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A Continuous Time Formation[*]
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A Continuous Time Formulation[*]

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Abstract

We study a class of continuous time optimal entry-exit decisions under uncertainty for a single firm and for a duopoly. Under very general hypotheses, the optimal policy for a single firm exists and is unique. This unique optimal policy is a barrier policy: A firm optimally enters or exits from an industry when the demand reaches certain barriers. In the context of a duopoly, there may exist multiple subgame perfect equilibria. We demonstrate how to identify a subgame perfect equilibrium in which both firms employ stationary barrier policies. In some of these stationary equilibria, a firm may exit even when the demand has been rising on the average.

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1   Introduction

Many important economic decisions are about the optimal timing to take certain actions. Examples of these decisions include the optimal timing to undertake a capital investment project, to exercise an American call option, to buy a house, and to refinance a mortgage. This paper focuses on a firm's decision to enter or to exit from an industry. The methods developed here are general, however, and can be applied to analyze optimal timing decisions in other contexts.

There is a vast economic literature on the entry and exit decisions of a firm, for which Wilson (1990) is a good recent survey. Much of the literature in entry-exit decisions of a firm uses discrete-time models. The notable exceptions are Dixit (1989), Fudenberg and Tirole (1986), and Ghemawat and Nalebuff (1985). Fudenberg and Tirole (1986) analyze a gaming situation between two competing firms with incomplete information when there is no exogenous uncertainty; while Ghemawat and Nalebuff (1985) study a similar situation with complete information. Dixit (1989) consider a single firm's entry and exit decision when the exogenous uncertainty is modeled by a Brownian motion.

The purpose of this paper is twofold. First, we use a general methodology in solving a single firm's optimal entry and exit problem in continuous time using the theory of martingales. Unlike Dixit (1989), who drew implications of his model using a particular parameterized example, we give general qualitative characterizations of a firm's optimal timing decision. Using a continuous time model when the choice variable is time is natural as time, after all, is continuous. More important, continuous time models are better able to yield clean analytic results. This has been evidenced in particular by continuous time models in finance; see Merton (1990). Here in this paper, we also get clean and general analytic results which are usually difficult to get using discrete time models.

Second, we go beyond a single firm's problem to analyze a duopolistic entry-exit game with complete information. The focus of our study here is to examine subgame perfect equilibria directly in continuous time without having to take limits of the outcomes of a sequence of discrete-time games. This is in contrast to the emerging literature on continuous time extensive form games; see, for example, Simon (1987) and Simon and Stinchcombe (1988).

The rest of this paper is organized as follows. Section 2 formulates the entry-exit problem in continuous time for a single firm. The uncertain demand is modeled by a standard Brownian motion. It is demonstrated that the optimal entry-exit policy is unique and is a barrier policy: The firm should enter if the demand increases to exceed a certain level and should exit afterwards when the demand decreases to reach another level.

Although presented in the context of a Brownian motion uncertainty, our methodology can be

used in any environment where the uncertainty is modeled by a Strong Markov process. In such a case, the optimal entry and exit decisions may no longer be barrier policies. Rather, a firm should enter the first time the demand reaches a possibly time-dependent Borel set and should exit afterwards the first time the demand reaches another possibly time-dependent Borel set. One advantage of using a Brownian motion to model uncertainty is that, besides the qualitative characterizations, the entry and exit barriers can also be analytically expressed.

Section 3 analyzes an exit game of a duopoly. The two firms in the industry, one strong and one weak in a sense to be formalized, are seeking the best time to exit. One subgame perfect equilibrium is the one in which the strong firm does not exit unless the demand is such that it cannot sustain even as a monopoly and as a result the weak firm always exits before the strong firm does. We also identify other candidates for a subgame perfect equilibria by characterizing the exact upper bound and lower bound on each firm's subgame perfect equilibrium strategies. Some interesting phenomena occur in these equilibria. For example, either firm may exit when the demand has been on the average increasing. This happens for the strong firm, for example, because the weak firm plays tough and would not exits until the demand falls significantly. The strong firm then trades off the potential of becoming a monopolist after the weak firm exits against the current duopoly losses. An increasing demand increases the expected waiting time for the strong firm to become a monopolist and is a bad news. Thus it exits when the demand reaches a critical level from below.

We continue in Section 4 to consider a game of an incumbent versus a potential entrant. We exhibit a subgame perfect equilibrium. In this equilibrium, as expected, the existence of a potential entrant makes the life span of the incumbent shorter than when it is a monopoly even though the incumbent may indeed remain as a monopoly throughout its lifetime. When the demand is low, the possibility of future duopoly competition limits the potential future monopoly profits. As a consequence the incumbent is less tolerant to the current monopoly losses than a monopoly facing no potential entrant and thus it exits earlier than a monopoly will even before the entrant enters. In addition, the entrant may not enter the industry even though the demand is above the level where both firms can be profitable as a duopoly. This is so because by waiting longer, the entrant may be able to enter after the incumbent exits. And the benefit from being a monopoly in the future outweighs the losses in the current duopoly profit.

Section 5 outlines how the analysis of Section 2 can be generalized to allow re-entry once a firm exits. Concluding remarks are in Section 6.

2   Single Firm Problems

We consider a single firm's decision to enter or to exit out of an industry, which is characterized by an uncertain demand. Formally we model the uncertain demand by taking the state space Ω to be the space of continuous functions of time from time 0 to infinity. Each $\omega \in \Omega$ represents a complete description of one possible demand over the time horizon $[0, \infty)$ and $\omega(t)$ is the demand at time t if the state is ω. The state-dependent demand can then be modeled by a coordinate process $X(\omega, t) = \omega(t)$, that is, the demand at time t is $X(\omega, t)$ if the state is ω.

The information the firm has at t contains the historical demands from time 0 to time t. Mathematically, this information is represented by the smallest sigma-field on Ω with respect to which $\{X(\omega, s); 0 \leq s \leq t\}$ is measurable and is denoted by $\mathcal{F}_t^o$. The information the firm has at "infinity" from observing the demand over time is $\mathcal{F}^o$, which is the smallest sigma-field finer than every $\mathcal{F}_t^o$ for all t. Let P be the probability on the measurable space $(\Omega, \mathcal{F})$ under which X is a (μ, σ)-Brownian motion starting from $X(0)$. That is, $X(t) = X(0) + \mu t + \sigma B(t)$, where B is a standard Brownian motion under P, $X(0)$ is a random variable independent of B, and μ and σ are constants. Note that in the above specification, we have used $X(t)$ and $B(t)$ to denote the random variables $X(\cdot, t)$ and $B(\cdot, t)$, respectively. Often, we will also use X_t interchangeably with $X(t)$.

To simplify the technical difficulty, we will "complete" the measurable space $(\Omega, \mathcal{F}^o)$ with respect to P and denote this space by $(\Omega, \mathcal{F})$.[1] Thus $(\Omega, \mathcal{F}, P)$ is a complete probability space.[2] We will also enlarge the information the firm has at time t to include not only the historical realization of demand from time 0 to time t but also all the demand scenarios described in $(\Omega, \mathcal{F}, P)$ that will only happen with zero probability. This information is the "completion" of $\mathcal{F}_t^o$ with respect to the completed probability space and is denoted by $\mathcal{F}_t$. It follows from Chung (1982, corollary to theorem 2.3.4) that $\{\mathcal{F}_t; t \in [0, \infty)\}$, the increasing family of sub-sigma-fields of $\mathcal{F}$, is right-continuous in that $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for all t. We assume that the probability measure P induces a family of conditional probability measures $\{P^x; x \in \Re\}$ such that, under P^x, X is a (μ, σ)-Brownian motion with initial state x or starting from x. All the almost surely statements will be with respect to P unless otherwise specified.

When the firm is already in the industry, it derives a profit rate of $\pi(X_t)$ at time t given the demand X_t, where $\pi(\cdot)$ is increasing[3] and nonconstant. We assume that $E_x[\int_0^\infty e^{-rt}|\pi(X_t)|dt] < \infty$

---

[1] The procedure of completion with respect to P is to generated a sigma-field $\mathcal{F}$ using $\mathcal{F}^o$ and all the subsets of P measure zero sets.

[2] A probability space is said to be complete if all the subsets of probability zero sets are measurable.

[3] We will use weak relations throughout. For example, increasing means nondecreasing, positive means nonnegative, lower than means no higher than and etc. When a relation is strict, we will use, for example, strictly increasing.

and that the process $\{e^{-rt}f(X_t), t \in \bar{\Re}_+\}$ is bounded below and above by two martingales, where

$$f(x) \equiv E_x\left[\int_0^\infty e^{-rt}\pi(X_t)dt\right], \tag{1}$$

$E_x[\cdot]$ is the expectation under P^x, $\bar{\Re}_+ \equiv \Re_+ \bigcup\{+\infty\}$, and we understand that at $t = \infty$, $e^{-rt}f(X_t)$ is set to be zero. By the fact that π is increasing and X is a Brownian motion, $f(x)$ is continuous and increasing. When the firm is not in the market, the profit rate is assumed to be zero. Entering or exiting from the market incur a one-time fixed cost. Let α denote the cost of entry and β be the cost of exit. For simplicity, we also assume that once a firm exits, it is prohibitively costly to re-enter. (This is not necessary. We will show in Section 5 how our results can be generalized to allow for the re-entry of a firm.) In order to avoid the trivial cases that the firm will either never enter or, once enter, will never exit, we assume that there exists a x^0 and y^0 with $-\infty < x^0 < y^0 < \infty$ such that $\pi(y) > -r\beta$ for $y > x^0$, and $\pi(y) < -r\beta$ for $y < x^0$, and $\pi(y) > r\alpha$ for $y > y^0$ and $\pi(y) < r\alpha$ for $y < y^0$.

We are interested in two problems a firm faces. First, what is the optimal time to exit from the industry if the firm is already in it? Second, given an optimal exit time of a firm, what is its optimal time to enter the industry? Formally, the first problem the firm faces is to find an optimal exit time, to maximize the expected discounted future profits:

$$\sup_{T \in \mathbf{T}} E\left[\int_0^T e^{-rt}\pi(X_t)dt - e^{-rT}\beta\right], \tag{2}$$

where $\mathbf{T}$ denotes the collection of all optional times[4], $E[\cdot]$ is the expectation under P, and $r > 0$ is the riskless interest rate. Note that, by the strong Markov property of X,[5] the objective function of the firm can be written as[6]

$$E\left[\int_0^T e^{-rt}\pi(X_t)dt - e^{-rT}\beta\right] = f(X_0) - E[e^{-rT}(f(X_T) + \beta)].$$

Given the discussion above, (2) is equivalent to

$$\inf_{T \in \mathbf{T}} E[e^{-rT}(f(X_T) + \beta)]. \tag{3}$$

Put

$$v(x) \equiv f(x) - \inf_{T \in \mathbf{T}} E_x[e^{-rT}(f(X_T) + \beta)]. \tag{4}$$

---

[4] $T : \omega \to \bar{\Re}_+$ is an optional time if $\{\omega \in \Omega : T(\omega) \leq t\} \in \mathcal{F}_t$ for all $t \in \Re_+$.

[5] X is said to have the Strong Markov property if for all optional time T and random variables Y independent of $\mathcal{F}_T$, we have $E[Y|\mathcal{F}_T] = E[Y|X_T]$ a.s., where $\mathcal{F}_T$, the sigma-field of events *prior to* T, consists of events A so that $A \bigcap \{T \leq t\} \in \mathcal{F}_t$.

[6] Throughout, we understand that $E[e^{-rT}g(X_T)] \equiv \int_{\{T<\infty\}} e^{-rT(\omega)}g(X(\omega, T(\omega)))P(d\omega)$ for any real-valued function g.

By the hypothesis that $e^{-rt}f(X_t)$ is bounded below by a martingale, $v(x)$ is finite for all x. Moreover, since f is continuous and X is a Brownian motion, v is a continuous function.

The second problem the firm faces is to find the optimal time to enter the industry knowing that it will behave optimally afterwards:

$$\sup_{T \in \mathbf{T}} E[e^{-rT}(v(X_T) - \alpha)], \tag{5}$$

where we have again used the strong Markov property of X. Put

$$\hat{v}(x) \equiv \sup_{T \in \mathbf{T}} E_x[e^{-rT}(v(X_T) - \alpha)]. \tag{6}$$

The hypothesis that $e^{-rt}f(X_t)$ is bounded above by a martingale implies that $\hat{v}(x)$ is finite for all x. Similar to v, $\hat{v}$ is also a continuous function.

A solution exists for either (3) or (5) means there exists an optional time that attains the infimum or supremum, respectively.

We will show below that the solutions to (3) and (5) exist and are characterized by two barriers l^* and u^*. The optimal exit time is the first time that the demand X is lower than l^* and the entry time is the first time that the demand is higher than u^*. When $l^* = -\infty$, the firm will never exit once it is in the industry and when $l^* = +\infty$, the firm exits at any level of demand and thus it will never enter to begin with and $u^* = +\infty$. Similarly, when $u^* = -\infty$, the firm enters at any level of demand; and when $u^* = +\infty$, the firm will never enter.

The next theorem shows the existence of a solution to (3) and this solution is a barrier policy.

Theorem 2.1 *There exists a solution to (3) and*

$$T^* \equiv \inf\{t \geq 0 : v(X_t) + \beta = 0\} = \inf\{t \geq 0 : X_t \leq l^*\}, \tag{7}$$

for some $l^* \in \Re$ with $\pi(l^*) \leq -r\beta$, is a solution.

PROOF. From the hypothesis that the process $\{e^{-rt}f(X_t), t \in \bar{\Re}_+\}$ is bounded below by a martingale, Theorem 1 of Huang and Li (1990) shows that there exists a solution to (3) and this solution is characterized by the first time that $e^{-rt}(f(X_t) + \beta)$ is equal to the largest regular submartingale dominated by it.[7] If these two processes never equal in certain state of nature, the firm will then never exit. We claim that this largest submartingale is $e^{-rt}\phi(X_t) \equiv e^{-rt}(f(X_t) - v(X_t))$.

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[7]A regular submartingale $\{Y_t; t \in \Re_+\}$ is an optional process so that for any bounded optional time T, $E[Y_T^-] < \infty$ and for all optional times $S \geq T$, $E[Y_S|\mathcal{F}_T] \leq Y_T$ a.s., where an optional process is a process measurable with respect to the sigma-field on $\Omega \times [0, \infty)$ generated by all the processes adapted to $\mathcal{F}$ having right-continuous paths.

First we show that $e^{-rt}\phi(X_t)$ is a regular submartingale dominated by $e^{-rt}(f(X_t) + \beta)$. It is easy to see that $v(x) \geq -\beta$ for all x. Thus $e^{-rt}\phi(X_t) \leq e^{-rt}(f(X_t) + \beta)$ a.s. Note that for any optional times $S \geq T$,

$$
\begin{aligned}
E[e^{-rS}\phi(X_S)|\mathcal{F}_T] &= E\left[e^{-rS} \inf_{\substack{\tau \in \mathbf{T} \\ \tau \geq S}} E[e^{-r(\tau-S)}(f(X_\tau) + \beta)|X_S]|\mathcal{F}_T\right] \quad a.s. \\
&\geq e^{-rT} \inf_{\substack{\tau \in \mathbf{T} \\ \tau \geq T}} E[e^{-r(\tau-T)}(f(X_\tau) + \beta)|\mathcal{F}_T] \quad a.s. \\
&= e^{-rT}(f(X_T) - v(X_T)) \quad a.s. \\
&= e^{-rT}\phi(X_T), \quad a.s.
\end{aligned}
$$

where we have used the strong Markov property of X. Thus $e^{-rt}\phi(X_t)$ is a regular submartingale dominated by $e^{-rt}f(X_t)$.

Second we claim that $e^{-rt}\phi(X_t)$ is the largest regular submartingale dominated by $e^{-rt}(f(X_t) + \beta)$. Suppose otherwise and let Y be a regular submartingale dominated by $e^{-rt}(f(X_t) + \beta)$ and $Y(\omega, t) > e^{-rt}\phi(X(\omega, t))$ on a set of (ω, t) whose projection onto Ω is of a strictly positive probability. Define

$$
S_n \equiv \inf\{t \geq 0 : Y(t) \geq e^{-rt}\phi(X_t) + \frac{1}{n}\}.
$$

It is easily verified that S_n is an optional time and by the hypothesis there exists an $n > 0$ so that $P_x\{S_n < \infty\} > 0$. Since $e^{-rt}(f(X_t) + \beta) \geq Y(t)$, we have

$$
e^{-rS_n}\phi(X_{S_n}) = \inf_{T \in \mathbf{T}, T \geq S_n} E[e^{-rT}(f(X_T) + \beta)|\mathcal{F}_{S_n}] \geq \inf_{T \in \mathbf{T}, T \geq S_n} E[Y(T)|\mathcal{F}_{S_n}] = Y(S_n) \quad a.s.,
$$

where the last equality follows from the fact that Y is a submartingale. However, $Y(S_n) \geq e^{-rS_n}\phi(X_{S_n}) + \frac{1}{n}$ with a strictly positive probability. Thus $Y(S_n) > e^{-rS_n}\phi(X_{S_n})$ with a strictly positive probability, which is a contradiction.

Third, we want to show that

$$
T^* \equiv \inf\{t \geq 0 : f(X_t) + \beta - \phi(X_t) = 0\}
$$

is a barrier policy, that is, T^* is the first time that X_t is less than or equal to a level l^*. Note that $f(x) - \phi(x) = v(x)$ and $v(x)$ is increasing since π is increasing and nontrivial and a Brownian motion has stationary and independent increments. Recalling that v is continuous and $v(x) \geq -\beta$, there must then exist a l^* so that

$$
T^* = \inf\{t \geq 0 : X_t \leq l^*\}.
$$

Finally, we are left to demonstrate that $l^* \in \Re$ and $\pi(l^*) \leq -r\beta$. To see this, we observe that T^* is a solution to (3) if and only if it is a solution to

$$\inf_{T \in \mathbf{T}} E[e^{-rT} \hat{f}(X_T)],$$

where

$$\hat{f}(x) \equiv E\left[\int_0^\infty e^{-rt}(\pi(X_t) + r\beta)dt\right].$$

Given this, T^* is a solution to (2) if and only if it is a solution to

$$\sup_{T \in \mathbf{T}} E\left[\int_0^T e^{-rt}(\pi(X_t) + r\beta)dt\right] = \hat{f}(X_0) - \inf_{T \in \mathbf{T}} E[e^{-rT} \hat{f}(X_T)].$$

That is, T^* is also an optimal exit time when the firm's profit function is $\pi + r\beta$ and there is no exit cost. In this case, it is clear that the firm will not exit unless it is making a loss; that is, when $\pi(l^*) + r\beta \leq 0$. Hence we have $\pi(l^*) \leq -r\beta$.

If $l^* = +\infty$, $\pi(y)$ must be less than $-r\beta$ for arbitrarily large y. This contradicts the hypothesis that $\pi(y) > -r\beta$ for $y > x^0$. Similarly, if $l^* = -\infty$, we must have $\pi(y) > -r\beta$ for arbitrarily small y and this contradicts the hypothesis that $\pi(y) < -r\beta$ for $y < x^0$. Hence $l^* \in \Re$. ∎

Using similar arguments and the hypothesis that $e^{-rt}f(X_t)$ is bounded from above by a martingale, one can show that solution to (5) is also characterized by a barrier: enter the industry if the demand X_t is greater than or equal to a given level u^*.

Theorem 2.2 *There exists a solution to (5) and*

$$T^e = \inf\{t \geq 0 : \hat{v}(X_t) = v(X_t) - \alpha\} = \inf\{t \geq 0 : X_t \geq u^*\}, \tag{8}$$

for some $u^* \in \Re$ with $\pi(u^*) \geq r\alpha$, is a solution.

Combining Theorem 2.1 and 2.4, the optimal entry time and exit time for the firm given that the firm is currently outside the industry are recorded below:

Theorem 2.3 *Suppose that the firm is currently outside the industry. Then the optimal entry time for the firm is T^e defined in (8) and the optimal exit time for the firm is*

$$T^x \equiv \inf\{t \geq T^e : X_t \leq l^*\}. \tag{9}$$

With the continuity and unbounded variation properties of a Brownian motion, we next show that the barrier policy characterized in Theorem 2.3 is the unique optimal entry-exit policy.

Theorem 2.4 *The T^e and T^x defined in (8) and (9) are the unique optimal entry and exit times.*[8]

PROOF.   We will show that T^* defined in (7) is the unique solution to (3). The arguments for proving the uniqueness of the optimal entry time are similar.

Proposition 2 of Huang and Li (1990) shows that if S is another optimal exit time, then $S \geq T^*$ a.s. Thus if $l^* = -\infty$, T^* must be the unique optimal exit time.

Assume therefore that $P\{S > T^*\} > 0$. Put

$$T_n \equiv \inf\{t \geq 0 : X_t \leq l^* - \frac{1}{n}\}.$$

By the fact that a Brownian motion has continuous and unbounded variation sample paths, it is easily seen that $T_n \downarrow T^*$ P-a.s. Thus $\{S > T^*\} = \bigcup_n \{S > T_n\}$. If we can show that, for every n, we have

$$E[e^{-rS}\phi(X_S)|\mathcal{F}_{T_n}] > e^{-rT_n}\phi(X_{T_n}) \quad a.s. \text{ on } \{S > T_n\}, \tag{10}$$

then we are done as this will imply

$$E[e^{-rS}\phi(X_S)|\mathcal{F}_{T^*}] > e^{-rT^*}\phi(X_{T^*}) \quad a.s. \text{ on } \{S > T^*\}$$

and hence

$$E[e^{-rS}(f(X_S) + \beta)] \geq E[e^{-rS}\phi(X_S)] > E[e^{-rT^*}\phi(X_{T^*})] = E[e^{-rT^*}(f(X_{T^*}) + \beta)],$$

and S is sub-optimal, where the first inequality follows as $e^{-rt}\phi(X_t)$ is dominated by $e^{-rt}(f(X_t)+\beta)$.

To show (10) let $\tau = \inf\{t \geq T_n : X_t = l^*\}$ and set $S_0 = T_n$, $S_2 = S \vee T_n$, and $S_1 = S_2 \wedge \tau$. Then for $i = 0, 1$,

$$e^{-rS_i}\phi(X_{S_i}) \leq E[e^{-rS_{i+1}}\phi(X_{S_{i+1}})|\mathcal{F}_{S_i}], \quad a.s., \tag{11}$$

since ϕ is a submartingale. Inequality (11) for $i = 0$ is strict on $\{S_1 > S_0\} = \{S > T_n\}$. To see this, we note that

$$
\begin{aligned}
& e^{-rS_0}\phi(X_{S_0}) - E[e^{-rS_1}\phi(X_{S_1})|\mathcal{F}_{S_0}] \\
= {} & e^{-rS_0}(f(X_{S_0}) + \beta) - E[e^{-rS_1}(f(X_{S_1}) + \beta)|\mathcal{F}_{S_0}] \\
= {} & E[\int_{S_0}^{\infty} e^{-rt}(\pi(X_t) + r\beta)dt|\mathcal{F}_{S_0}] - E[\int_{S_1}^{\infty} e^{-rt}(\pi(X_t) + r\beta)dt|\mathcal{F}_{S_0}] \\
= {} & E[\int_{S_0}^{S_1} e^{-rt}(\pi(X_t) + r\beta)dt|\mathcal{F}_{S_0}] < 0,
\end{aligned}
$$

---

[8] We used a different argument in an earlier version to prove this theorem. The current proof is suggested to us by an anonymous referee.

where the first equality follows since, on $\{S > T_n\}$, $\phi(X_{S_i}) = f(X_{S_i}) + \beta$ for $i = 0, 1$, and the inequality follows because $\pi(X_t) < -r\beta$ for $t \in [S_0, S_1)$. The two inequalities in (11) ($i = 0$ and $i = 1$) imply that on $\{S > T_n\}$,

$$
\begin{aligned}
E[e^{-rS}\phi(X_S)|\mathcal{F}_{T_n}] &= E[e^{-rS_2}\phi(X_{S_2})|\mathcal{F}_{S_0}] \\
&= E[E[e^{-rS_2}\phi(X_{S_2})|\mathcal{F}_{S_1}]|\mathcal{F}_{S_0}] \\
&\geq E[e^{-rS_1}\phi(X_{S_1})|\mathcal{F}_{S_0}] \\
&> e^{-rT_n}\phi(X_{T_n}),
\end{aligned}
$$

which is (10). ∎

In sum, we have shown that there exists a unique solution to a firm's entry and exit decision and this solution is a barrier policy. Note that in our demonstration of Theorems 2.1 and 2.2, we only used the Strong Markov property and time and spatial homogeneity of a Brownian motion, the latter of which allows us to conclude that v is increasing and time independent, thus the barrier policy, and the barriers are time independent. Thus these two theorems can be generalized easily to the cases where the demand is any optional process[9] having the Strong Markov property except that the barriers may be time dependent and may not be half-lines any more. Rather, the optimal policy will prescribe entry or exit when the demand enters or leaves a time-dependent Borel set. The proof of the uniqueness in Theorem 2.4 uses the fact that a Brownian motion has, in addition to the Strong Markov property, continuous and unbounded variation sample paths. But we did not use the distributional property of a Brownian motion, namely that its increments are Normally distributed. Thus the proof will work for any (nondegenerate) diffusion process.[10]

The advantage of using a Brownian motion to model uncertainty is that, besides the qualitative statements made above, we can explicitly calculate the optimal exit and entry barriers as demonstrated below. To determine the optimal barriers l^* and u^*, we first calculate the expected discounted profits for any barriers, using Harrison (1985, chapter 3). Let $T(y) \equiv \inf\{t \geq 0 : X_t = y\}$. Then

$$
\theta(x, y) \equiv E_x[e^{-rT(y)}] = \begin{cases} \exp[-a_*(x - y)] & \text{if } x \geq y, \\ \exp[-a^*(y - x)] & \text{if } y \geq x, \end{cases} \tag{12}
$$

and

$$
f(x) \equiv E_x\left[\int_0^\infty e^{-rt}\pi(X_t)dt\right] = \gamma \int_{-\infty}^\infty \pi(y)\theta(x, y)dy, \tag{13}
$$

---

[9]See footnote 7 for the definition of an optional process and we need the optionality of the demand in order to use the existence results of Huang and Li (1990).

[10]A diffusion process is a process having continuous sample paths and the Strong Markov property. A nondegenerate diffusion process also has unbounded variation sample paths.

where

$$\gamma \equiv \frac{1}{\sqrt{\mu^2 + 2\sigma^2 r}}, \tag{14}$$

$$a_* \equiv \sigma^{-2}[\sqrt{\mu^2 + 2\sigma^2 r} + \mu], \text{ and} \tag{15}$$

$$a^* \equiv \sigma^{-2}[\sqrt{\mu^2 + 2\sigma^2 r} - \mu]. \tag{16}$$

Let a discrete variable indicate whether the firm has entered the market (I) or not (O), and let $\bar{v}(x, l; I)$ and $\bar{v}(x, u, l; O)$ be the expected future profits under a barrier policy with parameters l and u when the current demand is x. It follows from the strong Markov property of X that

$$\bar{v}(x, l; I) = \begin{cases} f(x) - E_x[e^{-rT(l)}(f(X_{T(l)}) + \beta)] = f(x) - \theta(x, l)(f(l) + \beta) & \text{for } x \geq l, \\ -\beta & \text{for } x < l; \end{cases}$$

$$\bar{v}(x, u, l; O) = \begin{cases} E_x[e^{-rT(u)}(\bar{v}(X_{T(u)}, l; I) - \alpha)] = \theta(x, u)(\bar{v}(u, l; I) - \alpha) & \text{for } x \leq u, \\ \bar{v}(x, l; I) - \alpha & \text{for } x > u. \end{cases}$$

Thus, the values of the optimal barriers can be determined by finding an l^* maximizing $\bar{v}(x, l; I)$ and an u^* maximizing $\bar{v}(x, u, l^*; O)$. The optimal expected profits are $v(x) = \bar{v}(x, l^*; I)$ and $\hat{v}(x) = \bar{v}(x, u^*, l^*; O)$, respectively, depending upon whether the firm is already in the industry or not. These results are recorded in the following proposition.

Proposition 2.1 l^* *is the unique number satisfying*

$$h(l^*) \equiv -\int_{l^*}^{\infty} e^{-a^* z}[\pi(z) + r\beta]dz = 0, \tag{17}$$

and u^* *is the unique number satisfying*

$$\hat{h}(u^*) \equiv -\int_{l^*}^{u^*} e^{a_* z}[\pi(z) - r\alpha]dz + \frac{r}{a_*}(\alpha + \beta)e^{a_* l^*} = 0. \tag{18}$$

Moreover, $-\infty < l^* \leq x^0 < y^0 \leq u^* < \infty$*. In addition, the optimal expected profits are:*

$$\hat{v}(x) = \begin{cases} \theta(x, u^*)(v(u^*) - \alpha) & \text{if } x \leq u^*, \\ v(x) - \alpha & \text{if } x > u^*; \end{cases} \tag{19}$$

$$v(x) = \begin{cases} -\beta & \text{if } x < l^*, \\ f(x) - \theta(x, l^*)(f(l^*) + \beta) & \text{if } x \geq l^*. \end{cases} \tag{20}$$

Now we have completely solved the optimal entry and exit problem of a firm. The optimal policy is very simple. The firm should enter the industry if the demand rises to or above the level u^* and should exit afterwards when the demands falls to or below the threshold number l^*. Note that since $\pi(l^*) \leq -r\beta$, once in the industry, the firm will not exit until the instantaneous profit

rate becomes significantly negative. This is because there is a strictly positive probability that the demand will in the future be higher and produce positive profits. Thus the firm chooses to remain in the industry in anticipation of the rise in demand. This together with the exit cost and the discounting cause the firm to optimally delay the exit decision. Similarly, the firm does not enter the industry unless its instantaneous profit rate is significantly positive as there is a strictly positive probability that the demand will soon decline to make the profit negative and there is an entry cost. So the firm waits until the demand is sufficient high to enter.

The explicit expressions for l^* and u^* also allow us to derive the following comparative static:

Proposition 2.2 *Let $\pi_1 \geq \pi_2$ and $\beta_1 \geq \beta_2$, and let l_1^* and l_2^* be the corresponding optimal exit barriers for these two profit functions and exit costs, respectively. Then $l_1^* \leq l_2^*$. Moreover, l^* is decreasing in μ, σ and β and increasing in r, and u^* is increasing in σ and α and decreasing in r.*

A firm with a uniformly higher profit rate for all levels of demand than another firm will exit later. Also, the higher the expected increase in the demand, the later a firm will exit; while the higher the interest rate, the higher the exit barrier and the earlier the firm exits. The former is obvious and the latter follows since the firm does not exit immediately after the instantaneous profit becomes negative in the anticipation of future profits and an increase in the interest rate makes future profits less valuable. In addition, the larger the volatility of the demand, the lower the exit barrier. This is so since the exit option of the firm limits the downside risk of the uncertain demands and thus the added upside potential by an increase in σ makes the firm more willing to suffer current losses. Finally, an increase in the exit cost makes the firm stay in the market longer.

The comparative statics for the optimal entry time are less intuitive because a change of the parameters also affects the optimal exit time on which the optimal entry time depends. An increase in σ has two effects: the increase makes it more likely that the firm will suffer loss in the near future and in the mean time it depresses the exit barrier and thus increases the time span over which the firm will be making a negative instantaneous profits. In anticipation of the latter and because of the former, the firm increases its entry barrier and enters the industry later. So an increase in the volatility of the demands, may or may not increase the life span of the firm. An increase in the riskless interest rate also has two effects. First, it makes waiting to enter more costly for the firm. Second, it increases the exit barrier and thus makes the time span over which the firm will suffer losses shorter. The latter makes the firm afford to enter earlier and the former gives the firm incentive to enter earlier. The combined effects are that the optimal entry time is earlier. But since the optimal exit time is also earlier, it is unclear whether the total life span of the firm will be longer.

There is no clear direction of change in the optimal entry time when the firm has a uniformly higher profit rate or when μ increases. On the one hand, it makes waiting more costly. On the other hand, it increases the time span over which the firm will suffer losses by decreasing the exit barrier and thus creates an incentive for the firm not to enter until the demand is sufficiently high. These are two opposing effects. To illustrate the point, we consider a change in the profit rate. Let $\pi(z, \delta)$ be the instantaneous profit parameterized by δ. Assume $\pi(\cdot, \delta)$ is continuous, $\pi(z, \cdot)$ is continuously differentiable, and $\partial\pi(z, \delta)/\partial\delta \geq 0$ for all z. Thus, the profit rate is uniformly higher as δ increases. For simplicity also assume that $\alpha = \beta = 0$. From the first order condition (17), we obtain:

$$\frac{\partial l^*}{\partial \delta} = \frac{1}{\pi(l^*, \delta)} \int_{l^*}^{\infty} e^{-a^*(z-l^*)} \frac{\partial}{\partial \delta} \pi(z, \delta) dz \leq 0,$$

since $\pi(l^*, \delta) < 0$ and $\partial\pi(z, \delta)/\partial\delta \geq 0$. It then follows from (18) that

$$\frac{\partial u^*}{\partial \delta} = \frac{e^{-a_*(u^*-l^*)}}{\pi(u^*, \delta)} \left(\pi(l^*, \delta) \frac{\partial l^*}{\partial \delta} - \int_{l^*}^{u^*} e^{a_*(z-l^*)} \frac{\partial}{\partial \delta} \pi(z, \delta) dz \right).$$

Note that $\pi(u^*, \delta) > 0$. Thus, in the above expression inside the parentheses, the first term represents the effect due to the increased profit loss after entry because of the reduced exit barrier, which calls for an increase in u^*. The second term represents the cost of waiting due to an increased profit loss while waiting, which has a negative effect on the entry barrier u^*. If $\partial\pi(z, \delta)/\partial\delta = 0$ for $z \in [l^*, u^*]$ and $\partial\pi(z, \delta)/\partial\delta > 0$ for some non-trivial interval in (u^*, ∞), then $\partial u^*/\partial\delta > 0$, i.e., the firm will enter later with a higher profit rate. Notice that the negligible increments of the profit rate in the region $[l^*, u^*]$ implies an insignificant waiting cost but a significant profit loss after entry if there is a significant decrease in the exit barrier which could be caused by a higher profit rate in the region (u^*, ∞).

3 The Exit Game

We investigate in this section an duopolistic exit game. There are initially two firms in the industry facing a stochastic demand modeled by a Brownian motion. Denote by $\pi_{ij}(X_t)$ the profit rate for firm i if there are j firms in the market and the total demand in the industry is X_t, and by β_{ij} the corresponding exit cost, where $i = 1, 2$ and $j = 1, 2$. As in the single firm context, we assume that π_{ij} is increasing and nonconstant, and there exist x_{ij}^0 so that $\pi_{ij}(y) > -r\beta_{ij}$ for $y > x_{ij}^0$ and $\pi_{ij}(y) < -r\beta$ for $y < x_{ij}^0$. Moreover, $E[\int_0^\infty e^{-rt}|\pi_{ij}(X_t)|dt] < \infty$ and that the process $\{e^{-rt}f_{ij}(X_t), t \in \bar{\Re}_+\}$ is bounded below and above by two martingales, where f_{ij} is as defined in (1) by replacing π with π_{ij}. The profit for a firm in a duopoly situation is naturally less than that

in a monopoly situation, and exit is usually more costly for a monopolist. Thus we assume that $\pi_{i1}(y) > \pi_{i2}(y)$ for all y, and $\beta_{i1} \geq \beta_{i2}$. It then follows that $x_{i1}^0 < x_{i2}^0$.

A firm will exit from the industry when it is no longer profitable to remain. As the profit rate of a firm depends on whether it is a monopoly or a duopoly, its exit decision will certainly depend on the exit decisions made by the other firm. Thus a gaming situation occurs. In the analysis of this *exit game*, we will focus our attention on subgame perfect Nash equilibria in pure strategies and therefore need an extensive form specification of the game. We will assume that once a firm exits from the industry, it is prohibitively costly to re-enter. Note that once a firm becomes a monopoly, its optimal strategy afterwards should simply be its unique optimal exit time established in Section 2 in a single firm context. Therefore, the game will be completely specified if we designate at any time t and in any state ω, the strategy a firm follows given that its opponent is still in the industry.

Formally, let $\phi_i : \Omega \times \Re_+ \mapsto \overline{\Re}_+$ be the strategy of firm i, where $\phi_i(\cdot, t) : \Omega \to \overline{\Re}_+$ is an optional time with $\phi_i(\omega, t) \geq t$ P-a.s. and we recall that $\overline{\Re}_+$ denotes the extended positive real line. We also impose the following *regularity* conditions:

1. the set

$$A = \{(\omega, s) \in \Omega \times \Re_+ : \phi_i(\omega, s) = s, \, s \in \Re_+\} \tag{21}$$

   is progressively measurable;[11]

2. $\phi_i(\omega, t)$ is right-continuous in t.

For brevity of notation, we will often use $\phi_i(S)$ to denote $\phi_i(\omega, S(\omega))$ as a random variable for an optional time S. Our interpretation of ϕ_i is as follows: At any optional time S, if firm i and its opponent are both in the industry and its opponent will continue to be in the industry, firm i will not exit immediately in the states where $\phi_i(S) > S$ and will exit immediately in the states where $\phi_i(S) = S$. The two regularity conditions are about how $\phi_i(t)$ changes over time and their purposes will become clear later. Denote by Φ the space of all the mappings $\phi : \Omega \times \Re_+ \mapsto \overline{\Re}_+$ that satisfy these above conditions.

Given $\phi \in \Phi$ and an optional time S, we put

$$T(S; \phi) \equiv \inf\{t \geq S : \phi(\omega, t) = t\}. \tag{22}$$

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[11]A process Y is progressively measurable if, as a mapping from $\Omega \times \Re_+$ to $\Re$, its restriction to the time set $[0, t]$ is measurable with respect to the product sigma-field generated by $\mathcal{F}_t$ and the Borel sigma-field of $[0, t]$. The progressive sigma-field is the sigma-field on $\Omega \times \Re_+$ generated by all the progressively measurable processes. A subset of $\Omega \times \Re_+$ is progressively measurable if it is an element of the progressive sigma-field. A good reference for these is Dellacherie and Meyer (1978).

In words, $T(S; \phi)$ is the first time after S that the strategy ϕ instructs the firm to exit prior to the exit time of its opponent.

To make our interpretation of $\phi(S)$ and $T(S; \phi)$ precisely correct, however, we need to show that one is able to tell at each optional time S whether one should continue or exit according to $\phi(S)$ and $T(S; \phi)$. That is, we need to show $\phi(S)$ and $T(S; \phi)$ are optional times. It is for these properties we need the two regularity conditions.

Proposition 3.1 *Suppose that* $\phi \in \Phi$ *and* S *is an optional time. Then* $\phi(S)$ *is an optional time with* $\phi(S) \geq S$ *a.s., and* $T(S; \phi)$ *is also an optional time with* $T(S; \phi) \geq S$ *a.s. Moreover,* $\phi(T(S; \phi)) = T(S; \phi)$ *a.s.*

PROOF. To prove that $\phi(S)$ is an optional time it suffices to prove that $\phi(S) \wedge t$ is $\mathcal{F}_t$-measurable, or $\phi(S) \wedge t \in \mathcal{F}_t$ as S is an optional time. By the right-continuity of $\phi(\omega, t)$ in t, we know $\phi(t)$ is Borel measurable in t. By the composition of two mappings, it follows that $\phi(\omega, S(\omega) \wedge t) \wedge t \in \mathcal{F}_t$. Next note that

$$\begin{aligned} \phi(\omega, S(\omega)) \wedge t &= \phi(\omega, S(\omega) \wedge t) 1_{\{S(\omega) \leq t\}}(\omega) + \phi(\omega, S(\omega) \wedge t) 1_{\{S(\omega) > t\}}(\omega) \\ &= \phi(\omega, S(\omega) \wedge t) 1_{\{S(\omega) \leq t\}}(\omega) + t 1_{\{S(\omega) > t\}}(\omega) \qquad a.s. \end{aligned}$$

By the fact that $\{S(\omega) > t\} \in \mathcal{F}_t$ as S is an optional time, we know $\phi(S) \wedge t \in \mathcal{F}_t$ as $\phi(S)$ is an optional time. The assertion that $\phi(S) \geq S$ a.s. is obvious.

Next we want to show that $T(S; \phi)$ is an optional time. Observe that

$$T(S(\omega); \phi) = \inf\{t \geq 0 : (\omega, t) \in A \cap [S(\omega), \infty)\},$$

where A is defined in (21). By the hypothesis that A is a progressive set, and the fact that the stochastic interval $[S, \infty)$ is also a progressive set, then Dellacherie and Meyer (1978, IV.50) shows that $T(S; \phi)$ is an optional time.

Finally, the last assertion follows from the hypothesis that $\phi(t)$ is right-continuous in t. ∎

Note that the last assertion of the above proposition follows from the hypothesis that $\phi \in \Phi$ is right-continuous in t. In words, it says that a firm will indeed exit at $\phi(T(S; \phi)) = T(S; \phi))$ if both its opponent and itself still remain at S. This is related to the kind of intertemporal consistency discussed by Perry and Reny (1990) and Simon and Stinchcombe (1988).[12] To understand the necessity of this, it suffices to consider the following example. Let $\phi(t) = 1$ for all $t \in [0, 1/2]$ and $\phi(t) = t$ for all $t > 1/2$. That is, one should remain in the industry from time 0 to time 1/2 but

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[12]We thanks David Kreps for pointing this out to us.

exit immediately after time $1/2$. This specification implies that $T(0;\phi) = 1/2$, but $\phi(T(0;\phi)) = 1 \neq T(0;\phi)$. At time $1/2$, one is not sure what to do. His strategy at that time, $\phi(1/2) = 1$, instructs him to remain in the industry while his strategies after time $1/2$ tell him, however, to exit immediately! The right-continuity of $\phi(t)$ in t eliminates this possibility.

Before we turn our attention to the existence of a Nash equilibrium and a subgame perfect equilibrium, some notation is in order. Let l_{ij}^* be the unique optimal exit barrier in the single firm problem when the profit function is π_{ij} and the exit cost is β_{ij}. Then l_{ij}^* is the optimal exit barrier for firm i when there are always j firms in the industry during its life span. For example, l_{22}^* is the optimal exit barrier for firm 2 when it has no chance to be a monopoly. Given the ordering on π_{i1} and π_{i2}, and β_{i1} and β_{i2}, assumed earlier, Proposition 2.2 shows that $l_{i1}^* < l_{i2}^*$. We assume further that $l_{11}^* < l_{21}^*$ and $l_{12}^* < l_{22}^*$. That is, firm 1 is a stronger than firm 2 in that firm 1's exit barriers for both the monopoly case and the duopoly case are higher than those of firm 2. Note that these last assumptions can be insured by the hypothesis that $\pi_{1j} > \pi_{2j}$ and $\beta_{1j} \geq \beta_{2j}$. But this is not necessary. Figure 1 depicts one possible relative positions of l_{ij}^*'s. For convenience, we use $T_{ij}^*(S)$ to denote $\inf\{t \geq S : X_t \leq l_{ij}^*\}$ for any optional time S. Also, let f_{ij}, h_{ij}, and v_{ij} be the functions defined in (13), (17), and (20) respectively, with π replaced by π_{ij} and β replaced by β_{ij}.

We will use ϕ_{-i} to denote firm i's opponent's strategy. A *Nash equilibrium* of the extensive form exit game is a pair of strategies $(\phi_1, \phi_2) \in \Phi \times \Phi$ so that given ϕ_{-i}, ϕ_i solves:

$$\sup_{\phi \in \Phi} \quad E\left[\int_0^{T(0;\phi) \wedge T(0;\phi_{-i})} e^{-\tau t} \pi_{i2}(X_t) dt - e^{-\tau T(0;\phi)} \beta_{i2} 1_{\{T(0;\phi) \leq T(0;\phi_{-i})\}} \right.$$
$$\left. + e^{-\tau T(0;\phi_{-i})} v_{i1}(X_{T(0;\phi_{-i})}) 1_{\{T(0;\phi) > T(0;\phi_{-i})\}} \right], \tag{23}$$

for $i = 1, 2$.

Note that the action taken by firm i, or the *exit time* of firm i, in a Nash equilibrium (ϕ_1, ϕ_2) is an optional time

$$T(0;\phi_i) 1_{\{T(0;\phi_i) \leq T(0;\phi_{-i})\}} + T_{i1}^* 1_{\{T(0;\phi_i) > T(0;\phi_{-i})\}}.$$

That is, on the set $\{T(0;\phi_i) \leq T(0;\phi_{-i})\}$, where firm i exits first, it exits at $T(0;\phi_i)$; while on the set $\{T(0;\phi_i) > T(0;\phi_{-i})\}$, where the opponent exits first, firm i behaves like a monopoly.

A Nash equilibrium (ϕ_1, ϕ_2) is a *subgame perfect equilibrium*[13] if for any optional time S, $T(S;\phi_i)$ solves

$$\sup_{\phi \in \Phi} E[\int_S^{T(S;\phi) \wedge T(S;\phi_{-i})} e^{-\tau(t-S)} \pi_{i2}(X_t) dt - e^{-\tau(T(S;\phi)-S)} \beta_{i2} 1_{\{T(S;\phi) \leq T(S;\phi_{-i})\}}$$
$$+ e^{-\tau(T(S;\phi_{-i})-S)} v_{i1}(X_{T(S;\phi_{-i})}) 1_{\{T(S;\phi) > T(S;\phi_{-i})\}} | \mathcal{F}_S], \tag{24}$$

where $E[\cdot | \mathcal{F}_S]$ denotes the expectation conditional on $\mathcal{F}_S$.

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[13]Our definition of a subgame perfect equilibrium is a direct generalization of Selten (1975) to continuous time.

A subgame perfect equilibrium (ϕ_1, ϕ_2) is said to be *unique* if for any other subgame perfect equilibrium (ϕ_1', ϕ_2'), we have

$$T(S; \phi_i)1_{\{T(S;\phi_i) \leq T(S;\phi_{-i})\}} = T(S; \phi_i')1_{\{T(S;\phi_i') \leq T(S;\phi_{-i}')\}} \quad a.s. \tag{25}$$

for all optional time S. This definition of uniqueness seems rather weak as it does not require the uniqueness of the mappings ϕ_i. Rather it focuses on the uniqueness of the exit times taken by the two firms in any subgame. This, however, is the right sense of uniqueness. The significance of ϕ_i lies in the implied exit action taken by firm i in equilibrium. There can be two strategies ϕ_i and ϕ_i' which differ on a set of (ω, t) whose projection onto Ω is of a strictly positive probability. But they can imply the same exit times of the two firms in the subgame starting from an optional time S as long as (25) is true.

The optimizations of (23) and (24) look formidable as they are looking for complicated mappings $\phi \in \Phi$. The following proposition shows that (23) is equivalent to a much simpler problem in which the optimization is performed by looking for optional times.

Proposition 3.2 *Let* $S \in \mathbf{T}$. *For every* $\tau \in \mathbf{T}$ *with* $\tau \geq S$ *a.s., there exists a* $\phi \in \Phi$ *so that* $\tau = T(S; \phi)$ *a.s. Thus, (24) for a given* S *is equivalent to*

$$\sup_{\substack{\tau \in \mathbf{T} \\ \tau \geq S}} E\left[\int_S^{\tau \wedge T(S;\phi_{-i})} e^{-r(t-S)} \pi_{i2}(X_t)dt - e^{-r(\tau-S)}\beta_{i2}1_{\{\tau \leq T(S;\phi_{-i})\}} \right. $$
$$\left. + e^{-r(T(S;\phi_{-i})-S)}v_{i1}(X_{T(S;\phi_{-i})})1_{\{\tau > T(S;\phi_{-i})\}} \Big| \mathcal{F}_S \right]. \tag{26}$$

PROOF. Let $\tau \in \mathbf{T}$ with $\tau \geq S$ a.s. It suffices to show that there exists $\phi \in \Phi$ so that $T(S; \phi) = \tau$ a.s. Define

$$\phi(\omega, t) = 1_{[0,\tau(\omega)]}(\omega, t)\tau(\omega) + t1_{(\tau(\omega),\infty)}(\omega, t).$$

Then

$$A = \{(\omega, t) \in \Omega \times \Re_+ : \phi(\omega, t) = t, t \in \Re_+\} = [\tau(\omega), \infty).$$

It is known that the stochastic interval $[\tau(\omega), \infty)$ is progressive measurable; see Dellacherie and Meyer (1978, IV.62). Then $\phi \in \Phi$. It is also easily verified that $T(S; \phi) = \tau$ P-a.s. ∎

To characterize the subgame perfect equilibrium, we first note that the exit game we are considering satisfies the monotone property studied in Huang and Li (1990, Proposition 4) in the sense that the longer its opponent stays in the industry, the earlier a firm will exit as the best response.

Lemma 3.1 *For any optional time* S, *firm* i's *optimal exit time from* S *on is monotone decreasing in firm* j's *exit time from* S *on, for* $i = 1, 2$ *and* $j \neq i$.

Consequently, firm i's optimal exit time in response to an upper (lower) bound of its opponent's exit time is a lower (upper) bound of firm i's exit time. Note that starting from any time S, firm i's optimal exit time is $T_{i2}^*(S)$ if $T(S; \phi_{-i}) = \infty$, and is $T_{i1}^*(S)$ if $T(S; \phi_{-i}) = S$. Therefore, the optimal exit time of firm i in response to any feasible strategy of its opponent starting from any optional time S cannot be later than its monopoly exit time and cannot be earlier than its exit time when it is certain to remain a duopoly throughout its life span.

Proposition 3.3 *Let $(\phi_1, \phi_2) \in \Phi \times \Phi$ be a subgame perfect equilibrium. Then for all optional time S,*

$$T_{i2}^*(S) \leq T(S; \phi_i) 1_{\{T(S; \phi_i) \leq T(S; \phi_{-i})\}} + T_{i1}^*(S) 1_{\{T(S; \phi_i) > T(S; \phi_{-i})\}} \leq T_{i1}^*(S) \quad a.s.$$

Next we consider the optimal strategy of firm i when the other firm plays a simple strategy suggested in the single firm problem. Suppose firm j, $j \neq i$, does not exit until the demand is lower than or equal to a critical level l_j in every subgame, namely,

$$\phi_j(t) = \inf\{s \geq t : X_s \leq l_j\}. \tag{27}$$

Firm i's best response must solve (24) for any optional time S. The arguments identical to that in Theorem 2.1 show that there is a unique solution to (24) which is a barrier policy of the following form. At any optional time S when firm j is still present, firm i will exit when X_t enters the set $[u_i, l_{i2}^*]$ either from above or from below before it reaches $(-\infty, l_j]$ where $u_i \geq l_j$. Otherwise, X_t will reach the set $(-\infty, l_j]$ first and firm j will exit and firm i becomes a monopoly and will follows its unique optimal strategy thereafter. The value u_i is determined as follows.

It suffices to consider the case when $X_S = x \in (l_j, l_{i2}^*]$. By Harrison (1985, Chapter 3), firm i's payoff under the barrier policy depicted above when firm j plays (27) can be written as:

$$
\begin{aligned}
\bar{v}_i(x, u_i, l_j) &= f_{i2}(x) - E_x\left[e^{-rT(u_i)}(f_{i2}(u_i) + \beta_{i2})|T(u_i) \leq T(l_j)\right] \\
&\quad - E_x\left[e^{-rT(l_j)}f_{i2}(l_j)|T(u_i) > T(l_j)\right] + E_x\left[e^{-rT(l_j)}v_{i1}(l_j)|T(u_i) > T(l_j)\right] \\
&= f_{i2}(x) - \psi(x, u_i, l_j)(f_{i2}(u_i) + \beta_{i2}) - \psi(x, l_j, u_i)f_{i2}(l_j) + \psi(x, l_j, u_i)v_{i1}(l_j),
\end{aligned}
\tag{28}
$$

where we recall that $T(y) = \inf\{t \geq 0 : X_t = y\}$ for any scalar,

$$\psi(x, y, z) \equiv \frac{\theta(x, y) - \theta(x, z)\theta(z, y)}{1 - \theta(z, y)\theta(y, z)} \tag{29}$$

for $y \wedge z < x < z \vee y$, $x, y, z \in \Re$. Thus u_i must maximize (28). Direct computation yields:

$$\frac{\partial \bar{v}_i(x, u, l_j)}{\partial u} = C_i \cdot h_i(u, l_j),$$

where C_i is a strictly positive term and

$$h_i(x,y) \equiv \gamma \int_y^x (\pi_{i2}(z) + r\beta_{i2})(e^{a \bullet (z-y)} - e^{-a^\bullet(z-y)})dz + \beta_{i2} + v_{i1}(y), \qquad (30)$$

for $x, y \in \Re$ and for $i = 1, 2$. Using this expression, it is straightforward to verify the following proposition:

Proposition 3.4 *Let $j \neq i$ and suppose $\phi_j(t) = \inf\{s \geq t : X_s \leq l_j\} \ \forall t \in \Re_+$. Then firm i's unique optimal exit time in response to ϕ_j is*

$$\tau(t)1_{\{X_{\tau(t)} \in A_i\}} + T_{i1}^*(t)1_{\{X_{\tau(t)} \in A_j\}}, \qquad (31)$$

where $\tau(t) = \inf\{s \geq t : X_s \in A_i \cup A_j\}$, $A_i \equiv [u_i, l_i]$, $A_j \equiv (-\infty, l_j]$, $l_i = l_{12}^*$,

$$u_i \equiv \begin{cases} \inf\{y \geq l_j : h_i(y, l_j) \leq 0\}, & \text{if } \inf\{y \geq l_j : h_i(y, l_j) \leq 0\} \leq l_{i2}^*; \\ \infty, & \text{otherwise,} \end{cases}$$

and $h_i(\cdot, \cdot)$ is defined in (30).

When firm i finds itself playing a duopoly game at S with a demand $X_S \in (l_j, u_i)$, it knows that the opponent will not exit until the demand falls to the level l_j. So staying in the industry, firm i will incur losses as the current demand is lower than l_{i2}^*. But, remaining in the industry gives firm i the opportunity to become a monopolist if the demand falls to reach l_j and firm j exits. So firm i trades off this potential future gains with the current losses. Falling demand turns out to be a good news for firm i as it may become a monopolist sooner. On the other hand, a rising demand means firm i will suffer losses longer and is a bad news. On balance, when the demand rises to reach u_i, the prospect of the potential future monopoly profit becomes so dismal and firm i exits. Similarly, if $X_S \geq l_{i2}^*$, firm i exits when the demand falls below l_{i2}^*. Continuing on, firm i will suffer too much loss to be balanced out by the prospect of becoming a monopoly in the future.

The set $A_i = [u_i, l_{i2}^*]$ characterizes the situation when firm i exits as a duopoly. By Lemma 3.1, the tougher the opponent (a smaller l_j), the more likely that firm i exits as a duopoly, or equivalently, the larger the set A_i (a smaller u_i). In particular, if $l_j \leq l_{i1}^*$, firm j will only exit after firm i cannot sustain even as a monopolist. Thus firm i has no chance to become a monopolist and it should exit optimally the first time the demand is below l_{i2}^* and $u_i = l_j$. Now take $j = 1$ and $l_j = l_{11}^*$ for this case. Then $u_2 = l_{11}^*$ and $A_2 = [l_{11}^*, l_{22}^*]$. Firm 2's optimal exit time in respond to firm 1's strategy is to exit immediately when the demand falls below l_{22}^*.

On the other hand, if $l_j \geq l_{i2}^*$, then firm j exits before firm i does, and firm i will exit as a monopolist at T_{i1}^* or $u_i = \infty$. Now take $j = 2$ and $l_2 = l_{22}^*$; that is, firm 2 exits when the demand

falls below l_{22}^*. Then $u_1 = \infty$ and $A_1 = \emptyset$. Firm 1 never exits as a duopoly and its unique strategy is not to exit until the demand falls below l_{11}^*.

Combining the above, we have thus identified a subgame perfect equilibrium in which the "strong" firm (firm 1) always exits as a monopolist and the "weak" firm (firm 2) exits as a duopoly since $l_{11}^* < l_{21}^*$ and $l_{22}^* > l_{12}^*$.

Proposition 3.5 *Put, for all $t \in \Re_+$,*

$$
\begin{aligned}
\phi_1(t) &= T_{11}^*(t), \\
\phi_2(t) &= T_{22}^*(t).
\end{aligned}
$$

(ϕ_1, ϕ_2) is a subgame perfect Nash equilibrium.[14] *The expected discounted future profits in the equilibrium for the two firms, or equilibrium payoffs, are, respectively,*

$$
v_1(x) = \begin{cases} g_{12}(x, l_{22}^*) + \theta(x, l_{22}^*) v_{11}(l_{22}^*) & \text{if } x \geq l_{22}^*, \\ v_{11}(x) & \text{if } x < l_{22}^*; \end{cases} \tag{32}
$$
$$
v_2(x) = v_{22}(x), \tag{33}
$$

where v_{ij} is defined in (4) with π and β replaced by π_{ij} and β_{ij}, and where

$$
g_{ij}(x, y) \equiv f_{ij}(x) - \theta(x, y) f_{ij}(y). \tag{34}
$$

This equilibrium is a "stationary equilibrium" in that the strategies for the two firms at any time t is a "copy" of their strategies at time 0. In this case, one easily verifies that $T(S; \phi_i) = \phi_i(S)$ with probability one.

Proposition 3.5 and Proposition 3.3 also gives a trivial sufficient condition for the uniqueness of the subgame perfect equilibrium: if $l_{12}^* \leq l_{21}^*$, the equilibrium of Proposition 3.5 is the unique subgame perfect equilibrium. When $l_{12}^* \leq l_{21}^*$, the level of demand below which firm 2 cannot survive as a monopolist is even higher than that at which firm 1 cannot survive as a duopoly. So naturally, firm 2 always exits earlier than firm 1 in any subgame as firm 2 is much too weaker than firm 1 and thus we have a unique subgame perfect equilibrium.

However, other equilibria may arise if $l_{21}^* < l_{12}^*$. See Figure 1 for the assumed order of barriers l_{ij}^* in this case. If this is the case, when demand is between l_{21}^* and l_{12}^*, neither firm can survive as a duopoly and both can survive as a monopolist. Thus there may be more than one equilibrium. We now identify candidates of other equilibria.

---

[14]One easily verifies that $\phi_i \in \Phi$, for $i = 1, 2$.

Suppose that firm 2 does not exit until the demand is lower than or equal to l_{21}^* in every subgame, namely, $\phi_2(t) = T_{21}^*(t)$, which is an upper bound of firm 2's equilibrium exit time. By Proposition 3.4 firm 1's optimal exit time in response to firm 2's strategy is

$$\tau(t)1_{\{X_{\tau(t)} \in A_1\}} + T_{11}^*(t)1_{\{X_{\tau(t)} \in A_2\}} \tag{35}$$

where $A_1 = [u_1, l_{12}^*]$, $A_2 = (-\infty, l_{21}^*]$, and $\tau(t) = \inf\{s \geq t : X_s \in A_1 \cup A_2\}$. One can also verify that $u_1 > l_{21}^*$. As a consequence of Lemma 3.1, this exit time of firm 1 is a lower bound on its subgame perfect equilibrium exit times. This lower bound is a tighter lower bound than $T_{12}^*(S)$ as $u_1 > l_{21}^*$.

Suppose that u_1 above is equal to infinity. Then A_1 is an empty set and firm 1 does not exit as a duopoly in its response. Since its response here is a lower bound on its subgame perfect equilibrium exit times at any subgame, the equilibrium of Proposition 3.5 is the unique subgame perfect equilibrium. So we suppose that $u_1 \in (l_{21}^*, l_{12}^*]$. Arguments similar to those used to prove Theorem 2.1 show that, in response to firm 1's exit time defined in (35), firm 2's optimal exit time is a stationary two-barrier policy at an optional time S as a duopoly: exits when demand reaches A_3 before it reaches A_1, where $A_1 \equiv [u_1, l_{12}^*]$ as in (35), and $A_3 \equiv (-\infty, l_2] \cup [u_2, l_{22}^*]$ with $l_2\ (> l_{21}^*)$ and u_2 being two critical numbers; otherwise, exits when the demand falls below l_{21}^*. The interpretation of this two-barrier policy is similar to that for (31). When the demand is below l_{22}^* and firm 2 is a duopoly, it always trades off the potential of being a monopoly in the future against the current duopoly losses in its exit decision. Furthermore, this best response exit time for firm 2 becomes a new upper bound of its subgame perfect equilibrium exit times by Lemma 3.1 since (35) is a lower bound of firm 1's subgame perfect equilibrium exit times at any subgame. We repeat this procedure to generate tighter and tighter upper bounds on firm 2's and tighter and tighter lower bounds on firm 1's subgame perfect equilibrium exit times. If this procedure has a fixed point other than the equilibrium of Proposition 3.5, this fixed point is itself a subgame perfect equilibrium and provides the exact largest lower bound and the exact least upper bound, respectively, for firm 1's and firm 2's subgame perfect equilibrium exit times. This fact is recorded in the following proposition:

Theorem 3.1 *Suppose that there exist u_1^*, l_1^*, u_2^* and l_2^* with $l_{21}^* < l_2^* < u_1^* \leq l_1^* \leq l_{12}^* < u_2^*$ such that $h_1(u_1^*, l_2^*) = 0$, $h_1(u_2^*, l_1^*) = 0$, and $h_2(u_1^*, l_2^*) = 0$ where*

$$u_2^* = \begin{cases} \inf\{y \geq l_1^* : h_2(y, l_1^*) \leq 0\}, & \text{if } \inf\{y \geq l_1^* : h_2(y, l_1^*) \leq 0\} \leq l_{22}^*; \\ \infty, & \text{otherwise}, \end{cases}$$

and $h_i(x, y)$ are defined in (30) with the understanding that $\int_y^x = -\int_x^y$ for $x < y$. Then

$$
\begin{aligned}
\phi_1(t) &\equiv \tau(t)1_{\{X_{\tau(t)} \in A_1\}} + T_{11}^*(t)1_{\{X_{\tau(t)} \in A_2\}}, && t \in \Re_+, \\
\phi_2(t) &\equiv T_{21}^*(t)1_{\{X_{\tau(t)} \in A_1\}} + \tau(t)1_{\{X_{\tau(t)} \in A_2\}}, && t \in \Re_+,
\end{aligned}
\tag{36}
$$

is a subgame perfect equilibrium, where $\tau(t) \equiv \inf\{s \geq t : X_s \in A_1 \cup A_2\}$, $A_1 \equiv [u_1^*, l_1^*]$, and $A_2 \equiv (-\infty, l_2^*] \cup [u_2^*, l_{22}^*]$. Moreover, if (ϕ_1', ϕ_2') is another subgame perfect equilibrium, then $\phi_1(t) \leq T(t; \phi_1') \leq T_{11}^*(t)$ and $T_{22}^*(t) \leq T(t; \phi_2) \leq T_2(t)$ a.s.

The equilibrium of Theorem 3.1 is a "stationary equilibrium" in the sense that every $\phi_i(t)$ is a "copy" of $\phi_i(0)$. We can thus describe the equilibrium by looking at $\phi_i(0)$. We take cases, and also refer to Figure 1 in which the relative positions of the barriers are depicted. First, if $X_0 \geq l_{22}^*$, firm 2 exits when the demand decreases to l_{22}^* and firm 1 continues to l_{11}^* as a monopoly. Second, if $X_0 \in [u_2^*, l_{22}^*]$, firm 2 exits immediately and firm 1 is a monopoly throughout. Third, if $X_0 \in (l_1^*, u_2^*)$, firm 2 and firm 1 both stay on with the former making a negative profit. If the demand rises to reach u_2^*, it is a bad news for firm 2 as firm 1 will be in the industry for a long time. Thus firm 2 exits and firm 1 becomes a monopoly. On the other hand, if the demand drops to reach l_1^*, which is lower than l_{12}^*, firm 1 knows that firm 2 will not exit until either the demand reaches u_2^* from below or reaches l_{21}^* from above, so firm 1 exits as the prospect of being a monopoly in the future is gloomy and firm 2 becomes a monopoly. Fourth, if $X_0 \in [u_1^*, l_1^*]$, as firm 2 will continue until the demand decrease to l_2^*, firm 1 exits immediately. Fifth, if $X_0 \in (l_2^*, u_1^*)$, firm 1 exits when the demand increases to reach u_1^* before it decreases to l_2^* as the prospect of being a monopoly is gloomy. Otherwise, firm 2 exits when the demand reaches l_2^* and firm 1 continues as a monopoly. Finally, if $X_0 \leq l_2^*$, firm 2 exits immediately and firm 1 plays its monopoly strategy henceforth.

There are two interesting features of this equilibrium. First, either firm may exit when the demand has been increasing. This happens in two occasions. When demand is between l_1^* and u_2^*, the weak firm (firm 2) exits as the demand drifts up to reach u_2^* before it reaches down to l_1^*; or when the demand is between l_2^* and u_1^*, the strong firm (firm 1) may exit as the demand goes up to reach u_1^* before it decreases to reach l_2^*. Second, the strong firm exits before the weak firm does when the demand has been declining. This happens when the demand is between l_1^* and u_2^*.

The equations that determine u_1^*, l_2^*, l_1^*, and u_2^* are from the first order conditions for the two firms' optimization problems of (24). The existence of a solution to these equations with the desired order implies the existence of a stationary subgame perfect equilibrium in addition to the one in Proposition 3.5. Otherwise, A_1 becomes an empty set and A_2 becomes $(-\infty, l_{22}^*]$. Then one concludes that the lower bound on firm 1's subgame perfect equilibrium exit time at any subgame starting from an optional time S is $T_{11}^*(S)$ and the upper bound for firm 2's exit time is $T_{22}^*(S)$. As a consequence, there exists a unique subgame perfect equilibrium.

The following example demonstrates how one uses Theorem 3.1 to construct scenarios that imply either a unique or multiple subgame perfect equilibria.

Example 3.1 *Let $\beta_{ij} = 0$ for all i, j and*

$$\pi_{ij}(y) = \begin{cases} a_{ij}, & \text{if } y \geq x_{ij}^0; \\ -b_{ij}, & \text{if } y < x_{ij}^0, \end{cases}$$

be increasing step functions with $a_{i1} > a_{i2} > 0$, $b_{i2} > b_{i1} > 0$, and $x_{i1}^0 < x_{i2}^0$ for $i, j = 1, 2$. Also assume that $\mu < 0$.

We choose the parameters, a_{ij}, b_{ij} and x_{ij}^0, through the following steps:

1. *Choose l_{11}^*, l_{21}^*, l_{12}^*, and x_{11}^0 so that $l_{11}^* < l_{21}^* < l_{12}^*$, $l_{21}^* < x_{11}^0$, and $l_{12}^* - l_{21}^* < l_{21}^* - l_{11}^*$.*

2. *Let*

$$\eta(x) \equiv \frac{1}{a^*} e^{a^* x} + \frac{1}{a_*} e^{-a_* x}.$$

It can be shown that

$$\eta'(x) = e^{a^* x} - e^{-a_* x} \begin{cases} > 0, & \text{if } x > 0; \\ < 0, & \text{if } x < 0, \end{cases} \tag{37}$$

and for $\mu < 0$ (or $a^* > a_*$),

$$\frac{d}{dx}(\eta(x) - \eta(-x)) \begin{cases} > 0, & \text{if } x > 0; \\ < 0, & \text{if } x < 0. \end{cases} \tag{38}$$

Thus, for $\epsilon > 0$ sufficiently small, the following hold,

$$\eta(l_{21}^* - l_{11}^* + \epsilon) > \eta(-(l_{12}^* - l_{21}^* - 2\epsilon)), \tag{39}$$

and

$$\frac{\eta(l_{12}^* - l_{21}^* - \epsilon) - \eta(0)}{\eta(l_{12}^* - l_{21}^* - 2\epsilon) - \eta(0)} < \frac{\eta(l_{12}^* - l_{21}^*) - \eta(0)}{\eta(-(l_{12}^* - l_{21}^*)) - \eta(0)}, \tag{40}$$

since $\eta(l_{21}^* - l_{11}^*) > \eta(l_{12}^* - l_{21}^*) > \eta(-(l_{12}^* - l_{21}^*))$ by (37) and (38), also noticing that $l_{12}^* - l_{21}^* < l_{21}^* - l_{11}^*$ as determined in step 1.

Choose ϵ that satisfies (39), (40) and $0 < \epsilon < (l_{12}^* - l_{21}^*)/2$. Then, set b_{ij} so that

$$\frac{b_{12}}{b_{11}} = \frac{\eta(l_{21}^* - l_{11}^* + \epsilon) - \eta(0)}{\eta(-(l_{12}^* - l_{21}^* - 2\epsilon)) - \eta(0)}, \tag{41}$$

$$\frac{b_{22}}{b_{21}} = \frac{\eta(l_{12}^* - l_{21}^* - \epsilon) - \eta(0)}{\eta(l_{12}^* - l_{21}^* - 2\epsilon) - \eta(0)}. \tag{42}$$

Note that $b_{12} > b_{11}$ by (39) and $b_{22} > b_{21}$ by (37). Furthermore,

$$\frac{b_{22}}{b_{21}} < \frac{\eta(l_{12}^* - l_{21}^*) - \eta(0)}{\eta(-(l_{12}^* - l_{21}^*)) - \eta(0)} \tag{43}$$

by (40).

3. *Let* $l_{22}^* = l_{12}^* + \delta$, *where* $\delta > 0$ *is so chosen that*

$$\frac{b_{22}}{b_{21}} < \frac{\eta(l_{12}^* - l_{21}^*) - \eta(0)}{\eta(-(l_{22}^* - l_{21}^*)) - \eta(0)}. \tag{44}$$

The existence of such a δ *follows from (43) since* $\lim_{\delta \downarrow 0} l_{22}^* = l_{12}^*$.

4. *Finally, we choose* a_{ij} *and* x_{ij}^0 *to satisfy*

$$l_{ij}^* = x_{ij}^0 - \frac{1}{a^*}\ln(1 + d_{ij})$$

where $d_{ij} \equiv a_{ij}/b_{ij}$, *for* $i, j = 1, 2$.

Let $l_2^* = l_{21}^* + \epsilon$ *and* $u_1^* = l_{12}^* - \epsilon$. *Then* $h_1(u_1^*, l_2^*) = 0$ *by (41) and* $h_2(u_1^*, l_2^*) = 0$ *by (42). Also, for all* $y \in (l_{12}^*, l_{22}^*]$,

$$h_2(y, l_{12}^*) \geq e^{a \cdot l_2^*}(-b_{22}(\eta(-(l_{22}^* - l_{21}^*)) - \eta(0)) + b_{21}(\eta(l_{12}^* - l_{21}^*) - \eta(0))) > 0,$$

by (44). Therefore, $u_2^* = \infty$ *and* $l_1^* = l_{12}^*$. *So,* (T_1, T_2) *in (36) with above* u_i^* *and* l_i^* *is an equilibrium.*

It is much easier to construct the case in which there is a unique equilibrium. In the above example, after completion of step 1, we simply set

$$1 < \frac{b_{12}}{b_{11}} < \frac{\eta(l_{21}^* - l_{11}^*) - \eta(0)}{\eta(-(l_{12}^* - l_{21}^*)) - \eta(0)}.$$

This can be done since $\eta(l_{21}^* - l_{11}^*) > \eta(-(l_{12}^* - l_{21}^*))$. *Then, for any* $y \in [l_{21}^*, l_{12}^*]$

$$h_1(y, l_{21}^*) \geq h_1(l_{12}^*, l_{21}^*) = e^{a \cdot l_{21}^*}(-b_{12}(\eta(-(l_{12}^* - l_{21}^*)) - \eta(0)) + b_{11}(\eta(l_{21}^* - l_{11}^*) - \eta(0))) > 0,$$

and firm 1 will not exit before firm 2's longest possible exit time T_{21}^*. *The uniqueness of the equilibrium follows from Theorem 3.1.*

4   Game of an Incumbent Versus a Potential Entrant

In this section, we investigate the situation where firm 1 is initially in the market and has a single option to exit, while firm 2 is not in the market at the beginning and has options to enter and then exit. This is thus a game of an incumbent versus a potential entrant, henceforth abbreviated as simply the *entry game*.

Before we proceed formally, we note the following. First, in any subgame perfect equilibrium, once the two firms are in the industry in the same time, therehence, they must be playing a subgame perfect equilibrium in the exit game discussed in Section 3. Second, if firm 1 exits before firm 2 enters, then afterwards, firm 2 must follow its unique optimal single firm entry and exit decisions

characterized in Section 2. For simplicity, we assume that there exists a unique subgame perfect equilibrium for the exit game. By the analysis of Section 3, this unique equilibrium is the one in Proposition 3.5. Given this hypothesis and the two observations noted above, in analyzing the entry game with a focus on subgame perfect equilibria, we can restrict our attention on firm 1's exit decision before firm 2 enters and on firm 2's entry decision before firm 1 exits.

Let $\phi_1^b : \Omega \times \Re_+ \mapsto \overline{\Re}_+$ be firm 1's strategy before firm 2 enters, where $\phi_1^b(t)$ is an optional time with $\phi_1^b(t) \geq t$ a.s. In words, if at time t firm 2 has not entered and firm 1 has not exited, firm 1 will exit immediately if and only if $\phi_1^b(t) = t$. Similarly, let $\phi_2^e : \Omega \times \Re_+ \mapsto \overline{\Re}_+$ be firm 2's strategy before firm 1 exits, where $\phi_2^e(t)$ is an optional time with $\phi_2^e(t) \geq t$ a.s. for all $x \in \Re$. If at time t, firm 1 has not exited and firm 2 has not entered, firm 2 enters immediately if and only if $\phi_2^e(t) = t$. We assume that ϕ_1^b and ϕ_2^e are right-continuous in t and satisfy the measurability condition stipulated for ϕ_i in Section 3. Denote by Φ_1 and Φ_2 the space of all the possible ϕ_1^b and ϕ_2^e, respectively.

Define

$$T(S; \phi_1^b) = \inf\{t \geq S : \phi_1^b(t) = t\}$$

and

$$T(S; \phi_2^e) = \inf\{t \geq S : \phi_2^e(t) = t\}$$

for all $\phi_1^b \in \Phi_1$ and $\phi_2^e \in \Phi_2$. By Proposition 3.1, $T(S; \phi_1^b)$ and $T(S; \phi_2^e)$ are optional times.

A subgame perfect equilibrium of the entry game is composed of $(\phi_1^b, \phi_2^e) \in \Phi_1 \times \Phi_2$ so that $T(S; \phi_1^b)$ solves, for all optional time S,

$$\sup_{\substack{T \in \mathbf{T} \\ T \geq S}} E[\int_S^{T \wedge T(S;\phi_2^e)} e^{-r(s-S)} \pi_{11}(X_s)ds - e^{-r(T-S)} \beta_{11} 1_{\{T \leq T(S;\phi_2^e)\}}$$
$$+ e^{-r(T(S;\phi_2^e)-S)} v_1(X_{T(S;\phi_2^e)}) 1_{\{T > T(S;\phi_2^e)\}} | \mathcal{F}_S],$$

where v_1 is defined in (32), and $T(S; \phi_2^e)$ solves

$$\sup_{\substack{T \in \mathbf{T} \\ T \geq S}} E\left[e^{-r(T-S)}(v_2(X_T) - \alpha_{22}) 1_{\{T \leq T(S;\phi_1^b)\}} + e^{-r(T(S;\phi_1^b)-S)} \hat{v}_{21}(X_{T(S;\phi_1^b)}) 1_{\{T > T(S;\phi_1^b)\}} | \mathcal{F}_S \right],$$

where v_2 is defined in (33) and $\hat{v}_{21}$ is defined in (19) with π and α replaced by π_{21} and α_{21}, respectively.

The following proposition reports a stationary subgame perfect equilibrium for the entry game. The proof of this proposition, being similar to that of Theorem 3.1, is omitted. In this equilibrium firm 2 sets a critical level u_2^e such that it enters the market the first time demand is above u_2^e before

firm 1 exits and then plays the unique subgame perfect equilibrium of the exit game thereafter; otherwise it behaves optimally as a monopolist after firm 1 exits. The value u_2^e is determined so as to balance the marginal benefit of being a duopoly presently and the marginal benefit of being a future monopoly. Firm 1 sets a critical level l_1^b such that it exits if demand is below l_1^b before firm 2's entry; otherwise, it plays the unique subgame perfect equilibrium of the exit game after firm 2 enters.

We will use u_{21}^* and u_{22}^* to denote the entry barrier for firm 2 as a monopoly and as a duopoly, respectively, as described in (18) of Proposition 2.1. In addition, define

$$T_{21}^e(t) \equiv \inf\{s \ge t : X_s \ge u_{21}^*\}.$$

Proposition 4.1 *Put*

$$\phi_1^b(t) = \tau(t)1_{\{X_{\tau(t)} \in A_1\}} + T_{11}^*(\tau(t))1_{\{X_{\tau(t)} \in A_2\}}, \tag{45}$$

$$\phi_2^e(t) = T_{21}^e(\tau(t))1_{\{X_{\tau(t)} \in A_1\}} + \tau(t)1_{\{X_{\tau(t)} \in A_2\}}, \tag{46}$$

where $l_1^b \in (l_{11}^*, l_{12}^*)$, $u_2^e \in (u_{22}^*, \infty)$ are unique solutions to $\hat{h}_1(u_2^e, l_1^b) = 0$ and $\hat{h}_2(u_2^e, l_1^b) = 0$ with

$$\hat{h}_1(u, l) = \gamma \int_l^u (\pi_{11}(z) + r\beta_{11}) \left(e^{a^*(u-z)} - e^{-a_*(u-z)} \right) dz + \beta_{11} + v_1(u),$$

$$\hat{h}_2(u, l) = \gamma \left(- \int_{l_{22}^*}^u (\pi_{22}(z) - r\alpha_{22}) \left(e^{a_*(z-l)} - e^{-a^*(z-l)} \right) dz \right.$$

$$\left. + \left(\frac{r}{a_*} e^{a_*(l_{22}^* - l)} + \frac{r}{a^*} e^{-a^*(l_{22}^* - l)} \right) (\alpha_{22} + \beta_{22}) \right) + \hat{v}_{21}(l),$$

and

$$\tau(t) = \inf\{s \ge t : X_s \in A_1 \cup A_2\},$$

with $A_1 = (-\infty, l_1^b]$, and $A_2 = [u_2^e, +\infty)$. Then (ϕ_1^b, ϕ_2^e) is a subgame perfect equilibrium.

Three interesting observations can be made. First, in the equilibrium, $l_1^b > l_{11}^*$. Thus the existence of a potential entrant may force the incumbent to exit earlier than when there is no potential entrant and hence the incumbent has a shorter economic life time. This occurs when demand decreases to reach l_1^b before it increases to reach u_2^e. The potential of becoming a duopolist at higher demand levels in the future, makes firm 1 less tolerant to the incurrence of current losses.

Second, the entrant sets its entry level u_2^e before firm 1 exits higher than its duopoly entry level u_{22}^*. This is due to the opportunity of its becoming a monopolist if it waits long enough for firm 1 to exit. Therefore the entrant sacrifices its current duopoly profits in exchange of the potential monopoly profit in the future.

Finally, there are two possible results of the incumbent versus entrant game: "peace" or "war". By "peace", we mean that the entrant enters after the incumbent exits and both firms share the market by different time segment. If the entrant enters before the incumbent exits, then they battle as a duopoly in the market.

5   Generalizations

In contrast to Dixit (1989), the single firm entry-exit problem analyzed in Section 2 does not allow re-entry once a firm exits. Our approach can be generalized to count for re-entry, however.

Suppose that the cost of entry and re-entry is the same equal to α. The cost of exit is β. For simplicity, assume that π is bounded, strictly increasing, and continuous. Let $v(x; I)$ and $v(x; O)$ be the optimal expected profits when the current demand is x, and the firm is in the market and out of the market, respectively. Given that π is bounded and $r > 0$, $v(x; I)$ and $v(x; O)$ are bounded. When the firm is in the market, the problem the firm faces is to find an optimal exit time that solves:

$$v(x; I) = \sup_{T \in \mathbf{T}} E\left[\int_0^T e^{-rt} \pi(X_t) dt + e^{-rT}(v(X_T; O) - \beta) \right]. \tag{47}$$

When outside the market, the firm seeks an optimal entry time that solves:

$$v(x; O) = \sup_{T \in \mathbf{T}} E\left[e^{-rT}(v(X_T; I) - \alpha) \right]. \tag{48}$$

Note that (47) is equivalent to

$$\inf_{T \in \mathbf{T}} E\left[e^{-rT}(f(X_T) - v(X_T; O) + \beta) \right].$$

Assume to begin that $\bar{v}(x; I)$ and $\bar{v}(x; O)$ are continuous in x. Theorem 1 of Huang and Li (1990) shows that there exist solutions to (47) and (48). Assume that these solutions are barrier policies.

Let $\bar{v}(x, l; I)$ and $\bar{v}(x, u; O)$ denote the expected future profits, in and outside the market respectively, under a barrier policy with the exit and entry barriers, l and u. Then,

$$\bar{v}(x, l; I) = f(x) - \theta(x, l)f(l) + \theta(x, l)(\bar{v}(l, u; O) - \beta) \quad \text{for } x \geq l, \text{ and }, \tag{49}$$

$$\bar{v}(x, u; O) = \theta(x, u)(\bar{v}(u, l; I) - \alpha) \quad \text{for } x \leq u. \tag{50}$$

Letting $x = u$ in (49) and $x = l$ in (50), we can solve linear equations (49) and (50) to obtain:

$$\bar{v}(u, l; I) = f(u) - \frac{\theta(u, l)(f(l) + \beta) - \theta(u, l)\theta(l, u)(f(u) - \alpha)}{1 - \theta(u, l)\theta(l, u)}, \quad \text{and} \tag{51}$$

$$\bar{v}(l, u; O) = \frac{\theta(l, u)(f(u) - \alpha) - \theta(l, u)\theta(u, l)(f(l) + \beta)}{1 - \theta(l, u)\theta(u, l)}. \tag{52}$$

Thus, the expected future profits are given by (49) and (50) under a barrier policy with any pair of parameters (l, u). The search for the values of the optimal barriers is reduced to finding l and u that maximize $\bar{v}(x, u; O)$ and $\bar{v}(x, l; I)$. It can be shown that

$$\frac{\partial \bar{v}(x; \cdot)}{\partial u} = C_1 h_1(l, u), \quad \text{and} \quad \frac{\partial \bar{v}(x; \cdot)}{\partial l} = C_2 h_2(l, u),$$

where C_1 and C_2 are strictly positive terms,

$$h_1(l, u) \equiv \frac{1}{\gamma}(\alpha + \beta) - \int_l^u (\pi(z) - r\alpha)(e^{a_*(z-l)} - e^{-a^*(z-l)})dz, \tag{53}$$

$$h_2(l, u) \equiv -\frac{1}{\gamma}(\alpha + \beta) + \int_l^u (\pi(z) + r\beta)(e^{-a_*(u-z)} - e^{a^*(u-z)})dz, \tag{54}$$

and a_*, a_*, and γ are given by (14) - (16). Therefore, the optimal barriers must solve:

$$h_1(l^*, u^*) = 0, \quad \text{and} \quad h_2(l^*, u^*) = 0. \tag{55}$$

It is then straightforward to verify that $\bar{v}(x; I)$ and $\bar{v}(x; O)$ are indeed continuous functions of x as we have assumed earlier. Finally, we can use the principle of dynamic programming to verify that there do not exist non-barrier policies that strictly dominate the optimal barrier policy. This justifies our searching for an optimal barrier policy.

Finally, differentiating $h_1(l, u)$ with respect to u gives

$$\frac{\partial h_1(l, u)}{\partial u} = (e^{a_*(u-l)} - e^{-a^*(u-l)})(r\alpha - \pi(u)).$$

So, $h_1(l, u) > 0$ for $l < u < \pi^{-1}(r\alpha)$ since $\pi(\cdot)$ is strictly increasing and $h_1(l, l) = \gamma^{-1}(\alpha + \beta) \geq 0$ and $h(l, \cdot)$ is increasing for $l < u < \pi^{-1}(r\alpha)$. Thus, by (55)

$$u^* \geq \pi^{-1}(r\alpha) \geq \pi^{-1}(0).$$

Similarly, we can show that

$$l^* \leq \pi^{-1}(r\beta) \leq \pi^{-1}(0).$$

Given the above solution to a single firm's problem, we can then apply the technique developed in Section 3 to analyze a duopolistic entry-exit game, albeit more complicated.

6 Concluding remarks

We have analyzed optimal entry-exit decisions for a single firm and for firms behaving strategically, both in continuous time with Brownian motion uncertainty. Under very general hypotheses, we have demonstrated the existence and uniqueness of an optimal policy for a single firm. The optimal policy is a barrier policy and the entry and exit barriers are solutions to algebraic equations.

In a duopolistic context, we have worked directly in continuous time and shown ways to look for subgame perfect equilibria. One equilibrium is always the one in which the strong firm behaves like a monopoly throughout and thus the weak firm is always a duopoly in its lifetime. Other equilibria may exist, however. In these other equilibria, both the strong firm and the weak firm may exit even when the demand has been increasing on the average.

The method used in our analysis can potentially be useful to many other optimal economic decisions. Indeed, any situation in which dichotomous choices are made over time under uncertainty can be formulated as an optimal timing problem and our technique may apply.

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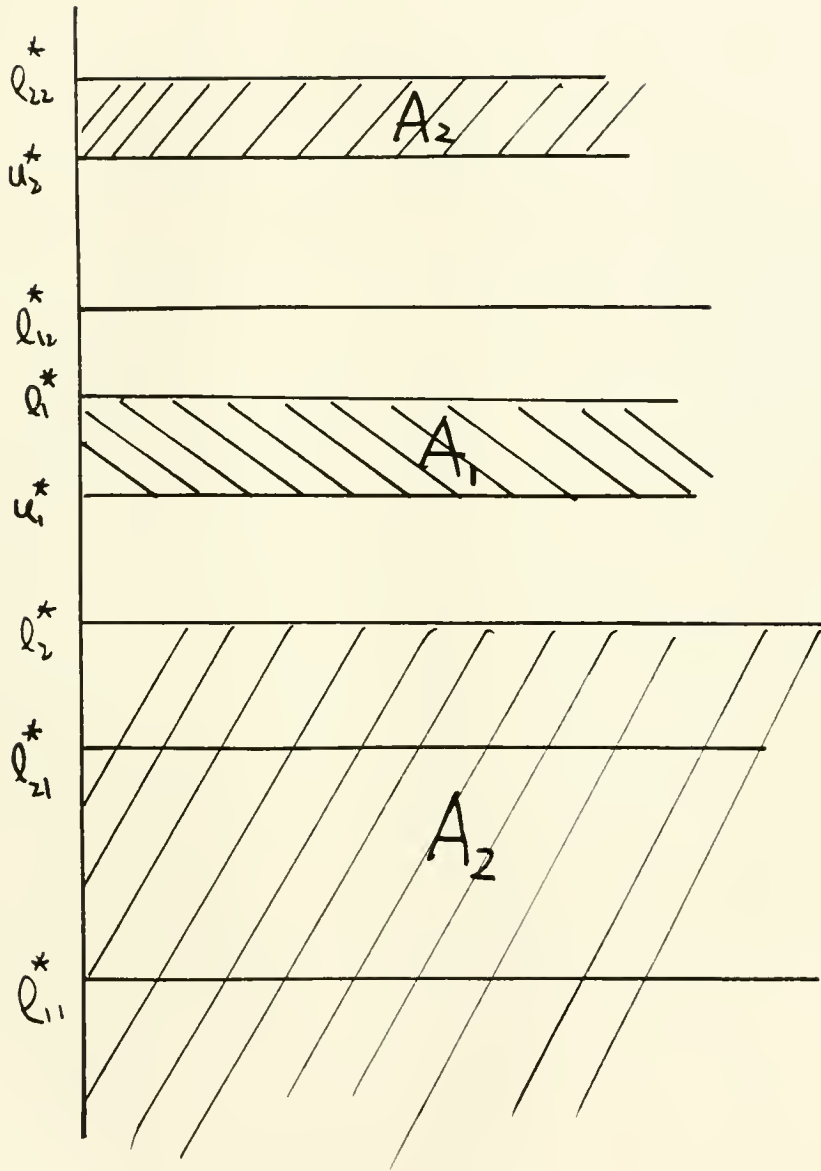
Figure 1.

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