MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.231: Physics of Solids I Due in Ses #25

Problem Set #5

Problem 1: Kinetic Energy

Show that the kinetic energy U_o of a three-dimensional gas of N free electrons at T=0 is given by $\frac{3}{5} N \varepsilon_F$.

Problem 2: Pressure and Bulk Modulus of an Electron Gas

a) Find the pressure of a free electron gas at T=0. Begin with the thermodynamic expression for the pressure based on the combined first and second law. Use the result from problem 1 and an expression for the volume derivative of ε_F. Show that P(T=0) can be expressed as $\frac{2}{3} U_o/V$.

b) Show that the bulk modulus $B = -V\frac{\partial P}{\partial V}$ of an electron gas at T=0 is $\frac{5}{3} P$ or $\frac{10}{9} U_o/V$. [Note that at T=0 the isothermal (constant T on the partial derivative) and the adiabatic (constant entropy) bulk moduli are equal due to the third law of thermodynamics.]

c) Use the data in table 6.1 of Kittel to estimate the electron contribution to the bulk modulus of sodium.

Problem 3: Chemical Potential of a Two-Dimensional Electron Gas

a) Find the density of states as a function of energy for a non-interacting free electron gas in two dimensions.

b) For this system it is possible to find an analytic expression for the temperature dependence of the chemical potential. Show that

$$\mu(T) = k_B T \ln\left[\exp(\varepsilon_F/k_B T) - 1\right],$$

where n is the number of electrons per unit area.

Problem 4: Density of States in a Superconductor

Consider electrons in a real metal whose Fermi surface is spherical with Fermi wavevector k_F and Fermi energy ε_F. We are interested only in states near the Fermi surface, and we can approximate their energies as follows:

$$\tilde{\varepsilon} \equiv \varepsilon - \varepsilon_F = v_F \hbar (k - k_F).$$

$\tilde{\varepsilon}$ is the energy measured relative to the Fermi energy (it can be positive or negative), and v_F is a parameter known as the group velocity at the Fermi surface.

a) Find an expression for v_F for a free non-interacting electron gas. Sometimes the proportionality between $\tilde{\varepsilon}$ and $(k - k_F)$ is expressed, not in terms of v_F, but in terms of an "effective mass" m^* for the excitations. Using the result you just obtained as a guide, write an equation for $\tilde{\varepsilon}$ in terms of a suitably defined m^*.

b) When the interaction which gives rise to superconductivity is turned on, the new energies $\tilde{\varepsilon}'$ are related to the old ones, $\tilde{\varepsilon}$, by the equations

$$\tilde{\varepsilon}' = +\sqrt{\tilde{\varepsilon}^2+\Delta^2} \quad \text{if } \tilde{\varepsilon}>0,$$

$$\tilde{\varepsilon}' = -\sqrt{\tilde{\varepsilon}^2+\Delta^2} \quad \text{if } \tilde{\varepsilon}<0,$$

Find the new density of states, $D(\tilde{\varepsilon}')$, in the vicinity of the Fermi surface.

Problem 5: Energy Transfer to a Fermion Gas

Consider a free, non-interacting gas of Fermions of mass M at T=0. Assume that a momentum $\vec{P} = \hat{x}p_x$ is given to the system (by the scattering of a neutron for example), and it goes into the creation of a single hole-particle pair.

a) For a given p_x, what is the *smallest* energy that can be imparted to the system by this process? Sketch this transition on a diagram of the Fermi sphere in momentum space.

a) For a given p_x, what is the *largest* energy that can be imparted to the system by this process? Sketch!

c) On a graph of energy, E, verses p_x show the band of energies that can be transferred to the system in this process. Show results for both $p_x < 2\hbar k_F$ and $p_x > 2\hbar k_F$.

Problem 6: Fermi Gases in Stars

Quantum mechanics plays an important role in the structure and evolution of stars. This problem is designed to introduce some of the important concepts, building on our ability to treat Fermi gases at absolute zero.

a) At the center of the Sun, the temperature is about 1.5×10^7 K and the number density of protons or electrons is about 1.0×10^{26} cm^{-3}. Use the ideal gas law to compute the pressure due to particles at the center of the Sun. Calculate the Fermi energy and the associated pressure of a zero temperature electron gas of this density. Is the electron gas at the center of the Sun degenerate?

b) When the fusion process in a star have have run their course and nuclear burning stops, the temperature begins to fall. The kinetic pressure of the particles and the radiation pressure both drop, allowing the gravitational forces to compress the star to higher densities. In some cases this process ends when the mass density reaches about 10^6 g–cm^{-3}. Further collapse is prevented by the pressure of the degenerate electron gas. What is the Fermi energy now? The star might still be as hot as 1.0×10^7 K. What would kT/ε_F be in that case? What is the Fermi pressure for the this electron gas?

c) The star described in b), a white dwarf, has an interesting relation between its mass and its radius. The relation can be found most easily by minimizing the total energy of the star, $E_{total} = E_{gravity} + E_{kinetic}$. The gravitational energy of a star is given by $-\zeta GM^2/R$ where ζ is a dimensionless constant of order unity which depends on how the density of the star varies with radial distance from the center. Write an expression for the kinetic energy of the electron gas (the proton contribution to the kinetic energy is much smaller) isolating its dependence on the total mass of the star M and on the star's radius R (assume a uniform electron density). Make a sketch of $E_{total}(R)$ which shows that there is a stable equilibrium point for some finite value of R. Find the equilibrium R and show that it is proportional to $M^{-1/3}$. Note that this implies that the volume is inversely proportional to the mass for a white dwarf star!

d) A problem with the calculation in c) is that the electron energies can become so high that speeds approach the speed of light. Calculate the velocity of an electron at the Fermi surface using the density given in c). The problem is more than simply computational. Use the results from problem 2a to show that the Fermi pressure for the classical electron gas is proportional to $n^{5/3}$. When the gas becomes relativistic the equation of state "softens" in that the pressure rises more slowly with density. Show that for an ultra-relativistic gas, where the dispersion curve is $\varepsilon(k) = c\hbar k$, the Fermi pressure is proportional to $n^{4/3}$.

e) A consequence of the relativistic softening of the Fermi pressure is that if the mass of the star is too great, and the densities become too large, the electron pressure can no longer withstand the gravitational pressure and the star collapses into an even more compact object, a neutron star or a black hole. As an indication of the problem, plot $E_{total}(R)$ for the ultra-relativistic white dwarf. What sort of equilibrium condition does this correspond to?