Admissible Nilpotent Coadjoint Orbits of p-adic Reductive Lie Groups

by

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ABSTRACT

The orbit method conjectures a close relationship between the set of irreducible unitary representations of a Lie group G over a local field, and admissible coadjoint orbits in the dual of the Lie algebra.

We define admissibility for nilpotent coadjoint orbits in $\mathfrak{p}$-adic reductive Lie groups, and compute the set of admissible orbits for a range of examples. We find that for unitary, symplectic, orthogonal, general linear and special linear groups defined over $\mathfrak{p}$-adic fields, the admissible nilpotent orbits coincide with the special orbits defined by Lusztig and Spaltenstein in connection with the Springer correspondence. We also show that for the $\mathfrak{p}$-adic Lie group of type G_2, all special orbits are admissible, and further that the minimal orbit is a nonspecial admissible orbit.

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To Ralph

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Contents

1 Introduction

A fundamental problem in the theory of Lie groups over local fields is the classification and construction of the irreducible unitary representations of a reductive Lie group G. One promising approach to this problem is known as the orbit method.

The orbit method seeks to associate irreducible unitary representations of G to coadjoint orbits in the dual of the Lie algebra of G. The former are algebraic objects; the latter have rich geometric structure. One can hope, and in many cases is able, to use this geometry to construct the associated representation. Kirillov provided the first success of the orbit method, with the following theorem.

Theorem 1.1 (Kirillov [K]). *Let G be a nilpotent connected simply connected real Lie group. There is a constructive bijection*

$$\hat{G}_u = \left\{ \begin{matrix} \textit{irreducible unitary} \\ \textit{representations of} \\ G \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \textit{coadjoint } G\textit{-orbits} \\ \textit{in } \mathfrak{g}^* \end{matrix} \right\} = \mathfrak{g}^*//G.$$

For example, if $G = \mathbb{R}$, then $\mathfrak{g}^* = \mathfrak{g}^*//G = \mathbb{R}$. The representation associated to the orbit $x \in \mathbb{R}$ is given by $\phi_x(y) = \exp(2\pi i x y)$ for each $y \in \mathbb{R}$. When G fails to be simply connected, however, the conclusion of the theorem can fail as well. For instance, let $G = \mathbb{R}/\mathbb{Z}$. Then we again have $\mathfrak{g}^* = \mathfrak{g}^*//G = \mathbb{R}$, but the unitary representations are only those ϕ_x with $x \in \mathbb{Z}$. In other words, not all coadjoint orbits arise in an orbit correspondence.

We thus seek to refine our statement of the orbit correspondence by describing those coadjoint orbits to which representations can conjecturally be associated. The first attempts in this direction led to the subset of *integral* orbits. They are defined as follows.

Fix an additive unitary character ψ of the base field F. For instance, if $F = \mathbb{R}$, we may choose $\psi(x) = \exp(2\pi i x)$.

Definition 1.2. An element $f \in \mathfrak{g}^*$ is *integral* if there exists a unitary representation τ of $G^f = \{g \in G \mid g \cdot f = f\}$ satisfying $\tau(\exp(X)) = \psi(f(X))$ for all X in $\mathfrak{g}^f \cap$ domain of exp.

This criterion is sufficient for the case of $G = \mathbb{R}/\mathbb{Z}$; in fact it is enough to obtain a stronger version of Kirillov's theorem.

Theorem 1.3 (Kirillov [K], Moore [Mo]). *Let G be nilpotent connected Lie group over a real or $\mathfrak{p}$-adic field. Then there is a constructive bijection*

$$\hat{G}_u \longleftrightarrow \left\{ \begin{matrix} \textit{integral coadjoint} \\ G\textit{-orbits in } \mathfrak{g}^* \end{matrix} \right\}.$$

Partial Proof. Let f be an element of $\mathfrak{g}^*$. The coadjoint orbit $G \cdot f \cong G/G^f$ is naturally a symplectic manifold: its tangent space $\mathfrak{g}/\mathfrak{g}^f$ at f carries the Kostant-Kirillov symplectic form ω_f, defined by

$$\omega_f(X,Y) = f([X,Y]) \qquad \text{for } X, Y \in \mathfrak{g}/\mathfrak{g}^f.$$

Suppose now that f is integral. To construct a representation of G associated to f, we choose a *real polarization* of $\mathfrak{g}$. A real polarization is a G^f-invariant subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing $\mathfrak{g}^f$, such that $\mathfrak{h}/\mathfrak{g}^f$ is a Lagrangian subspace of $\mathfrak{g}/\mathfrak{g}^f$. Such an $\mathfrak{h}$ always exists when G is nilpotent.

Since f is integral, we may define a representation of $H = \exp(\mathfrak{h})$ by

$$\sigma(\exp(X)) = \psi(f(X))$$

for all $X \in \mathfrak{h}$. Integrality guarantees the existence of σ on $\mathfrak{g}^f$; since here $\mathfrak{h}/\mathfrak{g}^f$ is an affine space, there is no obstruction to extending σ to all of $\mathfrak{h}$.

Finally, we induce this representation from H to G, and define $\rho(f, \mathfrak{h}) = \mathrm{Ind}_H^G \sigma$ to be the representation associated to the orbit $G \cdot f$.

What remains to be shown (and what we regrettably omit) is: that $\rho(f, \mathfrak{h})$ is irreducible and unitary; that $\rho(f, \mathfrak{h})$ is independent of the choice of representative f or polarization $\mathfrak{h}$; and that all irreducible unitary representations of G arise in this way. □

Kostant and Auslander were able to prove that integrality is the correct criterion for type I simply connected solvable real Lie groups as well. Nevertheless, the next example shows that integrality is not a sufficiently precise criterion to fully describe the desired set of orbits.

Let G be a reductive Lie group. Then we can identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via a nondegenerate G-invariant bilinear form on $\mathfrak{g}$. Thus for each $f \in \mathfrak{g}$, we obtain a Jordan decomposition $f = f_n + f_s$, where f_n is nilpotent, and f_s is semisimple. In other words, they act by nilpotent, respectively semisimple, endomorphisms in any algebraic representation.

It is easy to see that if f is nilpotent, then $f|_{\mathfrak{g}^f} \equiv 0$. Thus all nilpotent orbits are automatically integral. Not all nilpotent orbits arise in the orbit correspondence, however. One example is that of the minimal nilpotent orbit of the real symplectic group $Sp(2n, \mathbb{R})$ for $n > 1$. The only irreducible representations associated to this orbit are ones of the two-fold covering group of the symplectic group (the *metaplectic group*).

Hence we would like to find a refinement of integrality, one which gives a more restrictive condition on nilpotent orbits.

To motivate the refinement proposed by Duflo, let us consider just one more extension of Kirillov's result [LP].

Suppose G is real or $\mathfrak{p}$-adic Lie group containing a nilpotent connected normal subgroup N. Denote the Lie algebra of N by $\mathfrak{n}$. Let f be an integral element of $\mathfrak{n}^*$. By Theorem 1.3, we have an associated irreducible unitary representation (ρ, V_ρ) of N.

We want to construct an irreducible unitary representation of G associated to the coadjoint orbit $G \cdot f$, built out of ρ. We will need to use "Mackey theory," as defined by the following theorem.

Theorem 1.4 (Mackey [Ma, Theorem 8.1]). *Suppose G is a Lie group, and N a type I normal subgroup with a metrically smooth dual group. Let ρ be an irreducible unitary* *representation of N. For each $g \in G$, define the representation ρ^g of N via $\rho^g(n) = \rho(g^{-1}ng)$ for $n \in N$. Set*

$$G^\rho = \{g \in G \mid \rho^g \cong \rho\},$$

the stabilizer of ρ under G, and denote by $G \cdot \rho$ the G-orbit of ρ. Then we have a bijection

$$\left\{\begin{array}{l}\textit{irreducible unitary represen-}\\ \textit{tations } \sigma \textit{ of } G^\rho \textit{ for which } \sigma|_N\\ \textit{is a multiple of } \rho\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\textit{irreducible unitary represen-}\\ \textit{tations } \sigma' \textit{ of } G \textit{ such that } \sigma'|_N\\ \textit{is supported on } G\cdot\rho\end{array}\right\}.$$

The correspondence is given by $\sigma \mapsto Ind_{G^\rho}^G \sigma$.

In our case, we can use Theorem 1.3 to describe G^ρ as the quotient

$$G^\rho = G^f \ltimes N \big/ N^f,$$

where N^f embeds into the semidirect product as $n \mapsto (n, n^{-1})$.

The space V_ρ carries a distinguished *projective* representation $\pi \otimes \rho$ of $G^f \ltimes N$, where π is related to ρ by the equation

$$\pi(g)\rho(n) = \rho^g(n)\pi(g) \qquad \text{for all } \; g \in G^f, n \in N. \tag{1.1}$$

This determines π only up to a scalar of modulus one, and in general it is not possible to choose π to be a homomorphism. We must pass to a particular two-fold cover of G^f to get a (true) representation. The remarkable fact is that via the map

$$\mathrm{Ad}\colon G^f \to Sp(\mathfrak{n}/\mathfrak{n}^f),$$

the cocycle of the projective representation π coincides with that of the restriction to $Ad(G^f)$ of the Segal-Shale-Weil representation of $Sp(\mathfrak{n}/\mathfrak{n}^f)$ (see [W]). In other words, the two-fold cover in question is exactly the pullback of the metaplectic group $Mp(\mathfrak{n}/\mathfrak{n}^f)$ to G^f. Denote this cover by $(G^f)^{Mp}$. Thus, the projective representation $\pi \otimes \rho$ of $G^f \ltimes N$ leads to a (true) representation π^{Mp} of $(G^f)^{Mp}$ on V_ρ having the property that $\pi^{Mp} \otimes \rho$ is a representation of $(G^f)^{Mp} \ltimes N$.

This does not, however, give us a representation of G^ρ on that space. To see this, note that

$$G^\rho = (G^f)^{Mp} \ltimes N \big/ (N^f)^{Mp}.$$

The restriction of $\pi^{Mp} \otimes \rho$ to $(N^f)^{Mp}$ is nontrivial; it is given by a character τ^{-1}, where τ satisfies

1. $\tau(\varepsilon) = -1$, where ε is the nontrivial element of $(N^f)^{Mp}$ lifting the identity;

2. $\tau(\exp(X)) = \psi(f(X))$ for all $X \in \mathfrak{n}^f$.

Therefore, to get a representation σ of G^ρ with the desired property — that $\sigma|_N$ is a multiple of ρ — we must choose a representation θ of $(G^f)^{Mp}$ whose restriction to $(N^f)^{Mp}$ is given by τ. Then the exterior product of the representations $\theta \otimes 1$ and $\pi^{Mp} \otimes \rho$ gives such an irreducible unitary representation σ of G^ρ. (This representation θ is not unique. In fact, we get *all* irreducible unitary representations of G^ρ whose restriction to N are multiples of ρ from different choices of θ.)

We may now apply Mackey's theorem, to get an irreducible unitary representation of G. This completes our construction.

In 1980, Duflo generalized these ideas and proposed, for real Lie groups, the criterion of *admissibility* of an orbit. We extend it to include the $\mathfrak{p}$-adic case as well. It naturally arises from the question of the existence of the representations θ of the construction above, and is defined as follows.

Let G be an arbitrary Lie group over F, and let f be an element of $\mathfrak{g}^*$. Note that G^f preserves the symplectic form ω_f on $\mathfrak{g}/\mathfrak{g}^f$. Hence we may define a group $(G^f)^{Mp}$ such that the diagram

$$\begin{array}{ccc} (G^f)^{Mp} & \longrightarrow & Mp(\mathfrak{g}/\mathfrak{g}^f) \\ {\scriptstyle p}\downarrow & & \downarrow \\ G^f & \xrightarrow{Ad} & Sp(\mathfrak{g}/\mathfrak{g}^f) \end{array}$$

commutes. This is analogous to the metaplectic cover that arose in the construction above. Let ε denote the nontrivial element in the kernel of the map p.

Definition 1.5. An element $f \in \mathfrak{g}^*$ is *admissible* if there exists an irreducible unitary representation τ of $(G^f)^{Mp}$ satisfying

1. $\tau(\varepsilon) = -1$, that is, τ is a *genuine representation*;
2. $\tau(\exp(X)) = \psi(f(X))$ for all X in $\mathfrak{g}^f \cap$ domain of exp.

Such a pair (f, τ) is called an *admissible orbit datum.*

Duflo's work suggests that there should be a correspondence

$$\hat{G}_u \longleftrightarrow \{\text{admissible orbit data}\}\,.$$

Although there are known examples where this fails to give a bijection [V1], it is a close approximation to what we expect to hold true.

Let us now provide evidence that the admissibility criterion is a good one. Firstly, note that the theorems proven using integrality can be restated using admissibility. It turns out that in those cases, roughly speaking, the "integral τ" and the "admissible τ" are related by tensoring with the square root of a character related to an existing polarization.

Secondly, we note that for reductive Lie groups, admissibility is at least not a trivial criterion for nilpotent orbits $G \cdot f$. Instead, it becomes a question of the splitting of the covering group $(G^f)^{Mp}$ over G^f.

Thirdly, for *real* reductive Lie groups, there is an abundance of evidence linking admissible orbits and unitary representations. For the semisimple admissible orbits, we can use the known techniques of parabolic and cohomological induction to construct associated representations [V1]. For the case of nilpotent orbits, the situation is far less well understood. For one, there is as yet no definition for what it should mean for a representation to be "associated to" a nilpotent orbit, except in certain cases – such as those arising in the following theorem. There, a natural association of a representation to a nilpotent orbit comes out of the theory of primitive ideals and associated varieties.

Theorem 1.6 (Vogan [V2, Theorem 8.7]). *Suppose G is a real reductive Lie group. Suppose that a nilpotent orbit $G_{\mathbb{C}} \cdot f$ satisfies a certain nice algebraic condition. If π is an irreducible unitary representation of G associated to $G \cdot f$, then $G \cdot f$ is admissible.*

It is the case of nilpotent orbits in a reductive Lie group that we wish to investigate further. One hopes to generalize Theorem 1.6, and to understand how representations associated to nilpotent orbits might be constructed. We hope to gain a first insight into this problem by determining the set of admissible nilpotent orbits.

Inspiration [V1] from Lusztig's work with representations of Lie groups over finite fields, and from the Langlands' program, suggest a strong link between admissible nilpotent orbits and the so-called *special* orbits defined by Lusztig and Spaltenstein (see, for example, [Ca, §12.7]). We conjecture that, for nilpotent orbits in split reductive Lie groups over local fields, special should imply admissible. The converse cannot hold because, for example, all nilpotent orbits of a complex group are automatically admissible: the complex symplectic group is simply connected and hence admits no nontrivial covers.

Schwarz [Sch] computed the admissible nilpotent coadjoint orbits for the real classical groups. He proved that the special orbits coincided with the admissible orbits for all groups except the (nonsplit) unitary groups. In this last case, all orbits are special, but not all orbits are admissible.

In the present thesis, we extend Duflo's admissibility criterion to nilpotent orbits in $\mathfrak{p}$-adic Lie groups (as summarized above), and use this criterion to prove the following theorem (Theorems 5.10, 6.2, and 7.1).

Theorem 1.7. *Let F be a $\mathfrak{p}$-adic field. Let G be a reductive Lie group over F. Then*

1. *if $G = SL(n,F)$ or $G = GL(n,F)$, then all orbits are admissible; in particular, the admissible orbits coincide with the special orbits;*
2. *if $G = Sp(2n,F)$, $G = O(V)$ (an orthogonal group) or $G = SO(V)$ (a special orthogonal group), then the admissible orbits coincide with the special orbits;*

3. *if $G = U(V)$ (a unitary group) or $G = SU(V)$ (a special unitary group) then the admissible orbits coincide with the special orbits;*

4. *if G is a split group of type G_2, then all special orbits are admissible; the minimal orbit is admissible but not special.*

For the case of the group G_2, Savin [Sav] has constructed a representation of a three-fold cover of G associated to the minimal orbit. That this does not descend to a representation of G is a matter for further study: it suggests that admissibility alone is not enough to guarantee the existence of an associated representation. This is a fact known for certain examples of real Lie groups as well [V1]. Nevertheless, this theorem provides evidence that the admissibility criterion that we have defined for $\mathfrak{p}$-adic groups is the correct analog of Duflo's admissibility criterion, and that it gives a close approximation to the set of orbits arising in the orbit correspondence.

The first few sections of this paper treat the requisite background material. In Section 2, we fix our notation and terminology, and recall the notion of the Hilbert symbol and exponential map for $\mathfrak{p}$-adic fields. In Section 3, we outline Weil's construction of the Segal-Shale-Weil representation, its corresponding cocycle, and the metaplectic group. We omit all proofs; see [LV] and [Pe] for a complete treatment of the subject. We also consider the metaplectic covers of certain interesting subgroups of $Sp(W)$ and construct criteria for when these covers split.

In Section 4, we begin our discussion of admissibility. We define a criterion for the admissible nilpotent coadjoint orbits following Duflo [D] and Lion, Perrin [LP]. We then proceed to special cases. In Section 5 we consider the symplectic, orthogonal and unitary groups, using ideas of Moeglin in [M] for the classical groups. We proceed to the case of the special and general linear groups in Section 6. In Section 7, we classify the nilpotent orbits for the split $\mathfrak{p}$-adic group G_2, as well as presenting a treatment for the real case.

2 Background: $\mathfrak{p}$-adic Fields

By a *local field* we mean a locally compact, nondiscrete (commutative) field. The archimedean local fields are $\mathbb{R}$ and $\mathbb{C}$, the fields of real and complex numbers, respectively.

A nonarchimedean local field of characteristic zero is called a $\mathfrak{p}$-*adic field.* It is a finite algebraic extension of $\mathbb{Q}_p$, the p-adic completion of the field of rational numbers for some prime p. The nonarchimedean local fields of positive characteristic are the fields $\mathbb{F}_q((t))$, power series in one variable over the field with q elements. Although we are primarily interested in $\mathfrak{p}$-adic fields, we will often state results in the broader context of all local fields.

2.1 The Hilbert Symbol

We adopt the following notation for a nonarchimedean local field F:

val, val$_F$: the unique discrete valuation on F which gives a surjective map of F^* onto $\mathbb{Z}$, and sends 0 to ∞;

$\mathfrak{R}$: $= \{a \in F \mid \mathrm{val}(a) \geq 0\}$, the integer ring of F;

$\mathfrak{p}, \mathfrak{p}_F$: $= \{a \in F \mid \mathrm{val}(a) \geq 1\}$, the maximal ideal in $\mathfrak{R}$;

q: cardinality of the *residue field* $\mathfrak{R}/\mathfrak{p}$;

p: characteristic of the residue field, also called the *residual characteristic* of F;

e, e_F: $= \mathrm{val}_F(p)$, the degree of ramification of F over $\mathbb{Q}_p$.

In general, we will denote our base field by F. A quadratic extension of F is denoted $E = F(\omega)$, where $\omega \notin F$, but $\omega^2 \in F$. Then any element of E can be expressed as a sum $z = x + \omega y$, with x and y in F. With this notation, the *conjugate* of z is $\overline{z} = x - \omega y$. The *trace* of E over F is the map $\mathrm{Tr}_{E/F} : E \to F$ defined by $\mathrm{Tr}_{E/F}(z) = z + \overline{z}$. The *norm* (over F) of an element of E is defined as $\mathrm{N}_{E/F}(z) = z\overline{z}$.

Set $F^{*2} = \{a^2 \mid a \in F^*\}$; this is a subgroup of the multiplicative group of F. By way of example, the group F^*/F^{*2} has four elements when F is a $\mathfrak{p}$-adic field with p odd, but 2^{n-2} elements when F is a degree n extension of $\mathbb{Q}_2$. This gives the number of distinct quadratic extensions of F (with the trivial extension $F = F(1)$ included).

Let $(a/b)_F$ denote the *Hilbert symbol* of two nonzero elements a and b in F [N, III.5]. We recall some of its properties now.

Lemma 2.1. *Let F be a local field and a, b, c elements of F^*. The Hilbert symbol $(a/b)_F$ equals 1 if a is the norm of an element of $F(\sqrt{b})$, and -1 otherwise. It satisfies:*

1. $(ab/c)_F = (a/c)_F(b/c)_F$, $(a/b)_F = (b/a)_F$, $(a/a)_F = (a/-1)_F$;
2. $(a/1-a)_F = 1$;

3. $(a/b)_F = 1$ *for all* $b \in F^*$ *if and only if* $a \in F^{*2}$.

Also

4. $(a/b)_{\mathbb{C}} = 1$ *for all* $a, b \in \mathbb{C}$;

5. $(a/-1)_{\mathbb{R}} = sign(a)$.

6. If F *is nonarchimedean, of residual characteristic different from 2, then*

$$(a/-1)_F = (-1)^{\mathrm{val}(a)\,\frac{q-1}{2}}.$$

In particular, $(a/-1)_F = 1$ *for every* $a \in \mathfrak{R}^*$.

7. If $F = \mathbb{Q}_2$*, then* $(-1/-1)_F = -1$*, although* $(a/-1) = 1$ *for any* $a \in 1 + \mathfrak{p}^2$.

This final case of residual characteristic 2 is not well understood in general. We do have the following result, however, which will be useful for our study. Recall that e_F denotes the degree of ramification of the $\mathfrak{p}$-adic field F over $\mathbb{Q}_p$.

Lemma 2.2 (Fesenko, Vostokov [FV, VII, §4, Ex.6(c)]). *Let* F *be a* $\mathfrak{p}$*-adic field of residual characteristic 2, and let* $a, b \in F^*$. *If*

$$\mathrm{val}(a-1) + \mathrm{val}(b-1) > 2e_F,$$

then $(a/b)_F = 1$.

2.2 The Exponential Map

Suppose from now on that the characteristic of F is zero. Then one can formally define an exponential map $F \to F^*$ as an infinite power series centered at $0 \in F$. This series converges everywhere for $F = \mathbb{R}$ or $\mathbb{C}$, but for $\mathfrak{p}$-adic fields, it converges only on a neighborhood of zero. More precisely, we have the following theorem.

Theorem 2.3 ([N, III.1.2]). *Let* F *be a* $\mathfrak{p}$*-adic field of residual characteristic* p *and let* n *be an integer satisfying* $n > e_F/(p-1)$. *Then the power series*

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{2.1}$$

and

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

give mutually inverse isomorphisms (and homeomorphisms)

$$\exp\colon \mathfrak{p}_F^n \to 1 + \mathfrak{p}_F^n \quad \text{and} \quad \log\colon 1 + \mathfrak{p}_F^n \to \mathfrak{p}_F^n.$$

What is of interest to us here is the exponential map from a Lie algebra to the corresponding group. For linear groups, it is defined by the power series expansion (2.1), where this time we let $x = X$ denote an element of the Lie algebra $\mathfrak{g}$.

Lemma 2.4. *Suppose $G \subset GL(n, F)$ is a linear Lie group over a $\mathfrak{p}$-adic field F, with Lie algebra $\mathfrak{g} \subset End(F^n)$. Then the exponential map defines a continuous map from an open neighborhood of $0 \in \mathfrak{g}$ onto an open neighborhood of $1 \in G$.*

Proof. We need to understand the domain of the exponential map. Where it converges, it will be continuous. Let X be an element of $\mathfrak{g}$. There is some finite algebraic field extension of F, say $\widetilde{F}$, over which X has a complete Jordan decomposition. Specifically, there exists a matrix $h \in GL(n, \widetilde{F})$ such that $h^{-1}Xh = D + N$, where D is diagonal (with entries equal to the eigenvalues of X), N is strictly upper triangular (hence nilpotent), and $DN = ND$. It is easy to see that $\exp(X)$ converges if and only if $\exp(h^{-1}Xh) = h^{-1}\exp(X)h$ converges, which happens if and only if $\exp(D)$ converges.

Hence the convergence of the exponential map on X is equivalent to the convergence of the exponential map on the eigenvalues of X, which is determined by Theorem 2.3 above. Explicitly, each eigenvalue λ of X must satisfy

$$\mathrm{val}_{\widetilde{F}}(\lambda) > \frac{e_{\widetilde{F}}}{p-1}.$$

This domain might be viewed as a "tubular neighborhood" of the nilpotent cone in $\mathfrak{g}$. In particular, it is an open neighborhood of $0 \in \mathfrak{g}$. The same construction at the group level shows that the image of exp is an open neighborhood of $1 \in G$. $\square$

We make the following definition.

Definition 2.5. Let G_0 denote the open normal subgroup of G generated by the image of the exponential map.

When $F = \mathbb{R}$ or $\mathbb{C}$, G_0 coincides with the topological identity component of G. Recall, however, that $\mathfrak{p}$-adic groups are totally disconnected, so such a description does not apply.

If $\mathbb{G}$ is a simple, simply connected algebraic group defined over F and isotropic over F, and $G = \mathbb{G}(F)$ is its group of F-points, then $G_0 = G$ by a theorem of Kneser and Tits [Pl]. Such groups include $G = SL(n, F)$ and $G = Sp(2n, F)$, for example.

If $G = O(V, F)$ is an orthogonal group (the group of all transformations preserving a nondegenerate quadratic form on a vector space V), then $G_0 \subset SO(Q, F)$, as the following proposition shows.

Proposition 2.6. *Suppose G is a linear Lie group defined over a $\mathfrak{p}$-adic field F of residual characteristic p. Then for all $g \in G_0$, we have*

$$\det g \in \begin{cases} 1 + \mathfrak{p} & \textit{if } p \textit{ is odd;} \\ 1 + \mathfrak{p}^n & \textit{where } n \geq e_F + 1, \textit{ if } p = 2. \end{cases}$$

The stated result is weaker than what we are able to prove, but is sufficient for our applications.

Proof. Embed G in some $GL(n, F)$ and its Lie algebra $\mathfrak{g}$ in $\mathrm{End}(F^n)$. It suffices to prove the proposition for $g = \exp(X)$, where X is in the domain of the exponential map in $\mathfrak{g}$. We use the notation $F, \widetilde{F}, N, D$ of the proof of Lemma 2.4.

We wish to compute $\det(\exp X)$. Since $\det(\exp N) = 1$, we have

$$\det(\exp X) = \det(\exp D) = \exp(\mathrm{trace}(D)).$$

The trace lies in F; let $n = \mathrm{val}_F(\mathrm{trace}(D)) = \mathrm{val}_F(\mathrm{trace}(X))$. By Theorem 2.3, this expression converges if and only if $n > e_F/(p-1)$, in which case $\det(\exp X)$ takes values in $1 + \mathfrak{p}^n$. Since n is an integer and $e_F = \mathrm{val}_F(p) \geq 1$, this proves the proposition. □

Corollary 2.7. *If G is a linear Lie group defined over a $\mathfrak{p}$-adic field F, then* $(\det g/-1)_F = 1$ *for all $g \in G_0$. Moreover,* $\det g \neq -1$.

Proof. The second assertion follows from the equality $-1 = (p-1)+(p-1)p+(p-1)p^2+\cdots$, and the preceding theorem.

The first assertion follows immediately from Lemma 2.1[6] and Proposition 2.6 for the case of residual characteristic different from 2.

If F has residual characteristic 2, then we must apply Lemma 2.2. Since $-1 = 1 + 2 + 2^2 + 2^3 + \cdots$, we know $\mathrm{val}_F((-1) - 1) = \mathrm{val}_F(-2) = e_F$. On the other hand, if $g \in G_0$, then by Proposition 2.6, $\mathrm{val}_F(\det g - 1) \geq e_F + 1$. Hence the hypotheses are satisfies, and we conclude that $(\det g/-1)_F = 1$, as required. □

3 The Metaplectic Group

3.1 The Segal-Shale-Weil Representation

Let F be a local field, and ψ a nontrivial unramified unitary character of F. For convenience, not necessity, we assume the characteristic of F is different from 2; see [W]. Let W be a symplectic vector space over F, with symplectic form $\langle , \rangle$. The *Heisenberg group* $H(W)$ is the set $W \times F$ with multiplication given by:

$$(v, s) \cdot (w, t) = (v + w, s + t + \frac{1}{2}\langle v, w \rangle).$$

If l is a Lagrangian subspace of W, then $l \times F$ is an abelian subgroup of $H(W)$. We can define a representation of $H(W)$ by:

$$\pi_l = \mathrm{Ind}_{l \times F}^{H(W)} \mathrm{id} \times \psi.$$

By the Stone-von Neumann theorem [LV, 1.3.3 and A.1], this induced representation is irreducible and, up to equivalence, does not depend on the choice of Lagrangian l. Furthermore, any irreducible representation of $H(W)$ for which the center acts by this character ψ is equivalent to π_l.

Let $Sp(W)$ be the symplectic group of W, that is, the group of automorphisms of W preserving the symplectic form $\langle , \rangle$. We have a natural action of $Sp(W)$ on $H(W)$, given by

$$g \cdot (v, t) = (gv, t).$$

Define a new representation of $H(W)$ by

$$\pi_l^g(v, t) = \pi_l(gv, t);$$

it is equivalent to π_l by the Stone-von Neumann theorem. Hence there exists a unitary intertwining operator $T(g)$, unique up to a scalar of modulus one, such that

$$T(g) \circ \pi_l^g \circ T(g^{-1}) = \pi_l. \tag{3.1}$$

Denote the space of unitary operators on the space of π by $U(\pi)$. One can check that for any g_1, g_2 in $Sp(W)$, the operator $T(g_1)T(g_2) \in U(\pi)$ intertwines $\pi_l^{g_1 g_2}$ and π_l, and so must be a multiple of $T(g_1 g_2)$. Hence the map $g \mapsto T(g)$ defines a projective representation of $Sp(W)$, called the *Segal-Shale-Weil representation.*

We would like to be much more explicit in our understanding of this representation. Consider the bundle over $Sp(W)$ given by

$$\{(g, T(g)) \in Sp(W) \times U(\pi) \mid T(g) \text{ satisfies } (3.1)\}. \tag{3.2}$$

The fibre over each point is isomorphic to $\mathbb{C}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. There is a canonical section of this bundle, depending on the choice of l [LV, 1.6.9 and A.9]; denote it by $g \mapsto (g, R_l(g))$. For any g_1, g_2 in $Sp(W)$, we can find $c_l(g_1, g_2) \in \mathbb{C}^1$ such that

$$R_l(g_1)R_l(g_2) = c_l(g_1, g_2)R_l(g_1 g_2).$$

This defines $c_l \in H^2(Sp(W), \mathbb{C}^1)$, a 2-cocycle of $Sp(W)$. We use this section to identify our bundle (3.2) with the group $GMp(W) = Sp(W) \times \mathbb{C}^1$, where the multiplication in $GMp(W)$ is given by

$$(g_1, t_1) \cdot (g_2, t_2) = (g_1 g_2, t_1 t_2\, c_l(g_1, g_2)).$$

This $\mathbb{C}^1$-covering group of $Sp(W)$ is called the *Mackey obstruction group.* Note that the Segal-Shale-Weil representation gives rise to an honest representation of $GMp(W)$ via

$$(g, t) \mapsto tR_l(g).$$

This is called the *metaplectic representation.*

3.2 The Metaplectic Group

We now need an explicit formula for the 2-cocycle c_l.

For any three Lagrangian subspaces l_1, l_2, l_3 of W, let $\tau(l_1, l_2, l_3)$ denote the equivalence class of the quadratic form on $l_1 \oplus l_2 \oplus l_3$ given by:

$$(x_1, x_2, x_3) \longmapsto \langle x_1, x_2\rangle + \langle x_2, x_3\rangle + \langle x_3, x_1\rangle$$

(see [LV]). The invariant of this form we need to consider is called the *Weil index.*

The Weil index $\gamma(Q)$ of a nondegenerate quadratic form Q is a unitary character of the Witt group of F. It is defined by an integral equation in [W, §14], and has been computed explicitly for all local fields in, for example, [Pe, Appendix]. (Although this explicit form is quite complicated for $\mathfrak{p}$-adic fields, for $F = \mathbb{R}$, $\gamma(Q)$ is simply $\psi(\frac{1}{8}\text{signature of } Q)$.) Denote the quadratic form on F sending x to ax^2 by a. For any nondegenerate m-dimensional quadratic form Q on F, with discriminant $D \in F^*/F^{*2}$,

$$\gamma(Q)^2 = (D/-1)_F\, \gamma(1)^{2m}; \tag{3.3}$$

see [W, §28]. We extend γ to degenerate quadratic forms, by defining $\gamma(Q) = \gamma(Q/\text{rad}(Q))$ and $\gamma(0) = 1$.

Lemma 3.1 (Vergne [LV], Perrin [Pe], Ranga Rao [RR]). *Let W be a symplectic vector space. With respect to a choice l of Lagrangian subspace of W, the 2-cocycle of the Segal-Shale-Weil representation is given by*

$$c_l(g_1, g_2) = \gamma(\tau(l, g_1 l, g_1 g_2 l))$$

for any $g_1, g_2 \in Sp(W)$.

This cocycle, or the quadratic form $\tau(l_1, l_2, l_3)$, are variously called the Leray invariant, the Maslov index (over $\mathbb{R}$) and the Kashiwara index (over any local field). Weil described the cocycle only on an open subset of $Sp(W)$ in his original work [W, §43].

Vergne [LV, §1.7] gives a nice interpretation of the cocycle in terms of oriented Lagrangian planes for the real case; Perrin [Pe] describes an analogous construction in the local nonarchimedean case. We will use their results.

Before proceeding to the construction of the metaplectic group, let us illustrate the above construction with a particular case where the cocycle can be described concretely.

Let W be a symplectic space, and choose a decomposition $W = l \oplus l'$ of W into complementary Lagrangian subspaces. With respect to this decomposition, we can write any $g \in Sp(W)$ as a block matrix of the form

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

in the usual way.

Lemma 3.2 ([LV, 1.6.22, A.9]). *On the open set of $Sp(W) \times Sp(W)$ consisting of pairs of block matrices*

$$(g_1, g_2) = \left(\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \right)$$

such that C_1 and C_2 are invertible, the quadratic form $\tau(l, g_1 l, g_1 g_2 l)$ is given by the matrix

$$C_1^{-1}(C_1 A_2 + D_1 C_2) C_2^{-1} = A_2 C_2^{-1} + C_1^{-1} D_1. \tag{3.4}$$

The middle term on the left arises in the product $g_1 g_2$.

The proof is a straightforward calculation, and is omitted.

The final step in our construction is to find a smallest closed subgroup of $GMp(W)$ covering $Sp(W)$; in essence, reducing the cover to its non-trivial core. Note that when $F = \mathbb{C}$, $GMp(W)$ splits as a covering group for topological reasons, so we may exclude this case from now on.

Following [W, §43] and [LV, 1.7.8 and §A.16], we will define a map

$$\Psi : Sp(W) \longrightarrow \mathbb{C}^1$$

which satisfies

$$\Psi(g_1)\Psi(g_2) = c_l(g_1, g_2)^2\, \Psi(g_1 g_2). \tag{3.5}$$

Then the subgroup

$$Mp(W) = \{(g, t) \mid \Psi(g)^{-1} = t^2\}$$

of $GMp(W)$ is what we will call the *metaplectic group.* It is a non-trivial double cover of $Sp(W)$. The restriction of the metaplectic representation of $GMp(W)$ to $Mp(W)$ takes the form

$$(g, t) \longmapsto tR_l(g) = \Psi(g)^{-1/2} R_l(g).$$

It is also called the metaplectic representation.

To define such a map Ψ, we need the following lemma, which is not hard to prove.

Lemma 3.3 ([LV]). *For any $g \in Sp(W)$, consider the isomorphism*

$$\begin{aligned} \alpha_{l\mapsto gl} \colon l/(l \cap gl) &\longrightarrow (gl/(l \cap gl))^* \\ v &\longmapsto \langle \cdot, v \rangle. \end{aligned}$$

Choose orientations on the vector spaces l and $l \cap gl$; these give orientations on $l/(l \cap gl)$ and $(gl/(l \cap gl))^*$. If $l \neq gl$, then this map has a well-defined determinant $D(\alpha_{l\mapsto gl})$ modulo F^{*2}. When $l = gl$, set $D(\alpha_{l\mapsto gl}) = \det(g|_l)$, reflecting the extent to which g is orientation-preserving on l. These values $D(\alpha_{l\mapsto gl})$ depend only on g, and not on the choices made.

From the property (3.3) evolves the following proposition.

Proposition 3.4 (Vergne [LV], Perrin [Pe]). *For any $g \in Sp(W)$, the map*

$$\Psi(g) = (D(\alpha_{l\mapsto gl})/-1)_F \, \gamma(1)^{2(\dim l - \dim(l \cap gl))}$$

satisfies (3.5).

Note that this result is in some sense an extension of that in Lemma 3.2.

Remark 3.5. If -1 is a square in F, then in fact $\Psi(g) = 1$ for all $g \in Sp(W)$.

3.3 Coverings of Subgroups

Suppose we have a homomorphism $i \colon G \to Sp(W)$, where G is Lie group over F. Then we can define the *metaplectic cover* (with respect to i) of G to be the group $G^{Mp} = G^{Mp(W)}$ defined by the commutative diagram

$$\begin{array}{ccc} G^{Mp} & \longrightarrow & Mp(W) \\ \downarrow & & \downarrow p \\ G & \xrightarrow{\ i\ } & Sp(W). \end{array}$$

More explicitly, it is the subgroup of $G \times Mp(W)$ defined by

$$\{(g, x) \in G \times Mp(W) \mid i(g) = p(x)\}.$$

We say that the metaplectic cover of G (corresponding to W) *splits* if there is a section of this bundle, that is, a homomorphism π of G into $Mp(W)$ satisfying $p \circ \pi = i$.

In this section, we consider various kinds of groups G and maps $i \colon G \to Sp(W)$, with the intent of understanding the factors leading to a splitting of a metaplectic double covering group.

3.3.1 Symplectic Decompositions

In this section, we discuss the case where G preserves a decomposition of W into symplectic subspaces. Let ε represent the nontrivial element in the kernel of the projection map p.

Lemma 3.6. *Let W_1 and W_2 be two symplectic vector spaces, and set $W = W_1 \oplus W_2$. Then, viewing $Sp(W_1) \times Sp(W_2)$ as a subgroup of $Sp(W)$, we have*

$$(Sp(W_1) \times Sp(W_2))^{Mp(W)} = (Mp(W_1) \times Mp(W_2)) / \{(1,1), (\varepsilon, \varepsilon)\}.$$

Proof. Choose Lagrangian subspaces l_1 of W_1 and l_2 of W_2. Set $l = l_1 \oplus l_2$; then l is a Lagrangian subspace of W. An easy computation shows that for any two elements (g_1, g_2) and (h_1, h_2) in $Sp(W_1) \times Sp(W_2)$,

$$c_l((g_1, g_2), (h_1, h_2)) = c_{l_1}(g_1, h_1) c_{l_2}(g_2, h_2),$$

and

$$\Psi_l((g_1, g_2)) = \Psi_{l_1}(g_1) \Psi_{l_2}(g_2).$$

Thus the map

$$Mp(W_1) \times Mp(W_2) \longrightarrow (Sp(W_1) \times Sp(W_2))^{Mp(W)} \tag{3.6}$$

given by

$$((g_1, t_1), (g_2, t_2)) \mapsto ((g_1, g_2), t_1 t_2)$$

is a homomorphism. It is surjective, and its kernel is precisely the set $\{((g_1, t_1), (g_2, t_2)) \mid g_1 = 1, g_2 = 1, t_1 t_2 = 1\}$, as required. □

Proposition 3.7. *In the setting of Lemma 3.6, suppose we have a pair of group homomorphisms $i_1 \colon G_1 \to Sp(W_1)$ and $i_2 \colon G_2 \to Sp(W_2)$. Then the metaplectic cover of $G_1 \times G_2$ induced by the map $i_1 \times i_2$ into $Sp(W)$ splits if and only if each of the metaplectic covers $(G_1)^{Mp(W_1)}$ and $(G_2)^{Mp(W_2)}$ split.*

Proof. First suppose that we have a section

$$G_1 \times G_2 \to (Sp(W_1) \times Sp(W_2))^{Mp(W)}.$$

Given any element $(g, 1) \in G_1 \times G_2$, write its image under this homomorphism as

$$(g, 1) \mapsto ((i_1(g), i_2(1)),\ s(g, 1)).$$

By Lemma 3.6, the homomorphism s decomposes as $s(g, 1) = s_1(g) s_2(1)$, where s_1, s_2 are sections of $G_1^{Mp(W_1)}$, $G_2^{Mp(W_2)}$ respectively, determined *a priori* only up to sign. In this case, however, we can fix $s_2(1) = 1$, which leaves us with a well-defined homomorphism

$$\begin{aligned} G_1 &\to Mp(W_1) \\ g &\mapsto (i_1(g), s_1(g)). \end{aligned}$$

We construct a section of $G_2^{Mp(W_2)}$ in the same way.

Conversely, suppose we have sections s_1, s_2 of $G_1^{Mp(W_1)}$, $G_2^{Mp(W_2)}$, respectively. Composing $s_1 \times s_2$ with the projection homomorphism (3.6), we get a section of $(G_1 \times G_2)^{Mp(W)}$, as desired. □

Corollary 3.8. *Let G be a Lie group. In the setting of Lemma 3.6, suppose we have a pair of group homomorphisms $i_1\colon G \to Sp(W_1)$ and $i_2\colon G \to Sp(W_2)$. Assume the metaplectic cover of G induced by i_2 splits. Then the metaplectic cover of G induced by the map $i_1 \times i_2$ splits if and only if the metaplectic cover of G induced by i_1 splits.*

Proof. Suppose the metaplectic covers $G^{Mp(W_1)}$ and $G^{Mp(W_2)}$ of G induced by the maps i_1 and i_2, respectively, split. Then by Proposition 3.7, the metaplectic cover of $G \times G$ induced by the map $i_1 \times i_2$ splits, and hence in particular splits over the diagonal subgroup isomorphic to G.

Conversely, suppose the metaplectic covers $G^{Mp(W_2)}$ and $G^{Mp(W_1 \oplus W_2)}$ both split over G. The first part of the proof of Proposition 3.7 will serve us here: replace the element $(g,1) \in G_1 \times G_2$ with the diagonal element $(g,g) \in G \times G$, and the well-defined section $s_2(1) = 1$ with a section over all of G determined by the splitting of $G^{Mp(W_2)}$. The effect is the same: to remove the ambiguity of sign, and hence give a well-defined splitting of $G^{Mp(W_1)}$. □

The hypothesis in this corollary — that $G^{Mp(W_2)}$ splits — is essential. We conclude this section by showing that two nontrivial metaplectic double covers can induce a trivial cover over the diagonal subgroup.

Let Δ denote the diagonal embedding of $Sp(W)$ into $Sp(W) \times Sp(W)$.

Lemma 3.9. *Let W, W_1 and W_2 be symplectic vector spaces. Suppose we have homomorphisms $i_1\colon Sp(W) \to Sp(W_1)$ and $i_2\colon Sp(W) \to Sp(W_2)$, such that the metaplectic covers of $Sp(W)$ arising from each of i_1 and i_2 are nontrivial. Then the metaplectic cover of $Sp(W)$ induced by the map*

$$(i_1 \times i_2) \circ \Delta\colon Sp(W) \longrightarrow Sp(W_1 \oplus W_2) \tag{3.7}$$

splits.

Proof. Up to inner isomorphism, there is a unique nontrivial two-fold covering group of the symplectic group $Sp(W)$ [MVW, Ch2.II.1]. Hence our hypothesis implies that

$$Sp(W)^{Mp(W_1)} \cong Sp(W)^{Mp(W_2)} \cong Mp(W).$$

It then follows from Lemma 3.6 that the metaplectic cover of $Sp(W) \times Sp(W)$ induced by the map $i_1 \times i_2$ is isomorphic to $(Mp(W) \times Mp(W))/Z$, where Z is the central subgroup $\{(1,1),(\varepsilon,\varepsilon)\}$. The cover of $Sp(W)$ induced by (3.7) is the diagonal subgroup

$$\Delta(Mp(W) \times Mp(W))/Z = \{((g,t),(g,t')) \in Mp(W) \times Mp(W)\}/Z.$$

For each $g \in Sp(W)$, let $P(g)$ be a square root of $\Psi(g)^{-1}$, the character defining $Mp(W)$ (cf. (3.5)). Then we have a well-defined splitting homomorphism

$$Sp(W) \longrightarrow \Delta(Mp(W) \times Mp(W))/Z,$$

defined by sending g to the class modulo Z of the element $((g, P(g)), (g, P(g)))$. □

As a special and important case of these last two results, we record the following proposition.

Proposition 3.10. *Let W be a symplectic vector space, and W' an orthogonal vector space. Then $W \otimes W'$ is symplectic, and there is a natural map*

$$Sp(W) \longrightarrow Sp(W \otimes W') \tag{3.8}$$

given by $g(w \otimes w') = gw \otimes w'$. The metaplectic cover of the symplectic group $Sp(W)$ arising from this map splits if and only if $\dim W'$ is even.

Proof. We prove this by induction on the dimension of the orthogonal space W'. Suppose $\dim W' = 1$. Then $W \otimes W' \cong W$, since both are symplectic vector spaces of the same dimension. It follows that (3.8) is an isomorphism, and so the metaplectic cover of $Sp(W)$ is just $Mp(W)$, which does not split.

Now suppose we have the result for any dimension less than k. Let W' be an orthogonal vector space of dimension equal to k. Choose an orthogonal decomposition $W' = W_1 \oplus W_2$, with $\dim W_1 = 1$ and $\dim W_2 = k - 1$. Then $Sp(W)$ preserves the decomposition

$$W \otimes W' = (W \otimes W_1) \oplus (W \otimes W_2).$$

If $k-1$ is even, then apply Corollary 3.8 to deduce that the metaplectic cover of $Sp(W)$ does not split for k odd. If $k - 1$ is odd, then apply Lemma 3.9 to deduce that the metaplectic cover of $Sp(W)$ splits for k even. □

3.3.2 Lagrangian Decompositions

Now suppose that G preserves a decomposition of W into Lagrangian subspaces. Without loss of generality, identify G with its image under the homomorphism $i\colon G \to Sp(W)$.

Proposition 3.11. *Let W be a symplectic subspace over a $\mathfrak{p}$-adic field F, and G a closed linear subgroup of $Sp(W)$. Suppose G preserves a Lagrangian subspace $l \subset W$. Then the map*

$$\begin{array}{rcl} G_0 & \longrightarrow & Mp(W) \\ g & \longmapsto & (g, 1) \end{array} \tag{3.9}$$

defines a splitting of the metaplectic cover of G over G_0.

Proof. Since G preserves l, the quadratic form $\tau(l, g_1 l, g_1 g_2 l)$ is identically zero for all $g_1, g_2 \in G$. It follows that $c_l(g_1, g_2) = 1$, so (3.9) defines a homomorphism of G_0 into $GMp(W)$. It is certainly continuous.

To see that it takes values in $Mp(W)$, first note that $\dim l - \dim(l \cap gl) = \dim l - \dim l = 0$, so the $\gamma(1)$ term in Proposition 3.4 disappears. Moreover, since $G \subseteq Sp(W)$ is linear, we may apply Corollary 2.7 to deduce that in fact $(\det(g|_l)/-1)_F = 1$ for all $g \in G_0$. Thus we have $\Psi(g) = (\det(g|_l)/-1)_F = 1$ for all $g \in G_0$, which completes the proof. □

Although in this thesis we confine ourselves to linear Lie groups, it is worthwhile to note here an important case where a metaplectic splitting can be determined without using results from Section 2.2.

Proposition 3.12. *Let W be a symplectic subspace over a $\mathfrak{p}$-adic field F of residual characteristic different from 2, and G a Lie group. Suppose we have a map $i\colon G \to Sp(W)$, and two Lagrangian subspaces l and l' of W satisfying the following conditions:*

a) $l \cap l' = \{0\}$;

b) $i(G)$ preserves l and l';

c) there is an isomorphism $L\colon l \to l'$ intertwining the G-actions on l and l'.

Then the metaplectic cover G^{Mp} defined by i splits.

Proof. Choose dual bases of l and l', so that together they form a standard symplectic basis of W. With respect to this basis, the action of any element $g \in G$ is given by a matrix

$$\begin{bmatrix} A & 0 \\ 0 & {}^t\!A^{-1} \end{bmatrix}$$

where $A = i(g)|_l$ and ${}^t\!A^{-1} = i(g)|_{l'}$. Condition (c) above can then be stated as $L \circ A = {}^t\!A^{-1} \circ L$, from whence it follows that $\det(A) = \det(A)^{-1}$. Hence, $\det(i(g)|_l) = \pm 1$ for all $g \in G$. Since the residual characteristic of F is assumed different from 2, we can apply Lemma 2.1[5] to deduce that $(\det(i(g)|_l)/-1)_F = 1$. Hence the result follows from the proof of Proposition 3.11. □

3.3.3 Metaplectic Covers of $SL(2, F)$

We include one final observation about splittings of metaplectic covers over subgroups isomorphic to $SL(2, F)$. This particular result will be of use to us in Section 7.

First let us introduce some notation. Let W^k denote the unique k-dimensional irreducible representation of $SL(2, F)$. It carries an $SL(2, F)$-invariant bilinear form which is orthogonal if k is odd and symplectic if k is even. In the latter case, we write simply $Sp(k, F)$ for the corresponding symplectic group.

Proposition 3.13 (Vogan). *Let F be $\mathbb{R}$ or a $\mathfrak{p}$-adic field of residual characteristic not equal to 2, and let m be a positive integer. Corresponding to the action of $SL(2,F)$ on W^{2m}, we have a homomorphism*

$$SL(2,F) \to Sp(2m,F). \tag{3.10}$$

The metaplectic cover of $SL(2,F)$ induced by this map splits if and only if m is even.

Note that the embedding of $SL(2,F)$ into $Sp(2m,F)$ described here is not equivalent to the kind arising in Proposition 3.10, if $m > 1$.

Proof. The proof is by induction on m, using the isomorphism of $SL(2,F)$-spaces

$$W^{2m-1} \otimes W^2 \cong W^{2m} \oplus W^{2m-2}.$$

First note that if $m = 1$, then the map (3.10) is an isomorphism. By [W], the corresponding metaplectic cover is nontrivial. We prove here in detail the case $m = 2$; the general inductive step is no harder.

Consider the representation of $SL(2,F)$ given by

$$W = W^3 \otimes W^2 = W^4 \oplus W^2.$$

We compute the covering group $\widetilde{SL(2,F)} = SL(2,F)^{Mp(W)}$ in two different ways; comparing them determines the cover of $SL(2,F)$ induced from its action on W^4, as required.

Since W^3 is an orthogonal space, and W^2 symplectic, we have the following commutative diagram:

$$\begin{array}{ccccc} \widetilde{SL(2,F)} & \longrightarrow & (O(W^3) \times Sp(2,F))^{Mp(W)} & \longrightarrow & Mp(W,F) \\ \downarrow & & \downarrow & & \downarrow \\ SL(2,F) & \longrightarrow & O(W^3) \times Sp(2,F) & \longrightarrow & Sp(W,F). \end{array}$$

We first determine the pullback of $Mp(W,F)$ over the group $O(W^3) \times Sp(2,F)$. If we choose a Lagrangian l of W^2, then $W^3 \otimes l$ is a Lagrangian of W. Moreover, $O(W^3)$ preserves $W^3 \otimes l$, so the metaplectic cover induced by W splits over $O(W^3)_0$ by Proposition 3.11. On the other hand, applying Proposition 3.10, we see that this cover does not split over $Sp(2,F)$, since the orthogonal space W^3 is odd-dimensional. By [MVW, p34], the metaplectic group is the only nontrivial two-fold cover of the symplectic group. Hence we have the isomorphism

$$(O(W^3)_0 \times Sp(2,F))^{Mp(W)} \cong O(W^3)_0 \times Mp(2,F).$$

It follows that $\widetilde{SL(2,F)} \cong Mp(2,F)$.

On the other hand, using $W = W^4 \oplus W^2$ and Lemma 3.6, we have

$$\begin{array}{ccc} (Mp(4,F) \times Mp(2,F))/Z & \longrightarrow & Mp(W,F) \\ \downarrow & & \downarrow \\ Sp(4,F) \times Sp(2,F) & \longrightarrow & Sp(W,F), \end{array}$$

where Z denotes the diagonal central subgroup $\{(1,1),(\varepsilon,\varepsilon)\}$. Let Δ denote the diagonal embedding of $G \to G \times G$ for $G = SL(2,F)$, and let H be the group $SL(2,F)^{Mp(4,F)}$ that we wish to determine. This group H is defined by the diagram

$$\begin{array}{ccccc}
\widetilde{SL(2,F)} & \longrightarrow & (H \times Mp(2,F))/Z & \longrightarrow & (Mp(4,F) \times Mp(2,F))/Z \\
\downarrow & & \downarrow & & \downarrow \\
SL(2,F) & \xrightarrow{\Delta} & SL(2,F) \times SL(2,F) & \longrightarrow & Sp(4,F) \times Sp(2,F).
\end{array}$$

This group H is a double cover of $SL(2,F)$, so must be one of $Mp(2,F)$ or $SL(2,F) \times \{\pm 1\}$.

If $H = Mp(2,F)$, then by Lemma 3.9, $\widetilde{SL(2,F)}$ must be the trivial double cover of $SL(2,F)$. This contradicts our earlier computation.

Hence $H = SL(2,F)^{Mp(4,F)}$ is the trivial cover $SL(2,F) \times \{\pm 1\}$, as required. □

4 Admissibility of an Orbit

Let G be a reductive algebraic group defined over a local field F of characteristic zero. Let $\mathfrak{g}$ be the Lie algebra of G. Choose a nilpotent element f in $\mathfrak{g}^*$, the dual of the Lie algebra, and consider its orbit under the coadjoint action of G. This orbit $G \cdot f$ has a natural symplectic manifold structure, given on the tangent space at the base point f by the Kostant-Kirillov symplectic form

$$\omega(X_1, X_2) = f([X_1, X_2]),$$

where X_1 and X_2 are in $\mathfrak{g}/\mathfrak{g}^f$.

For our purposes it is convenient to work in the Lie algebra. Choose some faithful finite-dimensional representation ρ of $\mathfrak{g}$, and define a G-invariant nondegenerate bilinear form B on $\mathfrak{g}$ via

$$B(X_1, X_2) = \operatorname{Tr} \rho(X_1)\rho(X_2) \qquad \forall X_1, X_2 \in \mathfrak{g}.$$

Identify f with its preimage under the isomorphism $X \mapsto B(X, \cdot)$. Then we can choose a Lie triple in $\mathfrak{g}$ having f as its nilnegative element; that is, we can find an isomorphism

$$\phi \colon \mathfrak{sl}(2, F) \to \mathfrak{g}$$

sending Y to f, where $\{H, X, Y\}$ denotes the usual basis of $\mathfrak{sl}(2, F)$, with Lie bracket given by:

$$[H, X] = 2X \qquad [H, Y] = -2Y \qquad [X, Y] = H.$$

To simplify notation, we write ϕ for $\phi(\mathfrak{sl}(2, F))$.

The action of G^f on $\mathfrak{g}/\mathfrak{g}^f$ clearly preserves the symplectic form ω, and hence gives rise to a homomorphism (denoted *Ad* by abuse of notation)

$$Ad \colon G^f \longrightarrow Sp(\mathfrak{g}/\mathfrak{g}^f).$$

Recall the subgroup $G_0^f = (G^f)_0$ of Definition 2.5, and the character ψ of F fixed at the beginning of Section 3.

If f is nilpotent, then we can easily check that $f|_{\mathfrak{g}^f} \equiv 0$. Hence every nilpotent coadjoint orbit is integral, that is, there exists a character χ of G_0^f satisfying $\chi(\exp X) = \psi(f(X))$ for all X in the domain of the exponential map. Thus Duflo's condition for admissibility – that there exist a genuine character of the metaplectic double cover of G_0^f with this integrality property [D, p154] – reduces to the following in our case.

Definition 4.1. A nilpotent coadjoint orbit $G \cdot f$ is *admissible* if the metaplectic cover of G^f corresponding to $\mathfrak{g}/\mathfrak{g}^f$ splits over the subgroup G_0^f.

It is, however, unnecessary to consider all of G^f for admissibility. We can write $G^f = G^\phi \ltimes U(G^f)$, where G^ϕ is the centralizer of ϕ in G, and $U(G^f)$ is the unipotent radical of G^f [Ca, Prop. 5.5.9]. The latter term is a unipotent subgroup, isomorphic to a vector space (its Lie algebra) via the exponential map. It is easy to show that this implies that any double cover of $U(G^f)$ must split.

Hence the condition for admissibility can fail only over G_0^ϕ, and we restrict ourselves to considering the diagram

$$\begin{array}{ccc} (G^\phi)^{Mp} & \longrightarrow & Mp(\mathfrak{g}/\mathfrak{g}^f) \\ \downarrow & & \downarrow p \\ G^\phi & \xrightarrow{Ad} & Sp(\mathfrak{g}/\mathfrak{g}^f). \end{array}$$

There is one further reduction to make. Decompose $\mathfrak{g}/\mathfrak{g}^f$ into weight spaces under $\phi(H)$, the semisimple element of ϕ. We write

$$\mathfrak{g}/\mathfrak{g}^f = \bigoplus_i \left(\mathfrak{g}/\mathfrak{g}^f\right)[i].$$

It is easy to see that the $+1$ weight space of $\mathfrak{g}/\mathfrak{g}^f$ is equal to $\mathfrak{g}[1]$, and that $\mathfrak{g}[1]$ is a symplectic subspace of $\mathfrak{g}/\mathfrak{g}^f$ preserved by G^ϕ.

Remark 4.2. If $\mathfrak{g}[1] = 0$ then necessarily $\mathfrak{g}[2m+1] = 0$ for all integers m, and the orbit $G \cdot f$ is said to be *even.* In the classical Lie algebras, it is equivalent to the condition that in the partition type classification of the orbit, all parts have the same parity. In general, one can partially classify nilpotent orbits by their weighted Dynkin diagrams; the even orbits are those for which only even weights occur [CMcG].

Definition 4.3. Call a nilpotent coadjoint orbit $G \cdot f$ *pseudo-admissible* if the double cover of G^ϕ defined by

$$\begin{array}{ccc} (G^\phi)^{Mp} & \longrightarrow & Mp(\mathfrak{g}[1]) \\ \downarrow & & \downarrow p \\ G^\phi & \xrightarrow{Ad} & Sp(\mathfrak{g}[1]) \end{array}$$

splits over G_0^ϕ.

Proposition 4.4. *Suppose G is a linear Lie group over a $\mathfrak{p}$-adic field F. Then a nilpotent orbit $G \cdot f$ is admissible if and only if it is pseudo-admissible.*

Proof. Write $\mathfrak{g}/\mathfrak{g}^f = W \oplus \mathfrak{g}[1]$ where W denotes the sum of the remaining weight spaces. Both W and $\mathfrak{g}[1]$ are symplectic subspaces preserved by G^ϕ. Let us first prove that the metaplectic cover of G_0^ϕ induced by its map into $Sp(W)$ splits.

The subspaces

$$l = \sum_{i>1} \left(\mathfrak{g}/\mathfrak{g}^f\right)[i] \quad \text{and} \quad l' = \sum_{i<1} \left(\mathfrak{g}/\mathfrak{g}^f\right)[i]$$

are non-intersecting Lagrangians of W preserved by G^ϕ. Moreover, there is an obvious linear isomorphism of l onto l' commuting with the action of G^ϕ, defined by

$$(adY)^{i-1} \colon \left(\mathfrak{g}/\mathfrak{g}^f\right)[i] \to \left(\mathfrak{g}/\mathfrak{g}^f\right)[-i+2],$$

for all $i > 1$. Apply Proposition 3.11.

The equivalence of admissibility and pseudo-admissibility now follows from Corollary 3.8, applied to the symplectic decomposition $\mathfrak{g}/\mathfrak{g}^f = W \oplus \mathfrak{g}[1]$. $\square$

Thus the question of admissibility is determined on $\mathfrak{g}[1]$. It follows, for example, that even orbits are always admissible.

In subsequent sections, we analyze G^ϕ and $\mathfrak{g}[1]$ for particular choices of G.

5 Symplectic, Orthogonal and Unitary Groups

Let F be $\mathbb{R}$ or a $\mathfrak{p}$-adic field. We analyze the following cases together.

Symplectic Let V be a vector space over F equipped with a symplectic form $\langle , \rangle$. Then G is the symplectic group $Sp(V, F)$.

Orthogonal Let V be a vector space over F equipped with an orthogonal form $\langle , \rangle$. Then let G be either: the orthogonal group $O(V, F)$, the group of automorphisms of this form; or the special orthogonal group $SO(V, F)$, consisting of those automorphisms of determinant equal to 1.

Unitary Choose a quadratic extension $E = F(\omega)$ of F. Let V be a vector space over E equipped with a hermitian form $\langle , \rangle$, and denote by $U(V) \subset GL(V, E)$ the group of automorphisms of this form. Let G be either the unitary group $U(V)$, or its subgroup the special unitary group $SU(V)$.

We first fix some notation. If $\pi\colon \mathfrak{sl}(2, F) \to \mathrm{End}(V)$ is a representation, let

$$V[i] = \{v \in V \mid \pi(H)v = iv\}$$

denote its ith weight space under the semisimple element of $\mathfrak{sl}(2, F)$. Let W^m denote the unique (up to equivalence) irreducible representation of $\mathfrak{sl}(2, F)$ of dimension m. If $w_m \in W^m[m-1]$ is a nonzero highest weight vector of W^m, then a basis of the $\mathfrak{sl}(2, F)$-module W^m is

$$\{w_m, Yw_m, \ldots, Y^{m-1}w_m\}, \tag{5.1}$$

where we recall from Section 4 that $Y \in \mathfrak{sl}(2, F)$ is the lowering operator.

Choose a nilpotent orbit $G \cdot f$ and a Lie triple ϕ as in Section 4. Since $\mathfrak{g}$ acts on V, so does the subalgebra $\phi(\mathfrak{sl}(2, F))$. Decompose V into irreducible subrepresentations under ϕ. We can write

$$V = \bigoplus_{m \geq 1} V^m, \tag{5.2}$$

where V^m is the subspace of all copies of W^m in V.

Remark 5.1. This decomposition is strongly related to the partition classification of nilpotent orbits under $\overline{F}$. If $V^m \neq 0$, then m occurs in the partition for $G \cdot f$, with multiplicity equal to the number of times W^m occurs in V^m.

5.1 Structure of G^ϕ and $\mathfrak{g}[1]$

Our first step is to understand these V^m subspaces better. Define $V^{\langle m\rangle}$ to be the G^ϕ-space $\mathrm{Hom}_{\mathfrak{sl}(2,F)}(W^m, V)$, with the action of $g \in G^\phi$ on an element $T \in V^{\langle m\rangle}$ given by $(g \cdot T)(w) = g(T(w))$, for all $w \in W^m$. There is a G^ϕ-equivariant isomorphism

$$\begin{aligned} W^m \otimes V^{\langle m\rangle} &\rightarrow V^m \\ w \otimes T &\mapsto T(w), \end{aligned} \tag{5.3}$$

where G^ϕ acts only on the second factor of the tensor product.

The direct sum in (5.2) is orthogonal with respect to the form $\langle , \rangle$ on V, and hence we can consider the restriction of $\langle , \rangle$ to each V^m.

Recall that there is a nondegenerate bilinear form b_m, unique up to multiplication by a scalar, on each irreducible representation W^m of $\mathfrak{sl}(2,F)$. It is $\mathfrak{sl}(2,F)$-invariant, and has the property that for all $x \in W^m[i]$ and $y \in W^m[j]$, both nonzero,

$$b_m(x,y) \neq 0 \quad \Leftrightarrow \quad i = -j. \tag{5.4}$$

Lemma 5.2. *The form b_m on W^m is orthogonal if m is odd, and symplectic if m is even.*

Proof. Let us choose a basis for W^m of the form in (5.1). The $\mathfrak{sl}(2,F)$-invariance of b_m implies

$$\begin{aligned} b_m(w_m, Y^{m-1}w_m) &= -\, b_m(Yw_m, Y^{m-2}w_m) \\ &= \cdots \\ &= (-1)^{m-1} b_m(Y^{m-1}w_m, w_m), \end{aligned}$$

which by the nondegeneracy of b_m and property (5.4) is nonzero. It follows that b_m in symmetric if m is odd and skew-symmetric if m is even. □

Without loss of generality, we fix a scaling of b_m so that

$$b_m(w_m, Y^{m-1}w_m) = 1$$

for each m. (This depends of course on our choice of w_m made earlier.) We can now relate the form $\langle , \rangle|_{V^m}$ to a form on $V^{\langle m\rangle}$.

Lemma 5.3. *Define a nondegenerate bilinear form $(,)_m$ on $V^{\langle m\rangle}$ by*

$$(S,T)_m = \langle S(w_m), T(Y^{m-1}w_m)\rangle = \langle S(w_m), \phi(Y)^{m-1}T(w_m)\rangle$$

for all $S, T \in V^{\langle m\rangle}$. Then the form on $W^m \otimes V^{\langle m\rangle}$ defined by

$$(x \otimes S, y \otimes T) = b_m(x,y) \cdot (S,T)_m, \tag{5.5}$$

for $x, y \in W^m$ and $S, T \in V^{\langle m\rangle}$, is equivalent to $\langle , \rangle|_{V^m}$ via the isomorphism (5.3).

Proof. To prove the equivalence of the form defined in (5.5) and the form $\langle , \rangle$, we use (5.3) and compare them explicitly on a spanning set of the form

$$\{Y^k w_m \otimes T \mid 0 \le k \le m-1,\ T \in V^{\langle m \rangle}\}.$$

Let $\delta_{i,j}$ denote the Krœnecker delta function. Using the $\mathfrak{sl}(2, F)$-invariance of b_m, we have

$$\begin{aligned}(Y^k w_m \otimes S, Y^l w_m \otimes T) &\doteq b_m(Y^k w_m, Y^l w_m)\,(S,T)_m \\ &= (-1)^k b_m(w_m, Y^{k+l} w_m)\,\langle S(w_m), T(Y^{m-1} w_m)\rangle \\ &= (-1)^k\ \delta_{k+l,m-1}\,\langle S(w_m), T(Y^{m-1} w_m)\rangle.\end{aligned}$$

On the other hand, by the ϕ-invariance of $\langle , \rangle$, we have

$$\begin{aligned}\langle S(Y^k w_m), T(Y^l w_m)\rangle &= \langle \phi(Y^k) S(w_m), \phi(Y^{l-(m-1)}) T(Y^{m-1} w_m)\rangle \\ &= (-1)^k \langle S(w_m), \phi(Y)^{k+l-(m-1)} T(Y^{m-1} w_m)\rangle \\ &= (-1)^k\ \delta_{k+l-(m-1),0}\,\langle S(w_m), T(Y^{m-1} w_m)\rangle.\end{aligned}$$

So the two forms are equal, as required. □

Remark 5.4. It follows from Lemma 5.2 and Lemma 5.3 that this form $(,)_m$ is of the same type (symplectic, orthogonal or Hermitian) as $\langle , \rangle$ if m is odd, and of opposite type if m is even. (In the Hermitian case, take "opposite type" to mean skew-Hermitian.)

Now set $G^{\phi,m} = \mathrm{Aut}(V^{\langle m \rangle}, (,)_m) \to \mathrm{Aut}(V, \langle , \rangle) = G$. By the preceding lemma, we have a natural isomorphism

$$G^{\phi} = \prod_{m \ge 1} G^{\phi,m} \tag{5.6}$$

for G symplectic, orthogonal or unitary. For the special orthogonal and special unitary groups, G^{ϕ} is the subgroup given by the intersection of the product in (5.6) with G. In the question of admissibility, however, we shall see that this distinction is irrelevant: for the special orthogonal group, $G_0^{\phi} = \prod_{m \ge 1} G_0^{\phi,m}$ (see Section 5.3), so the splitting is the same; for the special unitary group, we shall see that the metaplectic cover of $(U(V)^{\phi})^{Mp}$ induced by its map into $Sp(\mathfrak{g}[1])$ splits completely (see Section 5.6), and therefore must also split completely over the subgroup $(SU(V)^{\phi})^{Mp}$. We will thus simplify our notation by writing (5.6) in all cases.

In Section 5.2, it will be convenient to identify $V^{\langle m \rangle}$ with each of the weight spaces $V^m[i]$ which are nontrivial, via the map

$$T \mapsto T(Y^{(m-1-i)/2} w_m). \tag{5.7}$$

This identification gives rise to a form $(,)_{m,i}$ on $V^m[i]$, which we describe in the following lemma.

Lemma 5.5. *Let m and i be integers satisfying*

$$m > 0, \qquad -(m-1) \le i \le m-1, \qquad \textit{and} \qquad i \equiv m-1 \mod 2.$$

Define a nondegenerate bilinear form $(,)_{m,i}$ on $V^m[i]$ as follows. Let x and y be arbitrary elements of $V^m[i]$, and let S and T be their respective inverse images in $V^{\langle m\rangle}$ under the isomorphism (5.7). Set

$$(x,y)_{m,i} = (-1)^{(m-1-i)/2}(S,T)_m.$$

Then the G^ϕ-equivariant isomorphism (5.7) intertwines the form $(,)_m$ on $V^{\langle m\rangle}$ and the form $(,)_{m,i}$ on $V^m[i]$.

In particular, $G^{\phi,m} = Aut(V^m[i], (,)_{m,i})$.

The proof is a straightforward calculation, which we omit.

5.2 Decomposition of $\mathfrak{g}[1]$ under the action of G^ϕ

Now that we have a more precise understanding of G^ϕ and the decomposition of V under ϕ, we proceed to the main problem of the section, namely, the decomposition of $\mathfrak{g}[1]$ under the adjoint action of G^ϕ.

When $G = Sp(V)$ or $G = O(V)$ (or $SO(V)$), $\mathfrak{g}$ is isomorphic to $\bigwedge^2 V$ and $\mathrm{Sym}^2 V$, respectively, via the G-equivariant isomorphisms generated by

$$v \otimes w \pm w \otimes v \longmapsto \langle \cdot, w\rangle v \pm \langle \cdot, v\rangle w. \tag{5.8}$$

A similar result holds for the case $G = U(V)$, as follows. Recall that $\overline{V}$ is isomorphic to V as a vector space over F, but has the conjugate E action. We write elements of $\overline{V}$ as $\overline{v}$, so that multiplication by an element λ of E takes the form

$$\lambda \cdot \overline{v} = \overline{\lambda v}.$$

Define $\bigwedge(V,\overline{V})$ to be the F-subspace of $V \otimes_E \overline{V}$ spanned by the skew elements $\{v \otimes \overline{w} - w \otimes \overline{v} \mid v, w \in V\}$. It is a vector space over F (not E), and is isomorphic to $\mathfrak{g}$ via the G-equivariant linear map generated by

$$v \otimes \overline{w} - w \otimes \overline{v} \longmapsto \langle \cdot, \overline{w}\rangle v - \langle \cdot, \overline{v}\rangle w. \tag{5.9}$$

The overlines on the right hand side are unnecessary, and will be omitted from now on.

Under the isomorphisms (5.8) and (5.9), the $+1$ weight space of $\mathfrak{g}$ under $\phi(H)$ can be identified with $\bigwedge^2 V[1]$, $\mathrm{Sym}^2 V[1]$, or $\bigwedge(V,\overline{V})[1]$, respectively. This holds for $SU(V)$ as well, since its Lie algebra differs from that of $U(V)$ only by a central part, which must lie in the *zero* weight space under $\phi(H)$.

Each of these identifications has a nice interpretation in terms of the weight spaces of V, as described in the following lemma.

Lemma 5.6. *For each pair (m, m') of positive integers, with m even and m' odd, the space*

$$\mathfrak{g}_{m,m'}[1] = \bigoplus_{i=-m+1}^{m-1} V^m[i] \otimes \overline{V^{m'}[-i+1]}$$

corresponds to a symplectic subspace of $\mathfrak{g}[1]$. *In fact,*

$$\mathfrak{g}[1] \cong \bigoplus_{\substack{m \text{ even} \\ m' \text{ odd}}} \mathfrak{g}_{m,m'}[1], \tag{5.10}$$

and this decomposition is G^ϕ-equivariant and orthogonal with respect to the symplectic form ω on $\mathfrak{g}[1]$. The action of G^ϕ on $\mathfrak{g}_{m,m'}[1]$ is given by the usual action of $G^{\phi,m} \times G^{\phi,m'}$ on each of the spaces $V^m[i] \otimes \overline{V^{m'}[-i+1]}$.

The notation $V^m[i] \otimes \overline{V^{m'}[-i+1]}$ is perhaps an awkward compromise: when V is an F-vector space, we mean $V^m[i] \otimes_F V^{m'}[-i+1]$; when V is an E-vector space, we mean $V^m[i] \otimes_E \overline{V^{m'}[-i+1]}$, viewed as an F-vector space by restriction of scalars.

Proof of Lemma 5.6. This is an exercise in understanding the isomorphisms (5.8) and (5.9). First note that by fixing the parity of m, we get an obvious, well-defined isomorphism between the space

$$\bigoplus_{\substack{m \text{ even} \\ m' \text{ odd}}} \left(\bigoplus_{\substack{i=-m+1 \\ i \text{ odd}}}^{m-1} V^m[i] \otimes \overline{V^{m'}[-i+1]} \right)$$

and $\bigwedge^2 V[1]$, $\mathrm{Sym}^2 V[1]$, or $\bigwedge(V, \overline{V})[1]$, respectively. This is the isomorphism in (5.10).

Let m, n be even integers, and m', n' odd integers (all positive). For $Z_1 \in \mathfrak{g}_{m,m'}[1]$ and $Z_2 \in \mathfrak{g}_{n,n'}[1]$, the form ω is given by

$$\omega(Z_1, Z_2) = \mathrm{Tr}\ \phi(Y)[Z_1, Z_2]. \tag{5.11}$$

We readily compute the Lie bracket of Z_1 with Z_2 via

$$[Z_1, Z_2]v = Z_1 Z_2 v - Z_2 Z_1 v$$

using (5.8) and (5.9). We give explicit formulas for the case $G = U(V)$; the other cases are similar.

Without loss of generality, we may assume

$$Z_1 = x_1 \otimes \overline{y_1} - y_1 \otimes \overline{x_1}$$
$$Z_2 = x_2 \otimes \overline{y_2} - y_2 \otimes \overline{x_2}$$

for some $x_1 \in V^m[i]$, $x_2 \in V^n[j]$, $y_1 \in V^{m'}[-i+1]$ and $y_2 \in V^{n'}[-j+1]$. Here, m and n are even, so i and j are odd.

Then we have

$$\begin{aligned}[Z_1, Z_2]v &= [x_1 \otimes \overline{y_1} - y_1 \otimes \overline{x_1}\,, x_2 \otimes \overline{y_2} - y_2 \otimes \overline{x_2}]v \\ &= \langle v, y_2\rangle\langle x_2, y_1\rangle x_1 - \langle v, y_2\rangle\langle x_2, x_1\rangle y_1 \\ &\quad - \langle v, x_2\rangle\langle y_2, y_1\rangle x_1 + \langle v, x_2\rangle\langle y_2, x_1\rangle y_1 \\ &\quad - \langle v, y_1\rangle\langle x_1, y_2\rangle x_2 + \langle v, y_1\rangle\langle x_1, x_2\rangle y_2 \\ &\quad + \langle v, x_1\rangle\langle y_1, y_2\rangle x_2 - \langle v, x_1\rangle\langle y_1, x_2\rangle y_2.\end{aligned} \tag{5.12}$$

Recall that $\langle V[i], V[j]\rangle \equiv 0$ unless $i = -j$. Hence all of the terms in (5.12) are zero, except in the following cases.

Case $i = -j + 2$: Then $\langle y_1, y_2\rangle$ can be nonzero; we have

$$[Z_1, Z_2]v = \langle v, x_1\rangle\langle y_1, y_2\rangle x_2 - \langle v, x_2\rangle\langle y_2, y_1\rangle x_1.$$

Case $i = -j$: Then $\langle x_1, x_2\rangle$ can be nonzero; we have

$$[Z_1, Z_2]v = \langle v, y_1\rangle\langle x_1, x_2\rangle y_2 - \langle v, y_2\rangle\langle x_2, x_1\rangle y_1.$$

To apply (5.11), choose an orthogonal basis $\{e_k\}$ of V. Set $\langle e_k, e_k\rangle = \alpha_k$ and recall that

$$\mathrm{Tr}\ \phi(Y)[Z_1, Z_2] = \sum_k \frac{1}{\alpha_k}\langle \phi(Y)[Z_1, Z_2]e_k, e_k\rangle. \tag{5.13}$$

Using the relation

$$\langle v, w\rangle = \sum_k \frac{1}{\alpha_k}\langle v, e_k\rangle\langle e_k, w\rangle,$$

we deduce the following.

Case $i = -j + 2$:

$$\omega(Z_1, Z_2) = \langle \phi(Y)x_2, x_1\rangle\langle y_1, y_2\rangle - \langle \phi(Y)x_1, x_2\rangle\langle y_2, y_1\rangle. \tag{5.14}$$

Case $i = -j$:

$$\omega(Z_1, Z_2) = \langle \phi(Y)y_2, y_1\rangle\langle x_1, x_2\rangle - \langle \phi(Y)y_1, y_2\rangle\langle x_2, x_1\rangle. \tag{5.15}$$

Finally, using the fact that $\langle V^m, V^n\rangle \equiv 0$ for any $m \neq n$, we conclude that (5.14) and (5.15) are identically zero unless $m = n$ and $m' = n'$. Thus, the images of $\mathfrak{g}_{m,m'}[1]$ are pairwise othogonal in $\mathfrak{g}[1]$.

The last statement of the lemma follows from the G^ϕ-equivariance of (5.8) and (5.9). □

See Figure 5.1 for a representation of the non-zero pairings within $\mathfrak{g}_{m,m'}[1]$, as determined by equations (5.14) and (5.15). It is clear how the conditions $i = -j + 2$ and $i = -j$ arise through the symmetry of the weight spaces about zero.

5.3 Finding G^ϕ-invariant Lagrangian Decompositions

We now have a decomposition of $\mathfrak{g}[1]$, and may apply the results of Section 3.3.1 to study the question of admissibility of $G \cdot f$ by studying the metaplectic cover of G_0^ϕ coming from each $\mathfrak{g}_{m,m'}[1]$ individually.

Following Mœglin [M, §1.3], we define

$$Y_{m,m'} = \bigoplus_{i \leq -1} V^m[i] \otimes \overline{V^{m'}[-i+1]}.$$

This is an isotropic subspace of $\mathfrak{g}_{m,m'}[1]$ (cf. Figure 5.1).

If $m < m'$, then $\dim_F Y = \frac{1}{2}\dim_F \mathfrak{g}_{m,m'}[1]$, so $Y_{m,m'}$ is a Lagrangian subspace of $\mathfrak{g}_{m,m'}[1]$. An element $(g, g') \in G^{\phi,m} \times G^{\phi,m'}$ preserves $Y_{m,m'}$, and acts on it block-diagonally, with its natural action on each F-vector space $V^m[i] \otimes \overline{V^{m'}[-i+1]}$. We are thus in the setting of Proposition 3.11, and conclude that the metaplectic cover corresponding to $Sp(\mathfrak{g}_{m,m'}[1])$ splits over $G_0^{\phi,m} \times G_0^{\phi,m'}$.

It remains to consider those $\mathfrak{g}_{m,m'}[1]$ with $m > m'$. In these cases, $Y_{m,m'}$ is isotropic but not maximally so. However, it does pair nondegenerately with the isotropic space

$$X_{m,m'} = \bigoplus_{i \geq 3} V^m[i] \otimes \overline{V^{m'}[-i+1]},$$

as one can deduce from Figure 5.1. The leftover piece, $V^m[1] \otimes \overline{V^{m'}[0]}$, although not itself symplectic, is G^ϕ-equivariantly isomorphic to a symplectic vector subspace of $\mathfrak{g}_{m,m'}[1]$, as described in the following lemma. Recall the bilinear forms $(,)_{m,i}$ on $V^m[i]$ defined in Lemma 5.5.

Lemma 5.7 ([M, §1.3]). *Suppose $m > m'$. Define*

$$\delta \colon V^m[1] \otimes \overline{V^{m'}[0]} \to \mathfrak{g}_{m,m'}[1]$$

by

$$x \otimes \overline{y} \mapsto \sum_{k=0}^{(m'-1)/2} (-1)^k \left(\phi(Y)^{-k} x \otimes \phi(Y)^k \overline{y}\right) \tag{5.16}$$

This is a $G^{\phi,m} \times G^{\phi,m'}$-equivariant isomorphism onto an orthogonal complement of $Y_{m,m'} \oplus X_{m,m'}$ in $\mathfrak{g}_{m,m'}[1]$. The form ω, restricted to $\mathfrak{g}_{m,m'}[1]$, pulls back under δ to a multiple of the form $Tr_{E/F}(\overline{(,)_{m,1}}\,(,)_{m',0})$ on $V^m[1] \otimes \overline{V^{m'}[0]}$.

Proof. It follows immediately from Figure 5.1 that $\omega(\mathrm{Im}(\delta), X_{m,m'}) \equiv 0$. One can also prove that $\omega(\mathrm{Im}(\delta), Y_{m,m'}) \equiv 0$ with an explicit calculation similar to that in the proof of

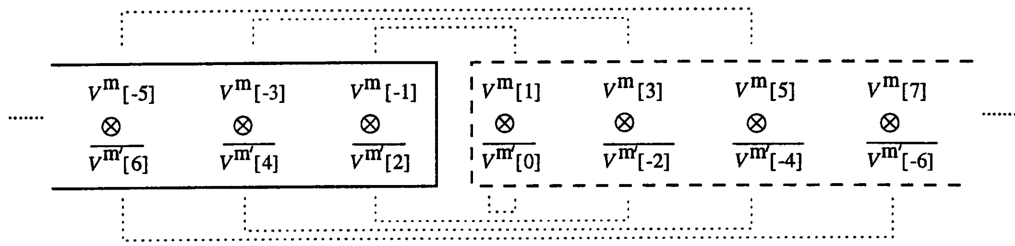

Figure 5.1: The symplectic vector space $\mathfrak{g}_{m,m'}[1]$. Dotted lines join spaces which admit a non-zero pairing under ω. Those along the bottom represent where (5.14) is nonzero, while those along the top represent where (5.15) is nonzero. The isotropic space $Y_{m,m'}$ is spanned by all the vector spaces within the thick solid line. The image of δ is a small subspace of the span of the spaces within the thick dashed line.

Lemma 5.6: use the G-equivariance of $\langle , \rangle$ to cancel non-zero terms in the alternating sum (5.16).

The final assertion can also be proven using an explicit calculation, which we include here. We consider the case where $G = U(V)$; the other cases are analogous.

Let $Z_1 = x_1 \otimes \overline{y_1}$ and $Z_2 = x_2 \otimes \overline{y_2}$ be two elements of $V^m[1] \otimes_E \overline{V^{m'}[0]}$. Then

$$\begin{aligned}\omega(\delta(Z_1), \delta(Z_2)) &= \sum_{k,l=0}^{(m'-1)/2} (-1)^{k+l}\, \omega\left(\phi(Y)^{-k}x_1 \otimes \phi(Y)^k\overline{y_1}\, , \phi(Y)^{-l}x_2 \otimes \phi(Y)^l\overline{y_2}\right) \\ &= \langle \phi(Y)x_2, x_1\rangle\langle y_1, y_2\rangle - \langle \phi(Y)x_1, x_2\rangle\langle y_2, y_1\rangle \quad \text{by (5.14)} \\ &= \overline{\langle x_1, \phi(Y)x_2\rangle}\langle y_1, y_2\rangle + \langle x_1, \phi(Y)x_2\rangle\overline{\langle y_1, y_2\rangle} \\ &= \mathrm{Tr}_{E/F}(\overline{\langle x_1, \phi(Y)x_2\rangle}\langle y_1, y_2\rangle) \\ &= (-1)^{(m-2)/2}(-1)^{(m'-1)/2}\mathrm{Tr}_{E/F}(\overline{(x_1, x_2)_{m,1}}\,(y_1, y_2)_{m',0})\end{aligned}$$

as required. □

We now wish to use the results of Section 3.3.1 with regards to the symplectic decomposition

$$\mathfrak{g}_{m,m'}[1] = (Y_{m,m'} \oplus X_{m,m'}) \oplus \delta(V^m[1] \otimes \overline{V^{m'}[0]}).$$

The first piece has a pair of G^ϕ-invariant, non-intersecting Lagrangians, intertwined by isomorphisms coming from the action of $\phi(Y)$, so we can apply Proposition 3.11 to conclude that the metaplectic cover over $G_0^{\phi,m} \times G_0^{\phi,m'}$ coming from its inclusion into $Sp(Y_{m,m'} \oplus X_{m,m'})$ splits. Thus to determine the splitting of the whole metaplectic cover it suffices to compute that on $\delta(V^m[1] \otimes \overline{V^{m'}[0]})$.

The second piece, however, does not admit a polarization invariant under the whole group $G^{\phi,m} \times G^{\phi,m'}$. We proceed on a case-by-case basis, for each of our groups G.

To reduce notation, we identify $V^m[1] \otimes \overline{V^{m'}[0]}$ and its image under δ.

5.4 Symplectic Group

Here, $G^{\phi,m}$ is an orthogonal group and $G^{\phi,m'}$ is a symplectic group (m even, m' odd, as usual) by Remark 5.4. Let $l \oplus l'$ be any Lagrangian decomposition of $V^{m'}[0]$; then $(V^m[1] \otimes l) \oplus (V^m[1] \otimes l')$ is a Lagrangian decomposition of $V^m[1] \otimes V^{m'}[0]$ preserved by $G^{\phi,m}$. We apply Proposition 3.11, to conclude that the corresponding metaplectic cover of the orthogonal group $G^{\phi,m}$ splits over $G_0^{\phi,m}$.

Since the product (5.6) in G^ϕ is direct, we can consider the case of the splitting over the symplectic group separately.

Construct the orthogonal vector space

$$U_{m'} = \bigoplus_{\substack{m>m' \\ m \text{ even}}} V^m[1].$$

Then $U_{m'} \otimes V^{m'}[0]$ is precisely the last remaining piece of $\mathfrak{g}[1]$ on which we need to determine the splitting of the metaplectic cover of $G^{\phi,m'}$: on all other direct summands of $\mathfrak{g}[1]$, either we have already proven that the metaplectic cover splits, or the action of $G^{\phi,m'}$ is trivial (and hence the splitting is automatic).

Now apply Proposition 3.10 to the symplectic vector space $V^{m'}[0]$ and the orthogonal space $U_{m'}$. Recall that $Sp(V^{m'}[0]) = G^{\phi,m'}$. We conclude that the metaplectic cover of $G^{\phi,m'}$ splits if and only if $\dim U_{m'}$ is even. This number $\dim U_{m'}$ is just the number of even parts m greater than m' (counted with multiplicity) in the partition type classification of the orbit (Remark 5.1).

To summarize: we have found that the metaplectic cover $(G^\phi)^{Mp(\mathfrak{g}[1])}$ splits over each of its orthogonal group components $G_0^{\phi,m}$ (m even). Over each of its symplectic group components $G^{\phi,m'}$ (m' odd), we have found that the cover splits if and only if the sum of the multiplicities in V of all the spaces W^m with m even and $m > m'$ is *even*. The number of such m less than m' does not affect the splitting.

5.5 Orthogonal Group

This case is exactly equivalent to that of a symplectic group. Interchange m and m' everywhere in the discussion, to conclude that the metaplectic cover splits only when the sum of the multiplicities in V of all the spaces $W^{m'}$ with m' odd and $m' < m$ is even for each fixed even m.

5.6 Unitary Group

Here, $(,)_{m,1}$ is a Hermitian form, and $(,)_{m',0}$ is skew-Hermitian by Remark 5.4. We apply the following lemma [Pr, §1].

Lemma 5.8. *Let F be a nonarchimedean local field of characteristic not equal to 2, and E a quadratic extension of F. Let V and W be vector spaces over E equipped, respectively,* *with a Hermitian form $(,)_V$ and a skew-Hermitian form $(,)_W$. The isometry groups $U(V)$ and $U(W)$ form a dual pair in $Sp(V \otimes_E W)$, where $V \otimes_E W$ is viewed as a symplectic vector space over F with symplectic form*

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \frac{1}{2} Tr_{E/F}((v_1, v_2)_V \overline{(w_1, w_2)_W}).$$

Then the metaplectic covers of $U(V)$ and $U(W)$ induced from $Mp(V \otimes_E W)$ both split.

Sketch of proof. It suffices to consider $U(V)$, by symmetry. Harris-Kudla-Sweet computed splittings

$$U(V) \times U(W) \longrightarrow GMp(V \otimes_E W)$$

explicitly in [HKS]. This $\mathbb{C}^1$-fold cover splits over $SU(V)$ by restriction. The group $SU(V)$ is equal to its commutator subgroup. Hence this map takes $SU(V)$ into the group of commutators of $GMp(V \otimes_E W)$. Since $GMp(V \otimes_E W)/Mp(V \otimes_E W)$ is abelian [MVW, Ch.2.II.9], it follows that the map in fact takes values in $Mp(V \otimes_E W)$, as required. Moreover, this splitting map is unique.

Next, include $U(1)$ into $U(W)$ via the inclusion of the 1-dimensional Hermitian space into W. The metaplectic cover of its image in $U(W)$ splits, not necessarily uniquely [Pr, Ch. 1].

These two splittings together give a splitting on $U(W) = SU(W) \rtimes U(1)$, since the uniqueness of the splitting over $SU(V)$ implies, in particular, that it will be normalised by the chosen splitting over $U(1)$. $\square$

Remark 5.9. Lemma 5.8 fails for $F = \mathbb{R}$; the metaplectic cover of $U(1)$ does not split in this case.

It follows that the metaplectic cover of $G^{\phi,m} \times G^{\phi,m'}$ induced from its inclusion into $Sp(V^m[1] \otimes \overline{V^{m'}}[0])$ must split for all $\mathfrak{p}$-adic fields F. Apply Corollary 3.8 to deduce that the metaplectic cover of G^{ϕ} induced from its action on all of $Sp(\mathfrak{g}[1])$ must split.

We have proven the following theorem.

Theorem 5.10. *Suppose F is a $\mathfrak{p}$-adic field. Let E be a quadratic extension of F.*

If $G = Sp(V, F)$, then a coadjoint orbit $G \cdot f$ is admissible if and only if, in the partition corresponding to the orbit, the number of even parts (counted with multiplicity) greater than any odd part is even.

If $G = O(V, F)$ or $SO(V, F)$, then a coadjoint orbit $G \cdot f$ is admissible if and only if, in the partition corresponding to the orbit, the number of odd parts (counted with multiplicity) less than any even part is even.

If $G = U(V) \subset GL(V, E)$, or $SU(V)$, then every coadjoint orbit is admissible.

For the case $F = \mathbb{R}$ and G symplectic or (special) orthogonal, these results have been obtained by Schwarz [Sch]. He also gave a complete answer for the real unitary group.

This characterization of the admissible orbits might seem no more than a peculiarity, if not for the notion of *special* nilpotent orbits, as defined by Lusztig and Spaltenstein. One of the descriptions of the set of special orbits in the classical groups is given in terms of the partition-type classification of nilpotent orbits (see [CMcG, §6.3]). Using Remark 5.1, we immediately deduce the following corollary.

Corollary 5.11. *Let G and F be as in Theorem 5.10. The admissible orbits under G coincide with the special orbits.*

6 Deciding Admissibility for the Special and General Linear Groups

Let F be a $\mathfrak{p}$-adic field, and let V be an n-dimensional vector space over F. The *general linear group* $GL(n, F)$ can be identified with the group of all automorphisms of V. The *special linear group* $SL(n, F)$ consists of all $g \in GL(n, F)$ with determinant 1. Similarly, we could identify the Lie algebra $\mathfrak{gl}(n, F)$ with $\mathrm{End}(V)$, and $\mathfrak{sl}(n, F)$ with all endomorphisms of trace 0. However, it will be more convenient for us to use the identification $\mathrm{End}(V) = V \otimes V^*$, where $V^* = \mathrm{Hom}_F(V, F)$. The action of a group element g on $f \in V^*$ is given by $(gf)(v) = f(g^{-1}v)$, for all $v \in V$. With respect to this action, the adjoint action of the group is given by $\mathrm{Ad}g(v \otimes f) = gv \otimes gf$.

We determine the admissibility of nilpotent orbits for both groups in this section. When there is no distinction to make, let $(G, \mathfrak{g})$ refer to either pair of Lie group and Lie algebra.

Let $G \cdot f$ be a nilpotent orbit in $\mathfrak{g}$, and let ϕ be a corresponding $\mathfrak{sl}(2, F)$-triple as in Section 4. Then $\phi = \phi(\mathfrak{sl}(2, F))$ is a subalgebra of $\mathfrak{g}$, and so acts on the space V. Decompose V into irreducible subrepresentations under this action, and further into weight spaces under $\phi(H)$, as in Section 5. We have

$$V = \bigoplus_{m \geq 1} V^m \cong \bigoplus_{m \geq 1} \bigoplus_{i=-m+1}^{m-1} V^m[i], \tag{6.1}$$

where V^m is the subspace of all copies of the m-dimensional irreducible representation of $\mathfrak{sl}(2, F)$ in V, and $V^m[i]$ denotes its ith weight space with respect to $\phi(H)$. The group G^ϕ will preserve each $V^m[i]$, since it intertwines the $\mathfrak{sl}(2, F)$-action on V. The space V^* decomposes in the same way, with the action on each component $(V^*)^m[i] \cong (V^m[-i])^*$ given by the adjoint (negative transpose).

The next step is to determine $\mathfrak{g}[1]$. Since any central part of $\mathfrak{gl}(n, F)$ lies in its zero weight space, $\mathfrak{g}[1]$ is the same for both $\mathfrak{gl}(n, F)$ and $\mathfrak{sl}(n, F)$. With respect to the identification of $\mathfrak{gl}(n, F)$ with $V \otimes V^*$, we define, for each pair of positive integers (m, m'), with m even and m' odd, a subalgebra

$$\mathfrak{g}_{m,m'}[1] = \bigoplus_{-m+1 \leq i \leq m-1} \left(V^m[i] \otimes (V^*)^{m'}[-i+1] \oplus V^{m'}[-i+1] \otimes (V^*)^m[i] \right).$$

Note that the terms $V^{m'}[-i+1]$, $(V^*)^{m'}[-i+1]$ are zero if $-i+1$ is not between $m'-1$ and $-m'+1$.

Lemma 6.1. *The space* $\mathfrak{g}[1]$ *decomposes as*

$$\mathfrak{g}[1] = \bigoplus_{(m,m')} \mathfrak{g}_{m,m'}[1], \tag{6.2}$$

where this sum runs over all pairs (m, m') *as above. This decomposition respects the symplectic form on* $\mathfrak{g}[1]$*, as well as the action of* G^ϕ*: each* $\mathfrak{g}_{m,m'}[1]$ *is a symplectic,* G^ϕ*-invariant subspace of* $\mathfrak{g}[1]$.

Proof. It is clear that (6.2) is an equality of vector spaces. We need to prove that, with respect to the symplectic form ω on $\mathfrak{g}[1]$, $\omega(\mathfrak{g}_{m,m'}[1], \mathfrak{g}_{n,n'}[1]) = 0$ whenever $(m, m') \neq (n, n')$ are pairs as above. As in the proof of Lemma 5.6, let $Z_1 \in \mathfrak{g}_{m,m'}[1]$, $Z_2 \in \mathfrak{g}_{n,n'}[1]$ be arbitrary. Without loss of generality, assume we have $Z_1 = v_1 \otimes f_1$ and $Z_2 = v_2 \otimes f_2$, where either $v_1 \otimes f_1 \in V^m[i] \otimes (V^*)^{m'}[-i+1]$ or $v_1 \otimes f_1 \in V^{m'}[-i+1] \otimes (V^*)^m[i]$. Similarly for $v_2 \otimes f_2$, with m, m', i replaced by n, n', j. To distinguish the many cases, let $m(v_k)$ (respectively $m(f_k)$) denote the dimension of the $\mathfrak{sl}(2, F)$-subspace in which v_k (respectively f_k) lies, for $k = 1, 2$.

Then we compute, for each $w \in V$,

$$\begin{aligned}
[Z_1, Z_2]w &= [v_1 \otimes f_1, v_2 \otimes f_2]w \\
&= (v_1 \otimes f_1)(v_2 \otimes f_2)w - (v_2 \otimes f_2)(v_1 \otimes f_1)w \\
&= (v_1 \otimes f_1)(f_2(w)v_2) - (v_2 \otimes f_2)(f_1(w)v_1) \\
&= f_2(w)f_1(v_2)v_1 - f_1(w)f_2(v_1)v_2.
\end{aligned}$$

Consequently, using (5.11) and summing over dual bases as in (5.13), we deduce

$$\begin{aligned}
\omega(Z_1, Z_2) &= \mathrm{Tr}\phi(Y)[Z_1, Z_2] \\
&= f_2(\phi(Y)v_1)f_1(v_2) - f_1(\phi(Y)v_2)f_2(v_1).
\end{aligned}$$

Recalling that $f(v) \neq 0$ only if $m(v) = m(f)$ and the $\mathfrak{sl}(2, F)$-weight of v is the negative of that of f, we immediately conclude that $\omega(v_1 \otimes f_1, v_2 \otimes f_2) = 0$ unless $m(v_1) = m(f_2)$ and $m(v_2) = m(f_1)$, as required.

Finally, note that the G^ϕ-equivariance of the decomposition is clear: G^ϕ in fact preserves each of the subspaces $V^m[i]$, for any integer $m > 0$. □

This lemma gives us a description of the group G^ϕ: it is wholly characterized by its preservation of the subspaces $V^m[i]$. Let $d_m = \dim(V^m[i]) = \dim((V^*)^m[i])$ denote the multiplicity of the m-dimensional irreducible representation of $\mathfrak{sl}(2, F)$ in V. If $G = GL(n, F)$, then $G^\phi = \prod_{m>0} GL(d_m, F)$; if $G = SL(n, F)$, then $G^\phi = S(\prod_{m>0} GL(d_m, F))$, the subgroup of all transformations of determinant equal to one. G^ϕ may be thought of as the set of block-diagonal matrices in G with respect to the decomposition (6.1) such that all blocks corresponding to the weight spaces of V^m (for any fixed m) are equal. Write an element g of G^ϕ as product element (g_m), as m runs over all nonzero V^m.

Finally, note that each $\mathfrak{g}_{m,m'}[1]$ contains, as complementary G^ϕ-invariant Lagrangians, the subspaces

$$l_{m,m'} = \bigoplus_{-m+1 \leq i \leq m-1} V^m[i] \otimes (V^*)^{m'}[-i+1]$$

and

$$l'_{m,m'} = \bigoplus_{-m+1 \leq i \leq m-1} V^{m'}[-i+1] \otimes (V^*)^m[i].$$

The determinant of the action of an element $g = (g_m)$ in G^ϕ on $l_{m,m'}$ is given by

$$\begin{aligned}\det(g|_{l_{m,m'}}) &= \prod_i \det(g_m \times (g_{m'})^*) \\ &= \prod_i \det(g_m)^{d_{m'}} \det((g_{m'})^*)^{d_m},\end{aligned}$$

where the product runs over all indices i giving rise to a nonzero space $V^m[i] \otimes (V^*)^{m'}[-i+1]$. It follows from Proposition 3.11 that the metaplectic cover of G_0^ϕ arising from its action on each of the symplectic spaces $\mathfrak{g}_{m,m'}[1]$ splits. By Corollary 3.8, this implies that the entire metaplectic double cover of G_0^ϕ corresponding to $\mathfrak{g}[1]$ splits, and that the orbit $G \cdot f$ is admissible.

We have proven the following theorem.

Theorem 6.2. *Let F be a $\mathfrak{p}$-adic field. For the groups $GL(n, F)$ and $SL(n, F)$, every nilpotent coadjoint orbit is admissible.*

All orbits under these groups are special, so we have the following immediate corollary.

Corollary 6.3. *Let F be a $\mathfrak{p}$-adic field. For the groups $GL(n, F)$ and $SL(n, F)$, the admissible orbits coincide exactly with the special orbits.*

7 The Exceptional Lie Group of Type G_2

To answer the question of admissibility of each of the nilpotent coadjoint orbits for G_2, we forgo the abstract approach of the preceding sections in favor of a case-by-case study.

Recall that the nilpotent orbits of any Lie group are partially classified by their weighted Dynkin diagrams [CMcG]. For the exceptional group G_2, there are 5 weighted Dynkin diagrams, each of which corresponds to one or more nilpotent coadjoint orbits over our field F. We anticipate that, as before, the admissibility of an orbit will be a function of its weighted Dynkin diagram, that is, of its stable orbit over an algebraically closed field.

In Table 7.1, we list the weighted Dynkin diagrams, and whether or not the given stable orbit is special.

Orbit	Diagram $\circ\!\!\Rrightarrow\!\!\circ$ α β	$\dim \mathcal{O}$	Special
$\{0\}$	0 0	0	yes
$\mathcal{O}_\alpha$	1 0	6	no
$\mathcal{O}_\beta$	0 1	8	no
$\mathcal{O}_{subreg}$	2 0	10	yes
$\mathcal{O}_{prin}$	2 2	12	yes

Table 7.1: Nilpotent Orbits in Type G_2 [CMcG, 128]

Note that $\mathcal{O}_{prin}$, $\mathcal{O}_{subreg}$ and $\{0\}$ are all even orbits, and hence admissible by Remark 4.2. The remainder of this section is devoted to deciding the admissibility of the remaining orbits. There is a unique minimal orbit in G [Sav]; we denote it again by $\mathcal{O}_\alpha$. Similarly for the $\mathcal{O}_\beta$: there is a unique 8-dimensional rational orbit.

In Section 7.1, we choose explicit $\mathfrak{sl}(2,F)$-triples for each orbit, and compute $\mathfrak{g}^\phi$ and $\mathfrak{g}[1]$ in the Lie algebra. We then find the subgroup G_0^ϕ corresponding to the Lie algebra $\mathfrak{g}^\phi$, and consider the question of admissibility of these orbits.

7.1 Explicit Triples for $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$

Let $\{\alpha,\beta\}$ be a set of simple roots for a Lie algebra of type G_2, chosen so that with respect to the Killing form κ on $\mathfrak{g}$,

$$\frac{\kappa(\alpha,\alpha)}{\kappa(\beta,\beta)} = 3.$$

Then the root system is the set

$$\{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(\alpha+2\beta), \pm(\alpha+3\beta), \pm(2\alpha+3\beta)\}.$$

See Figure 7.1.

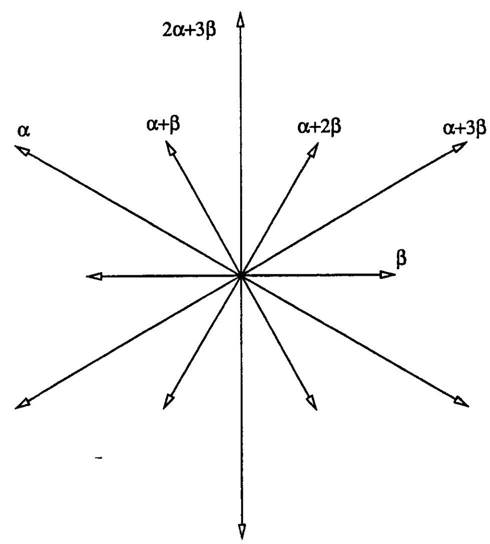

Figure 7.1: Root diagram for G_2. The labeled roots are exactly the positive roots, with respect to the choice of simple roots α and β.

Denote the root subspace of $\mathfrak{g}$ corresponding to a root γ by $\mathfrak{g}_\gamma$. For each positive root γ, choose root vectors $E_\gamma \in \mathfrak{g}[\gamma]$ and $F_\gamma \in \mathfrak{g}[-\gamma]$, normalized in a compatible way (see, for example, [Sav]). The minimal orbit $\mathcal{O}_\alpha$ is the orbit through F_α, and $\mathcal{O}_\beta$ is the orbit through F_β.

For $\mathcal{O}_\alpha$, we take ϕ to be the $\mathfrak{sl}(2)$-subalgebra of $\mathfrak{g}$ generated by E_α and F_α, with $Y = F_\alpha$ as the -2 weight vector. Then we can read the subspaces $\mathfrak{g}^\phi$ and $\mathfrak{g}[1]$ directly from the root diagram of G_2 (Figure 7.1). The algebra $\mathfrak{g}^\phi$ is given by the direct sum of all the trivial representations of ϕ in $\mathfrak{g}$, so is spanned by the vectors

$$\{E_{\alpha+2\beta},\ F_{\alpha+2\beta},\ H_{\alpha+2\beta} = [E_{\alpha+2\beta}, F_{\alpha+2\beta}]\}.$$

Thus the Lie subalgebra $\mathfrak{g}^\phi$ is isomorphic to $\mathfrak{sl}(2)$; in fact, it is the root-$\mathfrak{sl}(2)$ subalgebra corresponding to a short root. The vector space $\mathfrak{g}[1]$ is spanned by

$$\{E_{2\alpha+3\beta},\ E_{\alpha+\beta},\ F_\beta,\ F_{\alpha+3\beta}\}. \tag{7.1}$$

The adjoint action of $\mathfrak{g}^\phi$ on $\mathfrak{g}[1]$ is equivalent to the unique irreducible 4-dimensional representation of $\mathfrak{sl}(2)$. The vectors in (7.1) are weight vectors, listed in decreasing order of weights.

For $\mathcal{O}_\beta$, we take ϕ to be the $\mathfrak{sl}(2)$-subalgebra of $\mathfrak{g}$ generated by E_β and F_β, again with $Y = F_\beta$ as the -2 weight vector. In this case, $\mathfrak{g}^\phi$ is spanned by the vectors

$$\{E_{2\alpha+3\beta},\ F_{2\alpha+3\beta},\ H_{2\alpha+3\beta} = [E_{2\alpha+3\beta}, F_{2\alpha+3\beta}]\}$$

and $\mathfrak{g}[1]$ is spanned by

$$\{E_{\alpha+2\beta},\ F_{\alpha+\beta}\}. \tag{7.2}$$

So once again the Lie subalgebra $\mathfrak{g}^\phi$ is isomorphic to $\mathfrak{sl}(2)$, but this time to the root $\mathfrak{sl}(2)$ corresponding to a long root. Here, the adjoint action of $\mathfrak{g}^\phi$ on $\mathfrak{g}[1]$ is equivalent to the standard 2-dimensional representation of $\mathfrak{sl}(2)$; the vectors in (7.2) correspond to $+1$ and -1 weight vectors, respectively.

7.2 Admissibility for a $\mathfrak{p}$-adic Lie Group of type G_2

In this section, let G be the centerless Lie group of type G_2 over a $\mathfrak{p}$-adic field F. It is a linear group, so it is easy to compute the subgroup G^ϕ from its Lie algebra. In each of the two cases above, $\mathfrak{g}^\phi$ is isomorphic to $\mathfrak{sl}(2,F)$, so G_0^ϕ can only be $SL(2,F)$ or $PSL(2,F)$. Since in each case we have an irreducible even-dimensional representation of $\mathfrak{sl}(2,F)$ which lifts to the group, G_0^ϕ must be isomorphic to $SL(2,F)$.

To determine the admissibility of $\mathcal{O}_\beta$, the diagram we need to consider is

$$\begin{array}{ccc} SL(2,F)^{Mp} & \longrightarrow & Mp(2,F) \\ \downarrow & & \downarrow p \\ SL(2,F) & \xrightarrow{i} & Sp(2,F). \end{array}$$

The bottom map is an isomorphism, so the top map is as well. From [W], we know that the projection map p does not admit a splitting. Hence, the orbit $\mathcal{O}_\beta$ is not admissible.

The diagram we need to consider in the case of $\mathcal{O}_\alpha$ is

$$\begin{array}{ccc} SL(2,F)^{Mp} & \longrightarrow & Mp(4,F) \\ \downarrow & & \downarrow p \\ SL(2,F) & \xrightarrow{i} & Sp(4,F). \end{array}$$

This cover of $SL(2,F)$ splits by Proposition 3.13. Hence the minimal orbit $\mathcal{O}_\alpha$ is admissible.

We have proven the following theorem.

Theorem 7.1. *Let F be a $\mathfrak{p}$-adic field and let G be the adjoint group of type G_2 over F. Then*

1. *the principal, subregular and zero orbits are all admissible;*
2. *the 8-dimensional orbit is not admissible;*
3. *the minimal orbit is admissible.*

Corollary 7.2. *Let F be a $\mathfrak{p}$-adic field and let G be the adjoint group of type G_2 over F. Then all special orbits are admissible. The minimal orbit, which is not special, is admissible.*

7.3 Admissibility for Real Lie Groups of type G_2

First let G denote the linear group of type G_2 defined over $\mathbb{R}$. It has no center. We mimic the arguments used above for $\mathfrak{p}$-adic groups. The subgroups of G corresponding to the Lie subalgebras $\mathfrak{g}^{\phi}$ are isomorphic to $SL(2,\mathbb{R})$. Again, it follows immediately that the 8-dimensional orbit is not admissible for G, and from Proposition 3.13 that the minimal orbit is admissible for G.

Now consider the simply connected real Lie group of type G_2. It is the double cover of the adjoint group discussed above. It is not a linear group; in fact, the subgroup lifting the $\mathfrak{sl}(2,\mathbb{R})$ subalgebra corresponding to the long root is the nonlinear double cover of $SL(2,\mathbb{R})$, namely $Mp(2,\mathbb{R})$. It follows that the 8-dimensional orbit is admissible for G, because the diagram we need to consider is

$$\begin{array}{ccc} (M\bar{p}(2,\mathbb{R}))^{Mp} & \longrightarrow & Mp(2,\mathbb{R}) \\ \downarrow & & \downarrow{\scriptstyle p} \\ Mp(2,\mathbb{R}) & \xrightarrow{\ i\ } & Sp(2,\mathbb{R}). \end{array}$$

which splits by definition. Moreover, the minimal orbit, being admissible for the adjoint group, remains admissible for any covering group, and so is admissible here.

We have the following theorem, which is already known.

Theorem 7.3. *Let G be a real Lie group of type G_2 over $\mathbb{R}$. If G is simply connected, then all nilpotent coadjoint orbits are admissible. If G is adjoint, then only the 8-dimensional orbit fails to be admissible.*

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