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Generalization of the MV Mechanism

Jing Chen

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Abstract

Micali and Valiant proposed a mechanism for combinatorial auctions that is dominant-strategy truthful, guarantees reasonably high revenue, and is very resilient against collusions. Their mechanism, however, uses as a subroutine the VCG mechanism, that is not polynomial time.

We propose a modification of their mechanism that is efficient, while retaining their collusion resilience and a good fraction of their revenue, if given as a subroutine an efficient approximation of the VCG mechanism.

1 Introduction

Combinatorial Auctions 101 The following "summary" about combinatorial auctions is taken from [MV07], essentially verbatim.

A (non-Bayesian, $n \times m$) combinatorial-auction *context* is described as follows. There is a set of *players* $N = \{1, \ldots, n\}$ and a set of m goods G. A *valuation* is a function from G's subsets to $\mathbb{R}^+$, and each player i has a private valuation TV_i, which we refer to as i's *true valuation.* An *outcome* consists of (1) a *profile* (i.e., a vector indexed by the players) $P = P_1, \ldots, P_n$, where $P_i \in \mathbb{R}^+$ is the *price* to be paid by player i, and (2) an *allocation* $A = A_0, A_1, \ldots, A_n$, where A_i is the subset of goods allocated to player i, and A_0 the set of unallocated goods. For each outcome $\Omega = (A, P)$, the *utility* of player i is defined via his utility function u_i as follows: $u_i(TV_i, \Omega) = TV_i(A_i) - P_i$, that is, i's true value of the goods allocated to him minus the price he pays. Note that such a context is fully described by just N, G, and the true-valuation profile TV, which in turn determine the outcome space and the utility functions.

For such a context, a combinatorial-auction *mechanism* is a (possibly probabilistic) function $\mathcal{M}$ mapping a profile of valuations V to an outcome (A, P) such that A_i is empty and P_i is 0 whenever V_i is the null valuation.[1] An $n \times m$ context $\mathcal{C} = (N, G, TV)$ and an $n \times m$ mechanism $\mathcal{M}$ define a ($n \times m$) *combinatorial auction:* namely, the game $\mathcal{G} = (\mathcal{C}, \mathcal{M})$ envisaged to be played as follows. First, each player i (independently of the others) chooses a valuation BID_i on inputs TV_i, N, and G. Then, an outcome (A, P) is obtained by evaluating $\mathcal{M}$ on BID, the profile of all such valuations. We refer to the so chosen valuations as *bids*, to emphasize that they need not coincide with the players' true valuations. In such a game, a *strategy* is a (possibly probabilistic) way for a player to choose his bid. Say $\mathcal{M}$ is a *dominant-strategy truthful* (DST) mechanism, if for any player i, (1) bidding his true valuation is at least as good as any other strategy (in the sense of maximizing his own utility), no matter what bids the other players might choose; and (2) i cannot be charged more than he bids.

To emphasize the underlying mechanism $\mathcal{M}$, We consider $\mathcal{M}$ as two separate functions: an *allocation function* $\mathcal{M}_a$ and a *price function* $\mathcal{M}_p$, such that $\forall BID, \mathcal{M}(BID) = (\mathcal{M}_a(BID), \mathcal{M}_p(BID))$. For a probabilistic mechanism $\mathcal{M}$, the expected revenue generated by $\mathcal{M}$ on bid profile BID is $E[\sum_{i=1}^{n} \mathcal{M}_p(BID)_i]$. At last, if $C \subset N$, and V is a profile, then V_C is the sub-profile indexed by the players in C, that is, $V_C = \{V_i : i \in C\}$.

Social Welfare Notation The *social welfare* relative to a valuation profile V and an allocation A is denoted as $SW(V, A) \triangleq \sum_{i=1}^{n} V_i(A_i)$. If $A = \mathcal{M}_a(V)$ and the underlying mechanism $\mathcal{M}$ is clear from context, we use $SW(V)$ for short. For the true valuation profile TV, the notation is further shortened as $SW \triangleq SW(TV)$. The *maximum social welfare* relative to a valuation profile V is $MSW(V) = \max_{A \in \mathbb{A}(G)} SW(V, A)$, where $\mathbb{A}(G)$ is the set of all possible allocations of G. Again $MSW \triangleq MSW(TV)$. For any sub-profile of V, V_C, the notation is defined accordingly. For example, $SW_C \triangleq SW(TV_C)$,

[1] This guarantees that any player can "opt out" (i.e., win no goods and pay nothing) by bidding the null valuation.

$MSW_C \triangleq MSW(TV_C)$, etc. Particularly, for any $i \in N$, $SW_{-i} \triangleq SW_{N\setminus\{i\}}$, and MSW_{-i} is defined analogously.

1.1 The MV Mechanism

In [MV07], Micali and Valiant put forward a mechanism that we refer to as the MV mechanism. This mechanism is DST and generates expected revenue greater than $\frac{MSW_{-*}}{\log\min\{m,n\}}$ from any $n \times m$ combinatorial auction context, where "$*$" is the *star player* whose true valuation for some bundle, $S_* \subseteq G$, is higher than or equal to any player's valuation for any bundle, that is, $\forall i$ and $\forall S \subseteq G: \ TV_*(S_*) \geq TV_i(S)$. (Thus $MSW_{-*} \triangleq MSW_{N\setminus\{*\}}$). Given a bid profile BID, the MV mechanism works as follows. First it runs the VCG mechanism [V61, C71, G73] to get $VCG(BID) = (A', P')$. Then for each winner i, that is, a player to whom the VCG allocates a non-empty subset of goods ($A'_i \neq \emptyset$), the MV mechanism raises i's VCG price, P'_i, to a proper fraction of $MSW(BID_{-i})$. Specifically, they choose a scaling factor α from a continuous exponential distribution, allocate A'_i to i if and only if $P'_i + \alpha MSW(BID_{-i}) \leq BID_i(A'_i)$.

1.2 Computational Efficiency

The MV mechanism requires the exact computation of MSW and of all possible MSW_{-i}, quantities that have been shown to be NP-hard [RPH98] to compute, even in some very simple case. Thus, ultimately, the MV mechanism is not polynomial-time. Traditionally, game theory doesn't care about computational efficiency. But an efficient version of the MV mechanism will undoubtedly be more useful.

To discuss efficiency, one must decide on a suitable representation of valuations (i.e., bids). We assume that a valuation V is represented as a table, with each row corresponding to a subset of goods S and containing the value $V(S)$. Note that the computation of MSW is still NP-hard in this representation.

1.3 Our Contribution

We notice that, although the maximum social welfare is hard to compute exactly, it could possibly be efficiently approximated.

Definition 1. *Let $c > 1$ be a constant and M be a combinatorial-auction mechanism. We say that M is a c-*MSW mechanism*, if (1) M is DST, (2) M is polynomial-time, and (3) for any bid profile BID, $SW(BID, M_a(BID)) \geq MSW(BID)/c$. We refer to c as the* approximation ratio *of M.*

Notice that c-MSW mechanisms indeed exist in several contexts. For example, a $\sqrt{m}$-MSW mechanism exists for single-minded auctions [OS02][2]. Accordingly, we find

[2] A player i is *single-minded* if and only if there exists a single subset $S \subseteq G$ and $x \in \mathbb{R}^+$ such that for any $T \subseteq G$, $TV_i(T) = x$ whenever $S \subseteq T$ and 0 otherwise. A *single-minded* auction is an auction where all players are single-minded

it important to show that the MV mechanism can be slightly modified to achieve both revenue guarantee and computational efficiency. Specifically, we put forward the following theorem.

Theorem 1. *$\forall c > 1$, if there exists a c-MSW mechanism, there exists a DST and polynomial-time mechanism whose expected revenue is greater than $\frac{MSW_{-*}}{c \log \min\{m,n\}}$.*

2 The Modified MV Mechanism

The intuition is that instead of using the VCG mechanism, we use any c-MSW mechanism $\mathcal{M}'$. Also, instead of raising each winner i's price to a fraction of MSW_{-i}, we raise it to a fraction of $SW'(BID_{-i})$, the social welfare achieved by $\mathcal{M}'$ on input BID_{-i}. This is done by sampling the scaling factor α from a continuous exponential distribution, as in [MV07]. However, $\alpha SW'(BID_{-i})$ may not be sufficient to generate a good revenue, as in the worst case, $SW'(BID_{-i})$ is only a $1/c$ fraction of MSW_{-i} [3]. To generate as much revenue as possible, we act more aggressively and raise i's price to a fraction of $c \cdot SW'_{-i}$, which is an upper-bound of MSW_{-i}. Of course we need a balance some how to prevent the adjusted price from going too high so that most players fail to pay. This is achieved by changing the distribution of α a little so that this part is more conservative than before.

Given explicit knowledge of c, our mechanism $\mathcal{M}$ on input BID, computes the allocation and price (A, P) as follows:

1. Pick a scaling factor $\alpha \in [0, 1]$ as follows:
   (a) Let $\mu = \min\{m, n\}$, and $c_{m,n}$ solves the equation $e^{(x/c^2)-2} = x\mu$ such that $c_{m,n} > 2c^2$. Note that such a $c_{m,n}$ indeed exists and is unique, as discussed in Section 3.
   (b) $r \leftarrow [-(\frac{c_{m,n}}{c^2} - 2), 0]$.
   (c) With probability $p = \frac{1}{\frac{c_{m,n}}{c^2} - 1}$, $\alpha = 0$. With probability $1 - p$, $\alpha = e^r$.

2. Compute provisional allocation A' and corresponding price profile P' such that $(A', P') = \mathcal{M}'(BID)$. Let the set of provisional winners W' consist of all players that obtain a non-empty subset of goods in A'.

3. $\forall j \notin W'$, $A_j = \emptyset$ and $P_j = 0$. Furthermore, $\forall i \in W'$, Let $P''_i = P'_i + \alpha \cdot c \cdot SW'(BID_{-i})$. If $P''_i \leq BID_i(A'_i)$, then i becomes a final winner, $A_i = A'_i$ and $P_i = P''_i$. Otherwise $A_i = \emptyset$ and $P_i = 0$.

   Note that $SW'(BID_{-i})$ is the social welfare achieved by $\mathcal{M}'$ with input BID_{-i}, which can be efficiently evaluated from $\mathcal{M}'(BID_{-i})$.

[3] It may be sufficient, as the desired bound in Theorem 1 also contains a $1/c$ factor, but at least we don't know how to use it to do the proof.

3 Sketch of Proof

Without loss of generality, we assume that $c < \mu$. In fact, there exists a trivial μ-MSW mechanism $\mathcal{T}$: On input BID, $\mathcal{T}$ simply finds a player x and a subset of goods S_x, such that $\forall i$ and $\forall S \subseteq G : BID_x(S_x) \geq BID_i(S)$. $\mathcal{T}$'s allocation consists of assigning S_x to player x and the empty set to all other players. $\mathcal{T}$ imposes a price equal to the "second-highest bid" to x (i.e., $\mathcal{T}_p(BID)_x = \max_{i \neq x, S \subseteq G} BID_i(S)$), and price 0 to all other players [4]. It is easy to see that $\mathcal{T}$ is DST. Moreover, the social welfare generated by $\mathcal{T}$ is $SW(BID, \mathcal{T}_a(BID)) = BID_x(S_x)$. Notice that in the VCG mechanism, (1) the social welfare is $SW(BID, VCG_a(BID)) = MSW$, (2) there are at most μ winners and (3) for each winner i, $BID_i(VCG_a(BID)_i) \leq BID_x(S_x)$. Therefore we have $MSW \leq \mu BID_x(S_x)$, and we conclude that $\mathcal{T}$ is indeed a μ-MSW mechanism.

Recall $c_{m,n}$'s definition. W.l.o.g., $\mu \geq 2$. It is easy to verify that the continuous function $f(x) = e^{(x/c^2)-2} - x\mu$ is negative when $x = 2c^2$, positive when $x \geq 2c^2 \log(2\mu c^2) + 2c^2$, monotonically decreasing when $x \in (2c^2, c^2 \log \mu c^2 + 2c^2)$, and monotonically increasing when $x \in (c^2 \log \mu c^2 + 2c^2, 2c^2 \log(2\mu c^2) + 2c^2)$. Therefore the equation $f(x) = 0$ has a unique solution, $c_{m,n}$, when $x > 2c^2$. More precisely, $c_{m,n}$ belongs to the interval $(c^2 \log \mu c^2 + 2c^2, 2c^2 \log(2\mu c^2) + 2c^2)$. As $1 \leq c < \mu$, we know that $c^2 \log \mu < c_{m,n} < 2c^2 \log(2\mu^3) + 2c^2 = 6c^2 \log \mu + 4c^2 \leq 10c^2 \log \mu$.

Claim 1. $\mathcal{M}$ is DST.
Proof Sketch. This follows directly from the fact that $\mathcal{M}'$ is DST and the analysis in [MV07].

Claim 2. $\mathcal{M}$ generates expected revenue greater than or equal to $\frac{c \cdot MSW_{-*}}{c_{m,n}}$.

(Since $c_{m,n} = \Theta(c^2 \log \mu)$, this means that $\mathcal{M}$'s expected revenue is $O(\frac{MSW_{-*}}{c \log \mu})$.)

Proof Sketch. We prove that whenever BID is a valuation profile for a $n \times m$ auction, the expected revenue generated by $\mathcal{M}$ with input BID satisfies that

$$E[\sum_{i \in N} \mathcal{M}_p(BID)_i] \geq \frac{c \cdot MSW(BID_{-*})}{c_{m,n}}. \tag{1}$$

Claim 2 then follows from this equation and Claim 1.

The technique used to prove Equation 1 is similar to that in the proof of Theorem 2b in [MV07]. Recall that the proof there discusses the expected revenue generated by MV in two cases.

In the first case, the star player's bid for the bundle S'_* allocated to him is large enough, that is, $BID_*(S'_*) > P'_* + MSW(BID_{-*})$. (Note that S'_* may not be equal to S_*.) This implies that the star player is a provisional winner, i.e., $S'_* \neq \emptyset$, since the right part is always non-negative. Moreover, $*$ is also a final winner, as the highest possible price for him (the right part) is still less than his bid. Therefore in this case, the expected revenue generated by MV is lower-bounded only by the expected revenue generated by $*$, which already achieves the desired bound.

[4] Note that $\mathcal{T}$ is indeed polynomial-time using our representation of valuations.

In the complementary case, every provisional winner i's bid on the bundle S'_i allocated to him is not much larger than his provisional price P'_i, or in other words, P'_i is already a good approximation to $BID_i(S'_i)$. Combined with the price-raising scheme, the expected revenue generated by each provisional winner contributes a large enough fraction to the final revenue, and the desired bound follows.

Our detailed analysis is given below.

Case 1: The $*$ player's bid on S'_* allocated to him by $\mathcal{M}'$ on input BID satisfies $BID_*(S'_*) > P'_* + c \cdot SW'(BID_{-*})$. This implies that $*$ is a provisional winner as well as a final winner, using the same analysis as in [MV07]. Therefore we can also lower-bound the revenue of $\mathcal{M}$ by using the revenue of $*$ alone, and it is easy to show that

$$E[M_p(BID)_*] \geq \frac{c \cdot MSW(BID_{-*})}{c_{m,n}},$$

and we are done.

Case 2: $BID_*(S'_*) \leq P'_* + c \cdot SW'(BID_{-*})$. We claim that in this case, $\forall i \in W'$ with allocation S'_i and price P'_i,

$$BID_i(S'_i) \leq P'_i + c \cdot SW'(BID_{-i}). \tag{2}$$

This can be easily proven. If $i = *$, Equation 2 follows directly. $\forall i \neq *$, we know that

$$BID_i(S'_i) \leq BID_*(S_*) \leq MSW(BID_{-i}) \leq c \cdot SW'(BID_{-i}) \leq P'_i + c \cdot SW'(BID_{-i}),$$

where the first inequality follows from the definition of $*$ player and S_*, the second one is because $* \in N \backslash \{i\}$, and the third one is given by the fact that $\mathcal{M}'$ is a c-approximation mechanism.

Now we can use the technology used in the second case of [MV07]. First, $\forall i \in W'$, if $P'_i + e^{-(\frac{c_{m,n}}{c^2}-2)} \cdot c \cdot SW'(BID_{-i}) \leq BID_i(S'_i)$, then combining Equation 2, we have $-(\frac{c_{m,n}}{c^2} - 2) \leq \log \frac{BID_i(S'_i) - P'_i}{c \cdot SW'(BID_{-i})} \leq 0$, and following [MV07] we get

$$E[\mathcal{M}_p(BID)_i] \geq \frac{1}{\frac{c_{m,n}}{c^2} - 1} \left[BID_i(S'_i) - e^{-(\frac{c_{m,n}}{c^2}-2)} \cdot c \cdot SW'(BID_{-i}) \right]. \tag{3}$$

While if $P'_i + e^{-(\frac{c_{m,n}}{c^2}-2)} \cdot c \cdot SW'(BID_{-i}) > BID_i(S'_i)$, then $P'_i > BID_i(S'_i) - e^{-(\frac{c_{m,n}}{c^2}-2)} \cdot c \cdot SW'(BID_{-i})$. Therefore

$$E[\mathcal{M}_p(BID)_i] = \frac{P'_i}{\frac{c_{m,n}}{c^2} - 1} > \frac{1}{\frac{c_{m,n}}{c^2} - 1} \left[BID_i(S'_i) - e^{-(\frac{c_{m,n}}{c^2}-2)} \cdot c \cdot SW'(BID_{-i}) \right]. \tag{4}$$

That means in Case 2, Equation 3 is satisfied for all $i \in W'$. Summing this inequality up over all $i \in W'$, following [MV07], we get

$$E[\sum_{i \in N} \mathcal{M}_p(BID)_i] \geq \frac{c \cdot MSW_{-*}(BID)}{c_{m,n}}.$$

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