Massachusetts Institute of Technology
Department of Economics
Time Series 14.384

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Final Exam-Fall 1999

Instruction: This is a 180-minute open-book, open-notes exam.

1. Assume that ε_t and v_t are iid sequences of random variables, independent of one another, with $E\varepsilon_t^4 < \infty, Ev_t^4 < \infty$, $E\varepsilon_t = Ev_t = 0$ and $E\varepsilon_t^2 = Ev_t^2 = 1$. Let x_t be the stationary solution to the stochastic difference equation for all t

$$x_t = \phi x_{t-1} + v_t \text{ for } |\phi| < 1 \tag{1}$$

and let y_t be given as

$$y_t = x_t + \varepsilon_t. \tag{2}$$

a) Show that y_t is a stationary process. What is the univariate representation of y_t in terms of v_t and ε_t?

b) Find the spectral density of y_t.

c) Suppose you observe a sample of $\{x_t, y_t\}_{t=1}^n$ generated by (1) and (2). Consider the OLS estimator

$$\hat{\beta} = \frac{\sum_{t=2}^n y_t x_t}{\Sigma_{t=2}^n x_t^2}.$$

Find the probability limit of $\hat{\beta}$. Make suitable assumptions about initial conditions of (1) if necessary. Find the limit distribution of $\hat{\beta}$ after apropriate centering and rescaling. Is the OLS estimator efficient?

d) Now assume that ε_t and v_t are correlated. Find a consistent estimator of β and determine the limit distribution of your estimator.

2. Assume that $\varepsilon_t \in \mathbb{R}^2$ is iid$N(0, \Omega)$ where Ω is a positive definite, symmetric 2×2 covariance matrix. Assume that $z_t' = (y_t, x_t)$ is the stationary solution to the structural model

$$A_0 z_t = A_1 z_{t-1} + \varepsilon_t \text{ with } z_0 = 0. \tag{3}$$

The coefficient matrices A_0 and A_1 are given by

$$A_0 = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix}, \; A_1 = \begin{bmatrix} \phi & \beta \\ 0 & \theta \end{bmatrix}.$$

a) Find the reduced form for z_t. Express the variance Σ of the reduced form innovations in terms of Ω, A_0 and A_1 and find the reduced form coefficients Π in terms of A_0 and A_1.

b) Write down the likelihood function of the reduced form parameters.

c) Find the maximum likelihood estimators of the reduced form parameters.

d) How would you test the hypothesis that $\beta = 0$.

e) Find the ML estimator for the restricted reduced form when $\beta = 0$.

f) Assume that the reduced form parameters Π and Σ are indentified. Let the parameter space for $(\alpha, \beta, \phi, \theta)$ be $\mathbb{R}^4$. For which subsets of $\mathbb{R}^4$ are the structural parameters Ω, A_0 and A_1 identified? Do you need to impose restrictions on Ω? How would you estimate the structural parameters?

3. Let $x_t = \sum_{s=1}^{t} \varepsilon_s$ and ε_s ~iid$N(0,1)$ and $y_t = \sum_{s=1}^{t} v_s$ with v_s ~iid$N(0,1)$. Assume that $E\varepsilon_s v_t = 0$ for all t and s.

a) Are x_t and y_t stationary?

b) Consider a regression of y_t on x_t. Let $\hat{\beta}$ be the OLS estimator of that regression defined by

$$\hat{\beta} = \frac{\sum_{t=1}^{n} y_t x_t}{\sum_{t=1}^{n} x_t^2}.$$

Show that

$$\hat{\beta} \Rightarrow \frac{\int W_1(r) W_2(r) dr}{\int W_2(r)^2 dr}$$

where $W_1(r)$ and $W_2(r)$ are two independent standard Brownian motions on $C[0,1]$. How do you interpret this result.

c) Let the R^2 of the regression be given as

$$R^2 = \frac{\hat{\beta}^2 \sum_{t=1}^{n} (x_t - \bar{x})^2}{\sum_{t=1}^{n} (y_t - \bar{y})^2}$$

with $\bar{x} = n^{-1} \sum_{t=1}^{n} x_t$ and $\bar{y} = n^{-1} \sum_{t=1}^{n} y_t$. Find an expression for R^2 as $n \to \infty$.

d) Discuss the implications of your result for a standard F test of the hypothesis $\beta = 0$.

e) Now assume that $x_t = \sum_{s=1}^{t} \varepsilon_s$ and ε_s ~iid$N(0,1)$ and that

$$y_t = x_t + v_t$$

where v_s ~iid$N(0,1)$ and $E\varepsilon_s v_t = 0$ for all t and s. Consider the OLS estimator $\hat{\beta}$ from the regression of y_t on x_t where $\hat{\beta} = \sum_{t=1}^{n} y_t x_t / \sum_{t=1}^{n} x_t^2$. Is $\hat{\beta}$ consistent? Can you find a limit distribution for β after apropriate scaling and centering?

f) Continue to maintain the assumptions under e). How would you test the hypothesis that $\beta = 1$? Construct a test and indicate how you would obtain critical values under $H_0 : \beta = 1$.