Massachusetts Institute of Technology
Department of Economics
Time Series 14.384

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Final Exam-Fall 2001

Instructions: This is a 180-minute open-book, open-notes exam.

1. Assume that ε_t and η_t are sequences of *iid* $N(0,1)$ random variables with $E\varepsilon_t\eta_s = 0$ for all t and s. Let

$$x_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}$$

where $\sum_j |c_j| |j| < \infty$ and

$$u_t = \sum_{j=0}^{\infty} \gamma_j \eta_{t-j}$$

with $\sum_j |\gamma_j| |j| < \infty$. Assume that y_t is generated by

$$y_t = \beta x_t + u_t$$

and that we observe a sample of size n of observations $\{y_t, x_t\}_{t=1}^n$.

   a) Consider the OLS estimator
   $$\hat{\beta} = \frac{\sum_{t=1}^n x_t y_t}{\sum_{t=1}^n x_t^2}.$$
   Is $\hat{\beta}$ consistent? Find a limiting distribution of an appropriately centered and scaled version of $\hat{\beta}$. What is the variance of the limiting distribution of $\hat{\beta}$?

   b) Consider the random variable $z_t = x_t u_t$. Find the autocovariance function of z_t as a function of the parameters c_j and γ_j. Find the spectral density of z_t.

   c) Express the asymptotic variance of $\hat{\beta}$ found in a) in terms of the spectral densities of z_t and x_t (Hint: use the fact that $f_x(\lambda) = (2\pi)^{-1} \left|C(e^{-i\lambda})\right|^2$ where $C(L) = \sum_{j=0}^{\infty} c_j e^{-i\lambda j}$).

   d) How does your result in c) simplify if $\gamma_j = 0$ for $j \neq 0$?

   e) Derive a test of the hypothesis that u_t is an iid sequence against the alternative that $\gamma_j \neq 0$ for at least one $j \leq q$ where q is a fixed and known positive integer?

2. Let x_t be a weakly stationary process with $Ex_t = 0$. Consider the filtered process y_t where

$$y_t = \frac{1}{3}\left(x_t + x_{t-1} + x_{t-2}\right).$$

   a) Find the spectral density of y_t in terms of the spectral density $f_x(\lambda)$ of x_t and the power transfer function of the filter.

   b) Describe the properties of the filter by finding the gain function and the phase shift of the filter.

   c) Now consider the extracted 'cyclical' component y_t^c of x_t defined as
   $$y_t^c = x_t - y_t.$$
   Find the filter that transforms x_t into y_t^c. Assume that x_t is a unit root process of the form $x_t = x_{t-1} + u_t$ where u_t is iid $N(0,1)$ and $x_0 = 0$. Is y_t^c stationary? Prove your answer.

d) Consider the filtered series y_t^c. Discuss the optimality properties (or lack thereof) of y_t^c if the goal was to extract the components of x_t that generate the spectrum in a band $[a, b]$. How does your answer depend on a and b. (Hint: you can answer this part informally. You may find it useful to compare y_t^c to the approximately optimal filter according to Christiano and Fitzgerald when x_t is generated by

$$x_t = x_{t-1} + u_t - u_{t-1}.$$

Only spend time on this last point if you have time at the end of the exam.)

3. Assume that $x_t = \alpha x_{t-1} + \varepsilon_t - \theta\varepsilon_{t-1}$ where $\varepsilon_t \sim \text{iid} N(0, 1)$, $|\theta| < 1$ and $x_0 = 0$. Consider a regression of x_t onto x_{t-1}, ie. an estimator

$$\hat{\alpha} = \frac{\sum_{t=2}^{n} x_t x_{t-1}}{\sum_{t=2}^{n} x_{t-1}^2}.$$

a) Assume that $|\alpha| < 1$. Is $\hat{\alpha}$ consistent for the parameter value α? Prove your answer. Find a limiting distribution for $\hat{\alpha}$. Is $\hat{\alpha}$ asymptotically efficient?

b) Now assume that $\alpha = 1$. Is $\hat{\alpha}$ consistent in this case. Find the limiting distribution of $\hat{\alpha}$.

c) How could you use your distributional result in b) to approximate the finite sample bias of $\hat{\alpha}$. (Hint: no exact calculation is expected here. A rough outline of the procedure is sufficient).

d) Propose a test of $H_1 : |\alpha| < 1$ against the null hypothesis $H_0 : \alpha = 1$? Derive the limiting distribution of your test statistic under the null. How can you make the distribution of your test statistic nuisance parameter free?