

$$\mathscr{S}\left[\overline{\mathscr{S}\left(\bigcup_{\beta<\alpha} \mathscr{L}^\beta\right)}\right].$$

But from (29),

$$\mathscr{L}^\alpha \subset \mathscr{S}\left[\overline{\mathscr{S}\left(\bigcup_{\beta<\alpha} \mathscr{L}^\beta\right)}\right] \subset \mathscr{S}(\mathscr{L}^\alpha),$$

and applying $\mathscr{S}$ to both sides, we see that

$$\mathscr{S}(\mathscr{L}^\alpha) = \mathscr{S}\left[\overline{\mathscr{S}\left(\bigcup_{\beta<\alpha} \mathscr{L}^\beta\right)}\right]. \tag{30}$$

From Proposition B.5 and (30) we have the existence of a universal function for $\mathscr{S}(\mathscr{L}^\alpha)$ consistent with $\mathscr{S}(\mathscr{L}^\alpha)$. Q.E.D.

Proposition B.9 Let X be an uncountable Borel space. If $\beta < \alpha < \Omega$, then $\mathscr{L}_X^\beta$ is properly contained in $\mathscr{L}_X^\alpha$.

Proof We assume without loss of generality that $X = \mathscr{N}$ (Proposition B.7) and suppress the subscript $\mathscr{N}$. It is clear that for $\beta < \alpha$ we have $\mathscr{L}^\beta \subset$

$\mathscr{L}^\alpha$. Let L be a universal function for $\mathscr{S}(\mathscr{L}^\beta)$ consistent with $\mathscr{S}(\mathscr{L}^\beta)$ and define

$$A = \{z \in \mathscr{N} \mid z \in L(z)\}.$$

Then $A \in \mathscr{S}(\mathscr{L}^\beta)$. If $\mathscr{N} - A \in \mathscr{S}(\mathscr{L}^\beta)$, then for some $z_0 \in \mathscr{N}$ we have

$$\mathscr{N} - A = L(z_0).$$

If $z_0 \in A$, then $z_0 \notin L(z_0)$ and a contradiction is reached. If $z_0 \in \mathscr{N} - A$, then $z_0 \in L(z_0)$ and again a contradiction is reached. It follows that $\mathscr{N} - A \notin \mathscr{S}(\mathscr{L}^\beta)$. But $\mathscr{N} - A \in \mathscr{L}^\alpha$, so $\mathscr{L}^\beta$ is properly contained in $\mathscr{L}^\alpha$. Q.E.D.

Proposition B.10 Let X be a Borel space. The limit σ-algebra $\mathscr{L}_X$ is contained in $\mathscr{U}_X$ and

$$\mathscr{L}_X = \mathscr{S}(\mathscr{L}_X). \tag{31}$$

Indeed, $\mathscr{L}_X$ is the smallest σ-algebra containing the open subsets of X which satisfies (31).

Proof The result is trivial if X is countable, so assume that X is uncountable. It is clear that $\varnothing \in \mathscr{L}_X$ and $\mathscr{L}_X$ is closed under complementation, so we need only verify that $\mathscr{L}_X$ is closed under countable unions in order to show that it is a σ-algebra. If $Q_1, Q_2, \ldots$ is a sequence of sets in $\mathscr{L}_X$, then for some $\alpha < \Omega$, we have $Q_k \in \mathscr{L}_X^\alpha$ for every k. Then $\bigcup_{k=1}^\infty Q_k \in \mathscr{L}_X^\alpha \subset \mathscr{L}_X$.

We prove by transfinite induction that $\mathscr{L}_X^\alpha \subset \mathscr{U}_X$ for every $\alpha < \Omega$. This is clearly the case if $\alpha = 0$. If $\mathscr{L}_X^\beta \subset \mathscr{U}_X$ for every $\beta < \alpha$, where $\alpha < \Omega$, then by Lusin's theorem (Proposition 7.42), $\mathscr{S}(\bigcup_{\beta<\alpha} \mathscr{L}_X^\beta) \subset \mathscr{U}_X$. It follows that $\mathscr{L}_X^\alpha \subset \mathscr{U}_X$. Therefore $\mathscr{L}_X \subset \mathscr{U}_X$.

We now prove (31). As a result of Proposition 7.35(d), it suffices to prove that $\mathscr{L}_X \supset \mathscr{S}(\mathscr{L}_X)$. Let S be a Suslin scheme for $\mathscr{L}_X$. Since Σ is countable, there exists $\alpha < \Omega$ such that $S(s) \in \mathscr{L}_X^\alpha$ for every $s \in \Sigma$. Then $N(S) \in \mathscr{L}_X^{\alpha+1} \subset \mathscr{L}_X$, and (31) is proved.

Suppose $\mathscr{P}$ is a σ-algebra containing the open subsets of X which satisfies $\mathscr{P} = \mathscr{S}(\mathscr{P})$. Clearly, $\mathscr{B}_X = \mathscr{L}_X^0 \subset \mathscr{P}$. If $\mathscr{L}_X^\beta \subset \mathscr{P}$ for every $\beta < \alpha$, where $\alpha < \Omega$, then (14) implies that $\mathscr{L}_X^\alpha \subset \mathscr{P}$. Therefore $\mathscr{P}$ contains $\mathscr{L}_X$, which must be the smallest σ-algebra containing the open subsets of X and satisfying (31).
Q.E.D.

A major shortcoming of the analytic σ-algebra is that the composition of analytically measurable functions is not necessarily analytically measurable (cf. remarks following Proposition 7.50). However, the composition of limit-measurable functions is limit-measurable. We first give a formal definition of these terms and then prove the preceding statements.

Definition B.3 Let X and Y be Borel spaces, $D \subset X$, and $\mathscr{P}$ a σ-algebra on X. A function $f: D \to Y$ is said to be *$\mathscr{P}$-measurable* if $f^{-1}(B) \in \mathscr{P}$ for every $B \in \mathscr{B}_Y$. If $\mathscr{P} = \mathscr{L}_X$, we say that f is *limit-measurable*. The σ-algebra $\mathscr{P}$ is said to be *closed under composition of functions* if, whenever $f: X \to X$ is $\mathscr{P}$-measurable and $P \in \mathscr{P}$, then $f^{-1}(P) \in \mathscr{P}$.

In Definition B.3 there is no mention of a $\mathscr{P}$-measurable function g mapping X into a Borel space Y with which to compose f. If there were such a g, then to check that $g \circ f: X \to Y$ is $\mathscr{P}$-measurable, we would check that $f^{-1}[g^{-1}(B)]$ is $\mathscr{P}$-measurable for every $B \in \mathscr{B}_Y$. Since $g^{-1}(B) \in \mathscr{P}$, it suffices to check that $f^{-1}(P) \in \mathscr{P}$ for every $P \in \mathscr{P}$, which is the condition stated in Definition B.3. The stipulation in Definition B.3 that f have the same domain and range space is inconsequential as long as $\mathscr{P} = \mathscr{L}_X^\alpha$ for some $\alpha < \Omega$ or $\mathscr{P} = \mathscr{L}_X$ (see Proposition B.7). These are the only cases we consider. The closure of a σ-algebra under composition of mappings and the satisfaction of an equation like (31) are intimately related, as the following lemma shows.

Lemma B.8 Let X be a Borel space and let $\mathscr{P}$ be a σ-algebra on X. If $\mathscr{P}$ contains the analytic subsets of X and is closed under composition of functions, then

$$\mathscr{P} = \mathscr{S}(\mathscr{P}).$$

Proof If X is countable, the result is trivial, so we assume that X is uncountable. In light of Proposition 7.35(d), we need only prove that under the assumptions of the lemma we have $\mathscr{P} \supset \mathscr{S}(\mathscr{P})$. To do this, for an arbitrary Suslin scheme S for $\mathscr{P}$ we construct a $\mathscr{P}$-measurable function $f: X \to X$ and a set $P \in \mathscr{P}$ such that

$$f^{-1}(P) = N(S). \tag{32}$$

Let $\varphi: \mathscr{N} \to X$ be a Borel isomorphism from $\mathscr{N}$ onto X (Corollary 7.16.1), and let ψ be a one-to-one onto function from the set of positive integers to Σ. For $k = 1, 2, \ldots$, define $\bar{f}_k: \mathscr{N} \to \{1, 2\}$ by

$$\bar{f}_k(z) = \begin{cases} 1 & \text{if} \quad \varphi(z) \in S[\psi(k)], \\ 2 & \text{otherwise}, \end{cases}$$

and define $\bar{f}: \mathscr{N} \to \mathscr{N}$ by

$$\bar{f}(z) = [\bar{f}_1(z), \bar{f}_2(z), \ldots].$$

Finally, let $f: X \to X$ be given by $f = \varphi \circ \bar{f} \circ \varphi^{-1}$. We show that f is $\mathscr{P}$-measurable. This is equivalent to showing that $\bar{f} \circ \varphi^{-1}: X \to \mathscr{N}$ is $\mathscr{P}$-measurable. But $\bar{f} \circ \varphi^{-1}$ takes values in $\{(\zeta_1, \zeta_2, \ldots) \in \mathscr{N} | \zeta_n \leq 2 \, \forall n\}$ which has as a subbase the collection of open sets $\{R(k), \tilde{R}(k) | k = 1,2, \dots\}$, where

$$R(k) = \{(\zeta_1, \zeta_2, \dots) | \zeta_n \leq 2 \ \forall n \text{ and } \zeta_k = 1\}, \tag{33}$$
$$\tilde{R}(k) = \{(\zeta_1, \zeta_2, \dots) | \zeta_n \leq 2 \ \forall n \text{ and } \zeta_k = 2\}.$$

By the remark following Definition 7.6, the $\mathscr{P}$-measurability of the sets

$$\varphi(\bar{f}^{-1}[R(k)]) = S[\psi(k)], \qquad k = 1,2, \dots,$$
$$\varphi(\bar{f}^{-1}[\tilde{R}(k)]) = X - S[\psi(k)], \qquad k = 1,2, \dots,$$

implies the $\mathscr{P}$-measurability of $\bar{f} \circ \varphi^{-1}$. It follows that f is $\mathscr{P}$-measurable.

Define $P \subset X$ by

$$P = \bigcup_{z \in \mathscr{N}} \bigcap_{s < z} \varphi(R[\psi^{-1}(s)]),$$

where $R(k)$ is given by (33). Then P is an analytic subset of X, so $P \in \mathscr{P}$. We have

$$f^{-1}(P) = \varphi\left[\bigcup_{z \in \mathscr{N}} \bigcap_{s < z} \bar{f}^{-1}(R[\psi^{-1}(s)])\right]$$
$$= \bigcup_{z \in \mathscr{N}} \bigcap_{s < z} S(s) = N(S),$$

so (32) holds. Q.E.D.

Proposition B.11 Let X be a Borel space. The limit σ-algebra $\mathscr{L}_X$ is the smallest σ-algebra containing the analytic subsets of X which is closed under composition of functions.

Proof We show first that $\mathscr{L}_X$ is closed under composition of functions. It suffices to show that if $f: X \to X$ is $\mathscr{L}_X$-measurable, $\alpha < \Omega$, and $Q \in \mathscr{L}_X^\alpha$, then $f^{-1}(Q) \in \mathscr{L}_X$. If $\alpha = 0$, this is true by definition. Suppose that for some $\alpha < \Omega$ and for every $\beta < \alpha$ and $C \in \mathscr{L}_X^\beta$ we have $f^{-1}(C) \in \mathscr{L}_X$. We show that $f^{-1}(Q) \in \mathscr{L}_X$ for every $Q \in \mathscr{S}(\bigcup_{\beta < \alpha} \mathscr{L}_X^\beta)$, and this implies that $f^{-1}(Q) \in \mathscr{L}_X$ for every $Q \in \mathscr{L}_X^\alpha$. Choose $Q \in \mathscr{S}(\bigcup_{\beta < \alpha} \mathscr{L}_X^\beta)$ and let S be a Suslin scheme for $\bigcup_{\beta < \alpha} \mathscr{L}_X^\beta$ such that $Q = N(S)$. Then

$$f^{-1}(Q) = N(f^{-1} \circ S), \tag{34}$$

where $f^{-1} \circ S$ is the Suslin scheme defined by

$$(f^{-1} \circ S)(s) = f^{-1}[S(s)] \qquad \forall s \in \Sigma.$$

By the induction hypothesis, $f^{-1} \circ S$ is a Suslin scheme for $\mathscr{L}_X$, and we have from Proposition B.10 and (34) that $f^{-1}(Q) \in \mathscr{L}_X$.

The fact that $\mathscr{L}_X$ is the smallest σ-algebra containing the analytic subsets of X which is closed under composition of functions follows from Proposition B.10 and Lemma B.8. Q.E.D.

Corollary B.11.1 Let X, Y, and Z be uncountable Borel spaces. If $f: X \to Y$ and $g: Y \to Z$ are limit-measurable, then $g \circ f: X \to Z$ is limit-measurable. In particular, if f and g are analytically measurable, then $g \circ f$ is limit-measurable. It is possible to choose f and g to be analytically measurable so that $g \circ f$ is not analytically measurable.

Proof Proposition B.9 implies that $\mathscr{A}_X$, $\mathscr{A}_Y$, and $\mathscr{A}_Z$ are properly contained in $\mathscr{L}_X$, $\mathscr{L}_Y$, and $\mathscr{L}_Z$, respectively. Apply Proposition B.7 to the results of Proposition B.11. Q.E.D.

Using an argument similar to the first part of the proof of Proposition B.11, the reader may verify that if $f: X \to Y$ and $g: Y \to Z$ are analytically measurable, then $g \circ f$ is in fact $\mathscr{L}_X^2$-measurable. Indeed, one can show by induction that if f is $\mathscr{L}_X^m$-measurable and g is $\mathscr{L}_Y^n$-measurable, where m and n are integers, then $g \circ f$ is $\mathscr{L}_X^{m+n}$-measurable.

Let X be a Borel space, and for $Q \in \mathscr{U}_X$ define $\theta_Q: P(X) \to [0, 1]$ by

$$\theta_Q(p) = p(Q). \tag{35}$$

Then θ_Q is universally measurable (Corollary 7.46.1). If Q is Borel-measurable, then θ_Q is Borel-measurable (Proposition 7.25), and if Q is analytically measurable, then θ_Q is analytically measurable (Proposition 7.43). We consider the case when Q is $\mathscr{L}_X^\alpha$-measurable.

Proposition B.12 Let X be a Borel space. If $Q \in \mathscr{L}_X$, then θ_Q defined by (35) is $\mathscr{L}_{P(X)}$-measurable. In fact if $\alpha < \Omega$ and $Q \in \mathscr{L}_X^\alpha$, then θ_Q is $\mathscr{L}_{P(X)}^\alpha$-measurable.

Proof The last statement is true when $\alpha = 0$. If it is true for every $\beta < \alpha$, where $\alpha < \Omega$, and S is a Suslin scheme for $\bigcup_{\beta < \alpha} \mathscr{L}_X^\beta$, then for any $c \in R$, (98) of Chapter 7 holds, where $A = N(S)$ and $K(s)$ is defined by (92) of Chapter 7. For each $s \in \Sigma$, $K(s) \in \bigcup_{\beta < \alpha} \mathscr{L}_X^\beta$, so by the induction hypothesis, the set $\{p \in P(X) | p[K(s)] \geq c - (1/n)\}$ is in $\bigcup_{\beta < \alpha} \mathscr{L}_{P(X)}^\beta$. It follows from (98) of Chapter 7 and Proposition 7.35(b) that

$$\{p \in P(X) | p[N(S)] \geq c\} \in \mathscr{S}\left(\bigcup_{\beta < \alpha} \mathscr{L}_{P(X)}^\beta\right) \subset \mathscr{L}_{P(X)}^\alpha.$$

Thus, if $Q \in \mathscr{S}(\bigcup_{\beta < \alpha} \mathscr{L}_{P(X)}^\beta)$, then θ_Q is $\mathscr{L}_{P(X)}^\alpha$-measurable. The collection of sets Q for which θ_Q is $\mathscr{L}_{P(X)}^\alpha$-measurable forms a Dynkin system, so by the Dynkin system theorem (Proposition 7.24), θ_Q is $\mathscr{L}_{P(X)}^\alpha$-measurable for every $Q \in \mathscr{L}_X^\alpha$. This completes the induction step.

If $Q \in \mathscr{L}_X$, then for some $\alpha < \Omega$, $Q \in \mathscr{L}_X^\alpha$, so θ_Q is $\mathscr{L}_{P(X)}^\alpha$-measurable, and therefore θ_Q is $\mathscr{L}_{P(X)}$-measurable. Q.E.D.

B.5 Set Theoretic Aspects of Borel Spaces

The measurability properties of Borel spaces are closely linked to several issues in set theory which we have for the most part skirted. These issues are presented briefly here.

There is some controversy concerning the propriety of the axiom of choice and Cantor's continuum hypothesis in applied mathematics. The former is generally accepted and the latter is regarded with suspicion. The general axiom of choice says that given any index set A and a collection of nonempty sets $\{S_\alpha | \alpha \in A\}$, there is a function $f: A \to \bigcup_{\alpha \in A} S_\alpha$ such that $f(\alpha) \in S_\alpha$ for every $\alpha \in A$. We have used this axiom in Appendix A to construct examples. In particular, the set E of Example 1 of that appendix for which both E and E^c have p-outer measure one is constructed by means of the axiom of choice. We have also used this axiom to construct the set S in the proof of Lemma B.3, and this lemma was instrumental in proving that every uncountable Borel space is Borel-isomorphic to every other uncountable Borel space (Proposition B.3 and Corollary 7.16.1). However an alternative proof of Lemma B.3 which does not require the axiom of choice is possible, but is quite lengthy and will not be given.

The countable axiom of choice is the same as the general axiom except that the index set A is required to be countable. A paraphrase of this axiom is that given any countable collection of nonempty sets, one element can be chosen from each set. We have made extensive use of this axiom, such as in the choice, for each k, of a selector φ_k in the proof of Proposition 7.50(a). Indeed, much of real analysis and topology rests on the countable axiom of choice.

Solovay [S13] has shown that if the general axiom of choice is replaced by the weaker "principle of dependent choice," which is still stronger than the countable axiom of choice, then every subset of the real line may be assumed to be Lebesgue-measurable. A slight extension of this result shows that under these conditions every subset of any Borel space may be assumed to be universally measurable. Therefore, by choice of the proper axiom system, the measurability difficulties which are the subject of Part II can be made to disappear.

It is possible to show without the use of the axiom of choice that every uncountable Borel space X contains universally measurable sets which are not limit measurable. An unpublished proof of this is due to Richard Lockhart. If both the axiom of choice and the continuum hypothesis are adopted then it follows that $\mathscr{U}_X$ has a larger cardinality than $\mathscr{L}_X$. Since for each $\alpha < \Omega$, $\mathscr{B}_X \subset \mathscr{L}_X^\alpha$ and $\mathscr{B}_X$ has cardinality at least c, so does $\mathscr{L}_X^\alpha$. On the other hand, $\mathscr{L}_X^\alpha$ is contained in $\mathscr{S}(\mathscr{L}_X^\alpha)$ and there is a universal function for $\mathscr{S}(\mathscr{L}_X^\alpha)$, so the cardinality of $\mathscr{L}_X^\alpha$ is c. Now $\mathscr{L}_X = \bigcup_{\alpha < \Omega} \mathscr{L}_X^\alpha$, and the cardinality of the set of countable ordinals is less than or equal to c, so $\mathscr{L}_X$ has cardinality c. In contrast, under the assumption of the axiom of choice and Cantor's continuum hypothesis, $\mathscr{U}_X$ contains a set F of cardinality c which has measure zero with respect to every nonatomic probability measure [H5, Chapter III, Section 14]. Thus every subset of F is also in $\mathscr{U}_X$, and the cardinality of $\mathscr{U}_X$ is at least 2^c. It follows that $\mathscr{L}_X$ must be properly contained in $\mathscr{U}_X$.

Another relevant set theoretic work is that of Gödel [G1], who showed that it is consistent with the usual axioms of set theory to assume the existence of the complement of an analytic set in the unit square whose projection on an axis is not Lebesgue-measurable. This means that it is consistent with the usual axioms to assume the existence of an analytically measurable function $f:[0,1][0,1] \to R$ such that $f^*(x) = \inf_y f(x,y)$ is not Lebesgue measurable. This places a severe constraint on the types of strengthened versions of Proposition 7.47 which might be possible.

Appendix C

The Hausdorff Metric and the Exponential Topology

This appendix develops a metric topology on the collection of closed subsets (including the empty set $\varnothing$) of a compact metric space (X, d). We denote this collection of sets by 2^X. For $A \in 2^X$ and $x \in X$, define

$$d(x, A) = \min_{a \in A} d(x, a) \qquad \text{if} \quad A \neq \varnothing, \tag{1}$$

$$d(x, \varnothing) = \operatorname{diam}(X) = \max_{y, z \in X} d(y, z). \tag{2}$$

Definition C.1 Let (X, d) be a compact metric space. The *Hausdorff metric* ρ on 2^X is defined by

$$\rho(A, B) = \max\left\{\max_{a \in A} d(a, B), \max_{b \in B} d(b, A)\right\} \qquad \text{if} \quad A, B \neq \varnothing, \tag{3}$$

$$\rho(A, \varnothing) = \rho(\varnothing, A) = \operatorname{diam}(X) \qquad \text{if} \quad A \neq \varnothing, \tag{4}$$

$$\rho(\varnothing, \varnothing) = 0. \tag{5}$$

We have written max in place of sup in (3), since every set in 2^X is compact and $d(x, A)$ is a continuous function of x for every $A \in 2^X$. To see this latter property, consider a set $A \in 2^X$. If $A = \varnothing$, then the function $d(x, A)$ is constant and hence continuous. If $A \neq \varnothing$, then for $x, y \in X$ and $a \in A$ we have

$$d(x,a) \leq d(x,y) + d(y,a).$$

By taking the infimum of both sides over $a \in A$, we obtain

$$d(x,A) - d(y,A) \leq d(x,y).$$

By reversing the roles of x and y, we have

$$|d(x,A) - d(y,A)| \leq d(x,y) \qquad \forall x, y \in X, \tag{6}$$

which shows that $d(x, A)$ is a Lipschitz continuous function of x.

It is a tedious but straightforward task to verify that $(2^X, \rho)$ is a metric space, and this is left to the reader. We will prove that $(2^X, \rho)$ is a compact metric space. We first show some preliminary facts.

If A is a (not necessarily closed) subset of X, define

$$2^A = \{K \in 2^X | K \subset A\}.$$

We define two classes

$$\mathscr{G} = \{2^G | G \text{ is an open subset of } X\}, \tag{7}$$

$$\mathscr{K} = \{2^X - 2^K | K \text{ is a closed subset of } X\}. \tag{8}$$

To aid the reader, we will continue to denote points of X by lowercase Latin letters and subsets of X by uppercase Latin letters. Uppercase script letters will be used for subsets of 2^X, except for subsets of the form 2^A as defined above. In keeping with this practice, we denote open spheres in the two spaces as follows:

$$S_\varepsilon(x) = \{y \in X | d(x,y) < \varepsilon\},$$
$$\mathscr{S}_\varepsilon(A) = \{B \in 2^X | \rho(A,B) < \varepsilon\}.$$

Finally, classes of subsets of 2^X will be denoted by boldface script letters, as in the case of $\mathscr{G}$ and $\mathscr{K}$ defined above.

The topology obtained by taking $\mathscr{G} \cup \mathscr{K}$ as a subbase in 2^X is called the *exponential topology* and an extensive theory exists for it [K2, K3]. It can be developed for a nonmetrizable topological space X, but we are interested in it only when X is compact metric. In this case, the exponential topology is the topology generated by the Hausdorff metric, as we now show.

Proposition C.1 Let (X, d) be a compact metric space and ρ the Hausdorff metric on 2^X. The class $\mathscr{G} \cup \mathscr{K}$ as defined by (7) and (8) is a subbase for the topology on $(2^X, \rho)$.

Proof We first prove that when G is open and K is closed in X, then 2^G and $2^X - 2^K$ are open in $(2^X, \rho)$. If G or K is empty, then 2^G or $2^X - 2^K$, respectively, is easily seen to be open, so we assume G and K are nonempty. Suppose A is a nonempty closed subset of X and $A \in 2^G$. (The proof for $A = \varnothing$ is trivial.) Since A is compact, is a subset of G, and $X - G$ is closed, there exists ε with $0 < \varepsilon < \operatorname{diam}(X)$ such that

$$\min_{a \in A} d(a, X - G) \geq \varepsilon. \tag{9}$$

For $B \in \mathscr{S}_\varepsilon(A)$, we have $B \neq \varnothing$ and

$$\max_{b \in B} d(b, A) < \varepsilon. \tag{10}$$

From inequalities (9) and (10) we have that $B \subset G$. Hence $\mathscr{S}_\varepsilon(A) \subset 2^G$, and 2^G must be open. Turning to the case of $2^X - 2^K$ for K closed, we let $A \in 2^X - 2^K$ be nonempty. By definition, $A \notin 2^K$, so $A - K$ contains at least one point a_0. Since $X - K$ is open, we can find $\varepsilon > 0$ for which $S_\varepsilon(a_0) \subset X - K$. For $B \in \mathscr{S}_\varepsilon(A)$, we have

$$d(a_0, B) \leq \max_{a \in A} d(a, B) < \varepsilon,$$

which implies $B \cap S_\varepsilon(a_0) \neq \varnothing$ and $B \in 2^X - 2^K$. Therefore $\mathscr{S}_\varepsilon(A) \subset 2^X - 2^K$, and $2^X - 2^K$ is open.

Having thus shown that the sets 2^G and $2^X - 2^K$ are open in $(2^X, \rho)$ when G is open and K is closed, we must now show that given any open subset $\mathscr{G}$ of $(2^X, \rho)$ and any nonempty $A \in \mathscr{G}$, we can find open sets $G_1, G_2, \ldots, G_m$ and closed sets $K_1, K_2, \ldots, K_n$ in X for which

$$A \in 2^{G_1} \cap \cdots \cap 2^{G_m} \cap (2^X - 2^{K_1}) \cap \cdots \cap (2^X - 2^{K_n}) \subset \mathscr{G}.$$

Since $\mathscr{G}$ is open in $(2^X, \rho)$, there exists $\varepsilon > 0$ such that $\mathscr{S}_\varepsilon(A) \subset \mathscr{G}$. Since A is closed in the compact set X, there exist points $\{x_1, \ldots, x_n\}$ in A such that $A \subset \bigcup_{k=1}^n S_{\varepsilon/2}(x_k)$. Let

$$G_1 = \{x \in X \mid d(x, A) < \varepsilon\}$$

and

$$K_k = X - S_{\varepsilon/2}(x_k), \qquad k = 1, \ldots, n.$$

By construction, $A \in 2^{G_1}$ and, since for each k, $A \cap S_{\varepsilon/2}(x_k) \neq \varnothing$, we have $A \in 2^X - 2^{K_k}$. Therefore

$$A \in 2^{G_1} \cap (2^X - 2^{K_1}) \cap \cdots \cap (2^X - 2^{K_n}).$$

Suppose B is another set in $2^{G_1} \cap (2^X - 2^{K_1}) \cap \cdots \cap (2^X - 2^{K_n})$. The fact that $B \in 2^{G_1}$ implies

$$\max_{b \in B} d(b, A) < \varepsilon. \tag{11}$$

If for some $a_0 \in A$ we had $d(a_0, B) \geq \varepsilon$, then we would also have $S_\varepsilon(a_0) \subset X - B$. But for some $x_k \in A$, $a_0 \in S_{\varepsilon/2}(x_k)$ and this would imply in succession $S_{\varepsilon/2}(x_k) \subset X - B$, $B \subset K_k$, and $B \notin 2^X - 2^{K_k}$. This contradiction shows that

$$\max_{a \in A} d(a, B) < \varepsilon. \tag{12}$$

Inequalities (11) and (12) establish that $\rho(A, B) < \varepsilon$, and as a consequence

$$2^{G_1} \cap (2^X - 2^{K_1}) \cap \cdots \cap (2^X - 2^{K_n}) \subset S_\varepsilon(A) \subset \mathscr{G}. \qquad \text{Q.E.D.}$$

If a cover of a space contains no finite subcover, we say the cover is *essentially infinite*. To show that $(2^X, \rho)$ is compact when X is compact, we must show that no essentially infinite open cover of 2^X exists. As a consequence of the following lemma, this will be accomplished if we can show that the subbase $\mathscr{G} \cup \mathscr{K}$ contains no essentially infinite cover. We remind the reader that a topological space in which every open cover has a countable subcover is called Lindelöf, and in metrizable spaces this property is equivalent to separability.

Lemma C.1 Let Ω be a Lindelöf space and let $\mathscr{S}$ be a subbase for the topology on Ω. If there exists an essentially infinite open cover of Ω, then there exists one which is a subset of $\mathscr{S}$.

Proof Let $\mathscr{B}$ be the base for the topology on Ω constructed by taking finite intersections of sets in $\mathscr{S}$ and let $\mathscr{C}$ be an essentially infinite open cover of Ω. Each $C \in \mathscr{C}$ has a representation $C = \bigcup_{\alpha \in A(C)} B_\alpha$, where $B_\alpha \in \mathscr{B}$ for every $\alpha \in A(C)$. The collection $\bigcup_{C \in \mathscr{C}} \{B_\alpha | \alpha \in A(C)\}$ is an essentially infinite open cover of Ω, and, by the Lindelöf property, it contains a countable, essentially infinite, open subcover $\mathscr{D} = \{B_1, B_2, \ldots\}$. Each B_k has a representation $B_k = \bigcap_{j=1}^{n(k)} S_{kj}$, where $S_{kj} \in \mathscr{S}, j = 1, \ldots, n(k)$. If for each j the cover $\mathscr{D}_j = \{S_{1j}, B_2, B_3, \ldots\}$ is not essentially infinite, then there exists a finite subcollection $\bar{\mathscr{D}}_j$ which also covers Ω. But then

$$\{B_1\} \cup \left[\bigcup_{j=1}^{n(1)} (\bar{\mathscr{D}}_j - \{S_{1j}\})\right] \subset \mathscr{D}$$

is a finite subcover of Ω. This contradiction implies that for some index j_0, the cover $\mathscr{D}_{j_0}$ is essentially infinite. Denote $R_1 = S_{1j_0}$. In general, given $R_1, R_2, \ldots, R_n$ in $\mathscr{S}$ such that $B_k \subset R_k$, $k = 1, \ldots, n$, and $\{R_1, R_2, \ldots, R_n, B_{n+1}, B_{n+2}, \ldots\}$ is an essentially infinite open cover of Ω, we can use the preceding argument to construct $R_{n+1} \in \mathscr{S}$ for which $B_{n+1} \subset R_{n+1}$ and $\{R_1, R_2, \ldots, R_n, R_{n+1}, B_{n+2}, B_{n+3}, \ldots\}$ is an essentially infinite open cover of Ω. The collection $\{R_1, R_2, \ldots\}$ is an essentially infinite open cover contained in $\mathscr{S}$. Q.E.D.

Proposition C.2 Let (X, d) be a compact metric space and ρ the Hausdorff metric on 2^X. The metric space $(2^X, \rho)$ is compact.

Proof We first show that $(2^X, \rho)$ is separable. Since (X, d) is compact, it is separable. Let D be a countable dense subset of X and let

$$\mathscr{C} = \{\overline{S_{1/n}(x)} \mid x \in D,\ n = 1, 2, \ldots\}.$$

Let $\mathscr{D}$ consist of finite unions of sets in $\mathscr{C}$. Then $\mathscr{D}$ is countable and, as we now show, is dense in $(2^X, \rho)$. Given $A \in 2^X$ and $\varepsilon > 0$, choose a positive integer n satisfying $2/n < \varepsilon$. The collection of sets $\{S_{1/n}(x) \mid x \in D\}$ covers the compact set A, so there is a finite subcollection $\{S_{1/n}(x) \mid x \in F\}$ which also covers A and which satisfies $S_{1/n}(x) \cap A \neq \varnothing$ for every $x \in F$. The set $B = \bigcup_{x \in F} \overline{S_{1/n}(x)}$ is in $\mathscr{D}$ and satisfies $\rho(A, B) < \varepsilon$.

As a result of Proposition C.1, Lemma C.1, and the separability of $(2^X, \rho)$, to show that $(2^X, \rho)$ is compact we need only show that every open cover of 2^X which is a subset of $\mathscr{G} \cup \mathscr{K}$ contains a finite subcover of 2^X. Thus let $\{G_\alpha \mid \alpha \in A\}$ be a collection of open sets and $\{K_\beta \mid \beta \in B\}$ a collection of closed sets in X, and suppose

$$2^X = \left[\bigcup_{\alpha \in A} 2^{G_\alpha}\right] \cup \left[\bigcup_{\beta \in B} (2^X - 2^{K_\beta})\right].$$

Define the closed set $K_0 = \bigcap_{\beta \in B} K_\beta$. By definition, $K_0 \notin \bigcup_{\beta \in B}(2^X - 2^{K_\beta})$, so $K_0 \in \bigcup_{\alpha \in A} 2^{G_\alpha}$. Thus for some $\alpha_0 \in A$, we have $K_0 \in 2^{G_{\alpha 0}}$, i.e., $K_0 \subset G_{\alpha_0}$. This means that

$$X - G_{\alpha_0} \subset X - K_0 = \bigcup_{\beta \in B} (X - K_\beta),$$

and since $X - G_{\alpha_0}$ is compact, there exists a finite set $\{\beta_1, \beta_2, \ldots, \beta_n\} \subset B$ for which

$$X - G_{\alpha_0} \subset \bigcup_{k=1}^{n} (X - K_{\beta_k}). \tag{13}$$

To complete the proof, we show

$$2^X = 2^{G_{\alpha 0}} \cup \left[\bigcup_{k=1}^{n} (2^X - 2^{K_{\beta_k}})\right].$$

If $C \in 2^X$, then either $C \subset G_{\alpha_0}$, in which case $C \in 2^{G_{\alpha 0}}$, or else $C \cap (X - G_{\alpha_0}) \neq \varnothing$. In the latter case, (13) implies that for some k, $C \cap (X - K_{\beta_k}) \neq \varnothing$, i.e., $C \in 2^X - 2^{K_{\beta_k}}$. Q.E.D.

We now develop some convergence notions in $(2^X, \rho)$. Let $\{A_n\}$ be a sequence of sets in 2^X. Define

$$\overline{\lim_{n \to \infty}}\, A_n = \left\{x \in X \,\middle|\, \liminf_{n \to \infty} d(x, A_n) = 0\right\}, \tag{14}$$

$$\underline{\lim_{n \to \infty}}\, A_n = \left\{x \in X \,\middle|\, \limsup_{n \to \infty} d(x, A_n) = 0\right\}. \tag{15}$$

For example, if $X = [-1, 1]$ and $A_n = \{(-1)^n\}$, we have $\overline{\lim}_{n\to\infty} A_n = \{-1, 1\}$ and $\underline{\lim}_{n\to\infty} A_n = \varnothing$. If $X = [-1, 1]$ and $A_n = [-1/n, 1/n]$, we have

$$\overline{\lim_{n\to\infty}} A_n = \underline{\lim_{n\to\infty}} A_n = \{0\}.$$

Clearly we have $\underline{\lim}_{n\to\infty} A_n \subset \overline{\lim}_{n\to\infty} A_n$. It is also true that $\overline{\lim}_{n\to\infty} A_n$ and $\underline{\lim}_{n\to\infty} A_n$ are closed. To see this for $\overline{\lim}_{n\to\infty} A_n$, let $\{x_m\}$ be a sequence in $\overline{\lim}_{n\to\infty} A_n$ converging to x. Then from (6) we have for each m

$$\liminf_{n\to\infty} d(x, A_n) \le d(x, x_m) + \liminf_{n\to\infty} d(x_m, A_n) = d(x, x_m),$$

and since $d(x, x_m)$ can be made arbitrarily small by choosing m sufficiently large, we conclude that $x \in \overline{\lim}_{n\to\infty} A_n$. Replace $\liminf_{n\to\infty}$ by $\limsup_{n\to\infty}$ in the preceding argument to show that $\underline{\lim}_{n\to\infty} A_n$ is closed.

If $\overline{\lim}_{n\to\infty} A_n = \underline{\lim}_{n\to\infty} A_n$, we denote their common value by $\lim_{n\to\infty} A_n$. This notation is justified by the following proposition.

Proposition C.3 Let (X, d) be a compact metric space and ρ the Hausdorff metric on 2^X. Let $\{A_n\}$ be a sequence in 2^X. Then

$$\overline{\lim_{n\to\infty}} A_n = \underline{\lim_{n\to\infty}} A_n = A \tag{16}$$

if and only if

$$\lim_{n\to\infty} \rho(A_n, A) = 0. \tag{17}$$

Proof Assume for the moment that $A \neq \varnothing$ and suppose (16) holds. Then for each x in the compact set A, $d(x, A_n) \to 0$ as $n \to \infty$. Given $\varepsilon > 0$, let $\{x_1, \ldots, x_k\}$ be points of A such that the open spheres $S_{\varepsilon/2}(x_j), j = 1, \ldots, k$ cover A. Choose N large enough so that for all $n \ge N$

$$d(x_j, A_n) \le \varepsilon/2, \qquad j = 1, \ldots, k.$$

Now use the Lipschitz continuity [cf. (6)] of the function $x \to d(x, A_n)$ to conclude that

$$d(x, A_n) \le \varepsilon \qquad \forall x \in A.$$

This implies that

$$\lim_{n\to\infty} \max_{x\in A} d(x, A_n) = 0.$$

This equation and (3) imply that (17) will follow if we can show

$$\lim_{n\to\infty} \max_{y\in A_n} d(y, A) = 0. \tag{18}$$

If (18) fails to hold, then for some $\varepsilon > 0$ there exists a sequence $y_k \in A_{n_k}$ such that $n_1 < n_2 < \cdots$ and

$$d(y_k, A) \geq \varepsilon \qquad \forall k. \tag{19}$$

The compactness of X implies that $\{y_k\}$ accumulates at some $y_0 \in X$ which, by (19) and the continuity of $x \to d(x, A)$, must satisfy $d(y_0, A) \geq \varepsilon$. But $y_0 \in \overline{\lim}_{n\to\infty} A_n$ by (14), and this contradicts (16). Hence (18) holds.

Still assuming $A \neq \varnothing$, we turn to the reverse implication of the proposition. If (17) holds, then

$$\lim_{n\to\infty} d(x, A_n) = 0 \qquad \forall x \in A, \tag{20}$$

and

$$\lim_{n\to\infty} \max_{y \in A_n} d(y, A) = 0. \tag{21}$$

Equation (20) implies that

$$A \subset \underline{\lim}_{n\to\infty} A_n \subset \overline{\lim}_{n\to\infty} A_n. \tag{22}$$

If $x \in \overline{\lim}_{n\to\infty} A_n$, then by definition there exists a sequence $y_k \in A_{n_k}$ such that $n_1 < n_2 < \cdots$ and

$$\lim_{k\to\infty} d(x, y_k) = 0. \tag{23}$$

We have from (6) that

$$d(x, A) \leq d(x, y_k) + d(y_k, A),$$

and, letting $k \to \infty$ and using (21) and (23), we conclude $d(x, A) = 0$. Since A is closed, this proves $x \in A$ and

$$\overline{\lim}_{n\to\infty} A_n \subset A. \tag{24}$$

Combine (22) and (24) to obtain (16).

Assume finally that $A = \varnothing$. If (16) holds, then all but finitely many of the sets A_n must be empty, for otherwise one could find $y_k \in A_{n_k}, n_1 < n_2 < \cdots$, and $\{y_k\}$ would accumulate at some $y_0 \in \overline{\lim}_{n\to\infty} A_n$. If all but finitely many of the sets A_n are empty, then (5) implies that (17) holds. Conversely, if (17) holds and $A = \varnothing$, then (4) implies that all but finitely many of the sets A_n are empty. Equation (16) follows from (2), (14), and (15). Q.E.D.

For the proof of Proposition 7.33 in Section 7.5 we need the concept of a function which is upper semicontinuous in the sense of Kuratowski, or in abbreviation, upper semicontinuous (K).

Definition C.2 Let Y be a metric space and X a compact metric space. A function $F: Y \to 2^X$ is *upper semicontinuous* (K) if for every convergent sequence $\{y_n\}$ in Y with limit y, we have $\overline{\lim}_{n\to\infty} F(y_n) \subset F(y)$.

The similarity of Definition C.2 to the idea of an upper semicontinuous real or extended real-valued function is apparent [Lemma 7.13(b)]. Although we will not discuss functions which are lower semicontinuous (K), it is interesting to note that such a concept exists and has the obvious definition, namely, that the function $F: Y \to 2^X$ is *lower semicontinuous* (K) if for every convergent sequence $\{y_n\}$ in Y with limit y, we have $\underline{\lim}_{n\to\infty} F(y_n) \supset F(y)$. It can be seen from Proposition C.3 that a function $F: Y \to 2^X$ is continuous in the usual sense (where 2^X has the exponential topology) if and only if it is both upper and lower semicontinuous (K). We carry the analogy with real-valued functions even farther by showing that an upper semicontinuous (K) function is Borel-measurable, and the remainder of the appendix is devoted to this.

Lemma C.2 Let Y be a metric space and X a compact metric space. If $F: Y \to 2^X$ is upper semicontinuous (K), then for each open set $G \subset X$, the set

$$\{y \in Y | F(y) \subset G\} = F^{-1}(2^G) \tag{25}$$

is open.

Proof The openness of $F^{-1}(2^G)$ for every open G is in fact equivalent to upper semicontinuity (K), but we need only the weaker result stated. To prove it, we show that for G open, the set $F^{-1}(2^X - 2^G)$ is closed. If $\{y_n\}$ is a sequence in this set with limit $y \in Y$, then

$$F(y_n) \cap (X - G) \neq \varnothing, \qquad n = 1, 2, \ldots,$$

and so there exists a sequence $\{x_n\}$ in the compact set $X - G$ such that $x_n \in F(y_n)$, $n = 1, 2, \ldots$. This sequence has an accumulation point $x \in X - G$, and, by (14), $x \in \overline{\lim}_{n\to\infty} F(y_n)$. The upper semicontinuity (K) of F implies $x \in F(y)$, and so $F(y) \cap (X - G) \neq \varnothing$, i.e., $y \in F^{-1}(2^X - 2^G)$. Q.E.D.

Proposition C.4 Let Y be a metric space, (X, d) a compact metric space, and let 2^X have the exponential topology. Let $F: Y \to 2^X$ be upper semicontinuous (K). Then F is Borel-measurable.

Proof If $F: Y \to 2^X$ is upper semicontinuous (K) and G is an open subset of X, then $F^{-1}(2^G)$ is Borel-measurable in Y by Lemma C.2. If K is a closed subset of X, define open sets $G_n = \{x | d(x, K) < 1/n\}$. We have $K = \bigcap_{n=1}^{\infty} G_n$, and so a closed set A is a subset of K if and only if $A \subset G_n$, $n = 1, 2, \ldots$.

This implies $2^K = \bigcap_{n=1}^{\infty} 2^{G_n}$, and

$$F^{-1}(2^K) = \bigcap_{n=1}^{\infty} F^{-1}(2^{G_n})$$

is a G_δ-set, thus Borel-measurable in Y. It follows that for any set $\mathscr{G}$ in the subbase $\mathscr{G} \cup \mathscr{K}$ for the exponential topology on 2^X, $F^{-1}(\mathscr{G})$ is Borel-measurable in Y. By Proposition 7.1, any open set in 2^X can be represented as a countable union of finite intersections of sets in $\mathscr{G} \cup \mathscr{K}$ and so its inverse image under F is Borel-measurable. Q.E.D.

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Table of Propositions, Lemmas, Definitions, and Assumptions

Chapter 6

Chapter 7

Chapter 10

Chapter 11

Appendix A

Appendix B

Appendix C

Index

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U

W