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ALFRED P. SLOAN SCHOOL OF MANAGEMENT

LINEAR PROGRAMMING FOR FINANCIAL PLANNING

UNDER UNCERTAINTY

348-68

Stewart C. Myers

MASSACHUSETTS
INSTITUTE OF TECHNOLOGY
50 MEMORIAL DRIVE
CAMBRIDGE, MASSACHUSETTS 02139

Sloan School of Management

Massachusetts Institute of Technology

Cambridge, Massachusetts 02139

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The purpose of this paper is to propose, justify and explain the properties of a class of linear programming (hereafter "LP") approaches to long-term, corporate financial planning under uncertainty. The models discussed are novel in the following respects.

1. They are directly based on a theory of market equilibrium under uncertainty. Thus the capabilities of the model to deal with choice among risky assets and liabilities can be rigorously justified, assuming that the firm's objective is to maximize share price. Past linear programming models have been con-structed assuming certainty, and have dealt with some aspects of uncertainty through heuristic modifications.[1]

2. The models yield simultaneous solutions for the firm's optimal financing and investment decisions. The financing decision is not considered "with the investment decision given," nor vice-versa.

3. Some practical difficulties associated with the cost of capital concept are avoided. The traditional weighted average cost of capital does not appear in these LP models.

The first two characteristics should lead to some interest in the models as theory; the third, along with the ease of solution of

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[1] Probably the most important contributions are those of Weingartner [24] and Charnes, Cooper and Miller [4]. See also Weingartner's survey article [23].

LP problems, should generate interest in the models as practical decision-making tools. It is too soon, of course, to assess their ultimate fruitfulness in either respect.

The paper is organized as follows. The general linear format is explained in the next section. The key assumption justifying it is that the structure of security prices at equilibrium is best described by the class of security valuation models which imply risk-independence of financing and investment options. The following section examines a simple model in detail, and contrasts the LP approach with "traditional" approaches using the cost of capital. The outlines of a more comprehensive model are presented in the third section. Most aspects of this model are reasonably realistic, although it is not possible to deal with all aspects of financial management in an LP format. A concluding section summarizes the model's deficiencies. An appendix documents some practical difficulties associated with the cost of capital concept.

This working paper is a preliminary analysis, not as well-written as insightful as it should be. A revised treatment will follow.

I. THE LINEAR FORMAT FOR FINANCIAL PLANNING

We will consider the firm's financial planning problem in the following terms. The firm begins with a certain initial package of assets and liabilities. For a brand-new firm, this may be simply money in the bank and stock outstanding. For a going concern, the

package will be much more complicated. Any firm, however, has the
opportunity to change the characteristics of its initial package
by transactions in real or financial assets. The problem is to de-
termine which set of transactions for the initial period will maxi-
mize the firm's stock price.

We will be concerned primarily with long-lived assets and
liabilities, so the optimal transactions for the initial period will
reflect the firm's opportunities and strategy in subsequent periods.
Therefore, I have characterized the firm's problem as long-range
financial planning, even though tomorrow's decisions do not have to
be made today.

Assuming linearity, the firm's objective function is:

$$(1) \qquad \psi = \sum_{j=1}^{n} x_j A_j + \sum_{j=1}^{m} y_j F_j$$

where ψ = change in stock price

x_j = decision variable for the jth investment project --
i.e., j^{th} real asset option. $x_j = 1$ indicates
that the project is accepted.

A_j = change in stock price if project j is accepted;
in other words, project j's present value.

y_j = decision variable for the jth financing option.
$y_j = 1$ means that one dollar of financing is ob-
tained from the jth source.

F_j = the change in stock price per dollar of financing
obtained from source j.

What is implied by stating the firm's objective in this way?
First, we assume that acceptance of option j leads to a definite

change in stock price. In other words, a financing or investment option with uncertain returns does <u>not</u> have an uncertain value; the "market's" preferences are well-defined.[2]

Second, we assume that the change in stock price due to accepting option j is <u>independent</u> of management's decisions regarding other investment or financing options. Clearly, this assumption is crucial to the argument and requires close examination.

Are the Investment Options Mutually Interdependent?

If A_j is to be independent of decision variables for other projects, then the cash flows of project j cannot be causally related to what other assets are acquired. If this is true for all projects, 1, 2, . . . , n, then all are <u>physically</u> <u>independent</u> in the same sense G.M. and Ford shares are independent from the point of view of an investor: although these security's returns may be statistically related, Ford's actual future prices and dividends are not affected by whether or not the investor buys G.M. stock.

Assuming physical independence means that the linear format cannot not deal directly with an important class of capital budgeting problems. Suppose, for example, that investment options 1 and 2 are, respectively, a fleet of new trucks and a computer. If the trucks are purchased, then purchase of the computer will allow management to schedule usage of the trucks more efficiently. For this reason, the change in stock price if both projects 1 and 2 are accepted is greater than the sum of their present values

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[2]This assumption is innocuous, but worth stating because of the common assumption that present value should be regarded as a random variable under uncertainty.

separately considered. An interaction effect exists which cannot be treated directly in the LP format.

Writing the objective function as Eq. (1) also assumes that projects are risk-independent, in the sense that there are no statistical relationships among projects' returns such that some combinations of projects affect stock price by an amount different than the sum of their present values considered separately. In particular, risk-independence implies that there is no advantage to be gained by corporate diversification.

I have shown elsewhere[3] that risk-independence is a necessary condition for equilibrium in security markets. Naturally the proof rests on certain assumptions about the markets, of which the following are most important.

1. That equilibrium security prices conform to the time-state-preference model of security valuation, advanced by me in still another paper [14], or to certain other models which also imply risk-independence.[4]

2. That the risk characteristics of all investment options open to the firm are "equivalent" to those of securities or portfolios obtainable in the market. Two assets are equivalent when (all) investors are indifferent to holding one or the other, other factors (e.g., the scale of the securities' returns) being equal.[5]

3. That markets are perfect.

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[3]Myers [13].

[4]Although there is some controversy, it appears that the Sharpe-Lintner model of security valuation under uncertainty also implies risk independence. See [20], [9].

Lintner, however, has argued that investment projects are not risk-independent in his model -- specifically, that "the problem of determining the best capital budget of any given size is formally identical to the solution of a security portfolio analysis." [7], p. 65.

[5]For further explanation of this concept, see [13], pp.

The proof of risk-independence just cited was obtained by considering a bundle of risky assets, denoted by A, and two additional "projects" B and C. No restriction was placed on the distribution of B or C's cash flow over time, or on their risk characteristics. It was then shown, given the assumptions stated just above, that

$$(2) \qquad P_{AB} - P_A = P_{ABC} - P_{AC},$$

where P_A is the price per share if neither B or C is accepted, P_{AB} the price if only B is accepted, and so on. Equation (2) establishes that the change in stock price if B is adopted is independent of whether C is also adopted, and therefore that the projects are risk-independent.

Whether risk-independence is a property of actual security markets is a question that cannot be answered here, although it seems reasonable to expect at least a tendency toward this result. In any case, the implications of risk-independence are worth considering. Therefore, we shall assume it to exist for purposes of this paper.

Are Financing Options Mutually Independent?

If no restrictions are placed on the risk characteristics or pattern over time of B or C's cash flows, then Eq. (2) applies as well to financial assets as to real ones. Project B can be regarded as a bond issue, and C as a stock issue, without changing the proof in the slightest. Having assumed risk-independence among real assets,

it is no great step to assume further that financial assets are like-
wise risk-independent.

Physical independence among financial assets is another matter.
It is commonplace that the interest and principal payments on bonds
are affected by the size of the firm's equity base. A highly
levered firm may encounter difficulties servicing its debt, and
creditors will demand a higher promised yield in compensation. Con-
versely, returns to equity depend on commitments to creditors.
Therefore "debt," regarded as a single financing option, is not physi-
cally independent of equity issues.

The general strategy for handling this difficulty is to specify
a range of financing options where necessary and to add constraints
ruling out options made inappropriate by investment or other financ-
ing choices. Many different options can be grouped under the head-
ing "long term debt," for example, ranging from practically riskless
to highly speculative ones. Which of these options are feasible de-
pends on other financing and investment decisions. But financing
options can be treated as risk-independent if defined in this way.

However, before going into further detail on the constraints of
the LP problem, it may be helpful to give some concrete meaning to
the concept "present value" for financing options. (We usually ex-
amine the cost of financing, measured by the expected rates of re-
turn required by investors.)

Consider the firm's equilibrium stock price, $P(0)$, at the start
of period $t = 0$. $P(0)$ is the present value of the stream of dividends

$R(0)$, $R(1)$, . . . , $R(t)$, etc. Adoption of project j changes the dividend stream by B's cash flows, a_{j1}, a_{j2}, ... The cash flows are, of course, measured net of corporate income taxes.

Assuming project B is a real asset, the change in $P(0)$, or present value, associated with it is usually computed as

$$(3) \qquad PV_j = \Delta P(0) = \sum_{t=0}^{\infty} \frac{\overline{a}_{jt}}{(1 + \rho(j))^t}$$

where $\qquad \overline{a}_{jt}$ = the mean of $\widetilde{a}'_{jt}$, and

$\rho(j)$ = the required rate of return on a stream of cash flows with project j's risk characteristics.

At equilibrium, $\rho(j)$ is determined by the rate of return obtainable on securities with risk characteristics similar to project j's.

The rate $\rho(j)$ is not a weighted average cost of capital. The project's cash flows are assumed to affect the firm's dividends directly, without modification by any intervening financing arrangements. The relevant question is, "What is the market value of project j?" not "What is the market value of project j when financed, say, by 30 percent debt?"

Exactly the same procedure can be applied to determine the present value of financing options. They are unusual only in that the initial cash flow (f_{jo}) will usually be positive and future expected cash flows $(\widetilde{f}_{jt})$ negative or zero. As for real assets, the appropriate discount is determined by expected equilibrium rates of return

on other (financial) assets with similar risk characteristics.

The cash flows of the financing option are also assumed to af-
fect the firm's dividends directly -- there is no presumption that
proceeds of the financing are used to finance real assets.

Thus the treatments of real and financial assets are symmetri-
cal.

As a practical matter, computation of the present value per
dollar of a financing option is substantially eased by using the
following observation as a benchmark: in perfect markets, the pres-
ent value of all financing options is zero.

The proof of this statement is not at all difficult. By defini-
tion, all participants in perfect markets have access to the same
trading opportunities at the same prices, and no single participant
affects prices by his own actions. Thus a firm wishing to issue a
bond, for example, is forced to do so on exactly the same terms as
other firms (or individuals). Our hypothetical firm will be able to
issue bonds priced to yield the equilibrium market rate established
for bonds with its risk characteristics -- no more, no less. But
then the expected yield on the new bonds is exactly the (discount)
rate and their present value is zero.

This argument clearly can be applied to any type of generally
traded financial asset.

Financial markets are not absolutely perfect, of course, but it
is easiest to start with the presumption that $F_j = 0$, and then con-
how imperfections may change this figure. Some examples follow.

1. Costs of issue should be substracted from present value. In practice, this reduces the present value of both bond and stock issues, stock issues by the greater amount.

2. However, the tax advantages of corporate debt increase its present value. Thus F_j should reflect the present value of the tax "rebates" associated with a debt option j.

3. If there are special advantages to corporate debt vs. "personal" leverage, as critics of the Modigliani-Miller (MM) propositions have contended,[6] and if these advantages are sufficient to negate the propositions, then a further adjustment is necessary. The incremental dividends associated with the debt issue would tend to be discounted at a rate lower than the issue's expected yield.

4. Existence of a "clientele effect" reduces the present value of new stock issues. The effect exists if present stockholders value their holdings more highly than potential stockholders, who must therefore be paid a higher rate of return than present shareholders would require.[7]

    Measurement of the clientele effect is difficult because most firms use funds obtained from stock issues to undertake investment projects. Its impact on the present value of the issue is therefore the price discount necessary to market the shares less the present value of projects undertaken contingent on the issue. That is, if the projects' present value is positive, then the observed discount understates the impact of the clientele effect.

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[6]There has been an extended controversy on this matter: see Robichek & Myers [17], Ch. 3, for a review. Most of the points made by MM's opponents may be found in Durand [5].

[7]See Lintner [6].

Are Financing Options Independent of Investment Options, and Vice Versa?

The proof of risk-independence again applies, but certain inter-relationships must nevertheless be allowed for.

1. The firm's choice of assets determines the risk characteristics of its aggregate liabilities. Or, from a different point of view, we can say that the firm potentially can choose among a large number of financial assets, but that many combinations of real and financial options are infeasible -- e.g., Fledgling Electronics Corporation could not enjoy a 2:1 debt-equity ratio and simultaneously issue triple-A bonds.

2. The firm's financing strategy can affect the returns produced by its real assets. Most dramatic is the case of bankruptcy due to large debt-servicing requirements. The real costs apparently associated with bankruptcy[8] may be attributed to financing decisions, providing bankruptcy could have been avoided by a more conservative financial structure.

Investors will take the likelihood of bankruptcy into account in assessing the value of a firm's securities. They will also consider the possibility that management will incur real costs in avoiding bankruptcy if it seems imminent in some future contingency. Consequently, firms' market values will reflect the "financial risk" it undertakes. This is inconsistent with a linear objective function, since financial risk depends on the firm's overall financing and investment strategy, not simply on the individual options undertaken.

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[8]See Baxter [1], and Robichek and Myers [18], esp. pp. 15-22.

The LP format can be preserved in spite of these difficulties only if constraints can be used to express the interrelationships. The simplest arrangement is to require that total debt not exceed debt capacity, which in turn is related to the risk characteristics of the firm's real assets and the amount of equity backing provided. This provides a framework for assessing financial risk and, simultaneously, a means to insure that the optimal financing-investment package is internally consistent.

Of course, these debt capacity constraints can be written in a variety of specific forms. It probably is best to treat debt capacity not as an absolute constraint, but as a threshold, beyone which the present value of costs associated with possible bankruptcy are taken into account. On a still more sophisticated level, the decision-maker can easily specify a serie of thresholds. He can also choose among expressing the constraints in terms of stocks and flows, among various means of relating debt capacity to the firm's investment decision, and so on. Some examples of specific debt capacity constraints will be discussed in greater detail in Sections II and III.

A Comment on Risk-Independence and the Modigiliani-Miller Propositions

The statement that financing and investment decisions are risk-independent is closely related to the well-known Modigliani-Miller (MM) propositions. These require that "the cutoff rate [minimum permissable rate of return] for investment in the firm . . . will be completely unaffected by the type of security used to finance the investment."[9]

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[9][12], p. 288, MM intend this statement to apply only in a no-tax world. When corporate taxes exist, the cutoff rate in their model depends on financial leverage.

In the LP model MM's statement is true regardless of the tax environment -- true, that is, in terms of the objective function. Eq. (1) implies that the discount rates applied to financing and investment options are mutually independent. However, the firm's financing and investment decisions are related through the LP constraints. This will be made more clear in Sections II and III below.

As may be expected, quite similar arguments support risk-independence and the MM propositions. However, the two hypotheses are not identical. MM assert not only that financing and investment options are (risk) independent, but also that the present value of debt is zero (in a tax-free world) or equal to the present value of debt-related tax savings (in actuality). Their hypothesis is, therefore, disproved if the present value of debt financing is observed to be different from the present value of the associated tax savings. However, this observation would not necessarily imply that financing and investment options are risk-dependent. In other words, proof of the MM propositions is sufficient, but not necessary to prove risk-independence.[10]

Admittedly, disproof of the MM propositions could raise reasonable doubts about the existence of risk-independence, because the assumed market processes on which the two hypotheses are based are similar.

II. ANALYSIS OF A SIMPLE LP MODEL

The rudimentary example discussed in this section assumes the firm has open to it only one financing option, simply "debt," and that its financing problem is only to choose the stock of debt outstanding in each period from $t = 0$ to $t = H$, the horizon. However, the planned stock of debt cannot exceed "debt capacity" in any of these periods.

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[10]To see this, remember that the proof of risk independence is identical regardless of whether real assets, financial assets or both are considered. Since saying that investment proposals are risk-independent says nothing about whether the proposals' present values are large or small, saying that financing options are risk-independent likewise says nothing about which of these options are most valuable.

The LP problem is:

(4) $\underset{x_j, \ y_j}{\text{Max}}$ $\psi = \sum_{j=1}^{n} x_i A_i + \sum_{t=0}^{H} y_t F_t$

subject to:

$$\phi_t = Y_t - Z_t \leq 0, \qquad t = 0, \ 1, \ \ldots, \ H,$$

$$\phi_j = x_j - 1 \leq 0, \qquad j = 1, \ 2, \ \ldots, \ n.$$

Here Z_t, debt capacity for t, is assumed equal to the sum of debt capacities, Z_{jt}, of accepted projects at period t. Thus $Z_t = \sum_{j=1}^{n} x_j Z_{jt}$.

The Kuhn-Tucker conditions for the optimal solution are as follows.

(5) $\delta \psi / \delta x_j - \sum_{t=0}^{H} \lambda_t \frac{\delta \phi_t}{\delta x_j} - \lambda_j \frac{\delta \phi_j}{\delta x_j} \leq 0, \quad \text{all } j,$

$$\delta \psi / \delta y_t - \frac{\delta \phi_t}{\delta y_t} \leq 0, \quad \text{all } t.$$

The variables λ_t and λ_j are the imputed costs associated with the constraints ϕ_t and ϕ_j, respectively. Substituting for the partial derivatives, the conditions are

(5a) $A_j + \sum_{t=0}^{H} \lambda_t Z_{jt} - \lambda_j \leq 0,$

$$F_t - \lambda_t \leq 0 \ .$$

We assume, further, that corporate income is taxed, so that $F_t > 0$ for all t. Obviously the optimal solution will include as much debt as possible in every future period, and the constraints ϕ_t will be binding. The Kuhn-Tucker conditions also require, therefore, that $F_t - \lambda_t = 0$, or $F_t = \lambda_t$. This supports the further simplification

$$(6) \qquad A_j + \sum_{t=0}^{H} Z_{jt}F_t - \lambda_j \leq 0.$$

Equation (6) implies that the contribution of project j to stock price is measured by A_j, the "intrinsic" value of the project plus the present value of the additional debt the project supports. If $A_j + \sum Z_{jt}F_t > 0$ then the project should be accepted (if so, $\lambda_j > 0$ and Eq. (6) is an equality); if $A_j + \sum Z_{jt}F_t < 0$ then the project should be rejected (if so, $\lambda_j = 0$ and Eq. (6) is an inequality).

In a general way, this is equivalent to the usual doctrine that the weighted average cost of capital is a declining function of financial leverage, providing that reasonable debt limits are not exceeded. This doctrine implies that leveraged firms can undertake less valuable projects than unleveraged firms, which Eq. (6) also implies.

Nevertheless, there are important differences between even this simple LP model and the cost of capital approach. The cost of capital is usually computed as a single number reflecting (1) the risk characterics of the firm's existing assets and (2) the firm's existing financial structure, presumably appropriate to existing assets. This figure is used directly as a standard of profitability for new assets with risk characteristics similar to existing ones, and rather arbitrary adjustments are made for assets with dissimilar risk characteristics.

It is not easy to arrive at the correct adjusted rate purely by judgment, however. The adjustment should reflect not only (1) the risk characteristics of the project in question, but also (2) the amount of debt it will support (presumably riskier projects support less debt). The second factor is usually ignored.

Despite wide use of the cost of capital concept. There are only a few attempts[11] to provide a logically complete procedure for arriving at the required adjustments. In contrast, the LP approach takes projects' risk characteristics and debt capacities into account simultaneously and auto- matically. This is evident from the conditions for the optimal solution.

Further Comparison of LP and Cost of Capital Approaches

It will be of some interest to give a more precise idea of the range of situations in which the LP and cost of capital approaches are equivalent.

Starting with the simple LP model just described, we make three further assumptions.

1. That all investment projects under consideration are perpetuities. Thus $A_j = \bar{a}_j/$ (j) - I_j, where I is the initial investment, $\bar{a}_j$ is the ex- pected cash return required by the market for assets with j's risk characteristics.

2. That projects' debt capacities are the same in all future periods. Thus $Z_{jt} = Z_j$, a constant for all t.[12]

3. That the MM propositions hold. Thus the present value of the dollar's

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[11]See Solomon [19] and Tuttle and Litzenberger [22].

[12]Is this a reasonable assumption? It is hard to say because the concept "debt capacity" is unexplained. However, note that use of a constant risk- adjusted discount rate implicitly assumes that uncertainty associated with the cash flows $\bar{a}_{jt}$ increases with t. See [16]. Thus is could be argued that a normal project's debt capacity (assessed with information available at t=0) declines with time.

worth of debt outstanding in period t is the tax saving in t discounted to the present; $F = \dfrac{i\,T_c}{(1+i)^t}$ where T_c is the corporate income tax rate and i is the bondholders' required rate of return.

Under these assumptions, the optimal solution requires

$$\frac{\overline{a}_j}{\rho(j)} - I_j + Z_j \sum_{t=1}^{H} \frac{iT_c}{(1+i)^t} - \lambda_j \leq 0.$$

As H approaches infinity, the project's contribution, to stock price or "adjusted present value" is

$$(7) \qquad APV_j = \frac{\overline{a}_j}{\rho(j)} - I_j + Z_j T_c$$

The project's APV is positive only if its expected rate of return $\overline{a}_j/I_j$, is greater than the <u>cutoff rate</u> ρ^*; that is if:

$$(8) \qquad \overline{a}_j/I_j > \rho^* = \rho(j)(1-d_j T_c),$$

where $d_j = Z_j/I_j$.

This is exactly the cutoff rate recommended by MM, assuming project j has risk characteristics similar to the firm's existing assets.[13] Further,

$$(9) \qquad \rho^* = d_j(1-T_c)i + (1-d_j)k,$$

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[13]The MM propositions imply [12, p. 268] that V, the aggregate market value of the firm is

$$V = \overline{a}/\rho + T_c D,$$

where $\overline{a}$ is the expected after-tax cash flow of the firm, ρ the capitalization rate appropriate to this stream and D the stock of debt (consols) currently outstanding. A small increase dI in the scale of the firm's assets implies

$$\frac{dV}{dI} = \frac{1}{\rho} \cdot \frac{d\overline{a}}{dI} + T_c \frac{dD}{dI} .$$

This action is acceptable if $dV/dI > dI/dI = 1$. Thus the minimum acceptable rate of return $d\overline{a}/dI$ is

$$\rho^* = \frac{da}{dI} = \rho(1-T_c \frac{dD}{dI}),$$

which is equivalent to Eq. (8).

the weighted average, after-tax cost of capital, if d_j and $(1-d_j)$ are equal to the long-run propositions of debt and equity in the firm's capital structure, and if k, the expected rate of return on the firm's stock, behaves as MM predict.

If the MM propositions do not apply, use of a weighted average cost of capital corresponds to a somewhat different LP model. The only major change in assumptions is that F_t, the present value of debt, would be different than the MM propositions indicate.

The point is that these cases, in which the MM and/or weighted average cost of capital approaches arrive at the same present value for a project as the LP approach, is a rather special one. This does not imply the cost of capital approaches (either MM or weighted average) always lead to wrong decisions when the various special assumptions they require are relaxed. Nevertheless, their use can lead to wrong decisions in situations where the LP approach serves perfectly well.

The possible errors stemming from use of the cost of capital concept when investment options are not perpetuities are illustrated in Appendix A.

OUTLINE OF A "REALISTIC" MODEL

We now examine a model that is tolerably -- but, of course, not perfectly -- realistic. This is an interesting exercise because of the challenge of making the concept discussed above practically useful, and also because the discussion provides insight into some general problems of financial management.

The core of the model is shown in Table 1. Definitions of the

variables used in it appear in Table 2.

Characteristics of the "Core" Model

We can now explain the model part by part.

1. Compared to Eq. (4), the objective function has been ex-
   panded in three ways. First, the likelihood of a large
   number of distinct financing options is allowed for. In
   practice, financing options must be distinguished not only
   by the time at which financing is obtained, but also by
   the type of financing obtained -- e.g., stock issue vs. bond
   issue vs. term loan. Second, penalty costs are assessed
   when the LP solution violates certain limits (to be dis-
   cussed) of safety, convenience or practicality. The vari-
   ables S_t and $S_{\bar{t}}$ measure the amount by which the limit for
   period t is exceeded and q_t and $\bar{q_t}$ the cost (present value)
   per unit of excess.

   Third, the objective function includes $H + 1$ "margins
   of safety" as well as certain slack variables (e.g., L_t), which
   do not contribute directly to the aggregate present value of
   the financial plan, but do so indirectly by their role in the
   LP constraints.

2. The first constraints, Eqs. (10-2), are intended to require
   the program either to accept or reject each project. It
   does not do this with complete reliability, of course -- this
   would require an integer program. The decisionmaker using

TABLE 1

CORE OF A "REALISTIC" LP MODEL

(10-1)
$$\underset{x_j, y_j, S_t, L_t, T_t.}{\text{Max}} \quad \psi = \sum_{j=1}^{n} x_j A_j + \sum_{j=1}^{m} y_j F_j$$

$$+ \sum_{t=1}^{H+1} 0 \cdot (M_t^+ + M_t^- + L_t)$$

$$= \sum_{t=1}^{H+1} [q_t S_t + q_t S_t^-]$$

Subject to:

(10-2)
$$x_j \leq 1 \qquad\qquad j = 1, \ldots, n$$

(10-3)
$$\sum_{j=1}^{n} x_j \bar{a}_{jt} - \sum_{j \in D_1} y_j \bar{f}_{jt} \geq (M_t^+ - M_t^-)$$

(10-4)
$$\sum_{j=1}^{n} x_j \bar{a}_{jt} - \sum_{j \in D_1, D_2} y_j \bar{f}_{jt} \geq (M_t^+ - M_t^-) - S_t$$

(10-5)
$$\sum_{j=1}^{n} x_j \bar{a}_{jt} - \sum_{j=1}^{m} y_j \bar{f}_{jt} \geq (M_t^+ - M_t^-) - S_t - S_t^-.$$

(10-6)
$$\left(M_t^+ - M_t^-\right) + B_t + (1+r_L)L_{t-1} \geq \sum_{j=1}^{n} x_j (\bar{a}_{jt} - z_{jt})$$

(10-7)
$$S_t \leq \beta (M_t^+ - M_t^-)$$

(10-8)
$$S_t + S_t^- \leq \gamma (M_t^+ - M_t^-), \qquad \gamma > \beta$$

(10-9)
$$L_t = L_{t-1} + \sum_{j=1}^{n} x_j \bar{a}_{jt} - \sum_{j=1}^{m} y_j \bar{f}_{jt} + B_t$$

TABLE 1 (continued)

NOTE: Eqs. (10-3) to (10-9) apply for t=1, 2, . . ., H.

(10-10)   Constraints on net worth at the horizon
   .   period. Same format as Eqs. (10-3) to
   .   (10-8).
(10-15)

(10-16) $x_j, y_j, M_t^+, M_t^-, L_t, S_t, S_t^- \geq 0.$

TABLE 2

DEFINITIONS OF VARIABLES IN THE "REALISTIC" MODEL

x_j, y_j = decision variables for investment and financing options, respectively;

A_j = present value of project j;

F_j = present value per dollar of financing option j;

$\bar{a}_{jt}$ = the expected cash flow of the jth project in period t;

$\bar{f}_{jt}$ = the expected cash flow per dollar of the jth financing option in period t;

$\bar{a}_{j, H+1}$ = the expected value (i.e., market value) of project j at the end of the horizon period t = H;

$\bar{f}_{j, H+1}$ = expected market value of financing option j at the horizon;

M_t = the desired margin of safety by which financial obligations are met in period t -- in other words, desired expected net cash flow;

M_{H+1} = desired expected net worth at the end of horizon period;

$S_t + S_t^-$ = the amount by which the actual margin of safety for period t falls short of M_t.

q_t = the cost (present value) per dollar of falling short of the desired margin of safety for period t in amounts less than S_t^-;

q_t^- = the cost per dollar of falling short of the desired margin of safety for period t in amounts greater than S_j^-;

B_t = expected "autonomous" cash flow in period t, which may be either positive or negative;

B_{H+1} = expected market value of "autonomous" financing available at the horizon;

z_{jt} = "debt servicing capacity" of project j's cash flows at t;

$z_{j, H+1}$ = "debt capacity" associated with project j's market value ($\bar{a}_{j, H+1}$) at the end of the horizon period;

r_L = interest rate earned on liquid assets.

Lp is forced to accept the possibility that the program will rec-

ommend accepting, say, 3.1 percent of project j.

3. Eqs. (10-5) constrain expected net cash flow to be positive, with

desired margins of safety $(M_t^+ + M_t^-)$ in each future period. The

program can broach the desired safety margin by making S_t or S_t^- posi-

itive and incurring the penalty costs q_t or q_t^-.

The constraints are necessary because the present values of

most debt options will be positive, reflecting the present value

of taxes saved by deducting interest from corporate income. The LP

solution would, therefore, borrow in indefinitely large amounts if

unrestrained.

The specific form of these constraints requires further explana-

tion. First, it should be noted that they are expressed in terms

of expected values of cash flows. Contrast the simple model dis-

cussed above, in which constraints on borrowing are expressed by the

stocks of assets held and liabilities outstanding at various points

in time.[14]

Second, the slack variables L_t are interpreted as expected excess

cash. It is assumed that L_t is invested in liquid securities yield-

r_L per period.

4. The safety margins $(M_t^+ - M_t^-)$ are themselves variable. Their size

does not affect the objective function directly, but the LP routine

will attempt to make them as small as possible to ease the con-

straints (10-3, 4, 5).

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[14]Other ways of constraining borrowing are open to the decisionmaker, of
course. One promising approach is to express constraints in terms of reces-
sion cash flows, on the grounds that insolvency is most likely in this circum-
stance. In general, and if model simplicity were no object, the decision-
maker would constrain his financing-investment choice to provide a suitable
margin of safety in all relevant future states of nature. ("Suitable" can mean
negative -- it will not pay to make bankruptcy literally impossible.)

The constraints (10-6) are indended to set each period's desired margin at a reasonable level in light of the LP solution's asset and liability choices. The constraint may be put in words as follows:

$$
\begin{bmatrix} \text{Desired} \\ \text{Safety} \\ \text{Margin} \end{bmatrix} \quad \underline{\text{plus}} \quad \begin{bmatrix} \text{Autonomous} \\ \text{funds avail-} \\ \text{able or re-} \\ \text{quired} \end{bmatrix}
$$

$$
\underline{\text{plus}} \quad \begin{bmatrix} \text{Available} \\ \text{liquid} \\ \text{assets} \end{bmatrix} \quad \begin{matrix} \underline{\text{is at}} \\ \underline{\text{least}} \\ \underline{\text{equal to}} \end{matrix} \quad \begin{bmatrix} \text{The difference} \\ \text{between expected} \\ \text{cash flow of the} \\ \text{firm's assets } (\sum x_j \bar{a}_{jt} \\ \text{and their debt-} \\ \text{servicing capacity} \\ (\sum x_j z_{jt}). \end{bmatrix} .
$$

This statement doubtless makes sense intuitively. A firm can operate with a lower margin of safety if liquid assets, such as short-term securities, can be drawn upon; if autonomous funds are expected to be available, or if the firm's assets have large "debt servicing capacity."

"Autonomous" funds are included because, as a practical matter, not all of a firm's sources and uses of funds will be affected by the decision variables of the LP problem. The firm's existing assets will normally be expected to generate funds, and existing commitments to absorb them. Moreover, the firm may wish to insure that sufficient funds are available to undertake promising investments that are anticipated but cannot yet be identified. (This is an autonomous requirement for funds which will
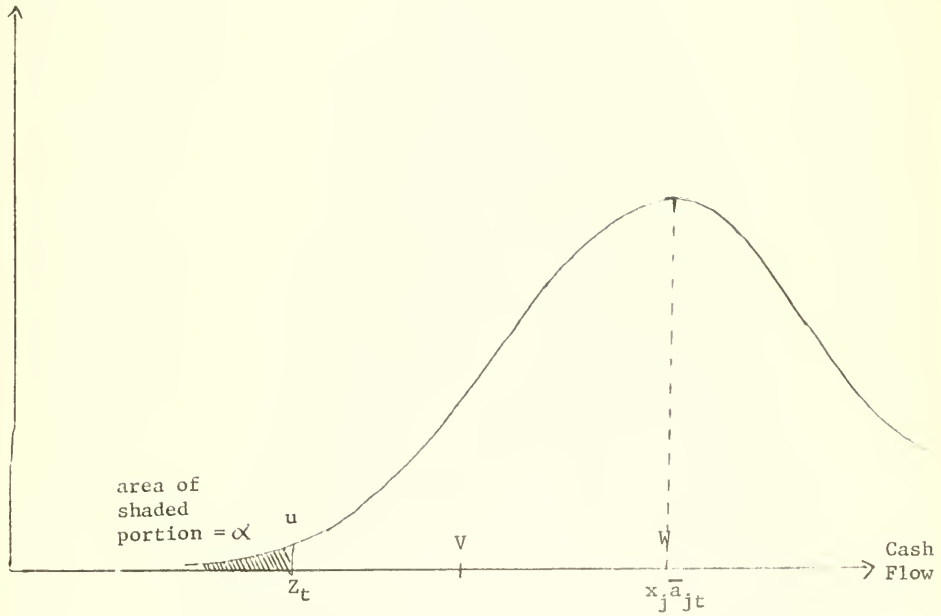
Figure 1

Determinants of Debt Servicing Capacity

reduce B_t or even make it negative.)

The idea of debt-servicing capacity also requires comment. It is defined as the lowest possible value of the aggregate cash flow generated by investment options accepted in the LP solution. More formally, "lowest possible value" means a number Z_t such that the probability of $\sum x_j a_{jt} > Z_t$ is $1 - \alpha$, α being an arbitrary confidence level close to zero, as is illustrated in Figure 1. We have, essentially, a problem of chance constrained programming.[15]

The firm desires a "cushion," including autonomous cash flow and liquid assets, equal to the difference of expected cash flow and debt-servicing capacity. In Figure 1, if autonomous funds and liquid assets amounted to UV then constraint (10-6) would set $(M_t^+ - M_t^-) = VW$.

Note that the safety margin can be negative if antonomous cash flow and liquid assets are large. This explains why the safety margin is expressed as $(M_t^+ - M_t^-)$ rather than simply M_t.

Unfortunately, reality and the LP format are not in good fit at the point at which debt servicing capacity is measured by $\sum x_j z_{jt}$. Obviously the desired "cushion" depends on the size and risk characteristics of the firm's planned investments. But it is not true that the possible deviation from the mean of aggregate cash flow is the sume of possible deviations for each

---

[15]See Byrne, Charnes, Cooper and Kortanek [2], and Charnes and Cooper [3].

planned project. It is not logically correct to express the firm's safety margin in terms of the simple weighted sum $\sum x_j(\overline{a}_{jt} - z_{jt})$, regardless of how the jth project's debt-servicing capacity for period t is defined.

Nevertheless, it seems likely that careful definition and measurement of z_{jt} will result in a fair approximation of the true relationship of the firm's investment choices to the risk of insolvency. This matter is under investigation.[16]

5. We have now defined, in concept at least, the <u>desired</u> margins of safety within which the LP solution is to be found. If the desired safety margins are actually observed in the LP solution, the probability of bankruptcy or even temporary insolvency should be negligible for all the periods $t = 1, \ldots, H$.

In actuality, the tax advantages of additional debt will usually be justified even at the expense of some likelihood of bankruptcy or insolvency. Therefore, the LP routine is allowed to violate the desired safety margins. However, such violations are charged with the estimated additional present value of the bankruptcy costs which may result.

Eq. (10-7) allows the safety margin for period t to be reduced by up to 100β percent. As this is done, S_t is increased at a cost per dollar of q_t. As the safety margin is further reduced ($S_t^- > 0$) a higher cost per dollar q_t^- is assessed.

---

[16]The most promising avenue of approach (assuming the LP format is retained) seems to be to regard the firm as a portfolio of projects subject to Sharpe's "diagonal model" of security performance. See [21]. This would postulate one common factor (company sales?) affecting all projects' cash flows. The "cushion" ($\overline{a}_{jt} - z_{jt}$) required for each project would be inversely related to the project's dependence on the common factor.

The cost relationship is shown graphically as the solid curve in Figure (2). In practical problems, a curve with more segments might provide a better fit to the true costs associated with financial risk.

6. Earlier in this paper it was emphasized that, although a firm potentially can issue a large number of financial assets, many combinations of real and financial assets are infeasible. The most common problem is that a newly issued bond can be anything from a blue-chip to a highly speculative security, depending on the characteristics of the firm's assets and the aggregate amount debt outstanding. Since the present values of blue chips and speculative bonds are likely to differ, the LP program should insure that each is issued only in appropriate circumstances.

This may be accomplished as follows. For each type of debt financing, define three financing options: low, medium and high-risk borrowing. Risk is assessed from the viewpoint of potential creditors. Thus, there will be low, medium and high-risk term loans; low, medium and high-risk bond issues, and so on.

We now consider all low-risk debt options, class D_1, separately. Eqs. (10-3) restrict the amount of low-risk debt issued to that amount which can be serviced retaining the full safety margin in each relevant future period.

Eqs. (10-4) in turn restrict aggregate low- and medium-risk debt (classes D_1 and D_2) to that amount which can be serviced within the less stringent margin $M_t^+ - M_t^- - S_t \leq (1-\beta)(M_t^+ - M_t^-)$.
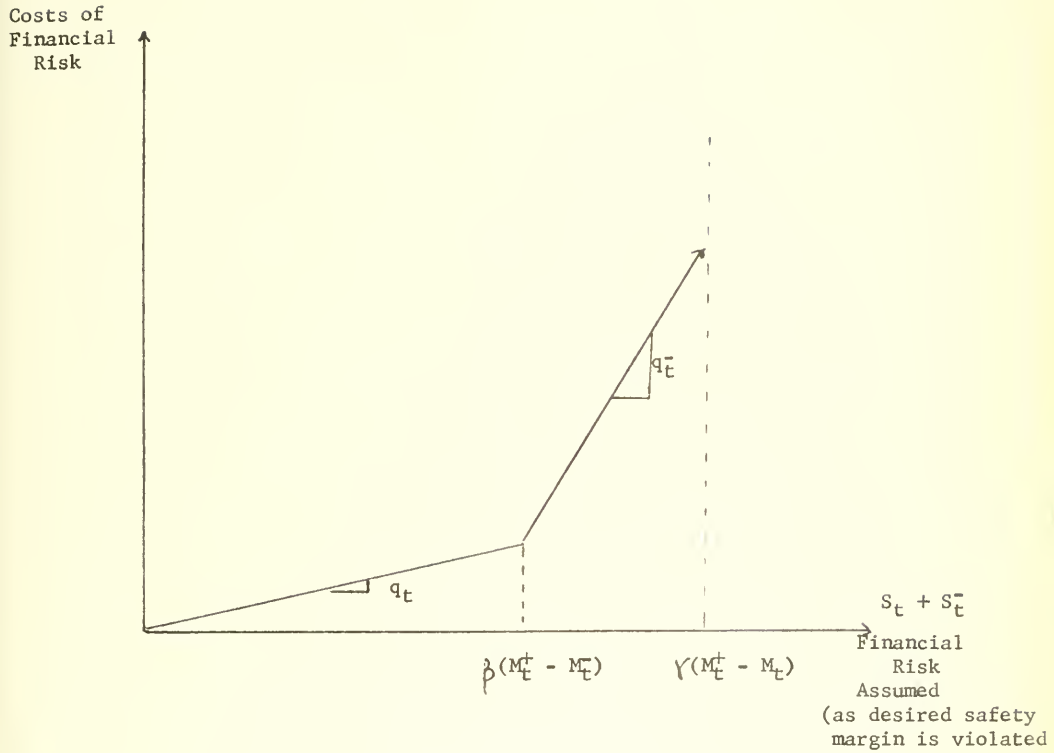
Figure 2

Costs of Violating Safety Margins

Eqs. (10-5), the least stringent standards, apply to all cash flows, including high-risk debt. These constraints were discussed above in detail.

It is necessary to insure that the assumed characteristics of debt options in the various classes are consistent with these restraints on their use.

7. We must also include a set of constraints for the $H + 1^{st}$ "period" to insure that the financial plan for the H time periods under consideration does not leave the firm in an undesirable terminal position. In the absence of such a restriction the program might, for example, recommend borrowing large amounts initially and still larger amounts in later periods to repay the principal and interest of the initial obligations -- much as the national debt is "rolled over" without provision for ultimate repayment. Constraints on cash flow in periods 1 through H would therefore not limit borrowing in the absense of supplementary constraints.

The form of constraints (10-9) to (10-14) is identical to (10-3) to (10-8), except that $\overline{a}_{j, H+1}$ and $\overline{f}_{j, H+1}$ should be interpreted as the stocks (market values) of the firm's assets and liabilities at the end of the horizon period. In other words, the restrictions are imposed on the planned net worth of additional financing and investment options undertaken due to the LP solution.

Other aspects of the model

There are a number of considerations excluded from the core of the LP

model which are, nevertheless, essential to a general model for long-range
financial planning. We will discuss the following: mutually exclusive or
contingent options, dividend policy, stock issues and non-financial con-
straints.

1. Means of dealing with mutually exclusive or contingent options

    have been fully discussed by Weingartner.[17] Two examples should

    convey the essence of the approach, however. If options i and j

    are mutually exclusive, we add the constraint

    $$x_i + x_j \leq 1 \quad .$$

    If option i is contingent on acceptance of j, then the con-

    straint is

    $$x_i - x_j \leq 0 \quad .$$

    Although this type of restriction usually applies to investment

    options, financing options may also be so restrained. For example,

    option j could be purchase of real estate and option i a mort-

    gage. It makes perfect sense to say that i is contingent on j.

    These constraints unfortunately do not prevent the potential

    problem of fractional projects accepted in the LP solution.

2. It is now widely accepted that dividend policy is irrelevant when

    perfect capital markets exist.[18] If this is taken as a first

    approximation, then dividend policy is easily handled in the LP

[17][24], pp. 32-34.

[18]For the original proofs of the proposition, see Miller and Modigliani
[10] and Lintner [6].

format. We treat each period's aggregate dividend payment as
an investment which yields no cash returns to the firm, but never-
theless has a net present value of zero. Then the LP program
will accept all investments with positive present values (possi-
bly including funds retained to establish "safety margins"), and
will treat dividend payments as a residual. This is the appropri-
ate strategy when dividend policy is irrelevant.

However, there are a number of reasons why dividend policy
may be regarded as relevant. First is the different rates at
which investors' capital gains and regular income are taxed.
This factor is ignored in the present model.

Second are the alleged market imperfections which lead in-
vestors to prefer high dividend payouts to low ones. Dividends
may be assigned positive, rather than zero, present values if this
is the case.

Third is the informational content of dividends. Changes in
dividends seem to be regarded as signals of changes in the firm's
long-run profitability. Therefore, dividends are cut only when
financial difficulties force it, and raised only when it is
reasonably clear that the increase can be maintained.

The easiest way to reflect the informational content of
dividends is to constrain aggregate dividends in period t to be no
less than in period t - 1. We may allow the program to violate
the constraint, but at a penalty cost.

Unfortunately this strategy is inappropriate when the firm

issues new equity. The informational content of dividends is really associated with dividends per share. Thus it is not appropriate to constrain aggregate dividends when the number of shares is variable. The number of shares outstanding, given an equity issue of fixed amount, depends on share price, among other things; share price in turn depends on the anticipated value of the firm's financing-investment package. These inter-relationships appear to make this problem irretrievably non-linear. I hope to consider how it can best be handled in an LP format, and whether the expense and complication of a less restricted format may be worthwhile on this score.

3. New issues of common stock deserve mention only because they tend to be subject to increasing returns to scale.

The present value of an equity issue of any size would be zero in a fully perfect market. However, the present value of actual issues must reflect registration and underwriting fees (largely fixed) and costs associated with the "clientele effect" -- i.e., the necessity to give new shareholders a particularly good "deal" to induce them to hold the firm's shares.

Figure 3 plots the probable present value of a stock issue vs. the gross amount of the issue (solid curve). This curve is not compatible with a linear objective function. Thus an approximation is necessary -- e.g., the dashed curve shown. This will be an acceptable approximation if fairly large equity issues are indicated by the LP solution, but misleading for small issues.
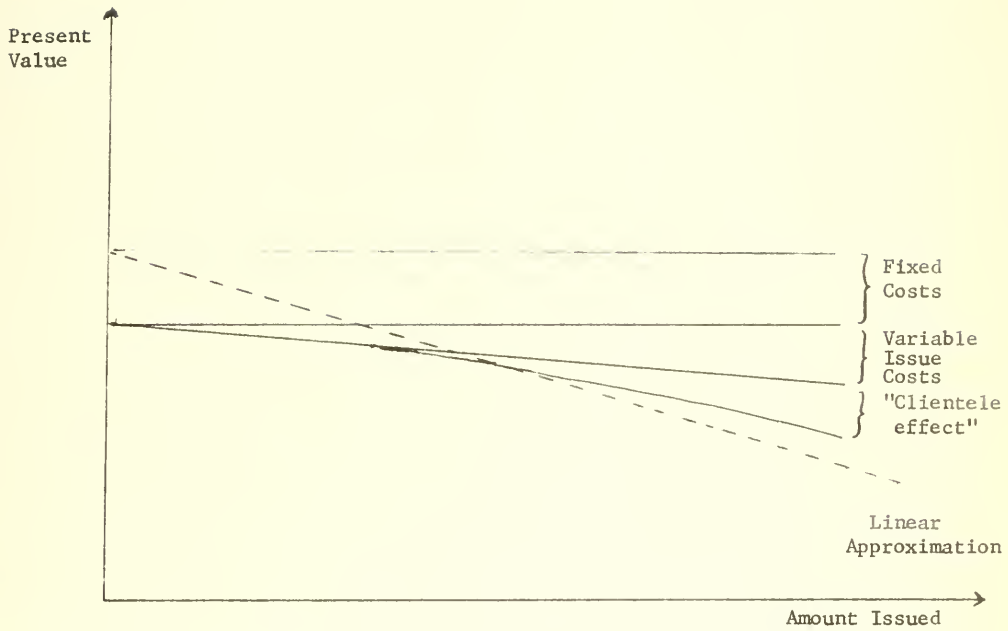
Figure 3

Present Value of an Equity Issue vs. Amount Issued

This difficulty is "built-in" and not amenable to easy solution.

4. Finally, in any practical context management will wish to add a
    variety of further constraints dictated by the special circum-
    stances in which the firm finds itself. Examples follow:

    a. Some borrowing options may impose constraints on management.
        E.g., restraints on working capital, dividend policy, and
        new investments are frequently written into term loans.

    b. There may be other scarce resources aside from capital,
        and the program may be used to ration these among proposed
        and existing projects. A common example is a shortage of
        technical or managerial talent.

    c. Other management goals may be reflected in constraints --
        for example, management may wish to insure that reported
        income does not fall, or that the firm's overall employment
        does not fluctuate too drastically.

CONCLUSION

The theory of financial management presented in this paper can be
reasonably criticized on at least two grounds.

1. Simplified view of security valuation -- The security valuation
    models consistent with the LP approach are easier to believe
    than prove, and also easier to disbelieve than disprove. But
    there is no doubt that their derivation ignores some potentially
    important considerations -- the observed capital market imperfec-
    tions, and particularly the different tax rates on regular income
    and capital gains. It cannot be proved that ignoring these factors

is a reasonable simplification.

2. The model yields a plan, not a strategy -- The LP solution is a
schedule of financing and investment choices which are fully
appropriate in only one state of nature among the many that can
occur -- the state in which all cash flows turn out equal to
their expected values. Other possible outcomes affect the LP
plan chiefly through the "safety margins" built into the solution.
But the plan does not consider how the firm's future financial
dicisions may be dependent on future events. It does not pro-
vide the optimal financial strategy. Ideally, dynamic program-
ming should be used.

This is, of course, not an exhaustive list of faults.

Judged by the kind of model we would like to use, then, the LP
approach is not that exciting. Judged by the current theory of financial
management (essentially based on the cost of capital concept) the LP
approach is potentially a significant improvement.

APPENDIX A

DIFFICULTIES ASSOCIATED WITH THE
COST OF CAPITAL CONCEPT

The point of this appendix is that evaluating projects with the cost of capital as usually measured is wholly reliable only when the projects are perpetuities. Strictly speaking, the point is not new,[19] but it deserves more emphasis than it has received.

Our analysis will be confined to the simple model of Section II above. We assume the MM propositions hold so that $F_t = \frac{iTc}{(1+i)^t}$. From Eq. (6), project j's APV, or net contribution to stock price, is

$$(A.1) \qquad APV_j = A_j + \sum_{t=0}^{H} [Z_{jt} iTc/(1+i)^t]$$

The derivation of the usual cost of capital measures (Eqs. (8) and (9)) from Eq. (6) assumes (1) that project j is a perpetuity and (2) that j's debt capacity, Z_{jt}, is constant over time. These assumptions imply that

$$(A.2) \qquad APV_j = \frac{\bar{a}_j}{\rho(j)} - I_j + Z_j T_c$$

As is apparent from their derivation, Eqs. (8) and (9) do not apply when assumptions (1) and/or (2) are violated. For our purposes, a simple example suffices. Consider a project requiring an investment I_j and returning $\bar{a}_{j1}$ at $t = 1$. From Eq. (A.1),

---

[19]Modigliani and Miller [11], p. 434, fn. 3.

$$\frac{APV_j}{I_j} = \frac{\overline{a}_{j1}/I_j}{1+\rho(j)} - 1 + \frac{d_j iTc}{1+i} .$$

Dropping subscripts, rearranging, and defining y as APV_j/I_j .

(A.3) $$\frac{a/I}{1+y} = 1 + \rho + \frac{(1+\rho)diTc}{(1+i)(1+y)}$$

We now define a "true" cost of capital, $\hat{\rho}$, as the rate which gives the correct APV when used to discount the projects cash flows:

$$\frac{a/I}{1+\hat{\rho}} - 1 = APV$$

(A.4)

$$\hat{\rho} = \frac{a/I}{1+APV} - 1$$

substituting in Eq. (A.3),

(A.5) $$\hat{\rho} = \rho - \frac{(1+\rho)diTc}{(1+i)(1+APV)} .$$

It is easy to verify that $\hat{\rho}$ is not the same as ρ^{*}:

$$\hat{\rho} \overset{?}{=} \rho^{*} = \rho(1-dTc)$$

If true, this would imply, using Eq. (A.5) that

$$\rho(1 + y + iy) = i ,$$

which is impossible if $\rho > i$ and $y = APV/I > 0$. Thus ρ^{*} will not

evaluate single-period investments correctly.[20]

The interested reader will not find it difficult to find other cases in which the usual cost of capital measures are inappropriate.

---

[20]It is natural to ask whether $\hat{\rho} = \rho(1-\frac{diTc}{1+i})$. Unfortunately, this is true only if $y = 1/\rho$, or if

$$APV/I = \frac{1}{\rho},$$

$$APV = \frac{1}{\rho}(I).$$

As reasonable values for $\frac{1}{\rho}$ are 3, 4 or larger, this is a decidedly special case.

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