Solution to the Theoretical Problems of Problem Set 1

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Problem 1: (a) $E(\varepsilon_t) = 0$. As $E(\varepsilon_t^2) = (Eu_t^2)(E\alpha_0 + \alpha_1 E\varepsilon_{t-1}^2) = \alpha_0 + \alpha_1 E(\varepsilon_{t-1}^2)$, after assuming stationarity, we obtain that,

$$E(\varepsilon_t^2) = \frac{\alpha_0}{1-\alpha_1} = Var(\varepsilon_t).$$

For the autocovariance, note that $E(\varepsilon_t \varepsilon_{t-k}) = 0, \forall k \geq 1$, as $E(u_t|\mathcal{F}_{t-1}) = 0$. Thus for the values for which the sequence is stationary we have that we obtain a process with a trivial autocovariance process.

At this stage in the solutions we need to be careful what is meant by a white noise process. In my opinion, one can mean two things when saying that a process is white noise: either the autocovariance is a trivial structure, or a mean 0 iid process. If we take to mean the first one, then the answer is that the process of interest is white noise for parameter values for which we can have a stationary process. These parameter values are to be determined at a later stage in the solutions.

It is easy to see that,

$$E(\varepsilon_t|\mathcal{F}_{t-1}) = 0, \text{ and that,}$$
$$E(\varepsilon_t^2|\mathcal{F}_{t-1}) = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2.$$

We clearly need that $\alpha_0 > 0$ and that $|\alpha_1| < 1$.

(b) As $Ey_t = \alpha_0 + \alpha_1 Ey_{t-1}$, we get that,

$$E(y_t) = \frac{\alpha_0}{1-\alpha_1}.$$

On a similar note, $E(y_t^2) = \frac{\alpha_0^2 + \alpha_1^2 + \frac{\alpha_0}{1-\alpha_1}}{1-\alpha_1^2}$, thus giving us that,

$$Var(y_t) = \frac{\alpha_0^2 + \alpha_1^2 + \frac{\alpha_0}{1-\alpha_1}}{1-\alpha_1^2} - \left(\frac{\alpha_0}{1-\alpha_1}\right)^2.$$

(d)Under the above restrictions, the sequence is stationary. To start with, note that $E(x_t) = 0$. Next, $E(\varepsilon_t^4) = 3(\alpha_0^2 + 2\alpha_0^2 \frac{\alpha_1}{1-\alpha_1} + \alpha_1^2 E(\varepsilon_{t-1}^4))$. Thus we get that,

$$E(\varepsilon_t^4) = \frac{3\alpha_0^2(1+alpha_1)}{(1-\alpha_1)(1-3\alpha_1^2)}.$$

Next, in order to compute the autocovariances, we get that $Ex_t^2 = Var(x_t) = \frac{2\alpha_0^2}{(1-\alpha_1)(1-3\alpha_1^2)}$. The last step of the proofs involves taking the above calculation one step further: $E(x_t x_{t-k}) = E(\alpha_0 + \alpha_1 \varepsilon_{t-1}^2)x_{t-k} = \alpha_1 E(x_{t-1}x_{t-k})$. Thus we have obtained that,

$$E(x_t x_{t-k}) = \alpha_1^k \times \frac{2\alpha_0^2}{(1-\alpha_1)(1-3\alpha_1^2)}.$$

Problem 2: (a) Note that as the root of the equation,

$$1 - 1.3L + 0.4L^2 = 0$$

are outside of the unit circle, we have that the process is stationary. Next, in order to compute the ACF, let us denote it by $a(\cdot)$. It can be seen that,

$$a(k) = 1.3a(k-1) - 0.4a(k-2).$$

Thus $a(k) = a_1 0.5^k + a_2 0.8^k$, by multiplying the original equation by x_{t-k}. Next, note that $E(x_t) = 0$. Also, $E(x_t\varepsilon_t) = E(\varepsilon_t^2) = 1$. Next, by multiplying the original equation by ε_{t-1} we obtain that,

$$E(x_t\varepsilon_{t-1}) = 1.3 - 1.2 = 0.1.$$

Lastly,

$$E(x_t\varepsilon_{t-2}) = 0.13 - 0.4 + 0.2 = -0.07.$$

Thus we have that,

$$E(x_t^2) = a_0 = 1.3a_1 - 0.4a_2 + 1.$$

By repeating this calculation, and rewriting the constraints, we then will obtain a linear equation system. After solving we will obtain that $a_1 = 1.1$ and that $a_2 = -0.4$. This completes the solution of part (a).

(b) The process is causal but not invertible as the polynomial on the right, namely $1 - 1.2L + 0.2L^2$, has a unit root.

(c) The $MA(\infty)$ representation for such a process does express everything in terms of the iid errors. First, note that the following formula holds,

$$\frac{1}{1-r} = \sum_{j\geq 0} r^j.$$

As $1 - 1.3L + 0.4L^2 = (1 - L/2) * (1 - \frac{4L}{5})$ we have that,

$$\frac{1}{1-1.3L+0.4L^2} = (\sum_j (L/2)^j)(\sum \frac{4L}{5}^j) = \sum_j L^j \times (0.5^j + 0.5^{j-1}0.8 + ... + 0.8^j) = \sum_j L^j 0.5^j \frac{0.6}{1-1.6^{j+1}}.$$

Therefore the $MA(\infty)$ representation for such a process will be given by the formula,

$$(1 - 1.2L + 0.2L^2) \times (\sum_j L^j 0.5^j \tfrac{0.6}{1-1.6^{j+1}}).$$

Thus, after finishing the multiplication, the above expresion simplifies to,

$-L^0 + L^1 * 1.1 + ... + L^j * (0.5^j \frac{0.6}{1-1.6^{j+1}} - 1.20.5^{j-1} \frac{0.6}{1-1.6^j} + 0.20.5^{j-2} \frac{0.6}{1-1.6^{j-1}})$.

This completes the solution for part (c).