NARROW-BAND NOISE PERFORMANCE OF PUMPED NONLINEAR SYSTEMS

by

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ABSTRACT

Circuit theory of linear time-invariant systems and the study of propagation of noise through such systems is a well-established discipline. Very little, however, is known about the noise performance of devices like harmonic multipliers, dividers, limiters, and systems consisting of such devices. In this thesis, we deal with the narrow-band noise performance of pumped nonlinear systems like this.

Penfield has developed a circuit theory for the study of propagation of small perturbations through such pumped nonlinear systems. These perturbations can be desired or undesired modulation, noise, or hum. In this thesis, it has been assumed that these perturbations are entirely due to the noise. Assumption is also made that the noise is narrow-band, and the signal-to-noise ratio at any point in the system is high. With these assumptions, circuit theory of small perturbations has been used in this thesis.

It has been shown that a noisy pumped nonlinear system can be considered as a multiport network, with each port exchanging power at only one frequency, and no two ports exchanging power at the same frequency. This multiport network describes only the terminal noise behavior of the device. Several methods of representing physical sources of noise in pumped nonlinear systems have been given. The concepts of exchangeable amplitude and phase noise powers have been developed, and a set of figures of merit for the system has been defined in terms of these exchangeable noise powers. Two other ways of characterizing the noise performance of pumped nonlinear systems have also been suggested. Some details of the analysis of noise performance of abrupt-junction varactor frequency multipliers and dividers have also been given.

The idea of lossless parametric imbeddings for multifrequency noisy networks has been introduced.

Finally, for multifrequency noisy networks, a set of matrices the eigenvalues of which remain invariant when the networks are subjected to linear transformations of different kinds has been presented.

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CHAPTER 1

INTRODUCTION

The quality of performance of any transducer is affected by the physical sources of noise within the transducer, and the sources of noise present in the source and load terminations. The noise performance of two-terminal-pair amplifiers has been studied and the concept of spot noise figure introduced by Friis and Fränz has played an essential role in communication practice [1]. There have been also many studies of noise in linear noisy networks [1], and statistical properties of noise through nonlinear devices [2].

The problem with which we shall be concerned is the noise performance of pumped nonlinear devices containing internal noise sources. The systems we have in mind are nonlinear, but operated in a periodic steady state, which is assumed to be known a priori. Examples of such systems are oscillators, frequency multipliers, dividers, limiters, modulators, and all linear networks.

Three major assumptions are made for such systems. The first is that the system is driven periodically by known voltages and currents which henceforth we shall call the carrier.[1] Second, we assume that the noise is bandlimited in a frequency

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[1] In some systems (such as frequency multipliers) the carrier may be at different frequencies at different parts of the system.

band surrounding the carrier. Third, we assume that the signal-to-noise ratio at any point in the system is high.

We have to mention here that a great deal is unknown about the noise performance of pumped nonlinear systems. Since the noise is assumed to be narrow-band, the noise performance of pumped nonlinear systems considered in this thesis will be called the "spot noise performance" of these systems.

A circuit theory for the study of propagation of small perturbations through pumped nonlinear systems has been developed by Penfield [3]. We start with a brief summary of this theory in Chapter 2. Several forms of representations have been developed for the modelling of physical sources of noise in pumped nonlinear systems in the remainder of this chapter.

Exchangeable noise power has been defined for linear systems [1], and the noise performance of these systems is usually characterized in terms of a noise figure defined in terms of these exchangeable noise powers. To characterize the noise performance of pumped nonlinear systems, we develop the concept of exchangeable amplitude and phase noise powers in Chapter 3. The values of these exchangeable noise powers are invariant to any linear lossless transformations. In this chapter we also define a set of figures of merit for these pumped nonlinear systems in terms of exchangeable noise powers. Two other ways of characterizing the noise performance of pumped nonlinear systems have also been described in Chapter 3.

In Chapters 4 and 5 we discuss in some detail the analysis of noise performance of abrupt-junction varactor frequency multipliers and dividers. The theory developed in Chapters 2 and 3 has been used in this chapter to investigate the noise performance of these devices. The figures of merit as defined in Chapter 3 have been evaluated and illustrated in Chapters 4 and 5 for the devices we have analyzed. Methods to analyse noise performance of parametric amplifiers driven by noisy pumps have also been given in this chapter.

Now most of the lossless nonlinear systems obey the Manley-Rowe relations. We start with a discussion of these relations in Chapter 6. A characteristic noise matrix has been defined for these pumped nonlinear systems in this chapter. The eigenvalues of this characteristic noise matrix have the dimensions of energy. In this chapter we also investigate a canonical form that can be obtained for these systems by lossless parametric imbedding.

It was shown by Haus and Adler that the eigenvalues of the characteristic noise matrix defined for linear noisy networks remain invariant when the network is imbedded in a linear lossless system. As far as terminal noise behavior is concerned, a pumped nonlinear system can be considered as a multiport multifrequency noisy network, with each port exchanging power at only one frequency, and no two ports exchanging power at the same frequency. In Chapter 7, we deal with the invariants that can be obtained by cascading a multifrequency noisy network with linear systems of different kinds. Typical examples of cascading networks are linear noiseless networks, linear and lossless networks, and linear lossless reciprocal networks.

As mentioned earlier in this chapter, very little is known about the noise performance of nonlinear systems. This thesis is expected to be a modest contribution to the study of noise performance of pumped nonlinear systems.

CHAPTER 2

REPRESENTATION OF NOISE SOURCES IN PUMPED NONLINEAR SYSTEMS

Spurious undesired signals are always present in systems and their components. These undesired signals are usually called noise. Since noise reduces the amount of information that can be transmitted with a specific signal power, quantitative measures of noise are often indispensable to engineering evaluation of systems.

Transducers performing signal processing such as amplification, frequency mixing, frequency multiplying, frequency shifting, etc., can be classified as two-ports for theoretical analysis. Several schemes have been used to represent noise at a given frequency ω_o in a linear two-port [1]. In this chapter such schemes will be developed for the representation of noise sources in pumped nonlinear systems. It will be shown that at each port, for each frequency of the carrier, it is necessary to have two equivalent internal noise sources rather than the one that is required in linear circuit theory. Frequency of the carrier is supposed to mean the frequency of the signal. It has also been shown that Rothe-Dahlke and Bauer-Rothe types of representations can be used for the characterization of noise sources in pumped nonlinear systems.

2.1. SOME CONSIDERATIONS OF PERIODICALLY DRIVEN NONLINEAR SYSTEMS+

Circuit theory of pumped nonlinear systems has been developed by Penfield [2]. The systems under consideration are nonlinear, but driven by a strong periodic signal. Examples of such systems are oscillators, frequency multipliers, limiters, discriminators, modulators, and systems consisting of such devices. It is of interest to enquire how small perturbations on the periodic driving are propagated through such systems, and to this end development of a circuit theory for these perturbations is desirable. In different contexts these perturbations could be desired or undesired modulation, noise, hum, or synchronizing signals. In this chapter we shall assume that the perturbations are entirely due to the noise present in the system. In general the random noise processes in such systems will, because of the periodic driving, be non-stationary, but various representations can be developed that are stationary, and hence can be described by spectral densities.

Consider a nonlinear system. Let us assume that the large signal voltages and currents at various points within the system are, by design, periodic with some frequency ω_o. Thus the voltage at some specific point within the network or across one of its terminal pairs, v(t), is of the form

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+This is a summary of Ref. [2].

$$\sum_{k=-\infty}^{\infty} V_k \, e^{jk\omega_o t} \tag{2.1}$$

where the V_k are the half-amplitude[1] Fourier coefficients, with $V_{-k} = V_k^*$. However, the actual voltage may deviate from (2.1) because of noise. Thus

$$v(t) = \sum_{k=-\infty}^{\infty} V_k \, e^{jk\omega_o t} + \delta v(t). \tag{2.2}$$

The circuit theory to be set up is one which describes the perturbations $\delta v(t)$ and relates it to similar perturbations of voltages and currents in other parts of the system.

The major restrictions of the theory are that the driving is periodic, that the perturbations are at frequencies close to the carrier, and that these perturbations are small.

In most systems of the type we are interested in, the carrier is a sine wave. It is convenient then to assume that the voltages and currents of the carrier are, at each port, sinusoidal[2]. Thus

$$v(t) = V_k \, e^{jk\omega_o t} + V_k^* \, e^{jk\omega_o t} + \delta v(t) \tag{2.3}$$

for some positive integer k.

---

[1]Note the use of half-amplitudes, rather than amplitudes or r.m.s. values.

[2]This assumption is no restriction on generality [2].

We now assume that the noise perturbations $\delta v(t)$ contain frequencies that are located in a band of width $2\omega_c$ centered about frequency $k\omega_o$ where $2\omega_c < \omega_o$. The perturbations are, therefore, bandlimited. Similar expressions can be written for currents and voltages at various places in the network. The various voltages like v(t) obey Kirchhoff's voltage law(KVL), and the various currents in the network obey Kirchhoff's current law(KCL). Furthermore, the carrier voltages and currents at various parts of the network obey KCL and KVL, and therefore the perturbations like $\delta v(t)$ and $\delta i(t)$ obey KCL and KVL.

Let us write $\delta v(t)$ as

$$\delta v(t) = 2\, v_c(t) \cos k\omega_o t + 2\, v_s(t) \sin k\omega_o t. \tag{2.4}$$

We can show that $v_c(t)$ and $v_s(t)$ are bandlimited about d.c. In Eq. (2.4) we have represented the perturbations $\delta v(t)$ in terms of two real slowly varying functions of time, $v_c(t)$ and $v_s(t)$, which are defined without regard to phase of the carrier. A similar decomposition can be done for the current perturbation $\delta i(t)$ in terms of two slowly varying currents $i_c(t)$ and $i_s(t)$. Similar decompositions can be done for all voltages and currents in the network. The voltages $v_c(t)$ so defined at various points in the network obey KVL, and the various voltages $v_s(t)$ also obey KVL. Similarly, the currents $i_c(t)$ and $i_s(t)$ obey KCL.

Let us write v(t) as

$$v(t) = V_k e^{jk\omega_o t} + V_k^* e^{-jk\omega_o t} + 2\, v_c(t) \cos k\, \omega_o t$$
$$+ 2\, v_s(t) \sin k\omega_o t$$
$$= 2 \,\text{Re} \left[|V_k|\, e^{j\emptyset} + v_c(t) - jv_s(t) \right] e^{jk\omega_o t}; \qquad (2.5)$$

and rewrite this in the form

$$v(t) = 2 \,\text{Re} \left[|V_k| + v_a(t) - jv_p(t) \right] e^{j(k\omega_o t + \emptyset)} \qquad (2.6)$$

where $v_a(t)$ and $v_p(t)$ are slowly varying functions of time[3], also bandlimited around d.c. The voltage $v_a(t)$ can be interpreted, since it is small, as a perturbations on the amplitude $|V_k|$ of the carrier. Similarly, since $v_p(t)$ is small, Eq. (2.6) can be rewritten in the form

$$v(t) = 2 \,\text{Re} \left[|V_k| + v_a(t) \right] e^{j[k\omega_o t + \emptyset + \emptyset_v(t)]} \qquad (2.7)$$

where the phase perturbations $\emptyset_v(t)$ is also slowly varying.

As pointed out earlier, it will be assumed in this chapter that the small perturbation voltage $\delta v(t)$ is actually a noise voltage. Let us call this $v_n(t)$, a sample function of a random process. We assume that each sample function $v_n(t)$ is band-limited and small. We now assume that even though the sample functions $v_n(t)$ may not be stationary, the physical source of

[3]The relationship between $v_a(t)$ and $v_p(t)$, and the previously defined $v_c(t)$ and $v_s(t)$, can be worked out easily [2].

the nonstationary properties of the noise is the periodic driving of the nonlinear system. With these assumptions, we can show that the random processes $v_c(t)$ and $v_s(t)$ are stationary in the wide sense.

Let us now investigate the "spot" frequency noise performance of these pumped nonlinear systems. In that case we may write

$$v_c(t) = V_c\ e^{j\omega t} + V_c^*\ e^{-j\omega t} \tag{2.8}$$

$$v_s(t) = V_s\ e^{j\omega t} + V_s^*\ e^{-j\omega t} \tag{2.9}$$

where V_c and V_s are random complex numbers.

The actual voltage v(t), then, is

$$v(t) = 2\ \mathrm{Re}\ [V_k + (V_c - jV_s)e^{j\omega t} + (V_c^* - jV_s^*)\ e^{-j\omega t}]\ e^{jk\omega_o t}. \tag{2.10}$$

In Eqs. (2.8) and (2.9) it is assumed that the power spectrum of the noise associated with the carrier at frequency $k\omega_o$ may be nonzero only at the four frequencies $\pm k\omega_o \pm \omega$. We shall call ω the frequency deviation. A typical spectrum of v(t) is shown in Fig. 2.1.

Equation (2.10) may be rewritten as

$$v(t) = 2\ \mathrm{Re}\left[V_k e^{jk\omega_o t} + V_{\alpha k} e^{j(k\omega_o+\omega)t} + V_{\beta k} e^{j(-k\omega_o+\omega)t}\right] \tag{2.11}$$

where

$$V_{\alpha k} = V_c - jV_s \tag{2.12}$$

$$V_{\beta k} = V_c + jV_s. \tag{2.13}$$

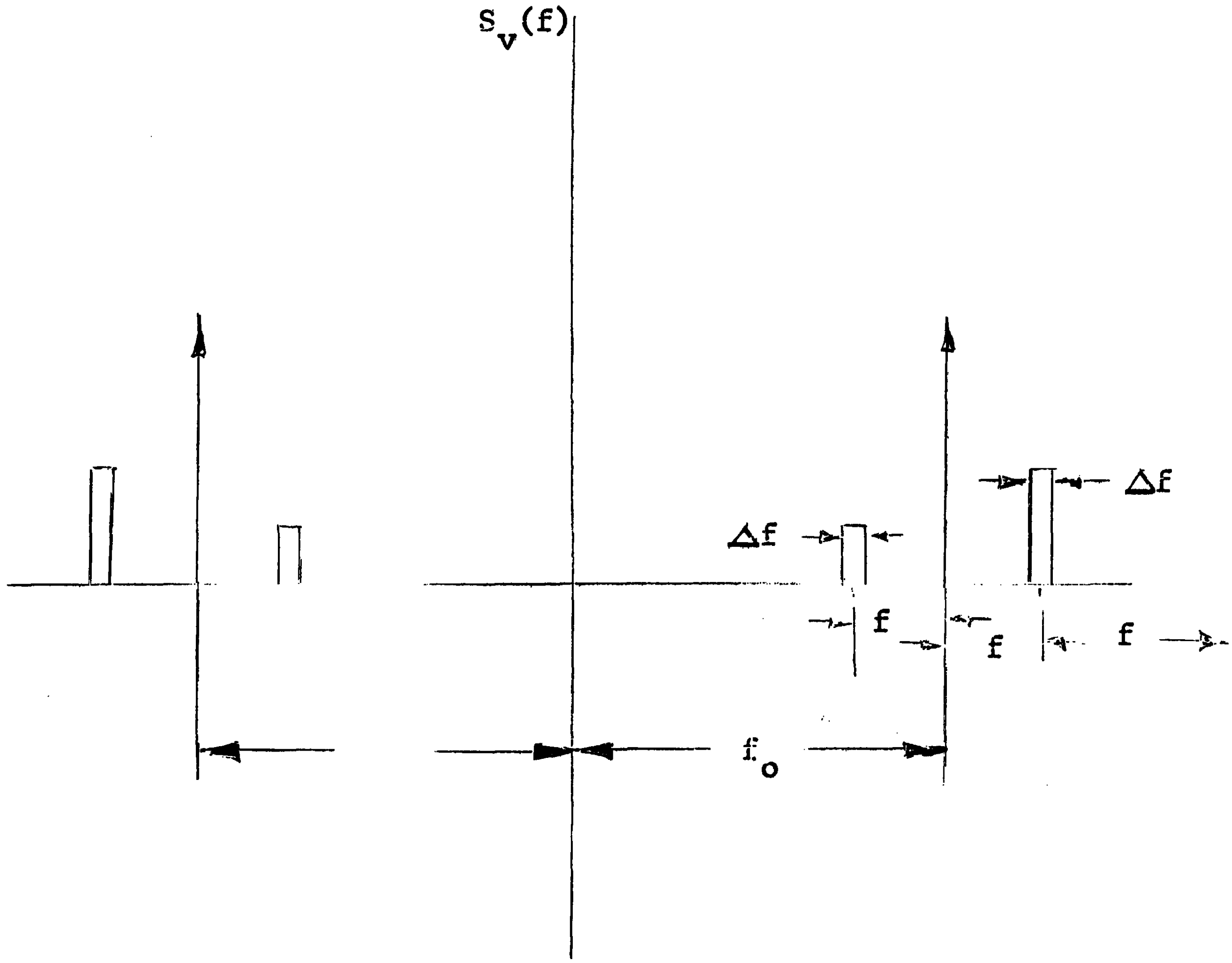

Figure 2.1. Spectrum of v(t)

Equation (2.11) shows explicitly the three frequencies $k\omega_o$, $k\omega_o+\omega$ and $-k\omega_o+\omega$. The two sidebands here are both higher in frequency than $k\omega_o$ and $-k\omega_o$, respectively, and therefore these representations are referred to as "uppersideband" representations. We will use representation of the form 2.11 in the rest of this work. Various other kind of representations can be used for v(t). All these and the mutual relations between them are given in Ref. [2].

It has also been shown [2] that the terminal noise behavior of a noiseless pumped nonlinear system may be written as

$$\begin{bmatrix} V_\alpha \\ V_\beta \end{bmatrix} = \begin{bmatrix} Z_{\alpha\alpha} & Z_{\alpha\beta} \\ Z_{\beta\alpha} & Z_{\beta\beta} \end{bmatrix} \begin{bmatrix} I_\alpha \\ I_\beta \end{bmatrix} \tag{2.14}$$

or

$$\underset{\sim}{V} = \underline{Z}\, \underset{\sim}{I}. \tag{2.15}$$

The matrix $\underline{Z}$ is a function of the operating point of the nonlinear system.

2.2. PHYSICAL SOURCES OF NOISE IN PUMPED NONLINEAR SYSTEMS

For the nonlinear systems that we shall consider in this chapter, we shall assume that the total voltage v(t) across the nonlinear element is related to the current i(t) through it by the equation

$$v(t) = F\left\{i(t)\right\} + n(t) \tag{2.16}$$

where $F\left\{i(t)\right\}$ is a functional of i(t), and n(t) is a noise voltage. For the pumped nonlinear systems we will also assume that the noise power in a frequency band surrounding the carrier at any particular frequency $k\omega_o$ is small. Let us also assume that nonzero carrier currents flow in the nonlinear system only at a finite number of frequencies[4] $0, \pm\omega_1, \cdots, \pm\omega_2, \cdots, \pm\omega_s$, where

$$\begin{bmatrix} \omega_1 \\ \vdots \\ \omega_i \\ \vdots \\ \omega_s \end{bmatrix} = \begin{bmatrix} \omega_o \\ \vdots \\ i\omega_o \\ \vdots \\ s\omega_o \end{bmatrix} . \tag{2.17}$$

According to Sec. 2.1 and Eq. (2.16), the "spot" frequency terminal noise behavior of this system at a frequency deviation ω is given by an equation of the form

[4]All these frequencies can be expressed in the form $\pm k\omega_o$, k an integer.

$$\begin{bmatrix} V_{\alpha o} \\ V_{\beta o} \\ \vdots \\ V_{\alpha 1} \\ V_{\beta 1} \\ \vdots \\ V_{\alpha i} \\ V_{\beta i} \\ \vdots \\ V_{\alpha s} \\ V_{\beta s} \end{bmatrix} = \begin{bmatrix} Z_{\alpha o \alpha o} & Z_{\alpha o \beta o} & \cdots & Z_{\alpha o \alpha 1} & Z_{\alpha o \beta 1} & \cdots & Z_{\alpha o \alpha i} & Z_{\alpha o \beta i} & \cdots & Z_{\alpha o \alpha s} & Z_{\alpha o \beta s} \\ Z_{\beta o \alpha o} & Z_{\beta o \beta o} & \cdots & Z_{\beta o \alpha 1} & Z_{\beta o \beta 1} & \cdots & Z_{\beta o \alpha i} & Z_{\beta o \beta i} & \cdots & Z_{\beta o \alpha s} & Z_{\beta o \beta s} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ Z_{\alpha 1 \alpha o} & Z_{\alpha 1 \beta o} & \cdots & Z_{\alpha 1 \alpha 1} & Z_{\alpha 1 \beta 1} & \cdots & Z_{\alpha 1 \alpha i} & Z_{\alpha 1 \beta i} & \cdots & Z_{\alpha 1 \alpha s} & Z_{\alpha 1 \beta s} \\ Z_{\beta 1 \alpha o} & Z_{\beta 1 \beta o} & \cdots & Z_{\beta 1 \alpha 1} & Z_{\beta 1 \beta 1} & \cdots & Z_{\beta 1 \alpha i} & Z_{\beta 1 \beta i} & \cdots & Z_{\beta 1 \alpha s} & Z_{\beta 1 \beta s} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ Z_{\alpha i \alpha o} & Z_{\alpha i \beta o} & \cdots & Z_{\alpha i \alpha 1} & Z_{\alpha i \beta 1} & \cdots & Z_{\alpha i \alpha i} & Z_{\alpha i \beta i} & \cdots & Z_{\alpha i \alpha s} & Z_{\alpha i \beta s} \\ Z_{\beta i \alpha o} & Z_{\beta i \beta o} & \cdots & Z_{\beta i \alpha 1} & Z_{\beta i \beta 1} & \cdots & Z_{\beta i \alpha i} & Z_{\beta i \beta i} & \cdots & Z_{\beta i \alpha s} & Z_{\beta i \beta s} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ Z_{\alpha s \alpha o} & Z_{\alpha s \beta o} & \cdots & Z_{\alpha s \alpha 1} & Z_{\alpha s \beta 1} & \cdots & Z_{\alpha s \alpha i} & Z_{\alpha s \beta i} & \cdots & Z_{\alpha s \alpha s} & Z_{\alpha s \beta s} \\ Z_{\beta s \alpha o} & Z_{\beta s \beta o} & \cdots & Z_{\beta s \alpha 1} & Z_{\beta s \beta 1} & \cdots & Z_{\beta s \alpha i} & Z_{\beta s \beta i} & \cdots & Z_{\beta s \alpha s} & Z_{\beta s \beta s} \end{bmatrix} \begin{bmatrix} I_{\alpha o} \\ I_{\beta o} \\ \vdots \\ I_{\alpha 1} \\ I_{\beta 1} \\ \vdots \\ I_{\alpha i} \\ I_{\beta i} \\ \vdots \\ I_{\alpha s} \\ I_{\beta s} \end{bmatrix} + \begin{bmatrix} n_{\alpha o} \\ n_{\beta o} \\ \vdots \\ n_{\alpha 1} \\ n_{\beta 1} \\ \vdots \\ n_{\alpha i} \\ n_{\beta i} \\ \vdots \\ n_{\alpha s} \\ n_{\beta s} \end{bmatrix} \quad (2.18)$$

where $V_{\alpha j}$ and $V_{\beta j}$ are the terminal noise voltages at frequencies $j\omega_o+\omega$ and $-j\omega_o+\omega$, respectively; and $I_{\alpha j}$ and $I_{\beta j}$ are the corresponding terminal noise currents. $n_{\alpha j}$ and $n_{\beta j}$ are the Fourier coefficients of n(t) at frequencies $j\omega_o+\omega$ and $-j\omega_o+\omega$, respectively.

Let us assume that the signal frequency at the input port of the transducer using the pumped nonlinear system is $m\omega_o$ and that at p^{th} port is $p\omega_o$[5] (see Fig. 2.2). Let us also assume that the terminal constraints at the other frequencies present in the nonlinear system are such that

$$\underset{\sim}{V}' = - \underline{Z}'' \underset{\sim}{I}' + \underset{\sim}{N}' \tag{2.19}$$

where $\underset{\sim}{V}'$ is a terminal noise voltage column matrix given by

$$\underset{\sim}{V}' = \begin{bmatrix} V_{\alpha o} \\ V_{\beta o} \\ \vdots \\ V_{\alpha i} \\ V_{\beta i} \\ \vdots \\ V_{\alpha(m-1)} \\ V_{\beta(m-1)} \\ V_{\alpha(m+1)} \\ V_{\beta(m+1)} \\ \vdots \\ V_{\alpha(p-1)} \\ V_{\beta(p-1)} \\ V_{\alpha(p+1)} \\ V_{\beta(p+1)} \\ \vdots \\ V_{\alpha s} \\ V_{\beta s} \end{bmatrix} \tag{2.20}$$

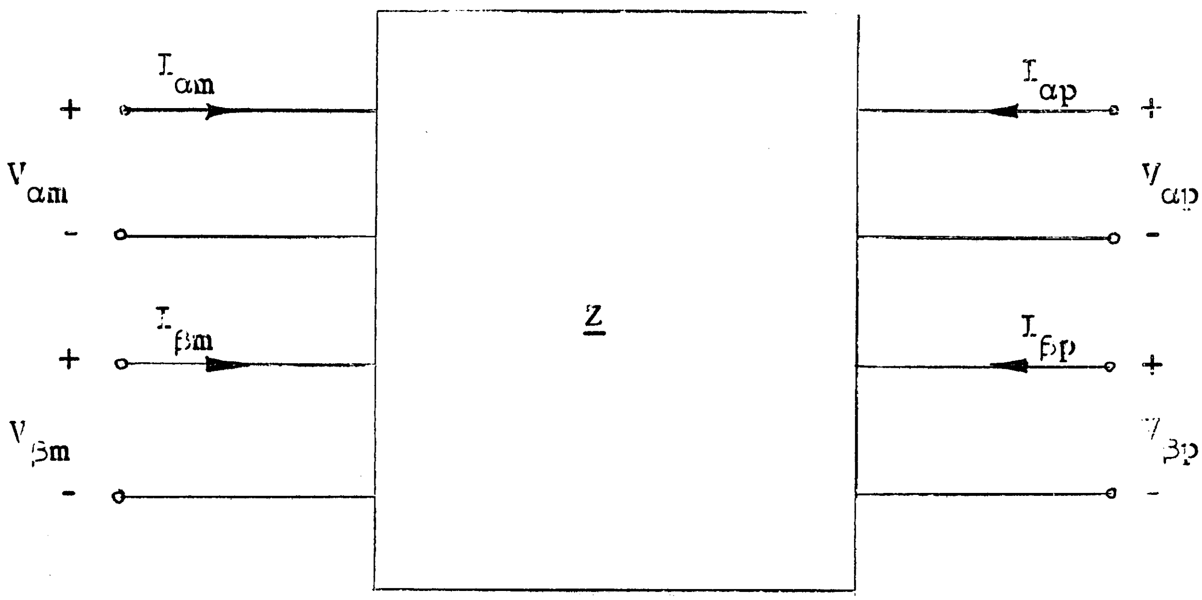

Figure 2.2. Pumped noisefree nonlinear two-port.

$\underset{\sim}{I}'$ and $\underset{\sim}{N}'$ are terminal current and noise matrices of the form similar to Eq. (2.20). $\underline{Z}''$ is the impedance matrix determined by the terminal constraints imposed on the system.

It may now be seen that by using Eqs. (2.18) and (2.19), we may obtain a relation between $V_{\alpha m}$, $V_{\beta m}$, $I_{\alpha m}$, $I_{\beta m}$, $V_{\alpha p}$, $V_{\beta p}$, $I_{\alpha p}$, and $I_{\beta p}$. In particular we can write

$$\begin{bmatrix} V_{\alpha m} \\ V_{\beta m} \\ V_{\alpha p} \\ V_{\beta p} \end{bmatrix} = \begin{bmatrix} Z'_{\alpha m\alpha m} & Z'_{\alpha m\beta m} & Z'_{\alpha m\alpha p} & Z'_{\alpha m\beta p} \\ Z'_{\beta m\alpha m} & Z'_{\beta m\beta m} & Z'_{\beta m\alpha p} & Z'_{\beta m\beta p} \\ Z'_{\alpha p\alpha m} & Z'_{\alpha p\beta m} & Z'_{\alpha p\alpha p} & Z'_{\alpha p\beta p} \\ Z'_{\beta p\alpha m} & Z'_{\beta p\beta m} & Z'_{\beta p\alpha p} & Z'_{\beta p\beta p} \end{bmatrix} \begin{bmatrix} I_{\alpha m} \\ I_{\beta m} \\ I_{\alpha p} \\ I_{\beta p} \end{bmatrix} + \begin{bmatrix} n''_{\alpha m} \\ n''_{\beta m} \\ n''_{\alpha p} \\ n''_{\beta p} \end{bmatrix} \tag{2.21}$$

2.3. REPRESENTATION OF NOISE IN PUMPED NONLINEAR SYSTEMS

It is the purpose of this section to develop different kind of representations for the internal noise sources present in a pumped nonlinear system.

Voltage Generator Type Model. It was shown in Sec. 2.2 that the terminal noise behavior of a pumped nonlinear system may be described by the Eq. (2.21).

An equivalent network to describe the terminal noise behavior of pumped nonlinear systems may, therefore, be obtained from Eq. (2.21) (see Fig. 2.3). This shows that, for the purpose of analysis, a pumped nonlinear system with internal noise sources may be separated into a noisefree four-port and four

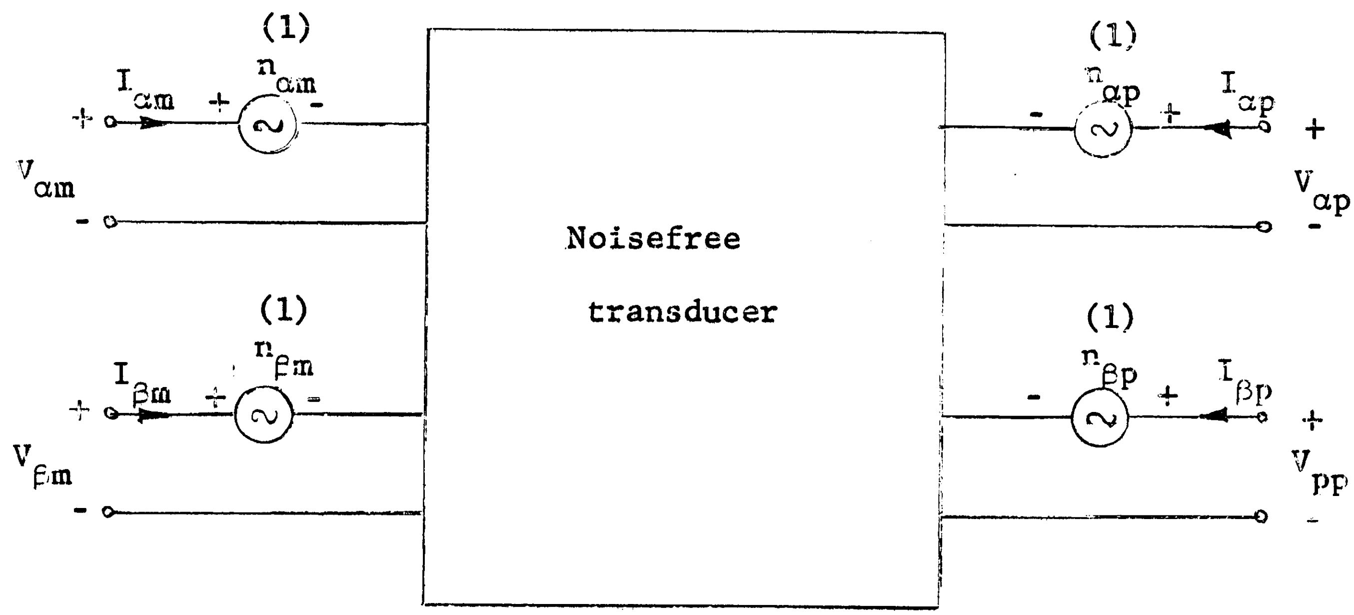

Figure 2.3. Separation of a pumped nonlinear four-port with internal noise sources into a noisefree four-port and external noise voltage generators.

external noise voltage generators. This representation is very similar to that obtained for linear noisy networks [1]. In the linear case, we can separate a two-port with internal noise sources into a noisefree two-port and two external noise voltage generators (see Fig. 2.4). The impedance matrix representation of such a device is given by

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} . \qquad (2.22)$$

<u>Current Generator Type Model</u>. Equation (2.21) can be written aslo in the form

$$\underset{\sim}{V}_{mp} = \underline{Z}_{mp} \underset{\sim}{I}_{mp} + \underset{\sim}{N}_{mp} . \qquad (2.23)$$

If the matrix $\underline{Z}_{mp}$ is nonsingular, we can write Eq. (2.23) as

$$\underset{\sim}{I}_{mp} = \underline{Z}_{mp}^{-1} \underset{\sim}{V}_{mp} + \underset{\sim}{N}'_{mp} \qquad (2.24)$$

where

$$\underset{\sim}{N}'_{mp} = - \underline{Z}_{mp}^{-1} \underset{\sim}{N}_{mp} = \begin{bmatrix} (N_{\alpha m})_i \\ (n_{\beta m})_i \\ (n_{\alpha p})_i \\ (n_{\beta p})_i \end{bmatrix} \qquad (2.25)$$

The equivalent circuit whose terminal noise behavior is given by Eq. (2.24) is shown in Fig. 2.5.

It must be mentioned here that the statistical properties

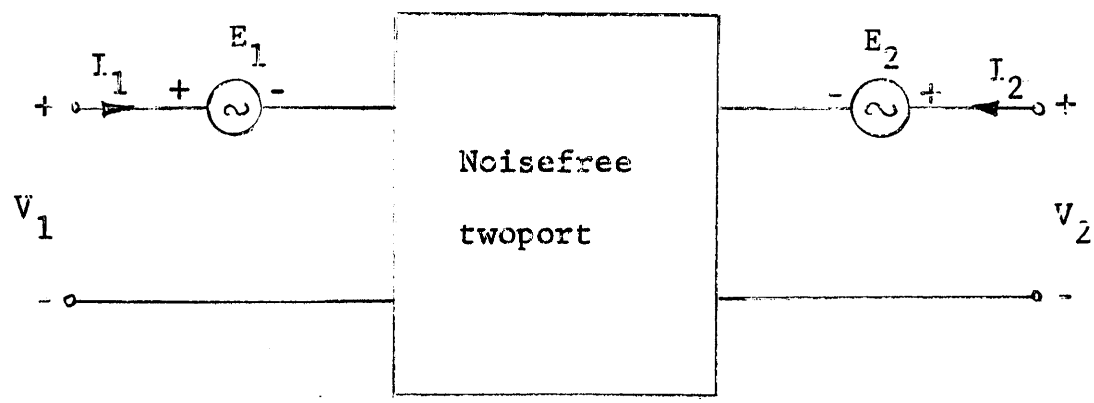

Figure 2.4. Separation of two-port with internal noise sources into a noisefree two-port and external voltage generators.

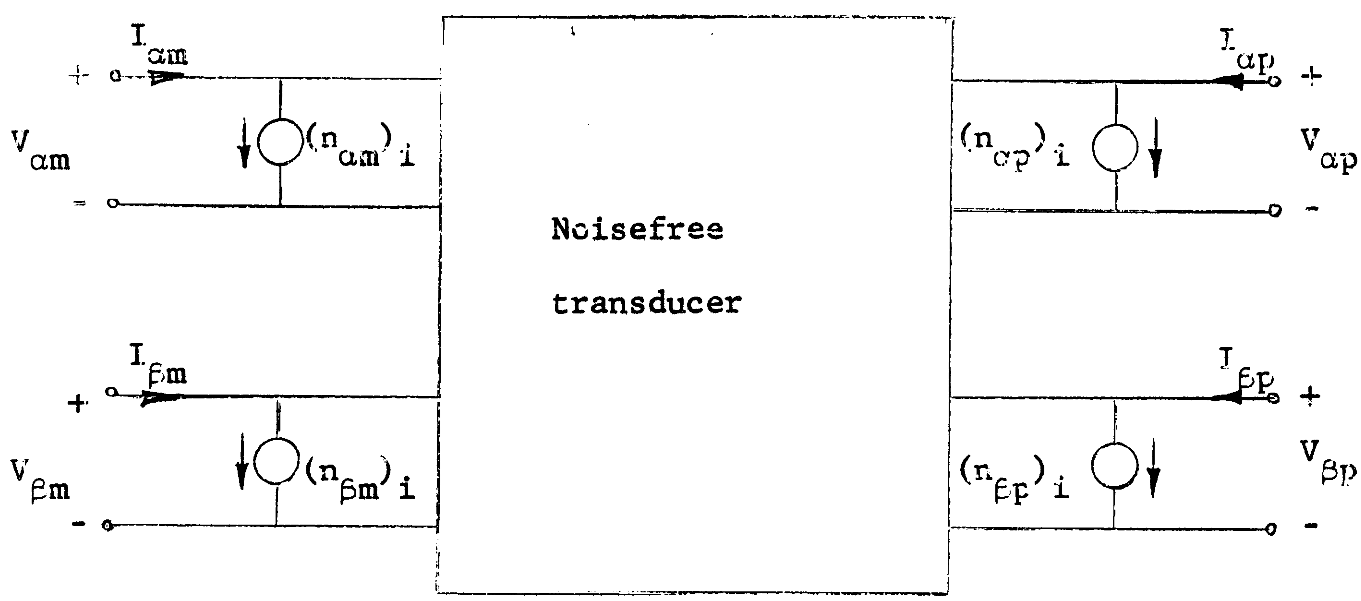

Figure 2.5. Separation of pumped nonlinear four-port with internal noise sources into a noisefree four-port and external noise current generators.

of the external noise voltage generators in Fig. 2.3 and those of the external current generators in Fig. 2.5 may be estimated by knowing the statistical properties of the noise voltage in Eq. (2.16). The statistical properties of n(t) may be estimated by knowing the physical sources of noise in the pumped nonlinear system.

2.4. ROTHE-DAHLKE TYPE MODEL FOR PUMPED NONLINEAR SYSTEMS

Consider a two-port linear noisy network with noise sources of unspecified origin. This two-port network can be represented by a number of different equivalent circuits [1]. An equivalent circuit of particular importance is that of Rothe-Dahlke [3]. This circuit, shown in Fig. 2.6, has both of the required noise generators at the input. For many purposes, especially calculating noise figure, this is convenient.

The question arises whether such a representation can be obtained for a pumped nonlinear system. The answer is in the affirmative.

From Eq. (2.21), we can write

$$\left[\begin{array}{c} \underset{\sim}{V}_m \\ \hline \underset{\sim}{V}_p \end{array}\right] = \left[\begin{array}{c|c} \underline{Z}'_{mm} & \underline{Z}'_{mp} \\ \hline \underline{Z}'_{pm} & \underline{Z}'_{pp} \end{array}\right] \left[\begin{array}{c} \underset{\sim}{I}_m \\ \hline \underset{\sim}{I}_p \end{array}\right] + \left[\begin{array}{c} \underset{\sim}{n}''_m \\ \hline \underset{\sim}{n}''_p \end{array}\right] \tag{2.26}$$

where

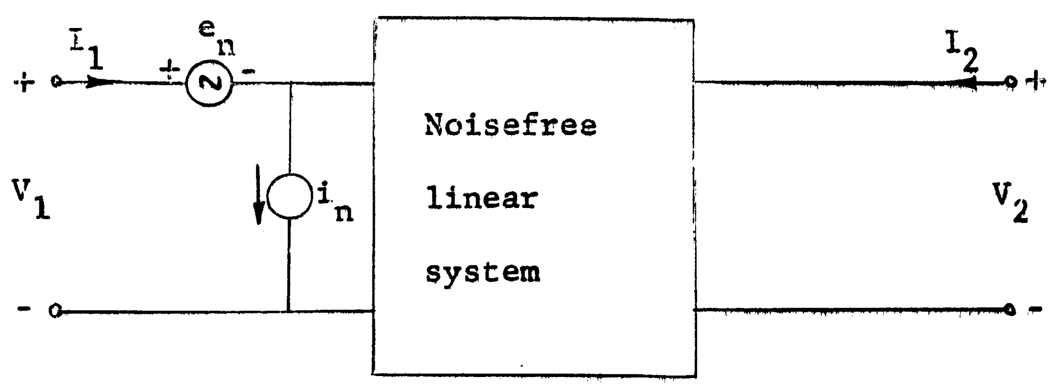

Figure 2.6. Rothe-Dahlke noise model for a noisy two-port network.

$$\underset{\sim}{V}_m = \begin{bmatrix} V_{\alpha m} \\ V_{\beta m} \end{bmatrix} \tag{2.27}$$

$$\underset{\sim}{V}_p = \begin{bmatrix} V_{\alpha p} \\ V_{\beta p} \end{bmatrix} \tag{2.28}$$

$$\underset{\sim}{I}_m = \begin{bmatrix} I_{\alpha m} \\ I_{\beta m} \end{bmatrix} \tag{2.29}$$

$$\underset{\sim}{I}_p = \begin{bmatrix} I_{\alpha p} \\ I_{\beta p} \end{bmatrix} \tag{2.30}$$

$$\underset{\sim}{n}''_m = \begin{bmatrix} n''_{\alpha m} \\ n''_{\beta m} \end{bmatrix} \tag{2.31}$$

$$\underset{\sim}{n}''_p = \begin{bmatrix} n''_{\alpha p} \\ n''_{\beta p} \end{bmatrix} \tag{2.32}$$

$$\underline{Z}'_{mm} = \begin{bmatrix} Z'_{\alpha m \alpha m} & Z'_{\alpha m \beta m} \\ Z'_{\beta m \alpha m} & Z'_{\beta m \beta m} \end{bmatrix} \tag{2.33}$$

$$\underline{Z}'_{mp} = \begin{bmatrix} Z'_{\alpha m \alpha p} & Z'_{\alpha m \beta p} \\ Z'_{\alpha m \alpha p} & Z'_{\beta m \beta p} \end{bmatrix} \tag{2.34}$$

$$\underline{Z}'_{pm} = \begin{bmatrix} Z'_{\alpha p\alpha m} & Z'_{\alpha p\beta m} \\ Z'_{\beta p\alpha m} & Z'_{\beta p\beta m} \end{bmatrix} \tag{2.35}$$

and

$$\underline{Z}'_{pp} = \begin{bmatrix} Z'_{\alpha p\alpha p} & Z'_{\alpha p\beta p} \\ Z'_{\beta p\alpha p} & Z'_{\beta p\beta p} \end{bmatrix} . \tag{2.36}$$

Equation (2.26) can be shown to be equivalent to the equation

$$\begin{bmatrix} \underset{\sim}{V}_m \\ \hline \underset{\sim}{I}_m \end{bmatrix} = \left[\begin{array}{c|c} \underline{A} & \underline{B} \\ \hline \underline{C} & \underline{D} \end{array}\right] \begin{bmatrix} \underset{\sim}{V}_p \\ \hline \underset{\sim}{I}_p \end{bmatrix} + \begin{bmatrix} \underset{\sim}{n}_v \\ \hline \underset{\sim}{n}_i \end{bmatrix} \tag{2.37}$$

where

$$\underline{A} = \underline{Z}'_{mm} \, \underline{Z}'^{-1}_{pm} \tag{2.38}$$

$$\underline{B} = \underline{Z}'_{mp} - \underline{Z}'_{mm} \, \underline{Z}'^{-1}_{pm} \, \underline{Z}'_{pp} \tag{2.39}$$

$$\underline{C} = \underline{Z}'^{-1}_{pm} \tag{2.40}$$

$$\underline{D} = - \underline{Z}'^{-1}_{pm} \, \underline{Z}'_{pp} \tag{2.41}$$

$$\underset{\sim}{n}_v = \underset{\sim}{n}''_m - \underline{Z}'_{mm} \, \underline{Z}'^{-1}_{pm} \, \underset{\sim}{n}''_p \tag{2.42}$$

and

$$\underset{\sim}{n}_i = - \underline{Z}'^{-1}_{pm} \, \underset{\sim}{n}''_p . \tag{2.43}$$

The system of Eq. (2.35) can be represented by the equivalent circuit of Fig. 2.7. In this representation the noise sources

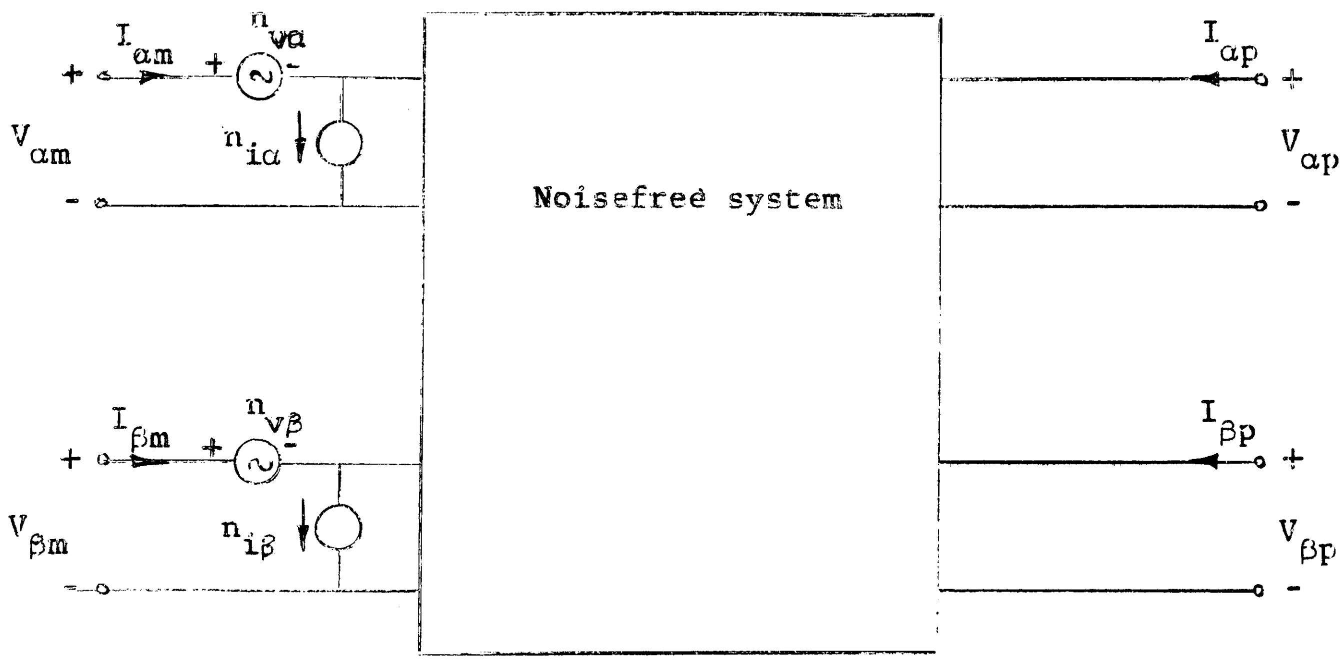

Figure 2.7. Equivalent circuit with two noise voltage sources and two noise current sources at the input.

appear only at the input of the transducer. For this representation two voltage and two current sources have been used.

By premultiplying by matrix $\underline{S}$ where

$$\underline{S} = \begin{bmatrix} \underline{A} & | & \underline{B} \\ -- & | & -- \\ \underline{C} & | & \underline{D} \end{bmatrix}^{-1} \tag{2.44}$$

we can write Eq. (2.35) as

$$\begin{bmatrix} \tilde{V}_p \\ --- \\ \tilde{I}_p \end{bmatrix} = \underline{S} \begin{bmatrix} \tilde{V}_m \\ --- \\ \tilde{I}_m \end{bmatrix} - \underline{S} \begin{bmatrix} \tilde{n}_v \\ --- \\ \tilde{n}_i \end{bmatrix}$$

$$= \underline{S} \begin{bmatrix} \tilde{V}_m \\ --- \\ \tilde{I}_m \end{bmatrix} + \begin{bmatrix} \tilde{n}'_v \\ --- \\ \tilde{n}'_i \end{bmatrix} \tag{2.45}$$

The equivalent circuit corresponding to Eq. (2.43) is given in Fig. 2.8. In this case the two noise current generators and two noise voltage generators follow the noisefree transducer.

Because of the apparent similarity of Fig. 2.7 to Fig. 2.6, representation of the form given in Fig. 2.7 will be called Rothe-Dahlke type model for pumped nonlinear systems. This model consists of a noisefree four-port preceded by two noise voltage and two current generators.

Let us write

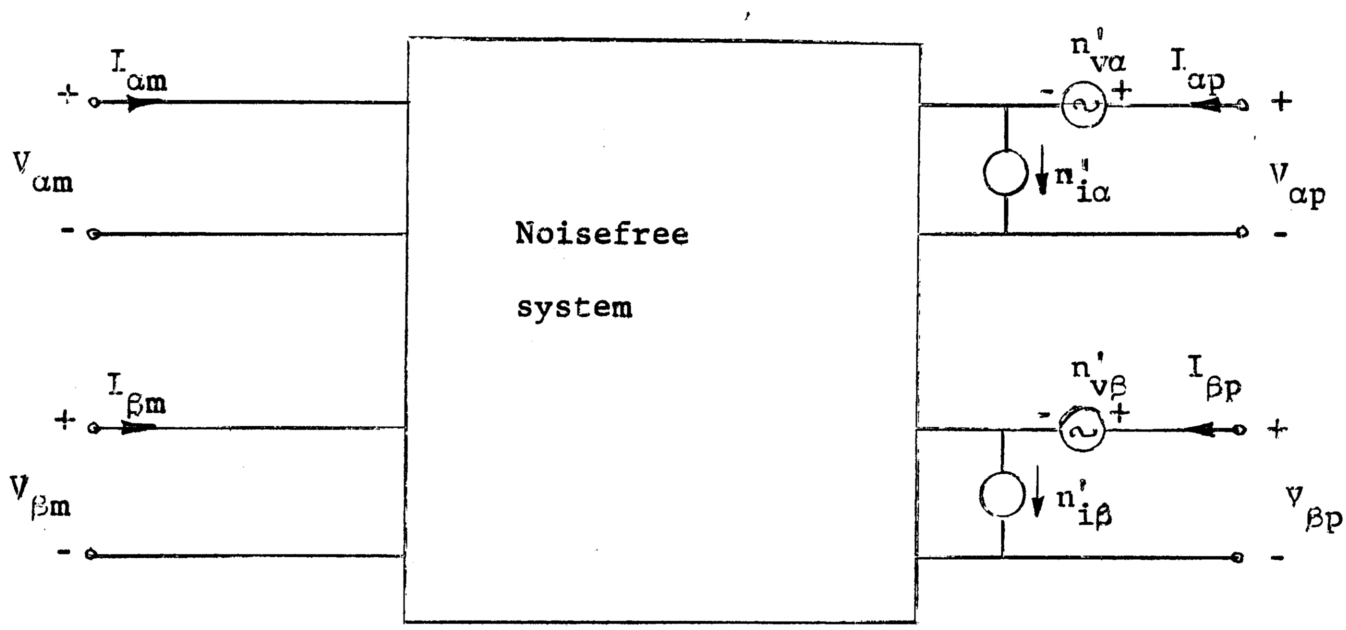

Figure 2.8. Equivalent circuit with two noise voltage sources and two current sources at the output port.

$$(\underset{\sim}{N})_1 = \begin{bmatrix} \underset{\sim}{n}_v \\ --- \\ \underset{\sim}{n}_i \end{bmatrix} = \begin{bmatrix} \underset{\sim}{n}''_m - \underline{Z}'_{mm} \, \underline{Z}'^{-1}_{pm} \, \underset{\sim}{n}''_p \\ \hline - \underline{Z}'^{-1}_{pm} \, \underset{\sim}{n}''_p \end{bmatrix} . \tag{2.46}$$

The statistical properties of $(\underset{\sim}{N})_1$ may, therefore, be estimated from the knowledge of statistical properties of $\underset{\sim}{n}''_m$ and $\underset{\sim}{n}''_p$. This statement is also true for the estimation of statistical properties of equivalent noise sources given in Fig. 2.8.

2.5. BAUER-ROTHE TYPE MODEL FOR PUMPED NONLINEAR SYSTEMS

The Rothe-Kahlke noise model for a linear noisy two-port transducer has two noise generators that may be correlated at the input. An alternate equivalent circuit, proposed by Bauer and Rothe [4], also has two noise generators at the input, but they are made to be uncorrelated. Because of this, the expression for noise figures for a linear transducer has a particularly simple form. This new model uses wave, or scattering variables [5]. The incoming and outgoing waves at the input of the linear system are

$$a_1 = \frac{V_1 + Z_\nu I_1}{\sqrt{Z_\nu + Z_\nu^*}} \tag{2.47}$$

and

$$b_1 = \frac{V_1 - Z_\nu I_1}{\sqrt{Z_\nu + Z_\nu^*}} \tag{2.48}$$

where V_1 and I_1 are the input voltage and current amplitudes, and Z_ν is a complex number with positive real part, the normalization impedance. In a similar way, the incoming and outgoing wave amplitudes at the output are a_2 and b_2, defined with a normalization impedance that is, in general, different from Z_ν, which was used at the input.

Wave noise generators are defined at the input, in terms of the generators in the Rothe-Dahlke model, Fig. 2.6, as

$$a_n = - \frac{e_n + Z_\nu i_n}{2\sqrt{\text{Re } Z_\nu}} \tag{2.49}$$

and

$$b_n = \frac{e_n - Z_\nu i_n}{2\sqrt{\text{Re } Z_\nu}} \ . \tag{2.50}$$

Thus, the linear system equations, including noise, are in terms of the two-port scattering matrix $\underline{S}$,

$$\begin{bmatrix} b_1 - b_n \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 + a_n \\ a_2 \end{bmatrix} \ . \tag{2.51}$$

An equivalent circuit of this representation is shown in Fig. 2.9. The wave generators are represented by directional couplers and ordinary sources.

We can also develop a Bauer-Rothe model for the representation of noise in pumped systems.

From Eq. (2.37) we have

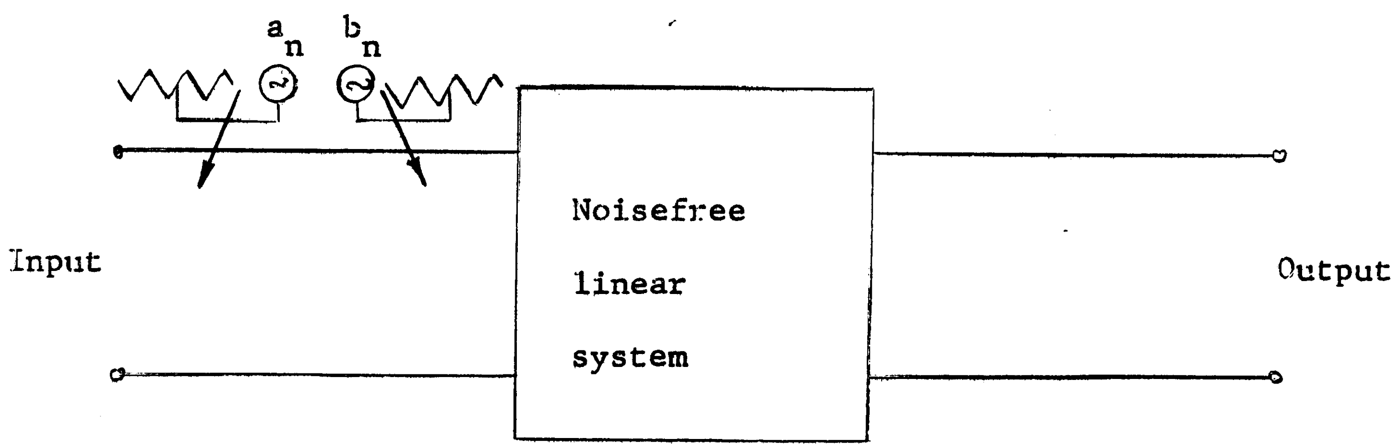

Figure 2.9. Baure-Rothe wave model of a noisy two-port network.

$$\begin{bmatrix} \underset{\sim}{V}_m \\ --- \\ \underset{\sim}{I}_m \end{bmatrix} = \left[\begin{array}{c|c} \underline{A} & \underline{B} \\ \hline \underline{C} & \underline{D} \end{array}\right] \begin{bmatrix} \underset{\sim}{V}_p \\ --- \\ \underset{\sim}{I}_p \end{bmatrix} + \begin{bmatrix} \underset{\sim}{n}_v \\ --- \\ \underset{\sim}{n}_i \end{bmatrix} . \tag{2.37}$$

This may be rewritten as

$$\begin{bmatrix} V_{\alpha m} \\ V_{\beta m} \\ I_{\alpha m} \\ I_{\beta m} \end{bmatrix} = \left[\begin{array}{c|c} \underline{A} & \underline{B} \\ \hline \underline{C} & \underline{D} \end{array}\right] \begin{bmatrix} V_{\alpha p} \\ V_{\beta p} \\ I_{\alpha p} \\ I_{\beta p} \end{bmatrix} + \begin{bmatrix} n_{\alpha v} \\ n_{\beta v} \\ n_{\alpha i} \\ n_{\beta i} \end{bmatrix} \tag{2.52}$$

Let Z_α and Z_β be the normalization impedances at ports 1-1' and 2-2', respectively (see Fig. 2.7). Z_α and Z_β are complex numbers with positive real parts.

The incoming and outgoing waves at the input of the pumped system are

$$a_{\alpha m} = \frac{V_{\alpha m} + Z_\alpha I_{\alpha m}}{\sqrt{Z_\alpha + Z_\alpha^*}} \tag{2.53}$$

$$a_{\beta m} = \frac{V_{\beta M} + Z_\beta I_{\beta m}}{\sqrt{Z_\beta + Z_\beta^*}} \tag{2.54}$$

$$b_{\alpha m} = \frac{V_{\alpha m} - Z_\alpha^* I_{\alpha m}}{\sqrt{Z_\alpha + Z_\alpha^*}} \tag{2.55}$$

and

$$b_{\beta m} = \frac{V_{\beta m} - Z_\beta^* I_{\beta m}}{\sqrt{Z_\beta + Z_\beta^*}} \tag{2.56}$$

In a similar way, we can define the incoming and outgoing wave amplitudes at the output of the pumped system. The normalization impedances used in this case, will in general, be different from Z_α and Z_β.

Let us now define wave noise generators at the input, in terms of the generators in the Rothe-Dahlke type model (see Fig. 2.7). These are

$$a_{\alpha n} = - \frac{n_{v\alpha} + Z_\alpha n_{i\alpha}}{2\sqrt{\mathrm{Re}\, Z_\alpha}} \tag{2.57}$$

$$a_{\beta n} = - \frac{n_{v\beta} + Z_\beta n_{i\beta}}{2\sqrt{\mathrm{Re}\, Z_\beta}} \qquad (2.58$$

$$b_{\alpha n} = \frac{n_{v\alpha} - Z_\alpha^* n_{i\alpha}}{2\sqrt{\mathrm{Re}\, A_\beta}} \tag{2.59}$$

and

$$b_{\beta n} = \frac{n_{v\beta} - Z_\beta^* n_{i\beta}}{2\sqrt{\mathrm{Re}\, Z_\beta}} \quad . \tag{2.60}$$

We now write

$$\underset{\sim}{N}_\alpha = \begin{bmatrix} b_{\alpha n} \\ a_{\alpha n} \end{bmatrix} \tag{2.61}$$

and

$$\underset{\sim}{N}_\beta = \begin{bmatrix} b_{\beta n} \\ a_{\beta n} \end{bmatrix} . \tag{2.62}$$

We may now show that by properly choosing Z_α and Z_β we can make the two matrices $\overline{\underset{\sim\alpha}{N}\;\underset{\sim\alpha}{N}^{+}}$ and $\overline{\underset{\sim\beta}{N}\;\underset{\sim\beta}{N}^{+}}$ diagonal. These values of Z_α and Z_β are given by

$$Z_\alpha = \frac{\overline{n_{\alpha v} n^*_{\alpha i}} - \overline{n^*_{\alpha v} n_{\alpha i}} + \left\{ (\overline{n_{\alpha v} n^*_{\alpha i}} - \overline{n^*_{\alpha v} n_{\alpha i}})^2 + 4\, \overline{|n_{\alpha v}|^2}\; \overline{|n_{\alpha i}|^2} \right\}^{1/2}}{2}$$

$$Z_\beta = \frac{\overline{n_{\beta v} n^*_{\beta i}} - \overline{n^*_{\beta v} n_{\beta i}} + \left\{ (\overline{n_{\beta v} n^*_{\beta i}} - \overline{n^*_{\beta v} n_{\beta i}})^2 + 4\, \overline{|n_{\beta v}|^2}\; \overline{|n_{\beta i}|^2} \right\}^{1/2}}{2}$$

Thus, the pumped system terminal equations, including noise, can be written as

$$\begin{bmatrix} b_{\alpha m} - b_{\alpha n} \\ b_{\beta m} - b_{\beta n} \\ b_{\alpha p} \\ b_{\beta p} \end{bmatrix} = \begin{bmatrix} S_{\alpha m\alpha m} & S_{\alpha m\beta m} & S_{\alpha m\alpha p} & S_{\alpha m\beta p} \\ S_{\beta m\alpha m} & S_{\beta m\beta m} & S_{\beta m\alpha p} & S_{\beta m\beta p} \\ S_{\alpha p\alpha m} & S_{\alpha p\beta m} & S_{\alpha p\alpha p} & S_{\alpha p\beta p} \\ S_{\beta p\alpha m} & S_{\beta p\beta m} & S_{\beta p\alpha p} & S_{\beta p\beta p} \end{bmatrix} \begin{bmatrix} a_{\alpha m} + a_{\alpha n} \\ a_{\beta m} + a_{\beta n} \\ a_{\alpha p} \\ a_{\beta p} \end{bmatrix} \tag{2.65}$$

$$= \underline{S} \begin{bmatrix} a_{\alpha m} + a_{\alpha n} \\ a_{\beta m} + a_{\beta n} \\ a_{\alpha p} \\ a_{\beta p} \end{bmatrix} . \tag{2.66}$$

The elements of the matrix $\underline{S}$ are functions of the matrices $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$.

Since the values of elements of $\underline{S}$ are not needed in our discussion, these relations are not given.

From Eq. (2.66), we can write an equivalent circuit as shown in Fig. 2.10. The wave generators are represented by directional couplers and ordinary sources. The values of these sources can be estimated by knowing the values of the sources shown in Rothe-Dahlke model in Fig. 2.7.

2.6. EXAMPLE

Let us now consider a nominally driven abrupt-junction varactor frequency doubler [6]. The model assumed for the varactor is given in Fig. 2.11. We will assume that the only physical source of noise within the varactor is the parasitic series resistance R_s. If the temperature of the diode is T_d, then the thermal noise due to R_s can be represented [2] at spot frequencies by

$$\overline{|e_n|^2} = 2\ kT_d\ R_s \triangle f \tag{2.67}$$

where $\triangle f$ is the frequency range of interest.

We shall now develop a voltage generator type model for the abrupt-junction varactor frequency doubler.

If Z_b is the bias loop impedance and n_b is the noise voltage source present in the bias loop, we may write the following equations for the doubler

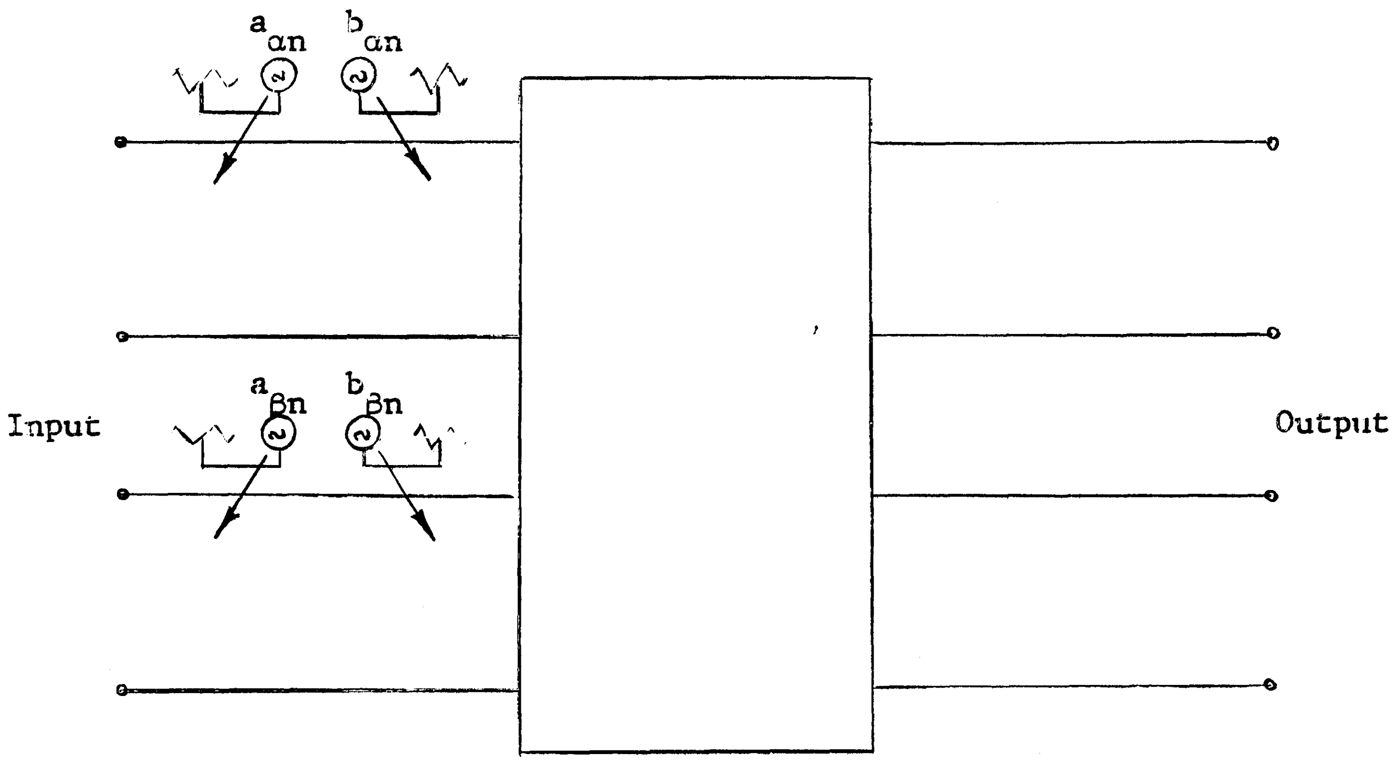

Figure 2.10. Bauer-Rothe type wave model of a noisy pumped system.

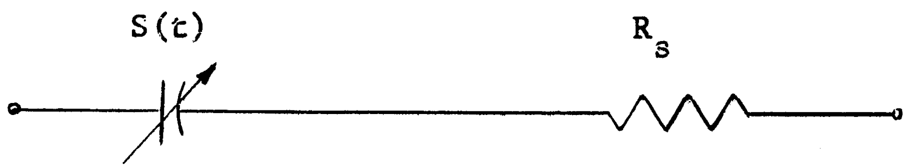

Figure 2.11. Equivalent circuit of a varactor.

$$
\begin{bmatrix} V_{\alpha 1} \\ V_{\beta 1} \\ V_{\alpha 2} \\ V_{\beta 2} \\ V_o \end{bmatrix}
=
\begin{bmatrix}
R_s + \frac{S_o}{j(\omega_o+\omega)} & \frac{S_2}{j(-\omega_o+\omega)} & \frac{S_{-1}}{j(2\omega_o+\omega)} & 0 & \frac{S_1}{j\omega} \\
\frac{S_{-2}}{j(\omega_o+\omega)} & R_s + \frac{S_o}{j(-\omega_o+\omega)} & 0 & \frac{S_1}{j(-2\omega_o+\omega)} & \frac{S_{-1}}{j\omega} \\
\frac{S_1}{j(\omega_o+\omega)} & 0 & R_s + \frac{S_o}{j(2\omega_o+\omega)} & 0 & \frac{S_2}{j\omega} \\
0 & \frac{S_{-1}}{j(-\omega_o+\omega)} & 0 & R_s + \frac{S_o}{j(-2\omega_o+\omega)} & \frac{S_{-2}}{j\omega} \\
\frac{S_{-1}}{j(\omega_o+\omega)} & \frac{S_1}{j(-\omega_o+\omega)} & \frac{S_{-2}}{j(2\omega_o+\omega)} & \frac{S_2}{j(-2\omega_o+\omega)} & R_s + \frac{S_o}{j\omega}
\end{bmatrix}
\begin{bmatrix} I_{\alpha 1} \\ I_{\beta 1} \\ I_{\alpha 2} \\ I_{\beta 2} \\ I_o \end{bmatrix}
+
\begin{bmatrix} n_{\alpha 1} \\ n_{\beta 1} \\ n_{\alpha 2} \\ n_{\beta 2} \\ n_o \end{bmatrix}
\tag{2.68}
$$

S_k is the k^{th} Fourier coefficient of the elastance S(t). ω is the frequency deviation and ω_o is the angular frequency of the input signal (see Fig. 2.12). Since the physical source of noise within the varactor is assumed to be the parasitic series resistance R_s we have

$$\overline{|n_{\alpha 1}|^2} = \overline{|n_{\beta i}|^2} = \overline{|n_{\alpha 2}|^2} = \overline{|n_{\beta 2}|^2} = 2\ R_s k T_d \triangle f. \tag{2.70}$$

Also the correlation between any two of $n_{\alpha 1}$, $n_{\beta 1}$, $n_{\alpha 2}$, $n_{\beta 2}$, and n_o is zero.

Let us now write (see Fig. 2.11)

$$\underset{\sim}{n}_i = \begin{bmatrix} n_{\alpha i} \\ n_{\beta i} \end{bmatrix} \tag{2.71}$$

$$\underset{\sim}{n}_o = \begin{bmatrix} n_{\alpha o} \\ n_{\beta o} \end{bmatrix} . \tag{2.72}$$

If m_o, m_1, and m_2 are the modulation ratios of the doubler and ω_c is its cutoff frequency [6], we can show from Eqs. (2.68) and (2.69) that

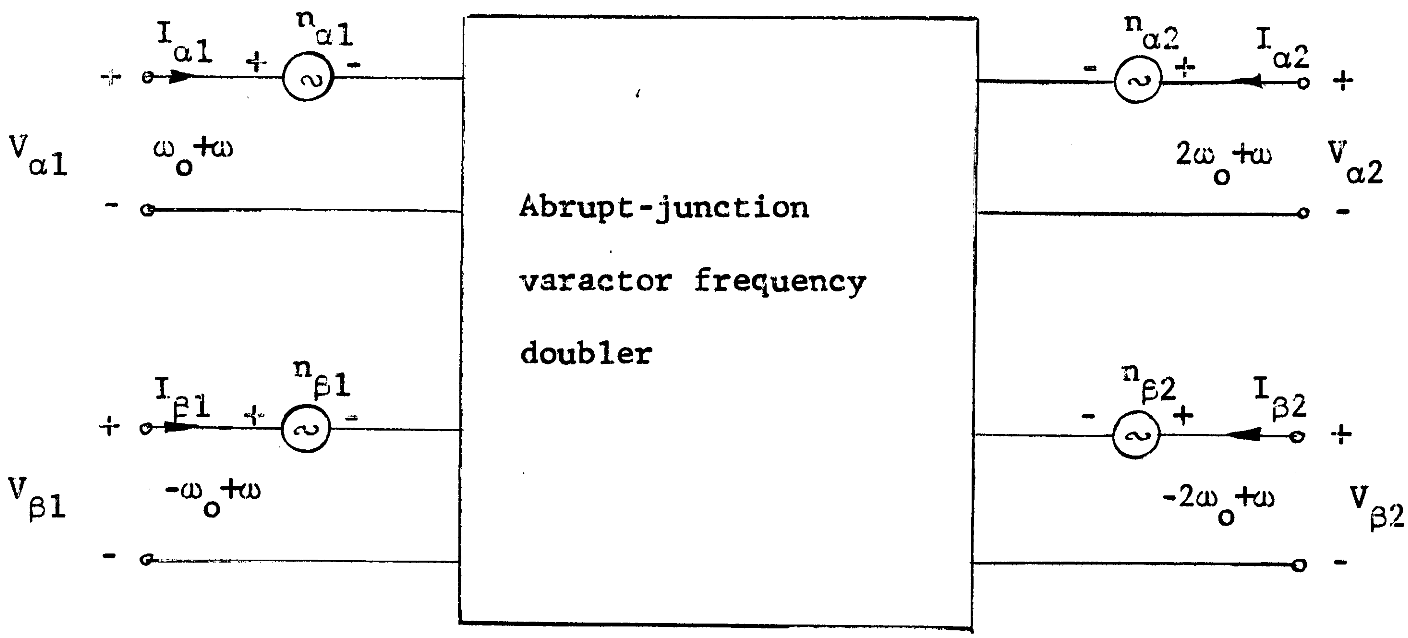

Figure 2.12. Voltage generator type model for varactor frequency doubler.

$$\overline{\underset{\sim}{n}_i\underset{\sim}{n}_i^{+}} = 2R_s kT_d\triangle f \begin{bmatrix} 1 + \dfrac{1 + \dfrac{\overline{|n_b|^2}}{2R_s kT_d\triangle f}}{\left|\dfrac{m_o}{m_1} + \dfrac{j(Z_b+R_s)}{m_1\left(\frac{\omega_c}{\omega}\right) R_s}\right|^2} & -\dfrac{1 + \dfrac{\overline{|n_b|^2}}{2R_s kT_d\triangle f}}{\left|\dfrac{m_o}{m_1} + \dfrac{j(Z_b+R_s)}{m_1\left(\frac{\omega_c}{\omega}\right) R_s}\right|^2} \\ -\dfrac{1 + \dfrac{\overline{|n_b|^2}}{2R_s kT_d\triangle f}}{\left|\dfrac{m_o}{m_1} + \dfrac{j(Z_b+R_s)}{m_1\left(\frac{\omega_c}{\omega}\right) R_s}\right|^2} & 1 + \dfrac{1 + \dfrac{\overline{|n_b|^2}}{2R_s kT_d\triangle f}}{\left|\dfrac{m_o}{m_1} + \dfrac{j(Z_b+R_s)}{m_1\left(\frac{\omega_c}{\omega}\right) R_s}\right|^2} \end{bmatrix} \tag{2.73}$$

$$\overline{\underset{\sim}{n}_o\underset{\sim}{n}_o^{+}} = 2R_s kT_d\triangle f \begin{bmatrix} 1 + \dfrac{1 + \dfrac{\overline{|n_b|^2}}{2R_s kT_d\triangle f}}{\left|\dfrac{m_o}{m_2} + \dfrac{j(Z_b+R_s)}{m_2\left(\frac{\omega_c}{\omega}\right) R_s}\right|^2} & -\dfrac{1 + \dfrac{\overline{|n_b|^2}}{2R_s kT_d\triangle f}}{\left|\dfrac{m_o}{m_2} + \dfrac{j(Z_b+R_s)}{m_2\left(\frac{\omega_c}{\omega}\right) R_s}\right|^2} \\ -\dfrac{1 + \dfrac{\overline{|n_b|^2}}{2R_s kT_d\triangle f}}{\left|\dfrac{m_o}{m_2} + \dfrac{j(Z_b+R_s)}{m_2\left(\frac{\omega_c}{\omega}\right) R_s}\right|^2} & 1 + \dfrac{1 + \dfrac{\overline{|n_b|^2}}{2R_s kT_d\triangle f}}{\left|\dfrac{m_o}{m_2} + \dfrac{j(Z_b+R_s)}{m_2\left(\frac{\omega_c}{\omega}\right) R_s}\right|^2} \end{bmatrix} \tag{2.74}$$

and

$$\overline{\underset{\sim}{n}_i \underset{\sim}{n}_o^+} = 2R_s kT_d \Delta f \left[\frac{1 + \dfrac{\overline{|n_b|^2}}{2R_s kT_d \Delta f}}{\left| \dfrac{m_o}{\sqrt{m_1 m_2}} + \dfrac{j(Z_b + R_s)}{\sqrt{m_1 m_2}\, \dfrac{\omega_c}{\omega} R_s} \right|^2} - \frac{1 + \dfrac{\overline{|n_b|^2}}{2R_s kT_d \Delta f}}{\left| \dfrac{m_o}{\sqrt{m_1 m_2}} + \dfrac{j(Z_b + R_s)}{\sqrt{m_1 m_2}\, \dfrac{\omega_c}{\omega} R_s} \right|^2} - \frac{1 + \dfrac{\overline{|n_b|^2}}{2R_s kT_d \Delta f}}{\left| \dfrac{m_o}{\sqrt{m_1 m_2}} + \dfrac{j(Z_b + R_s)}{\sqrt{m_1 m_2}\, \dfrac{\omega_c}{\omega} R_s} \right|^2} \quad \frac{1 + \dfrac{\overline{|n_b|^2}}{2R_s kT_d \Delta f}}{\left| \dfrac{m_o}{\sqrt{m_1 m_2}} + \dfrac{j(Z_b + R_s)}{\sqrt{m_1 m_2}\, \dfrac{\omega_c}{\omega} R_s} \right|^2} \right] \tag{2.75}$$

The methods of obtaining any other kind of model from this model are given in the preceding sections of this chapter.

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CHAPTER 3

CHARACTERIZATION OF NOISE PERFORMANCE OF PUMPED NONLINEAR SYSTEMS

This chapter deals with the study of noise performance of pumped nonlinear systems. Methods have been developed to characterize the noise performance of linear noisy systems [1,2]. In linear transducers one very significant question is the extent to which the transducer influences the signal-to-noise ratio over a narrow-band (essentially at one frequency) in the system of which it is a part. The term "spot-noise performance" has been used to refer to the effect of the transducer upon the single-frequency signal-to-noise ratio [1]. The concept of exchangeable powers has been developed for these devices, and meaningful definitions of noise figure, noise measure, and exchangeable powers have also been given.

In this chapter an attempt has been made to define meaningfully exchangeable amplitude and phase noise powers for the pumped nonlinear systems we are considering. The idea of linear lossless imbeddings has been used for this purpose. Meaningful noise figures have been defined by comparing output parameters with source parameters. This definition of noise figures has the advantage that the figures thus defined are invariant to any further linear lossless imbeddings one may wish to use.

Alternatively, a noise figure matrix has also been defined for these devices in terms of the noise voltage matrices and a gain matrix. A cascade formula has been obtained for the noise figure matrix thus defined.

Finally, the noise performance of pumped nonlinear systems has been characterized in terms of variances of input and output parameters. The figures defined in terms of these variances seem to have a lot of physical significance. It has, however, been shown that they are functions not only of the source impedance but also of the load impedance. This does not seem to be a very desirable feature.

It may be pointed out here that this set of characterization of noise performance of pumped nonlinear systems is by no means complete. It is, however, felt that these characterizations are adequate for our purpose.

3.1. DEFINITION OF EXCHANGEABLE AMPLITUDE AND PHASE NOISE POWERS

At a frequency ω_o of the carrier, an appropriate representation[1] of the terminal noise behavior of a pumped nonlinear system is

$$\begin{bmatrix} V_\alpha \\ \\ V_\beta \end{bmatrix} = \begin{bmatrix} Z_{\alpha\alpha} & Z_{\alpha\beta} \\ \\ Z_{\beta\alpha} & Z_{\beta\beta} \end{bmatrix} \begin{bmatrix} I_\alpha \\ \\ I_\beta \end{bmatrix} + \begin{bmatrix} n_\alpha \\ \\ n_\beta \end{bmatrix} \tag{3.1}$$

[1] In this chapter we shall use half-amplitudes, rather than amplitudes or r.m.s. values.

or

$$\underset{\sim}{V} = \underline{Z}\ \underset{\sim}{I} + \underset{\sim}{N} \tag{3.2}$$

as shown in Fig. 3.1 [3]. The noise is assumed to be band-limited and the signal-to-noise ratio is assumed to be high. V_s is the carrier voltage, and I_s is the carrier current. In this section we are going to consider only the terminal noise behavior of the device. The frequency deviation is assumed to be ω.

A linear lossless network whose impedance matrix at each "spot" frequency of interest is nonsingular is defined to be a nonsingular linear lossless network.[2]

Let the pumped nonlinear system be cascaded with a linear lossless network as shown in Fig. 3.2. Let also the system of Fig. 3.1 be described by

$$\underset{\sim}{V}' = \underline{Z}''\underset{\sim}{I}' + \underset{\sim}{N}' \tag{3.3}$$

in the amplitude-phase representation. The relations between amplitude-phase and α - β representations are given by [3]

$$\underset{\sim}{V}' = \underline{\lambda}_v\ \underset{\sim}{V} \tag{3.4}$$

$$\underline{Z}'' = \underline{\lambda}_v\ \underline{Z}\ \underline{\lambda}_i^{-1} \tag{3.5}$$

$$\underset{\sim}{I}' = \underline{\lambda}_i\ \underset{\sim}{I} \tag{3.6}$$

$$\underset{\sim}{N}' = \underline{\lambda}_v\ \underset{\sim}{N} \tag{3.7}$$

where

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[2] All linear lossless networks considered in this chapter will be assumed to be nonsingular.

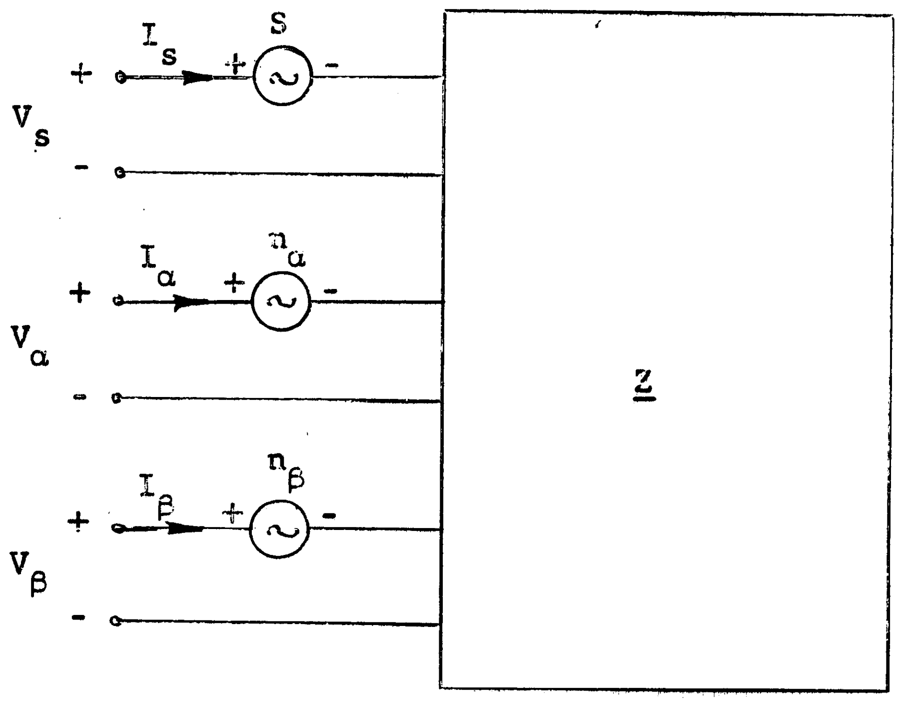

Figure 3.1. Equivalent representation of a pumped nonlinear system with internal noise sources.

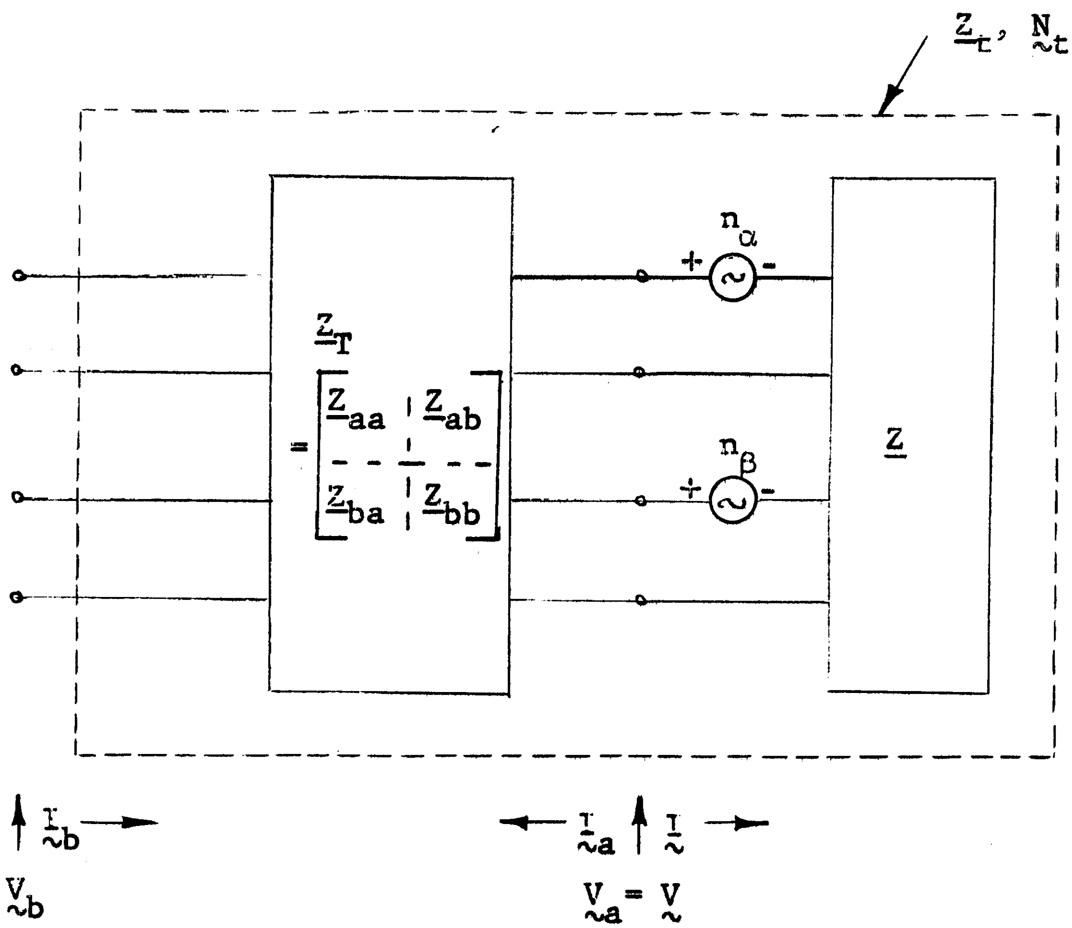

Figure 3.2. Pumped nonlinear system cascaded with a linear lossless network. Signal port is not shown in the figure.

$$\underline{\lambda}_v = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} e^{-j\phi_v} & 0 \\ 0 & e^{j\phi_v} \end{bmatrix} \quad (3.8)$$

and

$$\underline{\lambda}_i = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} e^{-j\phi_i} & 0 \\ 0 & e^{j\phi_i} \end{bmatrix}. \quad (3.9)$$

ϕ_v and ϕ_i are the phase angles of carrier voltage and carrier current at the frequency ω_o. It will be assumed that the load impedance for the carrier at frequency ω_o is purely resistive.[3] In that case, we have

$$\phi_v = \phi_i \quad (3.10)$$

and

$$\underline{\lambda}_v = \underline{\lambda}_i. \quad (3.11)$$

We can also show easily that

$$\underline{\lambda}_v^+ = \frac{1}{2} \underline{\lambda}_v^{-1}. \quad (3.12)$$

The pumped nonlinear system in the amplitude-phase representation is shown in Fig. 3.3. This system cascaded with a linear lossless network is shown in Fig. 3.4. Its terminal relations are given by Eq. (3.3); or we may write

[3]This assumption will also be made for the terminations at frequencies of the form $k\omega_o$, where k is an integer.

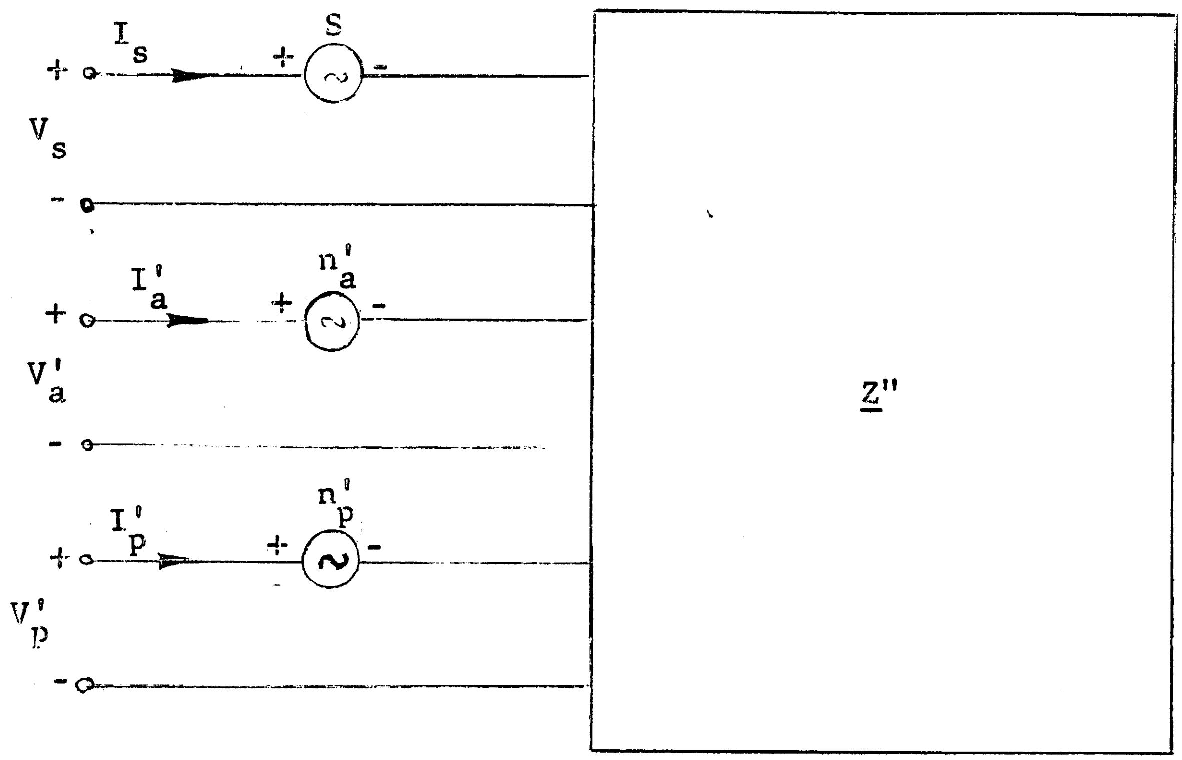

Figure 3.3. Equivalent representation of a pumped nonlinear system with internal noise sources in the amplitude-phase representation.

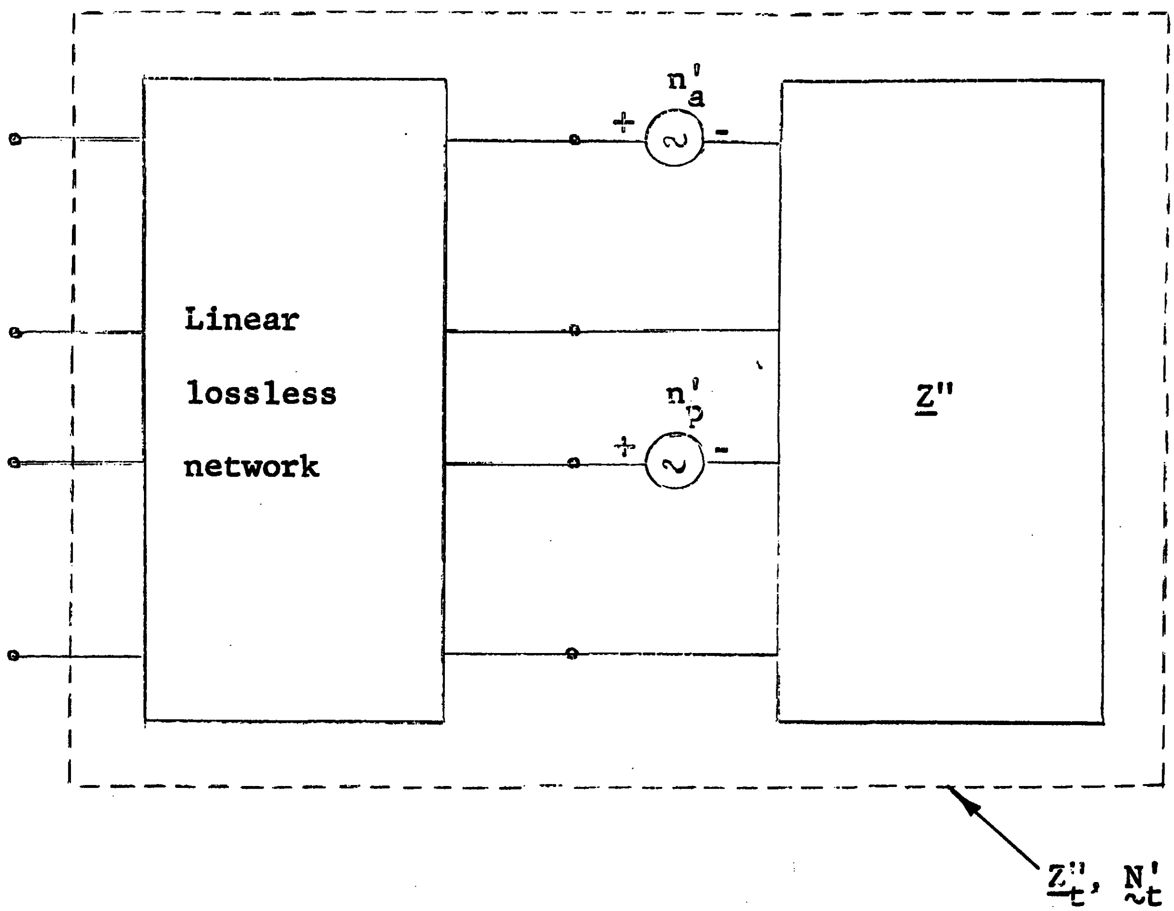

Figure 3.4. Pumped nonlinear system in the amplitude-phase representation cascaded with a linear lossless network. Signal port is not shown in the figure.

$$\begin{bmatrix} V'_a \\ \\ V'_p \end{bmatrix} = \begin{bmatrix} Z''_{aa} & Z''_{ap} \\ \\ Z''_{pa} & Z''_{pp} \end{bmatrix} \begin{bmatrix} I'_a \\ \\ I'_p \end{bmatrix} + \begin{bmatrix} n'_a \\ \\ n'_p \end{bmatrix} . \qquad (3.13)$$

n'_a, n'_p are complex random variables, the physical significance of which usually appears in their self- and cross-power spectral densities. A convenient summary of the power spectral densities is the matrix

$$\overline{\underset{\sim}{N}' \, \underset{\sim}{N}'^{+}} = \begin{bmatrix} \overline{|n'_a|^2} & \overline{n'_a \, n'^{*}_p} \\ \\ \overline{n'_p \, n'^{*}_a} & \overline{|n'_p|^2} \end{bmatrix} . \qquad (3.14)$$

Since the voltages of $\overline{\underset{\sim}{N}' \, \underset{\sim}{N}'^{+}}$ are noise voltages it can be argued on physical grounds that the matrix $\overline{\underset{\sim}{N}' \, \underset{\sim}{N}'^{+}}$ is positive definite [1].

When the pumped nonlinear system is cascaded with a linear lossless network, a new noise column matrix $\underset{\sim}{N}'_t$, and a new impedance matrix $\underline{Z}''_t$ are obtained. It is assumed that the linear lossless network does not affect the carrier.

We shall first find the new $\underline{Z}_t$ and $\underset{\sim}{N}_t$ matrices for the system in the α - β representation.

The analytical relation between the voltages and currents applied to the 4-port linear lossless network (it will be called the "transformation" network) of Fig. 3.2 can be written in the form

$$\underset{\sim}{V}_a = \underline{Z}_{aa} \underset{\sim}{I}_a + \underline{Z}_{ab} \underset{\sim}{I}_b \tag{3.15}$$

$$\underset{\sim}{V}_b = \underline{Z}_{ba} \underset{\sim}{I}_a + \underline{Z}_{bb} \underset{\sim}{I}_b . \tag{3.16}$$

The column vectors $\underset{\sim}{V}_a$ and $\underset{\sim}{V}_b$ comprise the terminal voltages applied to the transformation network on its two sides, and the column vectors $\underset{\sim}{I}_a$ and $\underset{\sim}{I}_b$ comprise the currents $\underset{\sim}{I}_T$ flowing into it. The four $\underline{Z}$ matrices in Eqs. (3.15) and (3.16) are each square and of second order. They make up the square fourth order matrix $\underline{Z}_T$ of the lossless transformation network. The condition of losslessness can be summarized in the following relations, which express the fact that the total time-average power P into the transformation network must be zero for all choices of the terminal currents:

$$P = \underset{\sim}{I}_T^{+} (\underline{Z}_T + \underline{Z}_T^{+}) \underset{\sim}{I}_T = 0, \text{ for all } \underset{\sim}{I}_T; \tag{3.17}$$

therefore

$$\underline{Z}_T + \underline{Z}_T^{+} = 0 \tag{3.18}$$

or

$$\underline{Z}_{aa} + \underline{Z}_{aa}^{+} = 0 \tag{3.19}$$

$$\underline{Z}_{ab} + \underline{Z}_{ba}^{+} = 0 \tag{3.20}$$

and

$$\underline{Z}_{bb} + \underline{Z}_{bb}^{+} = 0. \tag{3.21}$$

The original 2-port network, with impedance matrix $\underline{Z}$

and noise column matrix $\underset{\sim}{N}$, impose the following relation between the column matrices $\underset{\sim}{V}$ and $\underset{\sim}{I}$ of the voltages across, and the currents into, its terminals:

$$\underset{\sim}{V} = \underline{Z}\, \underset{\sim}{I} + \underset{\sim}{N}. \tag{3.2}$$

The currents $\underset{\sim}{I}$ into the 2-port network are, according to Fig. 3.2, equal and opposite to the currents $\underset{\sim}{I}_a$ into one side of the 4-port network. The voltages $\underset{\sim}{V}$ are equal to the voltages $\underset{\sim}{V}_a$. We thus have

$$\underset{\sim}{V} = \underset{\sim}{V}_a \tag{3.22}$$

$$\underset{\sim}{I} = - \underset{\sim}{I}_a. \tag{3.23}$$

Introduction of Eqs. (3.22) and (3.23) into Eq. (3.2) and application of the latter to Eq. (3.15) give

$$\underset{\sim}{I}_a = - (\underline{Z} + \underline{Z}_{aa})^{-1} \underline{Z}_{ab}\, \underset{\sim}{I}_b + (\underline{Z} + \underline{Z}_{aa})^{-1} \underset{\sim}{N}.$$

When this equation is substituted in Eq. (3.16), the final relation between $\underset{\sim}{V}_b$ and $\underset{\sim}{I}_b$ is determined:

$$\underset{\sim}{V}_b = \underline{Z}_t\, \underset{\sim}{I}_b + \underset{\sim}{N}_t \tag{3.24}$$

where

$$\underline{Z}_t = \underline{Z}_{bb} - \underline{Z}_{ba}\, (\underline{Z} + \underline{Z}_{aa})^{-1}\, \underline{Z}_{ab} \tag{3.25}$$

and

$$\underset{\sim}{N}_t = \underline{Z}_{ba}\, (\underline{Z} + \underline{Z}_{aa})^{-1}\, \underset{\sim}{N}. \tag{3.26}$$

Equation (3.24) is the matrix relation for the new pumped nonlinear system obtained by cascading the original one with a linear lossless system. Here $\underline{Z}_T$ is the new impedance matrix, and $\underset{\sim}{N}_t$ is the column matrix of the new open-circuit noise voltages in the α-β representation. By Eqs. (3.5) and (3.7), the new impedance matrix $\underline{Z}''_t$ and the new open-circuit noise voltage matrix $\underset{\sim}{N}'_t$ in the amplitude-phase representation are given by

$$\underline{Z}''_t = \underline{\lambda}_v \, \underline{Z}_t \, \underline{\lambda}_i^{-1} \tag{3.27}$$

where $\underline{\lambda}_v$ and $\underline{\lambda}_i$ are as given in Eqs. (3.8) and (3.9).

Let us now assume that the phase noise port in Fig. 3.4 is open-circuited (or short-circuited). The terminal relation at the amplitude noise port can now be written as (phase noise port open-circuited)

$$(V'_a)_b = (Z''_{aa})_t \, (I'_a)_b + (n'_a)_t . \tag{3.29}$$

The amplitude noise power that can be obtained from the system for an arbitrary amplitude noise terminal current $(I'_a)_b$ is given by

$$P_a = - \overline{[(V'_a)^*_b \, (I'_a)_b + (I'_a)^*_b \, (V'_a)_b]}$$

$$= - \Big[\overline{\Big\{(I'_a)_b + [(Z''_{aa})_t + (Z''_{aa})^*_t]^{-1} \, (n'_a)_t\Big\} [(Z''_{aa})_t + (Z''_{aa})^*_t]} \cdots$$

$$\cdots \overline{\Big\{(I'_a)_b + [(Z''_{aa})_t + (Z''_{aa})^*_t]^{-1} \, (n'_a)_t\Big\}^*}$$

$$- \overline{\Big\{(n'_a)^*_t \, [(Z''_{aa})_t + (Z''_{aa})^*_t]^{-1} \, (n'_a)_t\Big\}} \Big] . \tag{3.30}$$

The stationary value of P_a when $(I_a')_b$ is arbitrarily varied is clearly

$$P_{e,a} = \frac{\overline{(n_a')_t \, (n_a')_t^*}}{(Z''_{aa})_t + (Z''_{aa})_t^*} \, . \qquad (3.31)$$

We shall call this $P_{e,a}$ exchangeable amplitude noise power. This power can be obtained from the system by arbitrary variation of the terminal amplitude current.

This exchangeable amplitude noise power can be written in matrix form as

$$P_{e,a} = \frac{\overline{(n_a')_t \, (n_a')_t^*}}{(Z''_{aa}) + (Z''_{aa})_t^*} = \frac{\underset{\sim}{\xi}^+ \, \overline{\underset{\sim}{N}_t' \, \underset{\sim}{N}_t'^+} \, \underset{\sim}{\xi}}{\underset{\sim}{\xi}^+ \, (\underline{Z}_t'' + \underline{Z}_t''^+) \, \underset{\sim}{\xi}}$$

where the (real) column matrix $\underset{\sim}{\xi}$ may be represented as

$$\xi = \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix} . \qquad (3.33)$$

Let us now see whether there exists any stationary values of $P_{e,a}$ when the linear lossless network is arbitrarily varied. The variation of the linear lossless network in Fig. 3.4 now corresponds to variation of the transformation network through all possible forms. We wish to find the stationary values of $P_{e,a}$ corresponding to variation of the transformation network. To render explicit this variation, $\underset{\sim}{N}_t$ is first expressed in terms of the original $\underset{\sim}{N}'$ and $\underline{Z}''$. Accordingly,

$$\underset{\sim t}{N}' \overline{\underset{\sim t}{N}'^{+}} = \underline{\lambda}_v \overline{\underset{\sim t}{N} \underset{\sim t}{N}^{+}} \underline{\lambda}_v^{+}$$

$$= \underline{Z}''_{ba}(\underline{Z}'' + \underline{Z}_{aa})^{-1} \overline{\underset{\sim}{N}' \underset{\sim}{N}'^{+}} \left\{\underline{Z}''_{ba} (\underline{Z}'' + \underline{Z}''_{aa})^{-1}\right\}^{+}$$

$$= \underline{\tau}^{+} \overline{\underset{\sim}{N}' \underset{\sim}{N}'^{+}} \underline{\tau} \qquad (3.34)$$

where

$$\underline{\tau}^{+} = \underline{Z}''_{ba} (\underline{Z}'' + \underline{Z}''_{aa})^{-1}. \qquad (3.35)$$

Let us now express $\underline{Z}''_t$ in terms of $\underline{Z}''$. We have

$$\underline{Z}''_t + \underline{Z}''^{+}_t = \underline{\lambda}_v \underline{Z}_t \underline{\lambda}_i^{-1} + \underline{\lambda}_i^{+-1} \underline{Z}_t^{+} \underline{\lambda}_v^{+}$$

$$= \underline{Z}''_{ba}(\underline{Z}'' + \underline{Z}''_{aa})^{-1}(\underline{Z}'' + \underline{Z}''^{+})\left\{\underline{Z}''_{ba}(\underline{Z}'' + \underline{Z}''_{aa})^{-1}\right\}^{+}$$

$$= \underline{\tau}^{+} (\underline{Z}'' + \underline{Z}''^{+}) \underline{\tau}. \qquad (3.36)$$

It follows that

$$P_{e,a} = \frac{(\underset{\sim}{\xi}^{+}\underline{\tau}^{+}) \overline{\underset{\sim}{N}' \underset{\sim}{N}'^{+}} (\underline{\tau} \underset{\sim}{\xi})}{(\underset{\sim}{\xi}^{+}\underline{\tau}^{+})(\underline{Z}'' + \underline{Z}''^{+})(\underline{\tau} \underset{\sim}{\xi})} \qquad (3.37)$$

in which matrix $\underline{\tau}$ is to be varied through all possible values consistent with the lossless requirements upon the transformation network.

A new column matrix $\underset{\sim}{s}$ may be defined as

$$\underset{\sim}{s} = \underline{\tau} \underset{\sim}{\xi} = \begin{bmatrix} s_1 \\ \\ s_2 \end{bmatrix}. \qquad (3.38)$$

We may also write

$$\underset{\sim}{s}^{+} = [1 \quad 0] \left\{ \underline{Z}''_{ba} (\underline{Z}'' + \underline{Z}''_{aa})^{-1} \right\} \; . \tag{3.39}$$

Case I:

Let us now assume that ω/ω_o is not arbitrarily small. In this case we can show easily that

$$\underline{Z}''_{ba} = 2\, \underline{\lambda}_v\, \underline{Z}_{ba}\, \underline{\lambda}_v^{+}$$

$$= 2\, \underline{\lambda}_v \begin{bmatrix} m_1+jn_1 & 0 \\ 0 & m_2+jn_2 \end{bmatrix} \underline{\lambda}_v^{+} \tag{3.40}$$

and

$$\underline{Z}''_{aa} = 2\, \underline{\lambda}_v\, \underline{Z}_{aa}\, \underline{\lambda}_v^{+}$$

$$= 2\, \underline{\lambda}_v \begin{bmatrix} jx_1 & 0 \\ 0 & jx_2 \end{bmatrix} \underline{\lambda}_v^{+} \tag{3.41}$$

where x_1, x_2, m_1, m_2, n_1, and n_2 are arbitrary real numbers.

We can now write

$$\underline{Z}''_{aa} = 2\, \frac{1}{2} \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} jx_1 & 0 \\ 0 & jx_2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} j(x_1+x_2) & x_1-x_2 \\ -x_1+x_2 & j(x_1+x_2) \end{bmatrix} \; ; \tag{3.42}$$

and

$$(\underline{Z}'' + \underline{Z}''_{aa})^{-1} = \begin{bmatrix} Z''_{aa} + \frac{1}{2} j(x_1+x_2) & Z''_{ap} + \frac{1}{2}(x_1-x_2) \\ Z''_{pa} + \frac{1}{2}(x_2-x_1) & Z''_{pp} + \frac{1}{2} j(x_1+x_2) \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \dfrac{Z''_{pp} + \frac{1}{2} j(x_1+x_2)}{\triangle} & -\dfrac{Z''_{ap} + \frac{1}{2}(x_1-x_2)}{\triangle} \\ -\dfrac{Z''_{pa} + \frac{1}{2}(x_2-x_1)}{\triangle} & \dfrac{Z''_{aa} + \frac{1}{2} j(x_1+x_2)}{\triangle} \end{bmatrix}$$

$$= \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix} \tag{3.43}$$

where

$$\triangle = \begin{vmatrix} Z''_{aa} + \frac{1}{2} j(x_1+x_2) & Z''_{ap} + \frac{1}{2}(x_1-x_2) \\ Z''_{pa} + \frac{1}{2}(x_2-x_1) & Z''_{pp} + \frac{1}{2} j(x_1+x_2) \end{vmatrix} .$$

We can show that we can always make $\triangle$ nonzero by properly choosing x_1 and x_2.

From Eq. (3.40)

$$\underline{Z}''_{ba} = 2\,\underline{\lambda}_v \begin{bmatrix} m_1+jn_1 & 0 \\ 0 & m_2+jn_2 \end{bmatrix} \underline{\lambda}_v^{+}$$

$$= \frac{1}{2} \begin{bmatrix} (m_1+jn_1) + (m_2+jn_2) & -j(m_1+jn_1) + j(m_2+jn_2) \\ j(m_1+jn_1) - j(m_2+jn_2) & (m_1+jn_1) + (m_2+jn_2) \end{bmatrix} \qquad (3.46)$$

It, therefore, follows that

$$\underset{\sim}{s}^{+} = \underset{\sim}{\xi}^{+}\underline{\tau}^{+} = [1 \quad 0]\, \underline{Z}''_{ba}(\underline{Z}'' + \underline{Z}''_{aa})^{-1}$$

$$= \frac{1}{2}\Big[t_1 \left\{(m_1+jn_1) + (m_2+jn_2)\right\} + t_3 \left\{-j(m_1+jn_1) + j(m_2+jn_2)\right\}$$

$$+ t_2 \left\{j(m_1+jn_1) - j(m_2+jn_2)\right\} + t_4 \left\{(m_1+jn_1) + (m_2+jn_2)\right\}\Big] . \qquad (3.47)$$

By looking at Eq. (3.47) we may conclude that the elements s_1 and s_2 of the matrix $\underset{\sim}{s}$ take on all possible complex values as the lossless network is varied through all its allowed forms. Consequently, the stationary values of $P_{e,a}$ in Eq. (3.37) may be found by determining the stationary values of the expression

$$P_{e,a} = \frac{\underset{\sim}{s}^{+}\,\overline{\underset{\sim}{N}'\,\underset{\sim}{N}'^{+}}\,\underset{\sim}{s}}{\underset{\sim}{s}^{+}\,(\underline{Z}'' + \underline{Z}''^{+})\,\underset{\sim}{s}} \qquad (3.48)$$

as the complex column matrix $\underset{\sim}{s}$ is varied quite arbitrarily.

The solution of this problem is well-known in matrix theory [4]. The stationary values of the exchangeable amplitude noise power $P_{e,a}$ can be shown to be given by the eigenvalues of the matrix

$$\underline{M}_a = (\underline{Z}'' + \underline{Z}''^{+})^{-1} \overline{\underset{\sim}{N}' \underset{\sim}{N}'^{+}}. \tag{3.49}$$

In general there are two eigenvalues of matrix $\underline{M}_a$. The maximum of these eigenvalues will be defined by us to be the exchangeable amplitude noise power of the system. The exchangeable amplitude noise power of the system, therefore, is given by the maximum of the eigenvalues of the matrix $\underline{M}_a$. In this way we can meaningfully define exchangeable amplitude noise power.

In a similar way, it may be shown[6] that the stationary values of the exchangeable phase noise power $P_{e,a}$ are given by the eigenvalues of the matrix

$$\underline{M}_p = (\underline{Z}'' + \underline{Z}''^{+})^{-1} \overline{\underset{\sim}{N}' \underset{\sim}{N}'^{+}}. \tag{3.50}$$

The maximum of the eigenvalues of $\underline{M}_p$ will be defined as the exchangeable phase noise power of the system.

Also, $\underline{M}_a$ and $\underline{M}_p$ will be called the characteristic-noise matrices of the system.

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[6]In this case we open-circuit (or short-circuit) the amplitude noise port in Fig. 3.4.

Case II:

If ω/ω_o is arbitrarily small, it may be shown that

$$m_1 + jn_1 = m_2 + jn_2; \quad x_1 = x_2 \tag{3.51}$$

in Eqs. (3.40) and (3.41).

Accordingly,

$$\underline{Z}''_{ba} = \begin{bmatrix} m_1+jn_1 & 0 \\ 0 & m_1+jn_1 \end{bmatrix} \tag{3.52}$$

and

$$(\underline{Z}'' + \underline{Z}''_{aa})^{-1} = \begin{bmatrix} \dfrac{Z''_{pp}+jx_1}{\triangle} & \dfrac{-Z''_{ap}}{\triangle} \\ \dfrac{-Z''_{pa}}{\triangle} & \dfrac{Z''_{aa}+jx_1}{\triangle} \end{bmatrix} \tag{3.53}$$

where

$$\triangle = \begin{vmatrix} Z''_{aa}+jx_1 & Z''_{ap} \\ Z''_{pa} & Z''_{pp}+jx_1 \end{vmatrix} . \tag{3.54}$$

We may now write

$$\underset{\sim}{s}^+ = [1 \quad 0]\ \underline{Z}''_{ba}\ (\underline{Z}'' + \underline{Z}''_{aa})^{-1}$$

$$= \frac{1}{\Delta} (m_1 + jn_1) \; [1 \quad x_1] \begin{bmatrix} Z''_{pp} & -Z''_{ap} \\ j & 0 \end{bmatrix}$$

$$= \frac{1}{\Delta \lambda'_1} (m_1 + jn_1) \; [x'_1 \quad x_1 x'_1] \; \underline{A}$$

$$= \frac{1}{\Delta \lambda'_1} (m_1 + jn_1) \; [y_1 \quad y_2] \; \underline{A}$$

$$= \frac{1}{\Delta \lambda'_1} (m_1 + jn_1) \; \underset{\sim}{y}^+ \; \underline{A}$$

where

$$\underset{\sim}{y}^t = [y_1 \quad y_2] \tag{3.55}$$

$$\underline{A} = \begin{bmatrix} Z''_{pp} & -Z''_{ap} \\ j & 0 \end{bmatrix} \tag{3.56}$$

and x'_1 is an arbitrary real number.

Equation (3.48) may now be expressed as

$$P_{e,a} = \frac{\underset{\sim}{y}^t \; \underline{A} \; \overline{\underset{\sim}{N}' \; \underset{\sim}{N}'^+} \; \underline{A}^+ \; \underset{\sim}{y}}{\underset{\sim}{y}^t \; \underline{A}(\underline{Z}'' + \underline{Z}''^+)\underline{A}^+ \; \underset{\sim}{y}} \tag{3.57}$$

It can be shown that the matrices $\underline{A} \; \overline{\underset{\sim}{N}' \; \underset{\sim}{N}'^+} \; \underline{A}^+$ and $\underline{A}(\underline{Z}'' + \underline{Z}''^+)\underline{A}^+$ are Hermitian matrices. It, therefore, follows that

$$\underset{\sim}{y}^t \underline{A}\, \overline{\underset{\sim}{N}'\underset{\sim}{N}'^+}\, \underline{A}^+ \underset{\sim}{y} = \underset{\sim}{y}^t \left[\underline{A}\, \overline{\underset{\sim}{N}'\underset{\sim}{N}'^+}\, \underline{A}^+ + \left\{ \underline{A}\, \overline{\underset{\sim}{N}'\underset{\sim}{N}'^+}\, \underline{A}^+ \right\}^* \right] \underset{\sim}{y} \tag{3.58}$$

and

$$\underset{\sim}{y}^t \underline{A}(\underline{Z}'' + \underline{Z}''^+)\underline{A}^+ \underset{\sim}{y} = \underset{\sim}{y}^t \left[\underline{A}(\underline{Z}'' + \underline{Z}''^+)\underline{A}^+ + \left\{ \underline{A}(\underline{Z}'' + \underline{Z}''^+)\underline{A}^+ \right\}^* \right] \underset{\sim}{y}, \tag{3.59}$$

Accordingly,

$$P_{e,a} = \frac{\underset{\sim}{y}^t \left\{ \underline{A}\, \overline{\underset{\sim}{N}'\underset{\sim}{N}'^+}\, \underline{A}^+ + [\underline{A}\, \overline{\underset{\sim}{N}'\underset{\sim}{N}'^+}\, \underline{A}^+]^* \right\} \underset{\sim}{y}}{\underset{\sim}{y}^t \left\{ \underline{A}(\underline{Z}'' + \underline{Z}^+)\underline{A}^+ + [\underline{A}(\underline{Z}'' + \underline{Z}''^+)\underline{A}^+]^* \right\} \underset{\sim}{y}} \tag{3.60}$$

The column matrix $\underset{\sim}{y}$ in Eq. (3.60) is a real arbitrary vector the elements of which take on all possible values as the lossless network is varied. The stationary values of $P_{e,a}$ are, therefore, given by eigenvalues of the matrix

$$\underline{M}_a = \left\{ \underline{A}(\underline{Z}'' + \underline{Z}''^+)\underline{A}^+ + [\underline{A}(\underline{Z}'' + \underline{Z}''^+)\underline{A}^+]^* \right\}^{-1} \left\{ \underline{A}\, \overline{\underset{\sim}{N}'\, \underset{\sim}{N}'^+}\, \underline{A}^+ + [\underline{A}\, \overline{\underset{\sim}{N}'\, \underset{\sim}{N}'^+}]^* \right\} \tag{3.61}$$

In this case also the maximum of the eigenvalues of $\underline{M}_a$ will be defined as the exchangeable amplitude noise power of the system.

Similarly, it can be shown that the stationary values of the exchangeable phase noise power are given by eigenvalues of the matrix

$$\underline{M}_p = \left\{\underline{B}(\underline{Z}'' + \underline{Z}''^{+})\underline{B}^{+} + [\underline{B}(\underline{Z}'' + \underline{Z}''^{+})\underline{B}^{+}]^{*}\right\}^{-1} \left\{\underline{B}\ \overline{\underset{\sim}{N}'\underset{\sim}{N}'^{+}}\ \underline{B}^{+} + [\underline{B}\ \overline{\underset{\sim}{N}'\underset{\sim}{N}'^{+}}\ \underline{B}^{+}]^{*}\right\} \tag{3.62}$$

where

$$\underline{B} = \begin{bmatrix} -Z''_{pa} & Z''_{aa} \\ 0 & j \end{bmatrix} \tag{3.63}$$

We shall, therefore, define the maximum of the eigenvalues of $\underline{M}_p$ as the exchangeable phase noise power of the system.

In this section we have developed the concepts of exchangeable amplitude and phase noise powers for pumped nonlinear systems. These exchangeable noise powers are defined as the maximum of the powers that can be obtained from the system by cascading it with a linear lossless network. The values of these powers can not, therefore, be changed by the use of any further linear lossless networks. We have also showed that exchangeable amplitude and phase noise powers are the same for the system if ω/ω_o is not arbitrarily small. If ω/ω_o is arbitrarily small, the two exchangeable noise powers need not be the same as shown by Eqs. (3.61) and (3.62). We shall denote by λ_a the exchangeable amplitude noise power of the system, and by λ_p the exchangeable phase noise power.

Physical intuition requires that the values of λ_a and λ_p must be invariant to a linear lossless transformation that preserves the number of terminal pairs. This is indeed the case may be proved as follows. Suppose that the original system with characteristic-noise matrices $\underline{M}_a$ and $\underline{M}_p$ is cascaded with a 4-port linear lossless network as shown in Fig. 3.4. A new system is obtained with characteristic-noise matrices $\underline{M}_a'$ and $\underline{M}_p'$. The eigenvalues of $\underline{M}_a'$ and $\underline{M}_p'$ are the stationary values of the exchangeable amplitude and phase noise powers of the system obtained in a subsequent cascading of the type shown in Fig. 3.4. This second cascading network is completely variable. One possible variation removes the first 4-port cascading network. Accordingly, the stationary values of the exchangeable amplitude and phase noise powers do not change when a 4-port linear lossless network is cascaded with the system.

The results of this section can be summarized in the following three theorems.

<u>Theorem 3.1</u>. The stationary values of the exchangeable amplitude noise power that can be obtained from a pumped nonlinear system by cascading the system with a linear lossless network are given by the eigenvalues of the matrix

$$(1) \qquad \underline{M}_a = (\underline{Z}'' + \underline{Z}''^{+})^{-1} \, \overline{\tilde{N}'\tilde{N}'^{+}} \qquad (3.49)$$

when ω/ω_o is not arbitrarily small; and they are given by the eigenvalues of the matrix

$$(2) \quad \underline{M}_a = \left\{ \underline{A}(\underline{Z}'' + \underline{Z}''^{+})\underline{A}^{+} + [\underline{A}(\underline{Z}'' + \underline{Z}''^{+})\ \underline{A}^{+}]^{*} \right\}^{-1}$$

$$\left\{ \underline{A}\ \overline{\underset{\sim}{N}'\underset{\sim}{N}'^{+}}\ \underline{A}^{+} + [\underline{A}\ \overline{\underset{\sim}{N}'\underset{\sim}{N}'^{+}}\ \underline{A}^{+}]^{*} \right\} \tag{3.61}$$

when ω/ω_o is arbitrarily small. The matrix $\underline{A}$ is represented as

$$\underline{A} = \begin{bmatrix} Z''_{pp} & -Z''_{ap} \\ j & 0 \end{bmatrix} \tag{3.56}$$

Theorem 3.2. The stationary values of the exchangeable phase noise power that can be obtained from a pumped nonlinear system by cascading the system with a linear lossless network are given by the eigenvalues of the matrix

$$(1) \quad \underline{M}_p = (\underline{Z}'' + \underline{Z}''^{+})^{-1}\ \overline{\underset{\sim}{N}'\underset{\sim}{N}'^{+}} \tag{3.50}$$

when ω/ω_o is not arbitrarily small; and they are given by the eigenvalues of the matrix

$$(2) \quad \underline{M}_p = \left\{ \underline{B}(\underline{Z}'' + \underline{Z}''^{+})\underline{B}^{+} + [\underline{B}(\underline{Z}'' + \underline{Z}^{+})\underline{B}^{+}]^{*} \right\}^{-1}$$

$$\left\{ \underline{B}\ \overline{\underset{\sim}{N}'\underset{\sim}{N}'^{+}}\ \underline{B} + [\underline{B}\ \overline{\underset{\sim}{N}'\underset{\sim}{N}'^{+}}\ \underline{B}^{+}]^{*} \right\} \tag{3.62}$$

when ω/ω_o is arbitrarily small. The matrix $\underline{B}$ is represented as

$$\underline{B} = \begin{bmatrix} -Z''_{pa} & Z''_{aa} \\ 0 & j \end{bmatrix}. \tag{3.63}$$

Theorem 3.3. The eigenvalues of the characteristic-noise matrices $\underline{M}_a$, and $\underline{M}_p$ associated with a pumped nonlinear system are invariant to a linear lossless transformation of the form shown in Fig. 3.4.

As mentioned earlier, the exchangeable amplitude noise power of the system will be denoted by λ_a, and the exchangeable phase noise power by λ_p. The values of λ_a and λ_p do not change when the system is subjected to a linear lossless transformation of the type shown in Fig. 3.4.

3.2. CHARACTERIZATION OF NOISE PERFORMANCE OF PUMPED NONLINEAR SYSTEMS

For a linear transducer, the noise figure, at a specified output frequency, is defined as the ratio of the total noise power per unit bandwidth exchangeable at the output port when the only source of noise in the source network is thermal noise at standard temperature ($T_o = 290^o$ K) to that portion of the total noise power engendered at this frequency by the thermal noise of the source [1].

In this chapter, three sets of figures of merit have been proposed for pumped nonlinear systems. These seem to be adequate for our purpose.

Part I.

For a pumped nonlinear system, the concepts of exchangeable amplitude and phase noise powers have been developed in the previous section. We now propose the following definition for the noise figure matrix in terms of these exchangeable noise powers.

The noise figure matrix $\underline{F}$ for a pumped nonlinear system is defined as

$$\underline{F} = \begin{bmatrix} \dfrac{(S/\lambda_a)_{in}}{(S/\lambda_a)_{out}} \\ \\ \dfrac{(S/\lambda_p)_{in}}{(S/\lambda_p)_{out}} \end{bmatrix} \tag{3.64}$$

where $(S)_{in}$, $(\lambda_a)_{in}$, $(\lambda_p)_{in}$ are the signal power, the exchangeable amplitude noise power and the exchangeable phase noise power at the input port of the transducer, and $(S)_{out}$, $(\lambda_a)_{out}$, and $(\lambda_p)_{out}$ are the corresponding quantities at the output port.[7]

Let us now write $\underline{F}$ as

$$\underline{F} = \begin{bmatrix} F_A \\ \\ F_P \end{bmatrix} . \tag{3.65}$$

We can now make the following observations. The values of F_A and F_P do not change when linear lossless networks are interposed between the system and the source or between the system and the load. This is a very desirable feature. It can also be shown that F_A and F_P are equal for a linear

[7]It is assumed that the output signal frequency is different from the input signal frequency. If they are the same, we can treat the system as a linear transducer.

transducer when the only source of noise in the source network is thermal noise. Also the value of F_A or F_P in this case is identically equal to the noise figure defined for these transducers.

Part II.

The terminal noise behavior of a pumped nonlinear system, as shown in Fig. 3.3, is given by

$$\underset{\sim}{V}' = \underline{Z}'' \underset{\sim}{I}' + \underset{\sim}{N}'. \tag{3.3}$$

Let us assume that this is a source network. If this source network is used to drive another pumped nonlinear system (see Figs. 3.5, 3.6) the terminal noise relations of which are given by

$$\begin{bmatrix} \underset{\sim}{V}'_o \\ \hline \underset{\sim}{V}'_i \end{bmatrix} = \left[\begin{array}{c|c} \underline{Z}''_{oo} & \underline{Z}''_{oi} \\ \hline \underline{Z}''_{io} & \underline{Z}''_{ii} \end{array}\right] \begin{bmatrix} \underset{\sim}{I}'_o \\ \hline \underset{\sim}{I}'_i \end{bmatrix} + \begin{bmatrix} \underset{\sim}{n}'_o \\ \hline \underset{\sim}{n}'_i \end{bmatrix} \tag{3.66}$$

the terminal noise relations at the output of the resulting system are:

$$\underset{\sim}{V}'_o = \underline{Z}''_t \underset{\sim}{I}'_o + \underset{\sim}{N}'_t \tag{3.67}$$

where

$$\underline{Z}''_t = \underline{Z}''_{oo} - \underline{Z}''_{oi} (\underline{Z}'' + \underline{Z}_{ii})^{-1} \underline{Z}''_{io}$$

and

$$\underset{\sim}{N}'_t = \underset{\sim}{n}'_o + \underline{Z}''_{oi} (\underline{Z}'' + \underline{Z}''_{ii})^{-1} \left\{ \underset{\sim}{N}' - \underset{\sim}{n}'_i \right\} .$$

We shall now define a gain matrix $\underline{T}$ for the system as

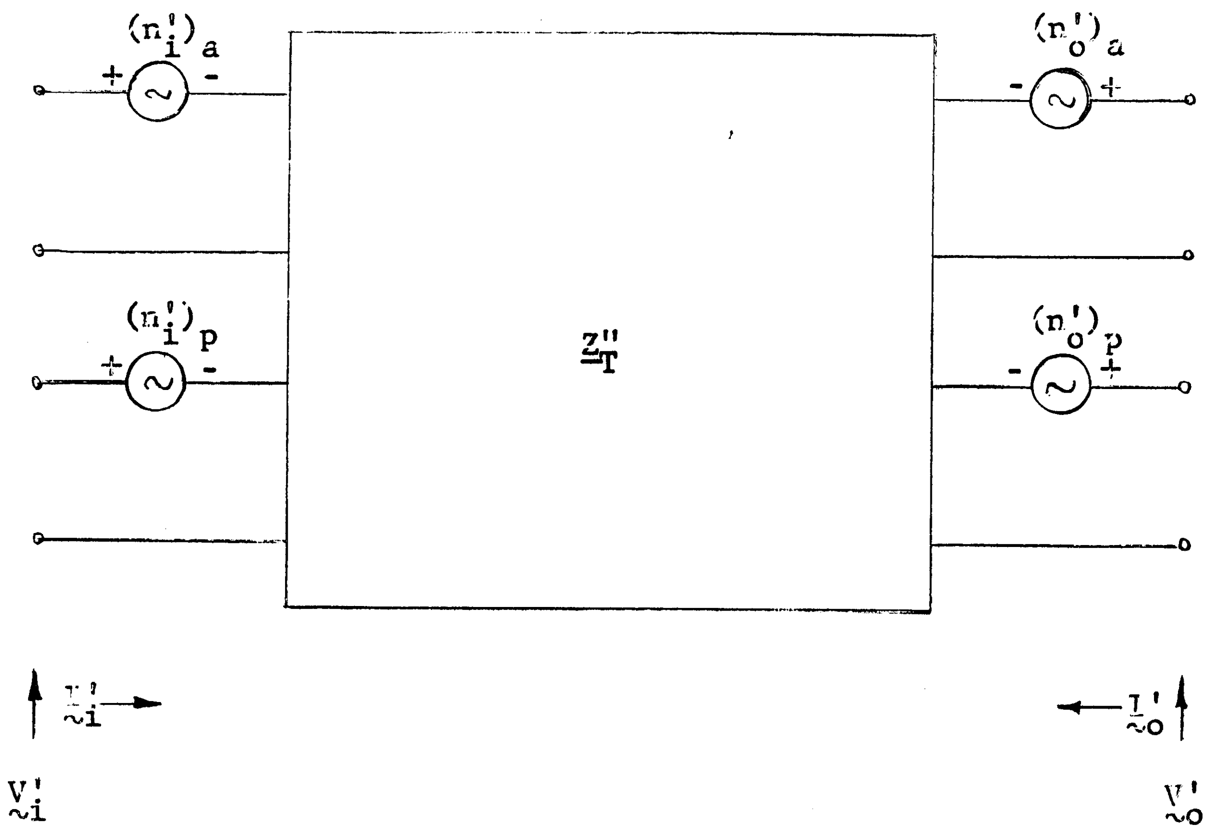

Figure 3.5. Pumped nonlinear system.

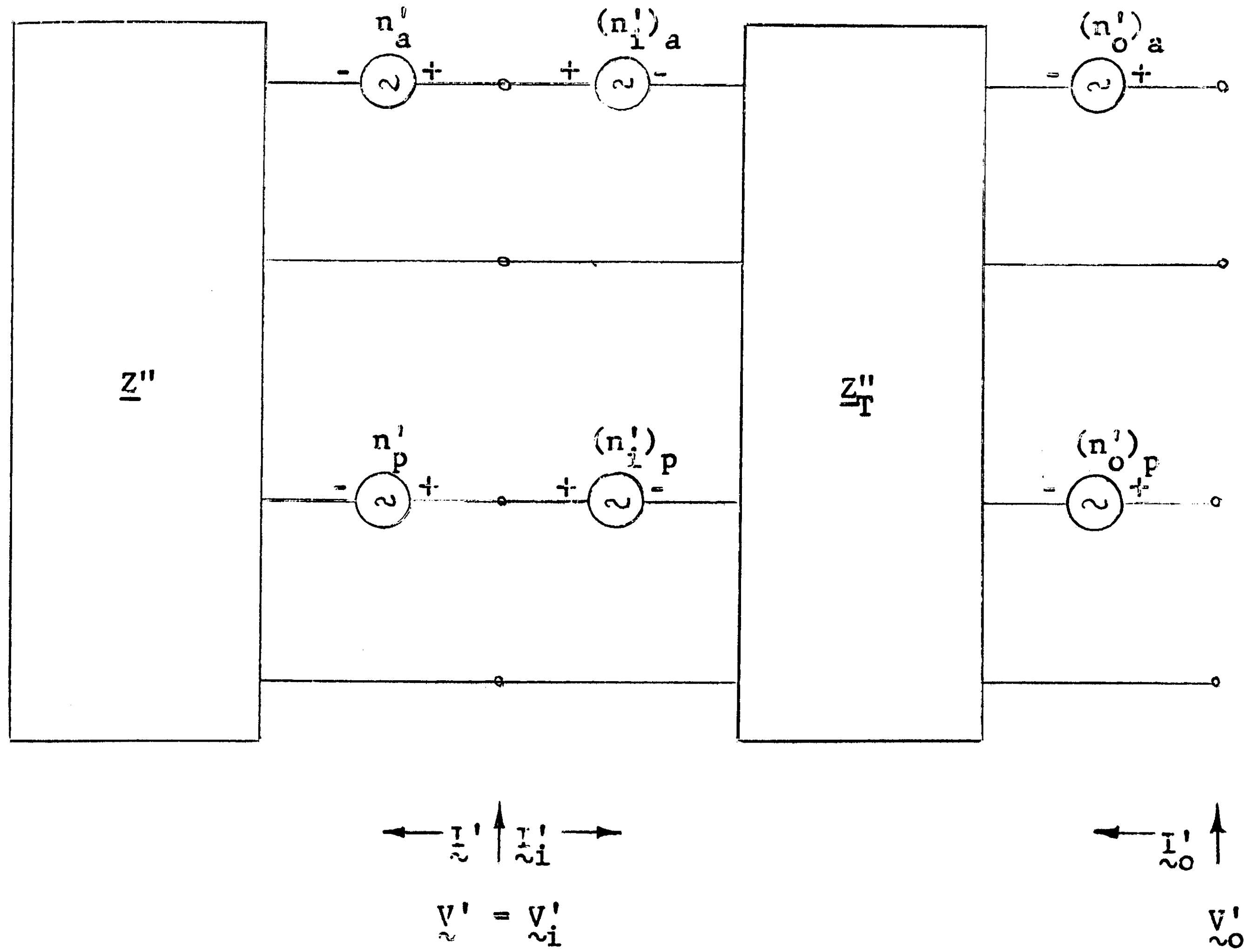

Figure 3.6. Pumped nonlinear system driven by a noisy source.

$$\underline{T} = \underline{Z}''_{oi} \, (\underline{Z}'' + \underline{Z}''_{ii})^{-1}. \tag{3.68}$$

If the driven nonlinear system does not contain any internal noise generators, Eq. (3.66) may be written as

$$\tilde{V}'_o = \underline{Z}''_t \, \tilde{I}'_o + \underline{T} \, \tilde{N}'. \tag{3.69}$$

We now propose the following definition of the noise figure matrix for the transducer we are considering. The noise figure matrix $\underline{F}'$ is defined as

$$\underline{F}' = \underline{T}^{-1} \, \overline{\tilde{N}'_t \, \tilde{N}'^{+}_t} \, \underline{T}^{+-1} \, \overline{\tilde{N}' \, \tilde{N}'^{+}}^{\,-1}. \tag{3.70}$$

We can show that Eq. (3.70) can be written as[8]

$$\underline{F}' = \underline{T}^{-1} \left[\underline{T} \, \overline{\tilde{N}'\tilde{N}'^{+}} \, \underline{T}^{+} + \overline{\tilde{N}'_i \, \tilde{N}'^{+}_i} \right] \underline{T}^{+-1} \, \overline{\tilde{N}' \, \tilde{N}'^{+}}^{\,-1} \tag{3.71}$$

where

$$\tilde{N}_i = \tilde{n}'_o - \underline{T} \, \tilde{n}'_i.$$

$\tilde{N}'_i$ will be called the noise matrix of the transducer.

According to Eq. (3.71)

$$\underline{F}' = \underline{I} + \underline{T}^{-1} \, \overline{\tilde{N}'_i \, \tilde{N}'^{+}_i} \, \underline{T}^{+-1} \, \overline{\tilde{N}' \, \tilde{N}'^{+}}^{\,-1}. \tag{3.72}$$

The matrix $\underline{F}'$ is a unity matrix if the pumped nonlinear system of Fig. 3.5 does not contain any internal noise

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[8] It is assumed that there is no correlation between the noise sources in the source network and those shown in Fig. 3.5.

generators. Let us now assume that the source is purely thermal.[9] In that case[10]

$$\overline{\underset{\sim}{N}' \underset{\sim}{N}'^{+}} = \begin{bmatrix} R_{in} k T_o \Delta f & 0 \\ 0 & R_{in} k T_o \Delta f \end{bmatrix}.$$

We may then write

$$\underline{F}' = \underline{I} + \frac{\underline{T}^{-1} \; \overline{\underset{\sim}{N}'_i \underset{\sim}{N}'^{+}_i} \; \underline{T}^{+-1}}{R_{in} k T_o \Delta f} \tag{3.73}$$

We can show that $\underline{T}^{-1} \; \overline{\underset{\sim}{N}'_i \underset{\sim}{N}'^{+}_i} \; \underline{T}^{+-1}$ is a positive definite matrix in all but cases of trivial interest. In that case the diagonal elements of $\underline{T}^{-1} \; \overline{\underset{\sim}{N}'_i \underset{\sim}{N}'^{+}_i} \; \underline{T}^{+-1}$ are always positive. The diagonal elements of $\underline{F}'$ will always be greater than unity; and the magnitudes of these elements will exhibit the noisy character of the device. This is another advantage of defining $\underline{F}'$ as in Eq. (3.70). It is also clear from Eq. (3.72) that $\underline{F}'$ will be a unity matrix if the pumped nonlinear system does not contain any internal noise generators.

Example. We showed in Chapter 2 that an equivalent circuit for a noisy pumped nonlinear system can be written as in Fig. 3.7.

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[9]This assumption is made in defining the noise figure for linear transducers.

[10]R_{in} is the real part of the internal impedance of the source, and T_o is the standard termperature.

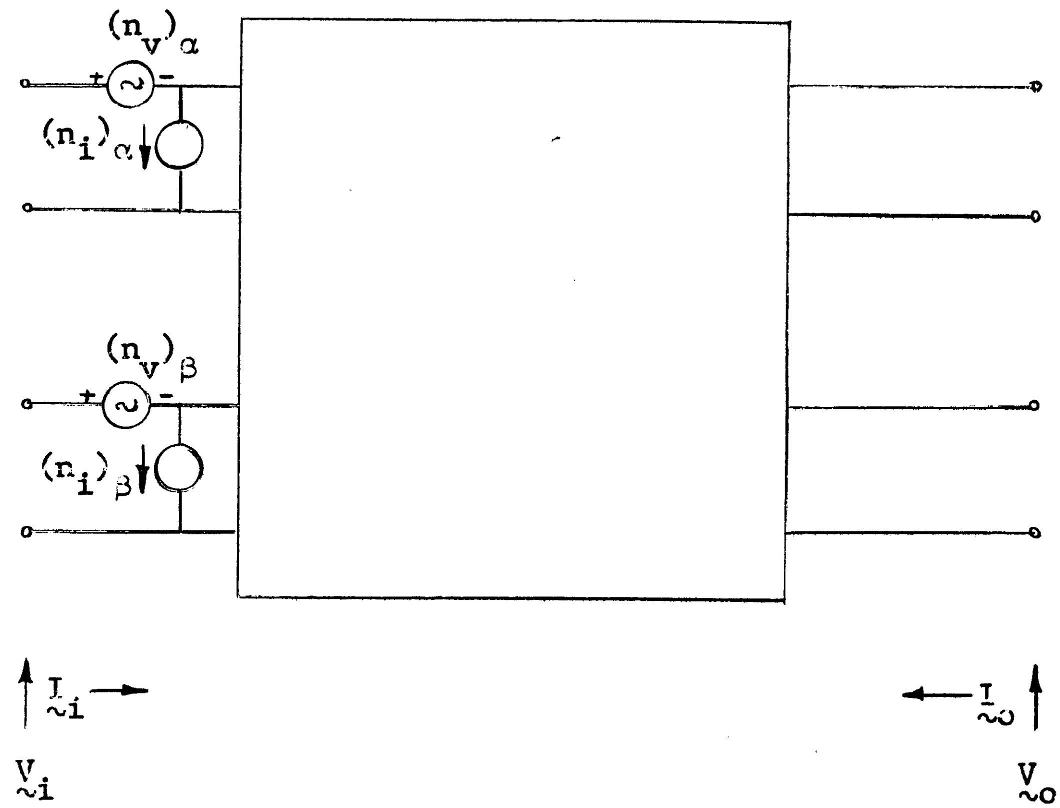

Figure 3.7. Pumped nonlinear transducer.

The terminal noise relations for this representation are given by

$$\begin{bmatrix} \underset{\sim}{V}_i \\ \hline \underset{\sim}{I}_i \end{bmatrix} = \begin{bmatrix} \underline{A} & \underline{B} \\ \hline \underline{C} & \underline{D} \end{bmatrix} \begin{bmatrix} \underset{\sim}{V}_o \\ \hline \underset{\sim}{I}_o \end{bmatrix} + \begin{bmatrix} \underset{\sim}{n}_v \\ \hline \underset{\sim}{n}_i \end{bmatrix} . \tag{3.74}$$

These relations are given in α-β representation. If the source network of Fig. 3.1 is used to drive this transducer (see Fig. 3.8), the terminal noise behavior at the output of the resulting system is given by an equation of the form

$$\underset{\sim}{V}_o = \underline{Z}_t \underset{\sim}{I}_o + \underset{\sim}{N}_t . \tag{3.75}$$

Equation (3.2) can be written as

$$[\ \underline{1} \mid -\underline{Z}\ ] \begin{bmatrix} \underset{\sim}{V} \\ \hline \underset{\sim}{I} \end{bmatrix} = \underset{\sim}{N} . \tag{3.76}$$

According to Fig. 3.8

$$\underset{\sim}{V} = V_i ; \ \underset{\sim}{I} = -\underset{\sim}{I}_i . \tag{3.77}$$

Accordingly

$$[\ \underline{1} \mid -\underline{Z}\ ] \begin{bmatrix} \underset{\sim}{V}_i \\ \hline \underset{\sim}{I}_i \end{bmatrix} = \underset{\sim}{N} .$$

Using Eq. (3.74), we now have

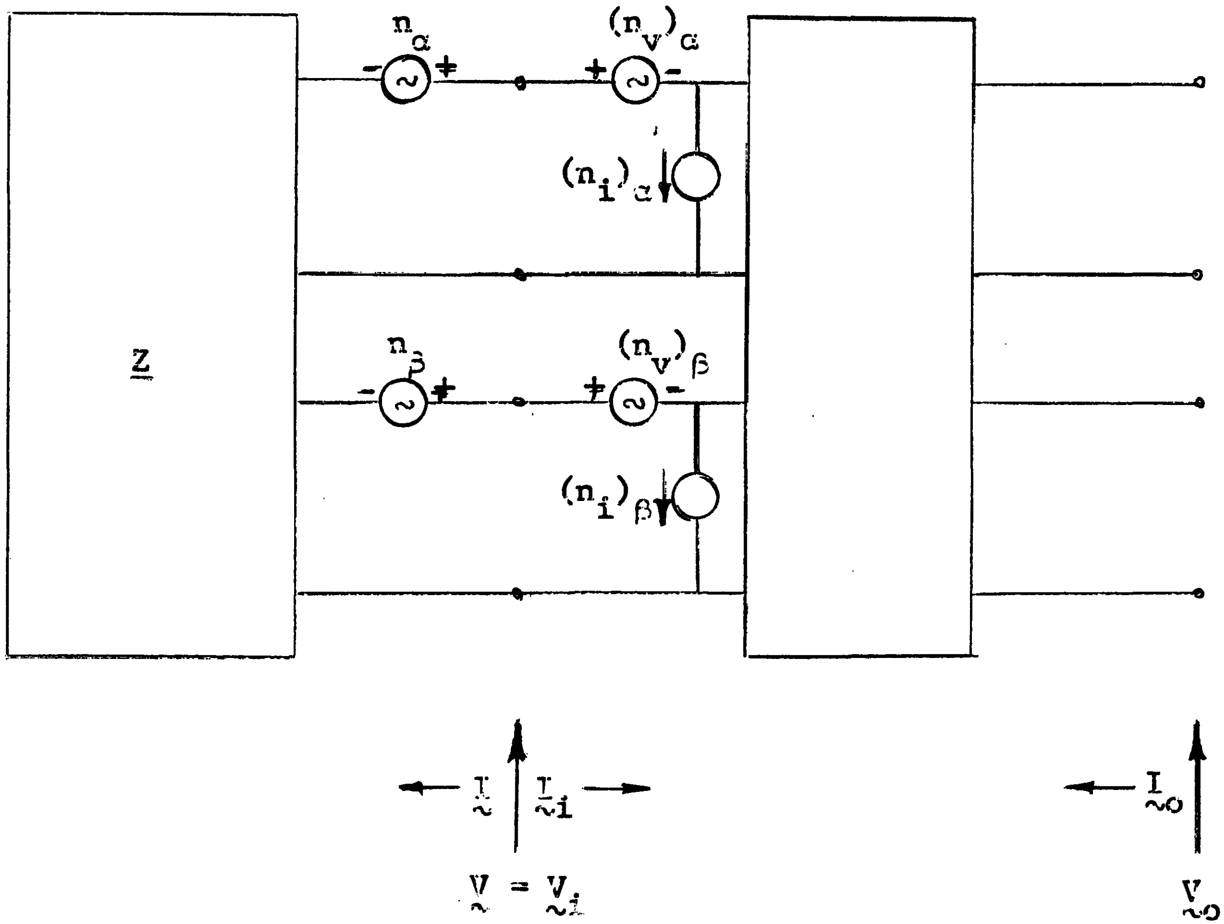

Figure 3.8. Pumped nonlinear transducer driven by a noisy source.

$$\left[\ \underline{1}\ \vdots\ -\underline{Z}\ \right]\left[\begin{array}{c:c}\underline{A} & \underline{B}\\ \hdashline \underline{C} & \underline{D}\end{array}\right]\left[\begin{array}{c}\underset{\sim}{V}_o\\ \hdashline \underset{\sim}{I}_o\end{array}\right] = \underset{\sim}{N} - \underset{\sim}{n}_v - \underline{Z}\,\underset{\sim}{n}_i. \tag{3.78}$$

Let us write

$$\underline{M} = \underline{A} + \underline{Z}\,\underline{C}. \tag{3.79}$$

Equation (3.79) can be shown to be equivalent to

$$\left[\ \underline{1}\ \vdots\ \underline{M}^{-1}\left\{\underline{B} + \underline{Z}\,\underline{D}\right\}\right]\left[\begin{array}{c}\underset{\sim}{V}_o\\ \hdashline \underset{\sim}{I}_o\end{array}\right] = \underline{M}^{-1}\,(\underset{\sim}{N} - \underset{\sim}{n}_v - \underline{Z}\,\underset{\sim}{n}_i). \tag{3.80}$$

Comparing Eqs. (3.75) and (3.80),

$$\underline{Z}_t = -\,\underline{M}^{-1}\left\{\underline{B} + \underline{Z}\,\underline{D}\right\} \tag{3.81}$$

$$\underset{\sim}{N}_t = \underline{M}^{-1}\,(\underset{\sim}{N} - \underset{\sim}{n}_v - \underline{Z}\,\underset{\sim}{n}_i). \tag{3.82}$$

Let us assume that

$$\underline{\lambda}_{v_o} = \frac{1}{2}\begin{bmatrix} 1 & 1\\ j & -j\end{bmatrix}\begin{bmatrix} e^{-j\phi_{v_o}} & 0\\ 0 & e^{+j\phi_{v_o}}\end{bmatrix} \tag{3.83}$$

$$\underline{\lambda}_{v_i} = \frac{1}{2}\begin{bmatrix} 1 & 1\\ j & -j\end{bmatrix}\begin{bmatrix} e^{-j\phi_{v_i}} & 0\\ 0 & e^{j\phi_{v_i}}\end{bmatrix} \tag{3.84}$$

where ϕ_{vo} and ϕ_{vi} are the carrier voltage phase angles at the input and output of the transducer respectively.

Using Eq. (3.28), we have

$$\underset{\sim}{N}_t' = \underline{\lambda}_{vo}\, \underline{M}^{-1}\, (\underset{\sim}{N} - \underset{\sim}{n}_v - \underline{Z}\, \underset{\sim}{n}_i) \qquad (3.85)$$

$$\underset{\sim}{N}' = \underline{\lambda}_{vi}\, \underset{\sim}{N}. \qquad (3.86)$$

Also, according to Eq. (3.68),

$$\underline{T} = \underline{\lambda}_{vo}\, \underline{M}^{-1}\, \underline{\lambda}_{vo}^{-1}. \qquad (3.87)$$

$\underline{F}'$ may now be written as

$$\underline{F}' = \underline{T}^{-1}\, \overline{\underset{\sim}{N}_t' \underset{\sim}{N}_t'^{+}}\, \underline{T}^{+-1}\, \overline{\underset{\sim}{N}' \underset{\sim}{N}'^{+}}^{\,-1}$$

$$= \underline{\lambda}_{vi}\, \underline{M}\, \underline{\lambda}_{v_o}^{-1}\, \underline{\lambda}_{v_o}\, \underline{M}^{-1} \left\{ \overline{\underset{\sim}{N}\underset{\sim}{N}^{+}} + \overline{[\underset{\sim}{n}_v + \underline{Z}\,\underset{\sim}{n}_i][\underset{\sim}{n}_v + \underline{Z}\,\underset{\sim}{n}_i]^{+}} \right\}$$

$$\underline{M}^{+-1}\, \underline{\lambda}_{v_o}^{+}\, \underline{\lambda}_{v_o}^{+-1}\, \underline{M}^{+}\, \underline{\lambda}_{v_i}^{+}\, \underline{\lambda}_{v_i}^{+-1}\, \overline{\underset{\sim}{N}\underset{\sim}{N}^{+}}^{\,-1}\, \underline{\lambda}_{v_i}^{-1}$$

$$= \underline{I} + \underline{\lambda}_{v_i}\, \overline{[\underset{\sim}{n}_v + \underline{Z}\,\underset{\sim}{n}_i][\underset{\sim}{n}_v + \underline{Z}\,\underset{\sim}{n}_i]^{+}}\, \overline{\underset{\sim}{N}\underset{\sim}{N}^{+}}^{\,-1}\, \underline{\lambda}_{v_i}^{-1}.$$

$$(3.88)$$

Equation (3.88) shows that $\underline{F}'$ is only a function of noise sources, $\underline{\lambda}_{vi}$, and $\underline{Z}$. This agrees very well with our physical intuition.

Cascade Formula. Let a source network whose terminal noise relations are given by Eq. (3.3) drive a transducer of the form shown in Fig. 3.6. The noise figure matrix $\underline{F}_1'$ is given by

$$\underline{F}_1' = \underline{I} + \underline{T}^{-1} \; \overline{\underset{\sim}{N}_i' \, \underset{\sim}{N}_i'^{+}} \; \underline{T}^{+-1} \; \overline{\underset{\sim}{N}' \, \underset{\sim}{N}'^{+}}^{\,-1} . \tag{3.72}$$

If this same source were driving a second transducer with gain matrix $\underline{T}_1$ and a noise matrix $(\underset{\sim}{N}_i')_1$, the noise figure matrix $\underline{F}_2'$ is given by

$$\underline{F}_2' = \underline{I} + \underline{T}_1^{-1} \; \overline{(\underset{\sim}{N}_i')_1 \, (\underset{\sim}{N}_i')_1^{+}} \; \underline{T}_1^{+-1} \; \overline{\underset{\sim}{N}' \, \underset{\sim}{N}'^{+}}^{\,-1} . \tag{3.89}$$

If the two transducers are put in cascade (it is assumed that the operating point of either of the transducers does not change due to this operation), and the combination is driven by the same source (see Fig. 3.9), the gain matrix of the combination is given by $\underline{T}_1 \, \underline{T}$ and the noise matrix by $\underline{T}_1 \, \underset{\sim}{N}_i' + (\underset{\sim}{N}_i')_1$.

The noise figure matrix $\underline{F}_{12}'$ of this combination can be written as

$$\begin{aligned}\underline{F}_{12}' - \underline{I} &= (\underline{T}_1 \, \underline{T})^{-1} \left\{ \underline{T}_1 \, \overline{\underset{\sim}{N}_i' \, \underset{\sim}{N}_i'^{+}} \, \underline{T}_1^{+} + \overline{(\underset{\sim}{N}_i')_1 \, (\underset{\sim}{N}_i')_1^{+}} \right\} (\underline{T}_1 \, \underline{T})^{+-1} \; \overline{\underset{\sim}{N}' \, \underset{\sim}{N}'^{+}}^{\,-1} \\ &= \underline{T}^{-1} \; \overline{\underset{\sim}{N}_i' \, \underset{\sim}{N}_i'^{+}} \; \underline{T}^{+-1} \; \overline{\underset{\sim}{N}' \, \underset{\sim}{N}'^{+}}^{\,-1} + \underline{T}^{-1} \left[\underline{T}_1^{-1} \; \overline{(\underset{\sim}{N}_i')_1 \, (\underset{\sim}{N}_i')_1^{+}} \; \underline{T}_1^{+-1} \right] \\ &\qquad \underline{T}^{+-1} \; \overline{\underset{\sim}{N}' \, \underset{\sim}{N}'^{+}}^{\,-1} .\end{aligned} \tag{3.90}$$

Using Eqs. (3.72) and (3.90),

$$\underline{F}_{12}' - \underline{I} = \underline{F}_1' - \underline{I} + \underline{T}^{-1} (\underline{F}_2' - \underline{I}) \; \overline{\underset{\sim}{N}' \, \underset{\sim}{N}'^{+}} \; \underline{T}_1^{+-1} \; \overline{\underset{\sim}{N}' \, \underset{\sim}{N}'^{+}}^{\,-1} . \tag{3.91}$$

This is the cascade formula we get when we put two

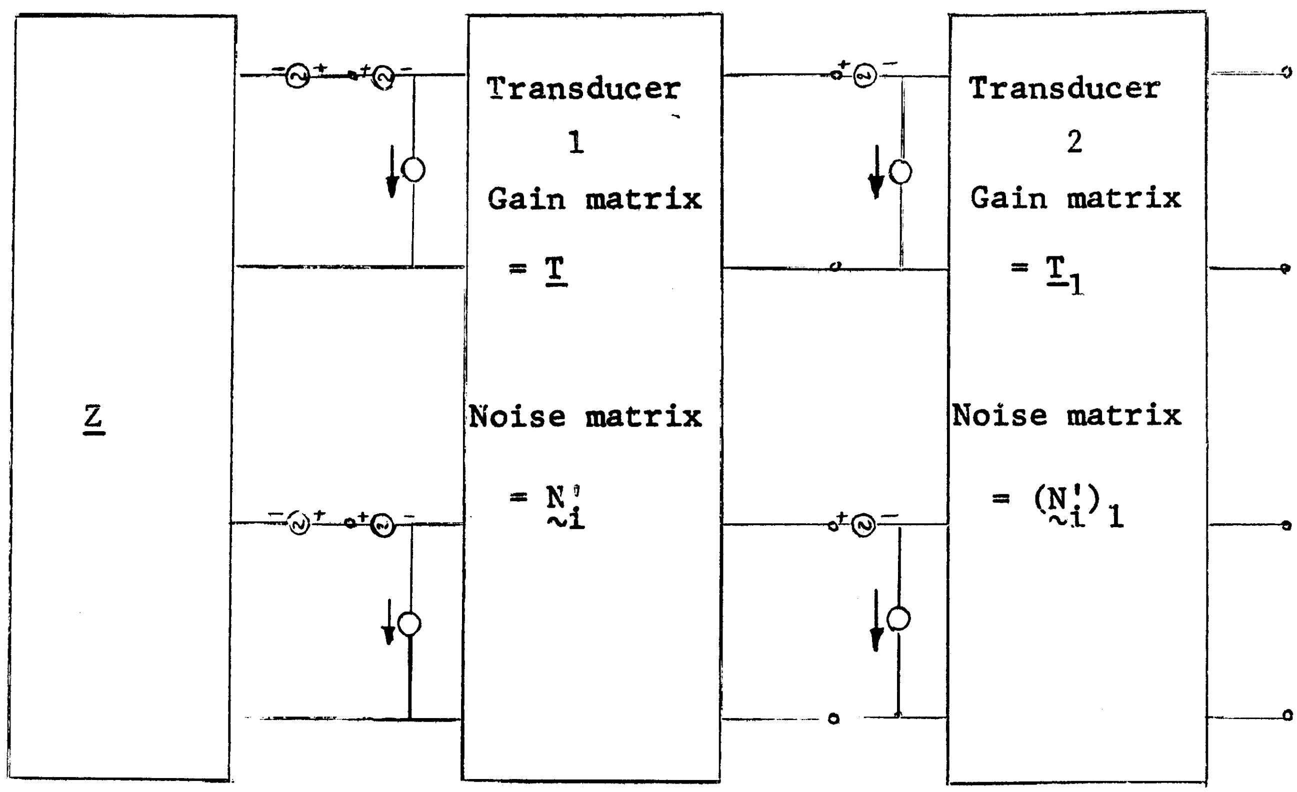

Figure 3.9. Two pumped nonlinear systems driven by a noisy source.

transducers in cascade as shown in Fig. 3.9. The noise figure matrix $\underline{F}'_{12}$ is expressed in terms of $\underline{F}'_1$, $\underline{F}'_2$, $\underline{T}$, and the noise power matrix of the source.

This expression is very similar to that obtained for linear transducers. If F'_1 and F'_2 are the noise figures of two linear two-port transducers and they are connected as shown in Fig. 3.10, we can show [1] that the noise figure F'_{12} of the resulting transducer is given by

$$F'_{12} - 1 = F'_1 - 1 + \frac{F'_2 - 1}{G} \tag{3.92}$$

where G is the exchangeable gain of the first transducer.

The noise figure matrix has been defined in this section with the aid of impedance formalism. If a different matrix representation (such as admittance matrix representation, chain matrix representation) is used it is easy to see that this noise figure matrix can be redefined in terms of these representations.

<u>Part III</u>.

It was mentioned in Chapter 2 that the total voltage v(t) around any frequency $\pm k\omega_o$ may be represented as

$$v(t) = 2\,\mathrm{Re}\left[V_k e^{jk\omega_o t} + V_{\alpha k} e^{j(k\omega_o+\omega)t} + V_{\beta k} e^{j(-k\omega_o+\omega)t}\right]. \tag{3.93}$$

We now write v(t) as

$$v(t) = 2\,\mathrm{Re}\left[|V_k| + (V_{ak}+j\,V_{pk})e^{j\omega t} + (V^*_{ak}+j\,V^*_{pk})e^{-j\omega t}\right]e^{j(k\omega_o t+\phi)} \tag{3.94}$$

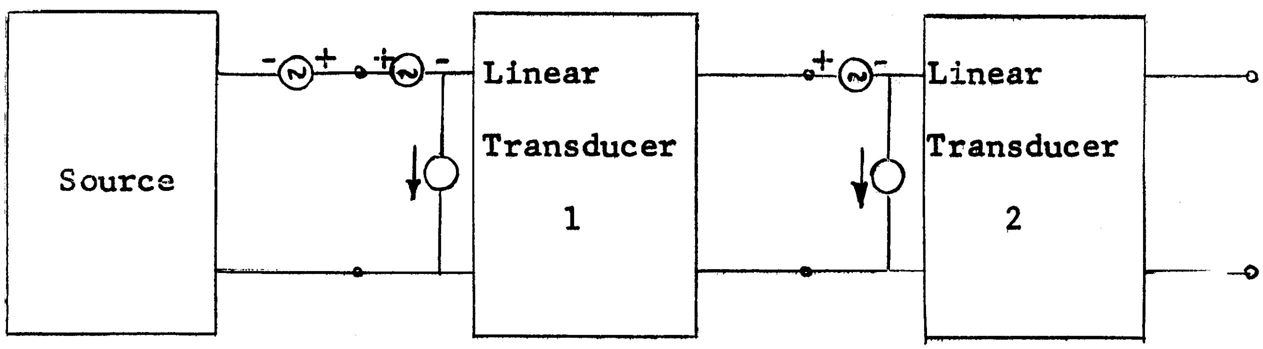

Figure 3.10. Combination of two linear transducers driven by a single source.

where

$$V_k = \left|V_k\right| e^{j\phi} \tag{3.95}$$

$$V_{ak} = \frac{V_{\alpha k}\ e^{-j\phi} + V_{\beta k}\ e^{j\phi}}{2}$$

$$V_{pk} = \frac{V_{\alpha k}\ e^{-j\phi} - V_{\beta k}\ e^{j\phi}}{2j} \tag{3.97}$$

Here, V_k is the carrier voltage component, and $V_{\alpha k}$ and $V_{\beta k}$ are the noise voltage components located at a frequency ω from the carrier.

Equation (3.94) can be written

$$v(t) = 2\ \mathrm{Re}\left[\left|V_k\right| + V_a(t) + j\ V_p(t)\right] e^{j(k\omega_o t+\phi)} \tag{3.98}$$

where

$$V_a(t) = V_{ak}e^{j\omega t} + V_{ak}^{*}\ e^{-j\omega t} \tag{3.99}$$

$$V_p(t) = V_{pk}e^{j\omega t} + V_{pk}^{*}\ e^{-j\omega t}. \tag{3.100}$$

Because of our assumption of high signal-to-noise ratio, v(t) can be written [5] as

$$v(t) \approx 2\left[\left|V_k\right| + V_a(t)\right]\ \cos\left[k\omega_o t + V_p(t)/\left|V_k\right| + \phi\right]. \tag{3.101}$$

We shall, therefore, refer to $V_a(t)$ as the amplitude noise, and to $V_p(t)$ as the phase noise.

The instantaneous amplitude of v(t) is given by

$$R(t) \approx 2\left[\left|V_k\right| + V_a(t)\right]\ ; \tag{3.102}$$

and the instantaneous angular frequency of v(t) is

$$\omega_i(t) \approx k\omega_o + V_p'(t)/|V_k| . \qquad (3.103)$$

The definition of instantaneous frequency adopted here is the time derivative of the instantaneous phase.

The normalized variance of the instantaneous amplitude of v(t) is given by

$$\sigma_R^2 = \overline{V_a^2\ (t)}/|V_k|^2 . \qquad (3.104)$$

The normalized variance of a stochastic variable is defined as the variance of the stochastic variable divided by the square of its mean. Also, the normalized variance of the instantaneous angular frequency of v(t) is

$$\sigma_\omega^2 = \overline{V_p'^2\ (t)}/|V_k|^2\ (k\omega_o)^2 . \qquad (3.105)$$

A functional representation of a transducer is given in Fig. 3.11. This representation assumes meaningful definitions of input impedance Z_{in} and load impedance Z_L for the transducer we are considering. We shall assume in the remainder of this section that such is the case for the transducers we are considering. It must be mentioned here that the representation given in Fig. 3.11 describes only the signal behavior of the device. When Re Z_{in} = Re Z_1, we shall say that the transducer is matched. It may be worthwhile to point out that Z_{in} can be a function of the input signal voltage V.

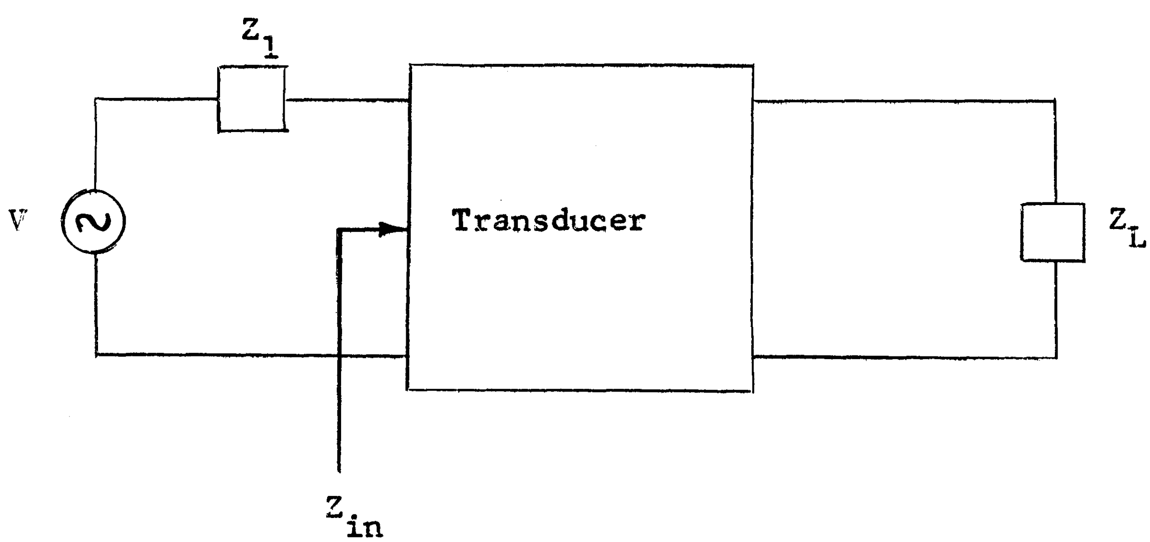

Figure 3.11. Transducer with an input impedance Z_{in} and a load impedance Z_L.

We define [6] amplitude figure of merit F''_A for the transducer as

$$F''_A = \frac{\text{Normalized variance of instantaneous amplitude of current in } Z_L}{\text{Normalized variance of instantaneous amplitude of current in } Z_1} \quad (3.106)$$

when the transducer is matched.

The frequency figure of merit F''_P is defined as

$$F''_P = \frac{\text{Normalized variance of instantaneous frequency of current in } Z_L}{\text{Normalized variance of instantaneous frequency of current in } Z_1} \quad (3.107)$$

when the transducer is matched.

It is assumed that the impedances Z_1 and Z_L are linear impedances. With this assumption, in defining F''_A and F''_P, the voltages instead of the currents across the impedances Z_L and Z_1 may be used. The two results will be the same.

The values of F''_A and F''_P will depend not only on the source and load impedances but also on the operating point. We feel that a convenient and useful operating point to define F''_A and F''_P is the point of optimum efficiency.

It must be noted that the values of F''_A and F''_P will change if we interpose a linear lossless network between source and transducer or between transducer and load. F''_A and F''_P have been defined in terms of the variances of input and output parameters. The variances of quantities like amplitude and frequency have a lot of physical significance. We may therefore say that a good transducer is characterized by low values of F''_A and F''_P.

The values of F''_A and F''_P may, therefore, enable us to compare and contrast the noise performance of different kinds of transducers; and may indicate the direction in which improved noise performance can be obtained.

Three different characterizations have been suggested in this chapter for describing the noise performance of pumped nonlinear systems. We shall mainly use F''_A and F''_P for analyzing and comparing the noise performance of harmonic generators and dividers considered in Chapters 4 and 5.

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CHAPTER 4

NOISE PERFORMANCE OF VARACTOR FREQUENCY MULTIPLIERS

Varactor frequency multipliers have found considerable application in the field of generating microwave signals for receiver local oscillators, parametric amplifier pumps, and other applications during the last few years.

Penfield, Rafuse, and others have analyzed many harmonic generators in which varactor diodes are used [1]. They have not investigated the effect of noise in these multipliers; the noise analysis of these devices forms the subject matter of this chapter.

The quality of performance of any transducer such as a harmonic generator is affected by the physical sources of noise within the transducer, and sources of noise in the source and load terminations of the transducer. There may be an operating point for a transducer at which the noise performance is optimum in some sense.

There have been made many studies of linear systems to determine this optimum noise performance [2], and some results concerning the statistical properties of noise through nonlinear devices have been published [3]. A great deal is unknown about the noise performance of nonlinear transducers such as harmonic generators.

The first order perturbation analysis developed in Chapter

2 has been used in this chapter to derive an explicit expression for the output signal of a harmonic generator with noise sources at several locations in the circuit.

In Chapter 3 three sets of figures of merit were defined. It was also mentioned that one of these sets of figures of merit defined in terms of variances of input and output parameters has the greatest physical significance. These figures of merit have been evaluated and explicitly expressed in terms of the known parameters in this chapter. The figures of merit thus obtained for these devices may enable us to compare and contrast the noise performance of various types of multipliers and multiplier chains.

Thus in this chapter, we have made an attempt to find out the manner in which the noise affects the signal in a harmonic generator, and to arrive at a sufficiently detailed understanding of the controlling parameters to indicate the direction in which improved noise performance can be obtained.

Only the abrupt-junction varactor frequency doubler has been treated in detail. Simplifying assumptions have been made for higher order abrupt-junction varactor frequency multipliers.

4.1. VARACTOR MODEL AND ASSUMPTIONS

Our varactor model, shown in Fig. 4.1, is a variable capacitance in series with a constant resistance R_s. We shall

Figure 4.1. Varactor model.

deal only with abrupt-junction varactor diodes that are nominally driven. Let Q_{max}, and Q_{min} be the maximum and minimum values of the charge between which the diode junction is driven, and Q_B and q_ϕ denote the charge at breakdown voltage and the charge at contact potential. The drive D is defined as

$$D = \frac{(Q_{max} - Q_{min})}{(Q_B - q_\phi)} \quad . \tag{4.1}$$

Now, $D = 1$ is the nominal drive; and $D > 1$ is for overdriving the junction. In this chapter our analysis has been given for $D = 1$ and it has been assumed that the varactor diode is of the abrupt-junction type. The analysis, however, may be easily extended to other types of junctions, and to other conditions of driving.

We assume that currents in the diode flow only at the input, output, and idler frequencies, and suitable external circuits prevent other currents from flowing.

The most important physical source of noise within the varactor seems to be the parasitic series resistance [1]. Other sources of noise such as shot noise and 1/f noise seem to be of secondary importance for many of the applications for which varactors are now used. For simplicity, we shall ignore all sources of noise within the varactor except the thermal noise associated with the parasitic series resistance R_s of the varactor.

To account for thermal noise generated in R_s, we use an equivalent noise voltage generator e_n with mean squared value,

$$\overline{|e_n|^2} = 4\, R_s k T_d \Delta f \qquad (4.2)$$

where T_d is the temperature of the diode, k is Boltzmann's constant, and Δf is the frequency band of interest.

We assume that the diode is driven periodically by voltages and currents that we choose to call the carrier/carriers. These voltages and currents can be determined by transducer analysis with no noise sources at any point in the transducer or in its terminations. This has been done by Penfield and Rafuse for many of the abrupt-junction varactor frequency multipliers that we are going to consider in this chapter [1]. We also assume that the noise affecting the signal at any point in the transducer is bandlimited in a frequency band surrounding the carrier and that the noise power is very small compared with the signal power in a band of frequencies surrounding the carrier. Let us assume that the large voltages and currents at various points in the transducer are, by design, periodic with some frequency ω_o. This entails no loss of generality for a harmonic generator.

There are various ways of characterizing a signal corrupted by narrow-band noise [1]. The characterization given in Chapter 2 will be used in this chapter. Let the noise corrupting the signal at any frequency $\pm k\omega_o$ have a power spectrum[1] centered around a frequency located ω away from the carrier. We shall call ω the "frequency deviation".

The total voltage $V_k(t)$ centered around any frequency $\pm k\omega_o$ will be represented as

$$V_k(t) = 2\ \mathrm{Re}\left[V_k e^{jk\omega_o t} + V_{\alpha k} e^{j(k\omega_o+\omega)t} + V_{\beta k} e^{j(-k\omega_o+\omega)t}\right] . \quad (4.3)$$

Here, V_k is the carrier voltage component, and $V_{\alpha k}$ and $V_{\beta k}$ are the noise voltage components located at a frequency ω from the carrier.

4.2. ANALYSIS OF THE DOUBLER

If the noise currents are allowed to flow at frequencies of the form $\pm\ \ell\omega_o+\omega$, the small signal equations of motion for a varactor may be expressed in the form [1]

$$\begin{bmatrix} \vdots \\ V_i \\ V_j \\ V_k \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & R_s + \frac{S_o}{j\omega_i} & \frac{S_{i-j}}{j\omega_j} & \frac{S_{i-k}}{j\omega_k} & \cdots \\ \cdots & \frac{S_{j-i}}{j\omega_i} & R_s + \frac{S_o}{j\omega_j} & \frac{S_{j-k}}{j\omega_k} & \cdots \\ \cdots & \frac{S_{k-i}}{j\omega_i} & \frac{S_{k-j}}{j\omega_j} & R_s + \frac{S_o}{j\omega_k} & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ I_i \\ I_j \\ I_k \\ \vdots \end{bmatrix} + \begin{bmatrix} \vdots \\ E_{ni} \\ E_{nj} \\ E_{nk} \\ \vdots \end{bmatrix} . \quad (4.4)$$

[1]The power spectrum of time function f(t) is given by the Fourier transform of time average of $f(t)f(t+\tau)$, where $-\infty<\tau<\infty$.

Here, the notation ω_ℓ refers to the frequency $\ell\omega_o+\omega$, and S_k is the Fourier coefficient of elastance S(t) at the frequency $k\omega_o$, $E_{n\ell}$ is the noise component attributable to R_s at the frequency $\ell\omega_o+\omega$. Note that

$$\overline{\left|E_{n\ell}\right|^2} = 2\ R_s kT_d \triangle f \tag{4.5}$$

and

$$\overline{E_{n\ell}E^*_{n\ell}} = 0 \qquad k \neq \ell. \tag{4.6}$$

Let us now investigate the noise performance of an abrupt-junction varactor frequency doubler. For an abrupt junction doubler that is nominally driven, only S_o, S_1, and S_2 may be nonzero. It can also be proved that by properly choosing the time origin we can make jS_1, and jS_2 real and positive [1].

If we denote by the subscripts α_1, β_1, α_2, β_2, and 0 the noise components of the quantities of interest at frequencies $\omega_o+\omega$, $-\omega_o+\omega$, $2\omega_o+\omega$, $-2\omega_o+\omega$, and ω, Eq. (4.4) becomes

$$\begin{bmatrix} V_o \\ V_{\alpha 1} \\ V_{\beta 1} \\ V_{\alpha 2} \\ V_{\beta 2} \end{bmatrix} = \begin{bmatrix} R_s + \frac{S_o}{j\omega} & \frac{S_{-1}}{j(\omega_o+\omega)} & \frac{S_1}{j(-\omega_o+\omega)} & \frac{S_{-2}}{j(+2\omega_o+\omega)} & \frac{S_2}{j(-2\omega_o+\omega)} \\ \frac{S_1}{j\omega} & R_s + \frac{S_o}{j(\omega_o+\omega)} & \frac{S_2}{j(-\omega_o+\omega)} & \frac{S_{-1}}{j(+2\omega_o+\omega)} & 0 \\ \frac{S_{-1}}{j\omega} & \frac{S_{-2}}{j(\omega_o+\omega)} & R_s + \frac{S_o}{j(-\omega_o+\omega)} & 0 & \frac{S_1}{j(-2\omega_o+\omega)} \\ \frac{S_2}{j\omega} & \frac{S_1}{j(\omega_o+\omega)} & 0 & R_s + \frac{S_0}{j(2\omega_o+\omega)} & 0 \\ \frac{S_{-2}}{j\omega} & 0 & \frac{S_{-1}}{j(-\omega_o+\omega)} & 0 & R_s + \frac{S_o}{j(-2\omega_o+\omega)} \end{bmatrix} \begin{bmatrix} I_o \\ I_{\alpha 1} \\ I_{\beta 1} \\ I_{\alpha 2} \\ I_{\beta 2} \end{bmatrix} + \begin{bmatrix} n_o \\ n_{\alpha 1} \\ n_{\beta 1} \\ n_{\alpha 2} \\ n_{\beta 2} \end{bmatrix} \tag{4.7}$$

where $n_{\alpha 1}$ is the Fourier coefficient of the thermal noise voltage resulting from R_s at the frequency $\omega_o+\omega$, etc.

Let the input, output, and bias terminations of the doubler be such that

$$\begin{bmatrix} V_{\alpha 1} \\ V_{\beta 1} \\ V_{\alpha 2} \\ V_{\beta 2} \\ V_o \end{bmatrix} = - \begin{bmatrix} R_1+jX_{\alpha 1} & & & & \\ & R_1+jX_{\beta 1} & & 0 & \\ & & R_2+jX_{\alpha 2} & & \\ & & & R_2+jX_{\beta 2} & \\ & 0 & & & R_o \end{bmatrix} \begin{bmatrix} I_{\alpha 1} \\ I_{\beta 1} \\ I_{\alpha 2} \\ I_{\beta 2} \\ I_o \end{bmatrix} + \begin{bmatrix} n'_{\alpha 1} \\ n'_{\beta 1} \\ 0 \\ 0 \\ n'_o \end{bmatrix} . \quad (4.8)$$

Here, $n'_{\alpha 1}$, and $n'_{\beta 1}$ are the noise voltages associated with the input termination at frequencies $\omega_o+\omega$, and $-\omega_o+\omega$, respectively; and n'_o is the noise source present in the bias loop. The statistical properties of these noise sources are either known or can be estimated by physical measurements.

Penfield and Rafuse [1] have shown that tuning the output circuit gives near optimum efficiency for the nominally driven abrupt-junction doubler.

In that case

$$X_{\alpha 1} = \frac{S_o}{\omega_o} \left\{ 1 + \frac{\omega}{\omega_o} \right\} \quad (4.9)$$

$$X_{\beta 1} = \frac{S_o}{\omega_o} \left\{ - 1 + \frac{\omega}{\omega_o} \right\} \quad (4.10)$$

$$X_{\alpha 2} = \frac{S_o}{2\omega_o} \left\{ 1 + \frac{\omega}{2\omega_o} \right\} \tag{4.11}$$

$$X_{\beta 2} = \frac{S_o}{2\omega_o} \left\{ - 1 + \frac{\omega}{2\omega_o} \right\}. \tag{4.12}$$

We assume that $\omega/\omega_o << 1$. Let us neglect the terms in ω/ω_o and its higher powers. Let us also assume that

$$R_s << \frac{S_o}{\omega_o} \frac{\omega}{\omega_o} . \tag{4.13}$$

Since the input and output are tuned, the output amplitude noise current I_{a2} and the output phase noise current I_{p2} are given by [4]

$$I_{a2} = \frac{1}{2} (I_{\alpha 2} + I_{\beta 2}) \tag{4.14}$$

$$I_{p2} = \frac{1}{2} j \ (I_{\alpha 2} - I_{\beta 2}). \tag{4.15}$$

Using Eqs. (4.7), (4.8), (4.14), and (4.15), we can show that

$$I_{a2} = \frac{1}{2} \left[-\left(R_1 + R_s + \frac{|S_2|}{\omega_o} \right) (n_{\alpha 2} + n_{\beta 2}) + \frac{|S_1|}{\omega_o} \left\{ (n'_{\alpha 1} + n'_{\beta 1}) - (n_{\alpha 1} + n_{\beta 1}) \right\} \right] \Big/ \left[(R_2 + R_s)\left(R_1 + R_s + \frac{|S_2|}{\omega_o} \right) + \frac{|S_1|^2}{2\omega_o^2} \right] \tag{4.16}$$

$$I_{p2} = \frac{1}{2} j \left[\frac{|S_1|}{\omega_o} \left\{ (n'_{\alpha 1} - n'_{\beta 1}) - (n_{\alpha 1} - n_{\beta 1}) - \left(R_1 + R_s - \frac{|S_2|}{\omega_o} \right) (n_{\alpha 2} - n_{\beta 2}) \right\} + 2 \frac{|S_1|}{\omega_o} \frac{\omega_o}{\omega} \left[\frac{|S_1|}{\omega_o} + \frac{|S_2|}{|S_1|} \left(R_1 + R_s - \frac{|S_2|}{\omega_o} \right) \right] I_o \right] \Bigg/ \left[(R_2 + R_s) \left(R_1 + R_s - \frac{|S_2|}{\omega_o} \right) + \frac{|S_1|^2}{2\omega_o^2} \right] \tag{4.17}$$

where

$$I_o = \left[n'_0 - n_o - \frac{|S_1|}{D\omega_o} \left\{ R_2 + R_s + \frac{|S_2|}{2\omega_o} \right\} \left\{ (n'_{\alpha 1} + n'_{\beta 1}) - (n_{\alpha 1} + n_{\beta 1}) \right\} \left[\frac{|S_1|^2}{2\omega_o^2} - \frac{|S_2|}{2\omega_o} \left(R_1 + R_s + \frac{|S_2|}{\omega_o} \right) \right] + (n_{\alpha 2} + n_{\beta 2})/D \right] \Bigg/ \left[R_o + R_s - j \frac{S_o}{\omega} \right] \tag{4.18}$$

$$D = (R_2 + R_s) \left(R_1 + R_s + \frac{|S_2|}{\omega_o} \right) + \frac{|S_1|^2}{2\omega_o^2} .$$

Equations (4.16-4.19) explicitly express the output noise current of the doubler in terms of the input noise voltage and other sources of noise that may be present at several locations in doubler circuit.

4.3. FIGURES OF MERIT FOR THE DOUBLER

It was shown in Chapter 3 that there are various ways of characterizing the noise performance of pumped nonlinear systems.

For the harmonic generators, the figures of merit defined in terms of the variances seems to have the greatest physical significance. The reason is that the smaller is the variance of output amplitude or frequency the better is the frequency multiplier. Therefore, a good frequency multiplier is characterized by small values of F''_A and F''_P.

Let us assume that the statistics of the noise sources associated with the input termination of the doubler are given by

$$\overline{\left|n'_{\alpha 1} + n'_{\beta 1}\right|^2} = 4\, R_1 k T_a \triangle f \tag{4.20}$$

$$\overline{\left|n'_{\alpha 1} - n'_{\beta 1}\right|^2} = 4\, R_1 k T_p \triangle f \tag{4.21}$$

$$\overline{(n'_{\alpha 1} - n'_{\beta 1})(n'_{\alpha 1} + n'_{\beta 1})^*} = \rho \sqrt{\overline{\left|n'_{\alpha 1} - n'_{\beta 1}\right|^2}\ \overline{\left|n'_{\alpha 1} + n'_{\beta 1}\right|^2}}\,. \tag{4.22}$$

Using the results of Section 4.2, we can show that

$$F_A - 1 = \frac{3 + 4m_2 \dfrac{\omega_c}{\omega_o} + 4\dfrac{T_d}{T_a}\left(1 + m_2 \dfrac{\omega_c}{\omega_o}\right)\left\{1 + 4\dfrac{m_2^2}{m_1^2}\left(1 + \dfrac{1}{m_2}\dfrac{\omega_o}{\omega_c}\right)^2\right\}}{\left(1 + 2m_2 \dfrac{\omega_c}{\omega_o}\right)} \tag{4.23}$$

$$F_P - 1 = \frac{1}{1 + m_2 \dfrac{\omega_c}{\omega_o}}\left[\frac{T_d}{T_p}\left\{1 + \frac{4}{m_1^2}\left(\frac{\omega_o}{\omega_c}\right)^2\right\} + \frac{16}{4\left(1 + \dfrac{R_o}{R_s}\right)^2\left(\dfrac{\omega}{\omega_c}\right)^2 + 1}\right.$$

$$\frac{1}{m_1^2}\left(m_1^2 + 2m_2\frac{\omega_o}{\omega_c}\right)^2 \left\{ \frac{\overline{n_o'^2}}{4R_s kT_p \triangle f} + \frac{T_d}{T_p} \right.$$

$$\left[1 + \frac{\left(\frac{m_1^2 + 2m_2^2}{2m_1}\right)^2 \left(\frac{\omega_c}{\omega_o}\right)^2 + 4\,\frac{m_2^4}{m_1^4}\left(1 + \frac{2m_2^2 - m_1^2}{2m_2}\,\frac{\omega_c}{\omega_o}\right)^2}{\left(1 + 2m_2\frac{\omega_c}{\omega_o}\right)^2} \right]$$

$$+ \frac{T_a}{T_p}\left[\frac{\left(1 + m_2\frac{\omega_c}{\omega_o}\right)\left(\frac{m_1^2 + 2m_2^2}{2m_1}\right)\left(\frac{\omega_c}{\omega_o}\right)^2}{\left(1 + 2m_2\frac{\omega_c}{\omega_o}\right)^2} - \frac{\left(\frac{m_1^2 + 2m_2^2}{2}\right)\left(1 + m_2\frac{\omega_c}{\omega_o}\right)\frac{\omega_c}{\omega_o}}{\left(2m_2 + m_1^2\frac{\omega_c}{\omega_o}\right)\left(1 + 2m_2\frac{\omega_c}{\omega_o}\right)} \right]^2$$

$$\left. \left[\left(1 + \frac{R_o}{R_s}\right)\left(\frac{\omega}{\omega_o}\right) \mathrm{Re}\,\rho + \frac{1}{2}\left(\frac{\omega_c}{\omega_o}\right) \mathrm{Im}\,\rho\right] \sqrt{\frac{T_a}{T_p}} \right\} \tag{4.24}$$

where m_1, and m_2 are the modulation ratios of the doubler corresponding to the point of optimum efficiency, and ω_c is the cutoff frequency of the varactor [1].

F_A, therefore, is only a function of T_a, T_d and the operating point of the transducer. There is, therefore, no phase-to-amplitude conversion in a doubler, at least to the first degree of approximation. Note also that the bias noise does not add anything to the output amplitude noise.

Now F_p-1 may also be written

$$F_p - 1 = C_1 \frac{T_d}{T_p} + C_2 \frac{\overline{|n_o|^2}}{4R_s k T_p \triangle f} + C_3 \frac{T_a}{T_p} + C_4 \sqrt{\frac{T_a}{T_p}} \tag{4.25}$$

where C_1, C_2, C_3, and C_4 are constants. The source of each term in F_p may now be identified.

The first term, $C_1\ T_d/T_p$, is contributed by the internal noise caused by R_s and the action of the varactor as a doubler. The term $C_2\ \overline{|n_o'|^2}/4R_s kT_d \triangle f$ is due to the noise source present in the bias network. The third term $C_3\ T_a/T_p$, is due to amplitude-to-phase conversion through the bias loop of the network. The fourth term, $C_4\ \sqrt{T_a/T_p}$, is due to the conversion of amplitude-phase crosspower at the input to phase noise power at the output.

It is worth pointing out that as bias impedance goes to infinity C_2-C_4 go to zero; and the constant C_1 reduces to some other constant C_1'.

Sometimes it is desirable to have the low-frequency values of F_A and F_p. By "low frequency" we mean

$$\omega_o/\omega_c << 1. \tag{4.26}$$

These can be shown to be

$$F_A - 1 = \left(12.50 + 19.87 \frac{T_d}{T_a}\right) \frac{\omega_o}{\omega_c} \tag{4.27}$$

and

$$F_p - 4 = \frac{1.967}{4\left(1+\frac{R_o}{R_s}\right)^2\left(\frac{\omega}{\omega_c}\right)^2+1}\frac{T_a}{T_p} + \frac{T_d}{T_p}\left(50 + \frac{50.4}{4\left(1+\frac{R_o}{R_s}\right)^2\left(\frac{\omega}{\omega_c}\right)^2+1}\right)\frac{\omega_o}{\omega_c}$$

$$- \frac{11.20}{4\left(1+\frac{R_o}{R_s}\right)^2\left(\frac{\omega}{\omega_c}\right)^2+1}\frac{\omega_o}{\omega_c}\left[\left(1+\frac{R_o}{R_s}\right)\left(\frac{\omega}{\omega_o}\right)\text{Re}\,\rho + \frac{1}{2}\frac{\omega_c}{\omega_o}\text{Im}\,\rho\right]\sqrt{\frac{T_a}{T_p}}$$

$$+ \frac{34.6}{4\left(1+\frac{R_o}{R_s}\right)^2\left(\frac{\omega}{\omega_c}\right)^2+1}\frac{\omega_o}{\omega_c}\frac{\overline{|n_o'|^2}}{4kT_pR_s\Delta f}\,. \qquad (4.28)$$

The modulation ratios of the doubler can be obtained for the case of optimum efficiency by using a digital computer. We assumed that

$$T_a/T_d = T_p/T_d = 1 \qquad (4.29)$$

$$R_o = \infty \qquad (4.30)$$

$$\rho = 0.$$

With these assumptions, the values of F_A'' and F_P'' have been obtained for the doubler. These are illustrated in Figs. 4.2 and 4.3.

4.4. ANAYLSIS OF HIGHER ORDER ABRUPT-JUNCTION VARACTOR FREQUENCY MULTIPLIERS

The analysis of noise performance of abrupt-junction varactor frequency multipliers is very similar to that given for the doubler. In analyzing the noise performance of these multipliers, in addition to the assumptions made in Secs. 4.2 and 4.3, the following assumptions are also made. All the idler terminations are assumed to be tuned and lossless. The bias source impedance R_o is assumed to be infinity. This very much simplifies the analysis.

4.5. FIGURES OF MERIT FOR THE HIGHER ORDER VARACTOR FREQUENCY MULTIPLIERS

For the 1-2-3 tripler, 1-2-4 quadrupler, 1-2-4-5 quitupler, 1-2-3-6, and 1-2-4-6 sextupler, and 1-2-4-8 octupler, the figures of merit can be expressed in terms of the modulation ratios of the varactor. These modulation ratios for the case of optimum efficiency can be obtained by using a digital computer.[2] These modulation ratios can then be used to evaluate amplitude and frequency figures of merit.

The results obtained for the 1-2-3 tripler, 1-2-4 quadrupler, 1-2-3-6, and 1-2-4-6 sextupler, and 1-2-4-8 octupler are illustrated in Figs. 4.4 and 4.5. The figures of merit

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[2]We are very grateful to Mr. Bliss L. Diamond of the M.I.T. Lincoln Laboratory for making available to us the values of these modulation ratios [5].

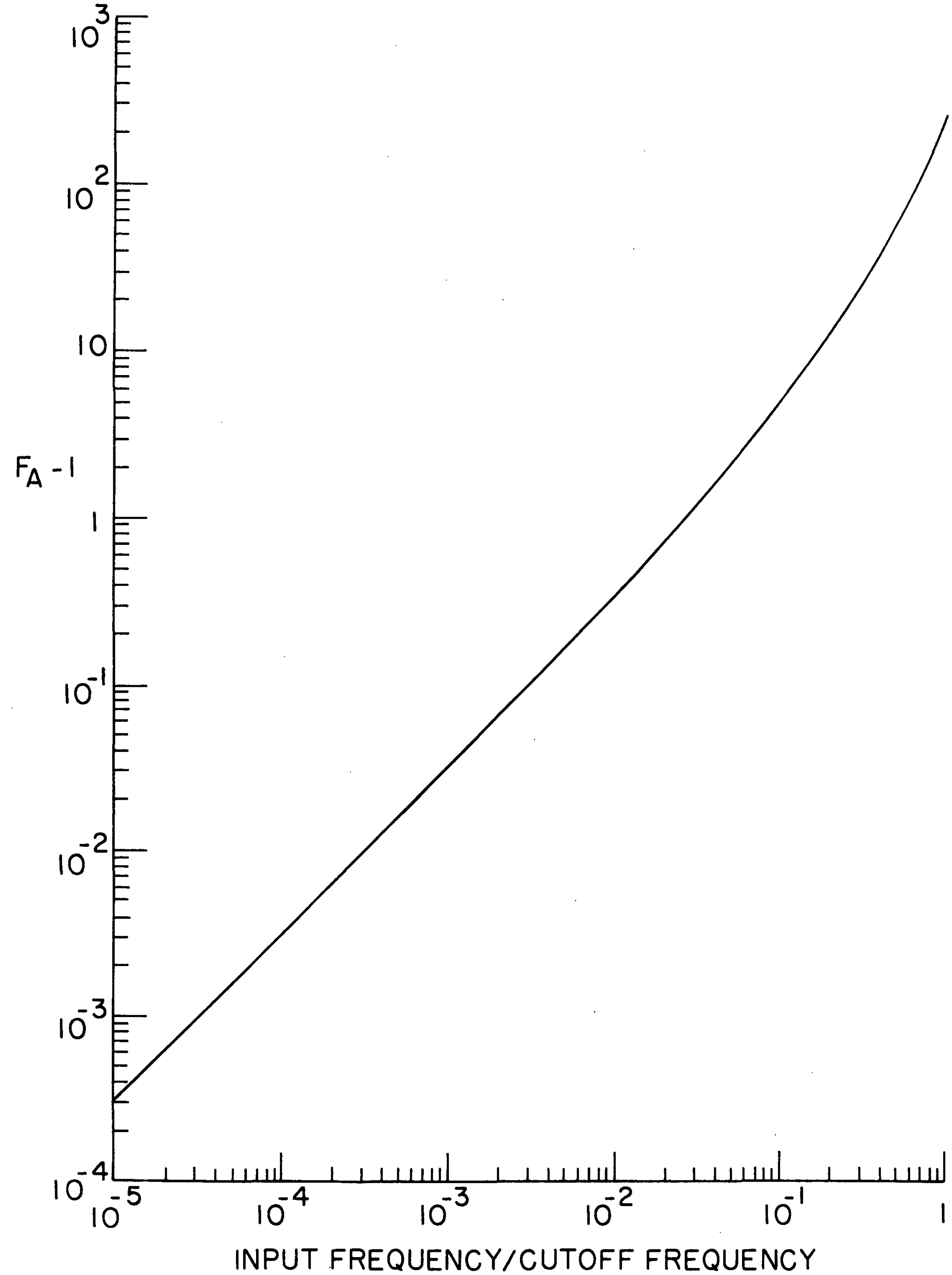

Figure 4.2. Amplitude figure of merit for the doubler.

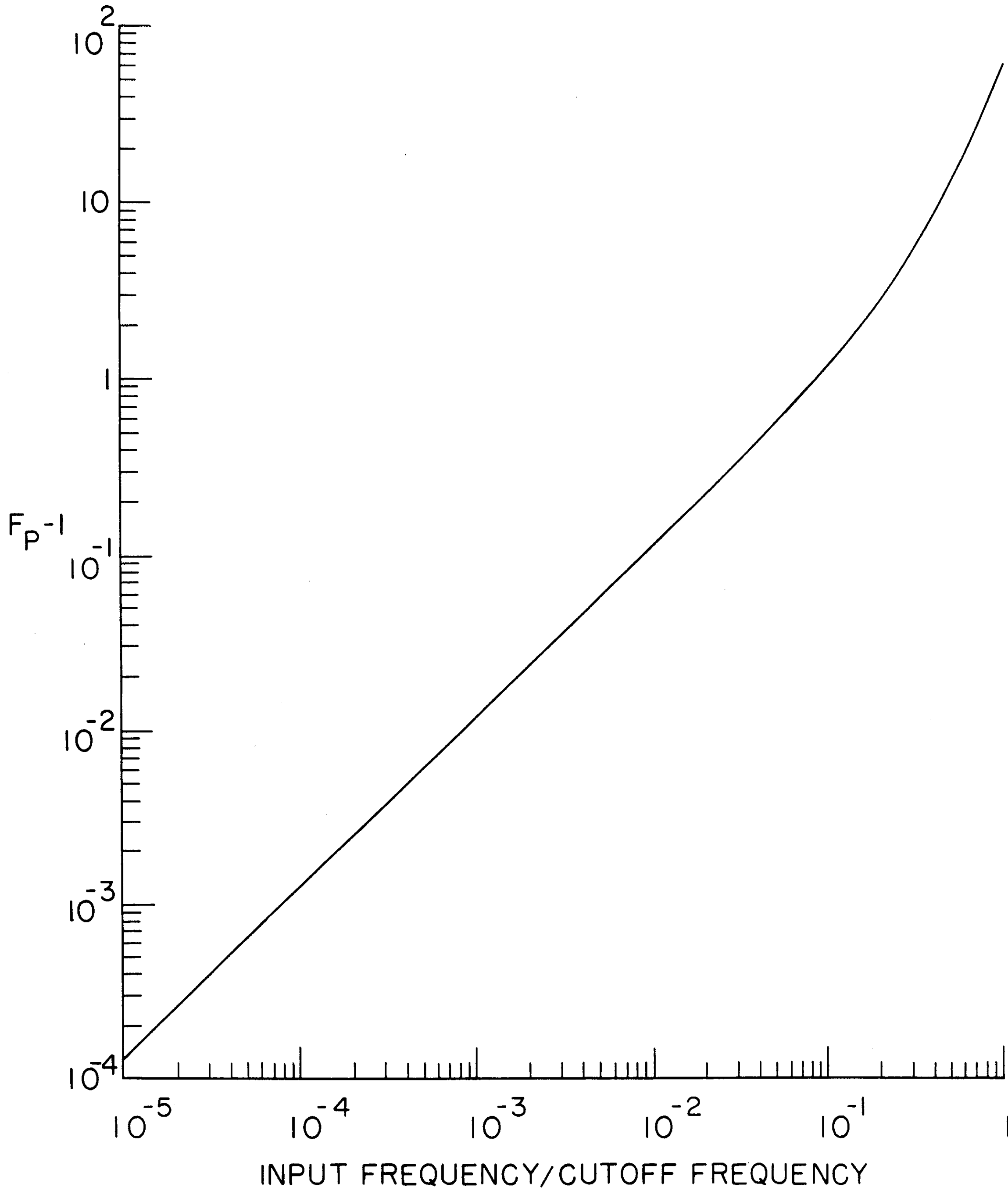

Figure 4.3. Frequencv figure of merit for the doubler.

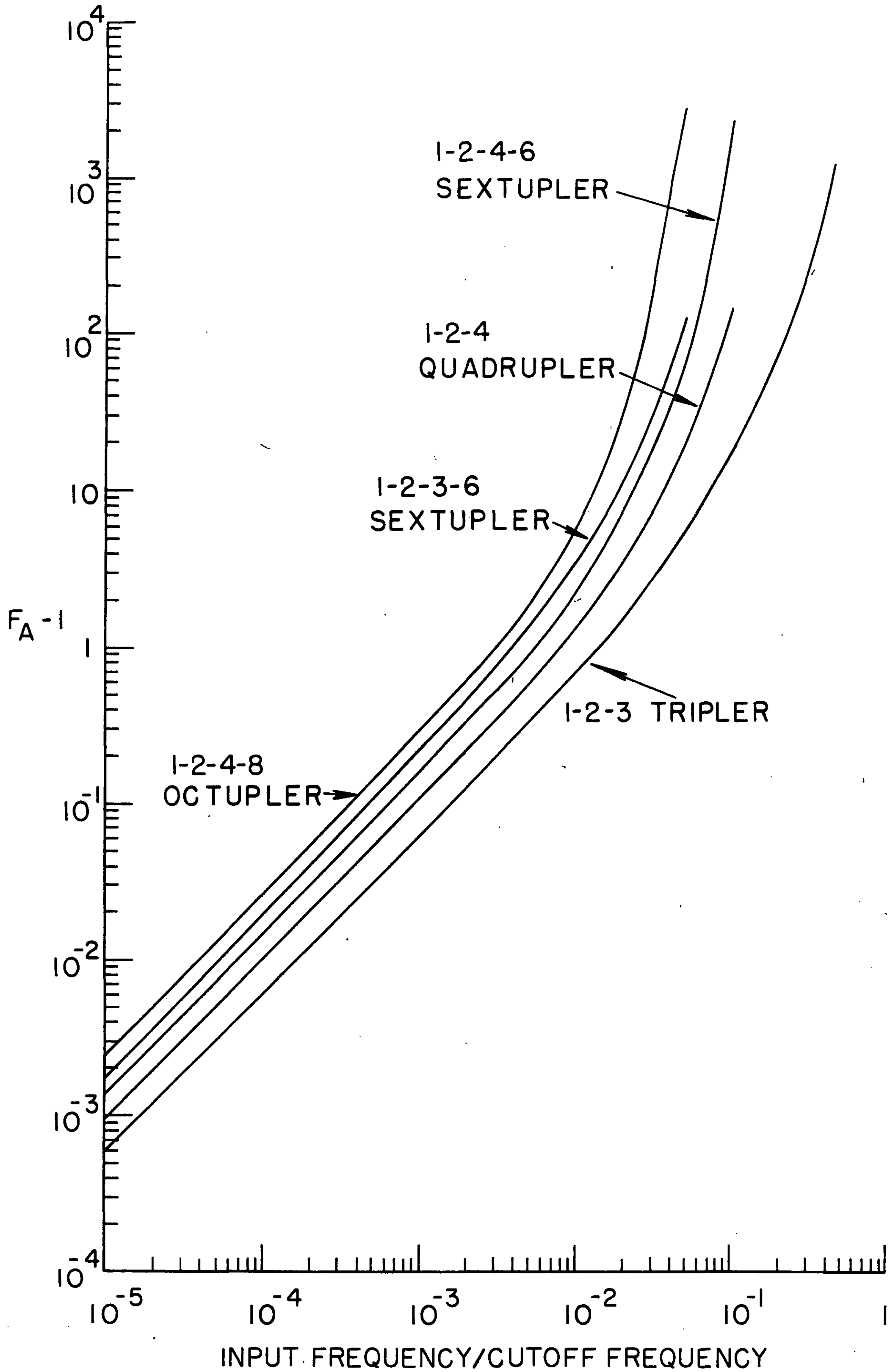

Figure 4.4. Amplitude figure of merit for the multipliers.

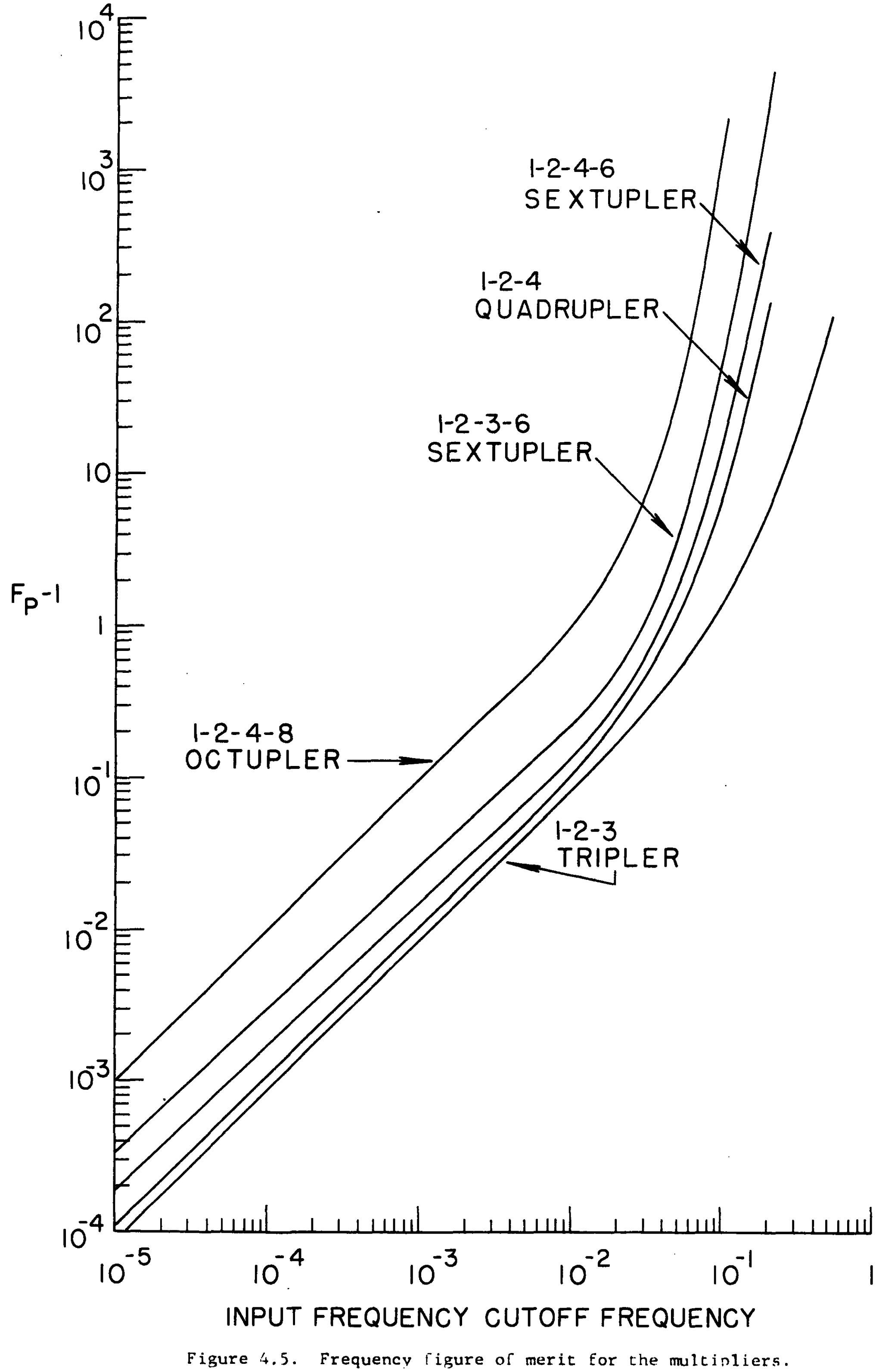

Figure 4.5. Frequency figure of merit for the multipliers.

for the 1-2-4-5 quintupler are shown in Figs. 4.6 and 4.7.

By looking at Figs. 4.2-4.7, the following conclusions may be drawn about the noise performance of the doubler, and higher order multipliers, we have considered in this chapter.

1. The minimum values of amplitude and frequency figures of merit are unity for a varactor frequency multiplier.

2. Finite bias source impedance leads to amplitude-to-phase but not phase-to-amplitude conversion, at least to the first degree of approximation.

3. Finite bias source impedance also leads to conversion of amplitude-phase cross noise power at the input into phase noise power (but not amplitude noise power) at the output, to the first degree of approximation.

4. In case we retain higher order terms in ω/ω_o, there is amplitude-to-phase and phase-to-amplitude conversion, as well as the conversion of amplitude-phase cross noise power at the input into amplitude and phase noise power at the output.

4. The higher the value of ω_o/ω_c, the poorer is the noise performance of the multiplier. For values of $\omega_o/\omega_c < 10^{-3}$, the values of F_A and F_P are very near to unity in case of lossless idler terminations.

We may note here that it is possible to find the point of optimum noise performance for these multipliers by using a digital computer. This has not been done by us in this chapter.

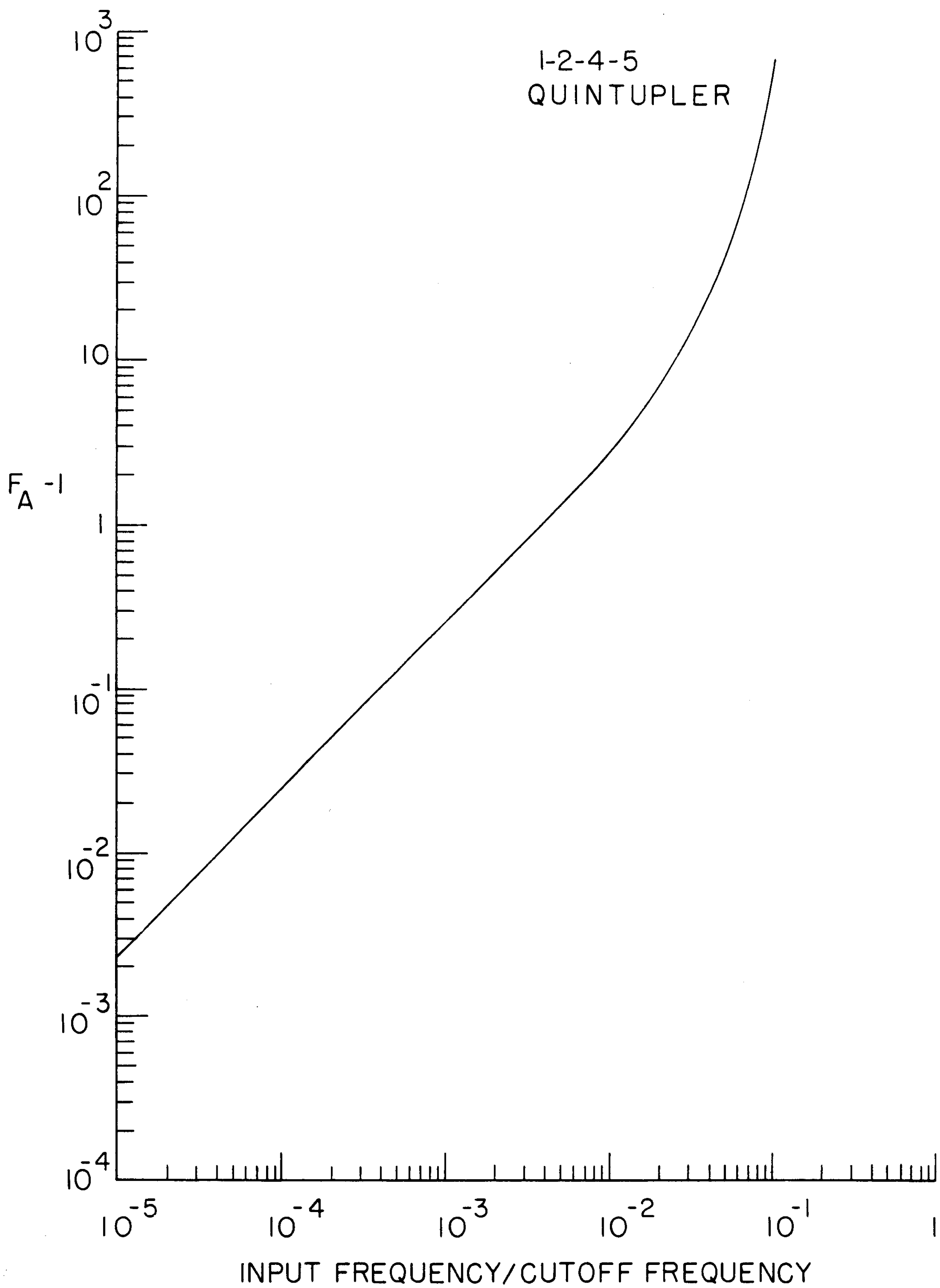

Figure 4.6. Amplitude figure of merit for the quintupler.

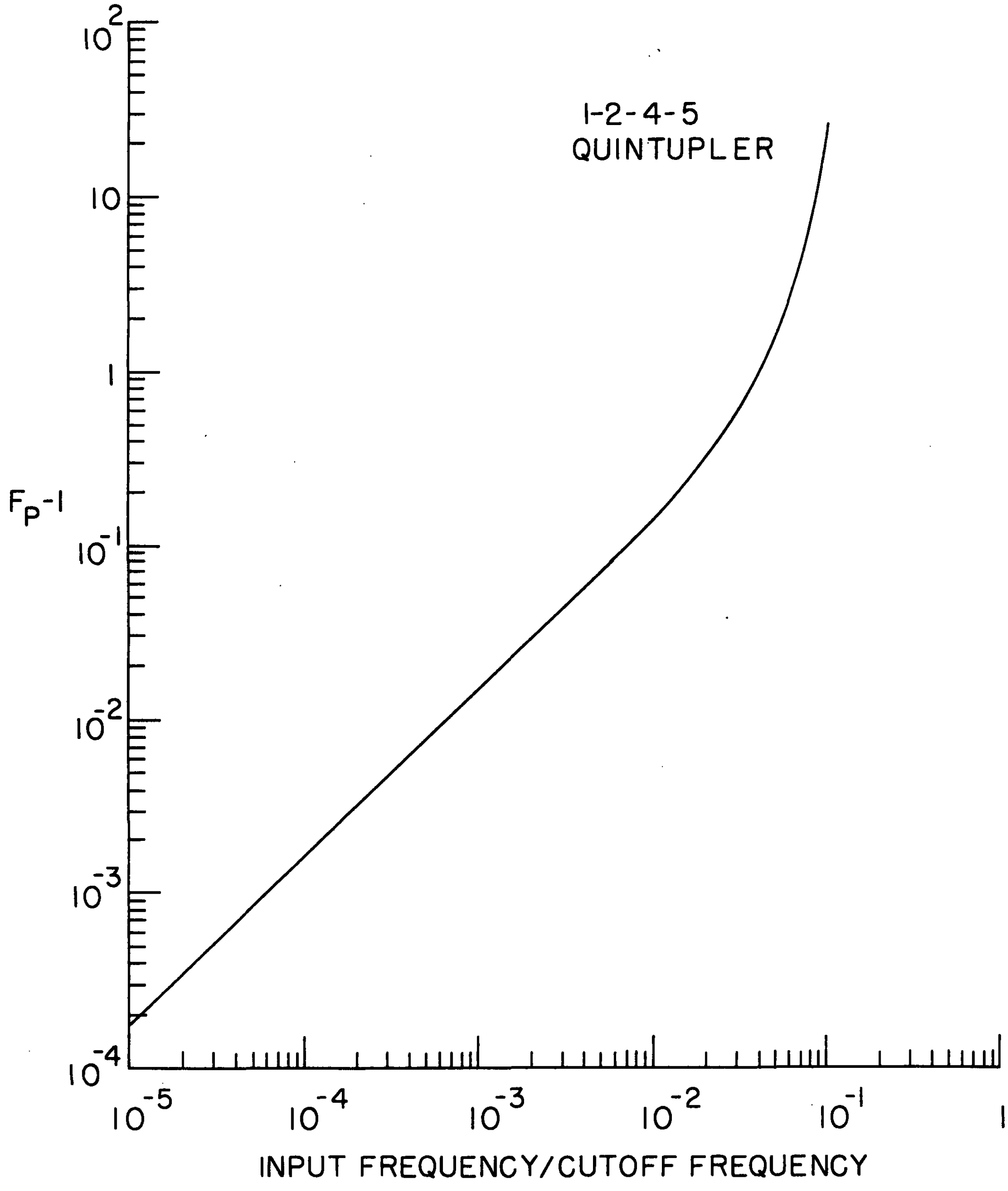

Figure 4.7. Frequency figure of merit for the quintupler.

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CHAPTER 5

NOISE PERFORMANCE OF VARACTOR FREQUENCY DIVIDERS AND PARAMETRIC AMPLIFIERS

In this chapter we start with the analysis of noise performance of a divide-by-two circuit using a varactor diode. The methods are very similar to those outlined in Chapter 4.

The techniques developed in Chapter 2 and Chapter 4 are then used to evaluate the noise performance of parametric amplifiers which are driven by noisy pumps. It has been shown for such amplifiers that only the amplitude noise present in the pump affects the amplifier noise performance, the phase noise does not.

5.1. NOISE PERFORMANCE OF DIVIDE-BY-TWO CIRCUIT

The noise analysis of a divide-by-two circuit using a varactor diode can be done in a manner similar to that outlined in Section 4.2. It was assumed that the bias source impedance R_o is infinite while calculating the values of F''_A and F''_P for this circuit. Let m_1 and m_2 be the modulation ratios of the divide-by-two circuit for the case of optimum efficiency. We can show that F''_A and F''_P are given by

$$F''_A = \left[1 + \frac{4m_2}{m_1^2}\frac{\omega_o}{\omega_c}\right]^2 \left[1 + \frac{T_d}{T_a}\frac{\left\{1 + \frac{4}{m_1^2}\left(\frac{\omega_o}{\omega_c}\right)^2\left(2 + \frac{m_1^2}{4m_2}\frac{\omega_o}{\omega_c}\right)^2\right\}}{1 + \frac{m_1^2}{4m_2}\frac{\omega_c}{\omega_o}}\right] \tag{5.1}$$

$$F''_P = 4m_2\left(\frac{\omega_c}{\omega_o} + \frac{4m_2}{m_1^2}\right)$$

$$\left[\frac{\frac{m_1^2}{4}\left(\frac{\omega_c}{\omega_o}\right)^2\left[\frac{T_d}{Tp} + 1 + \frac{m_1^2}{4m_2}\frac{\omega_c}{\omega_o}\right] + \frac{T_d}{T_p}\left[2 + \frac{m_1^2}{4m_2}\frac{\omega_c}{\omega_o}\right]^2}{\left\{2m_2\frac{\omega_c}{\omega_o}\left[2 + \frac{m_1^2}{4m_2}\frac{\omega_c}{\omega_o}\right] + \frac{1}{2}m_1^2\left(\frac{\omega_c}{\omega_o}\right)^2\right\}^2}\right] \tag{5.2}$$

These figures of merit are illustrated in Figs. 5.1 and 5.2.

5.2. NOISE PERFORMANCE OF VARACTOR PARAMETRIC AMPLIFIERS

It is usually assumed in the analysis of noise performance of parametric amplifiers that the pump is noiseless. This condition is usually not satisfied in practice. Let us now assume that the pump is noisy; and S_1 is the first elastance coefficient of the varactor at frequency ω_o of the pump. Let us also assume that S_α and S_β are the α and β components [1] of the elastance noise associated with S_1. Let V_s, I_s, V_i, and I_i be the signal voltages and currents at the signal

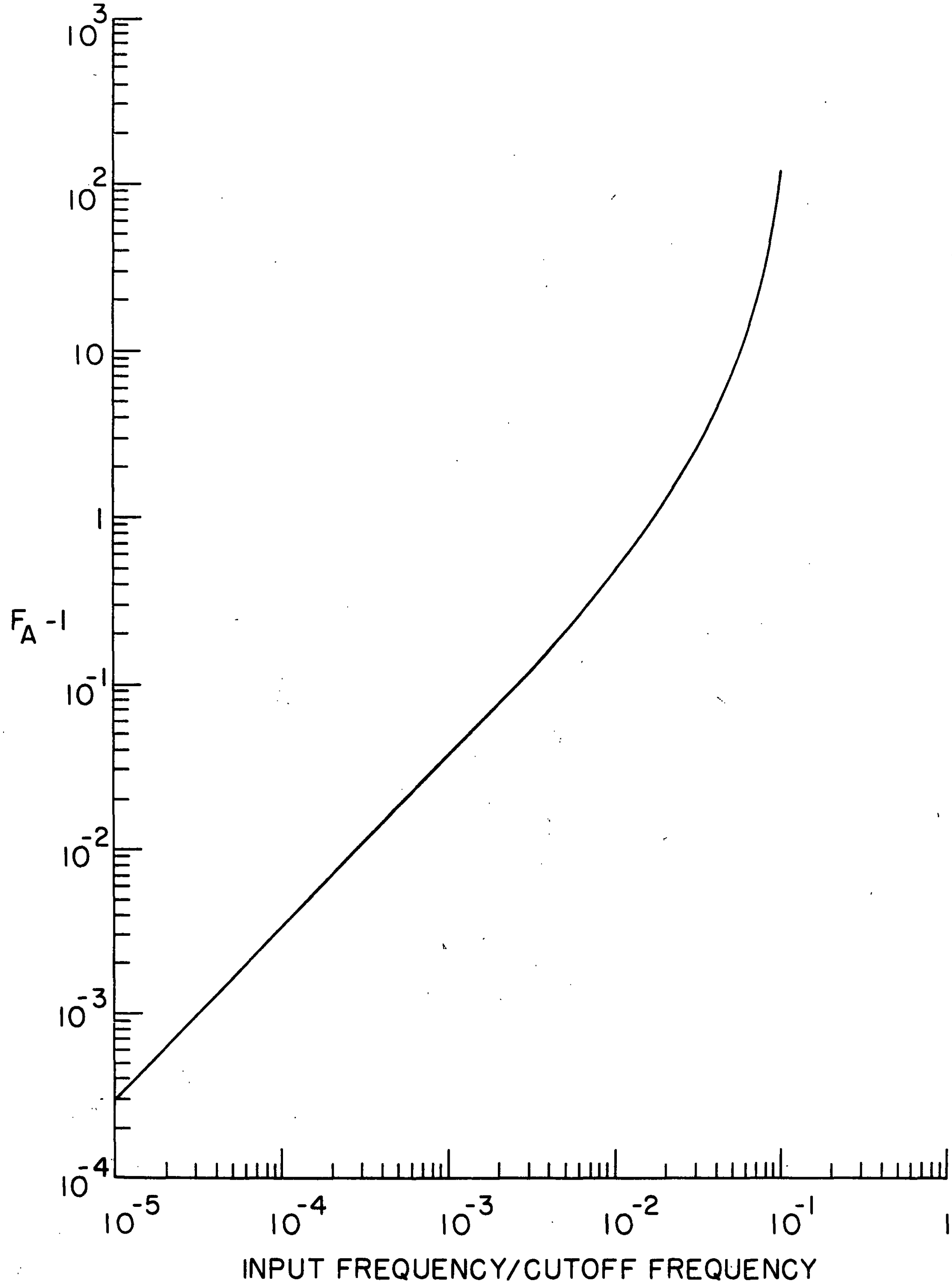

Figure 5.1. Amplitude figure of merit for divide-by-two circuit.

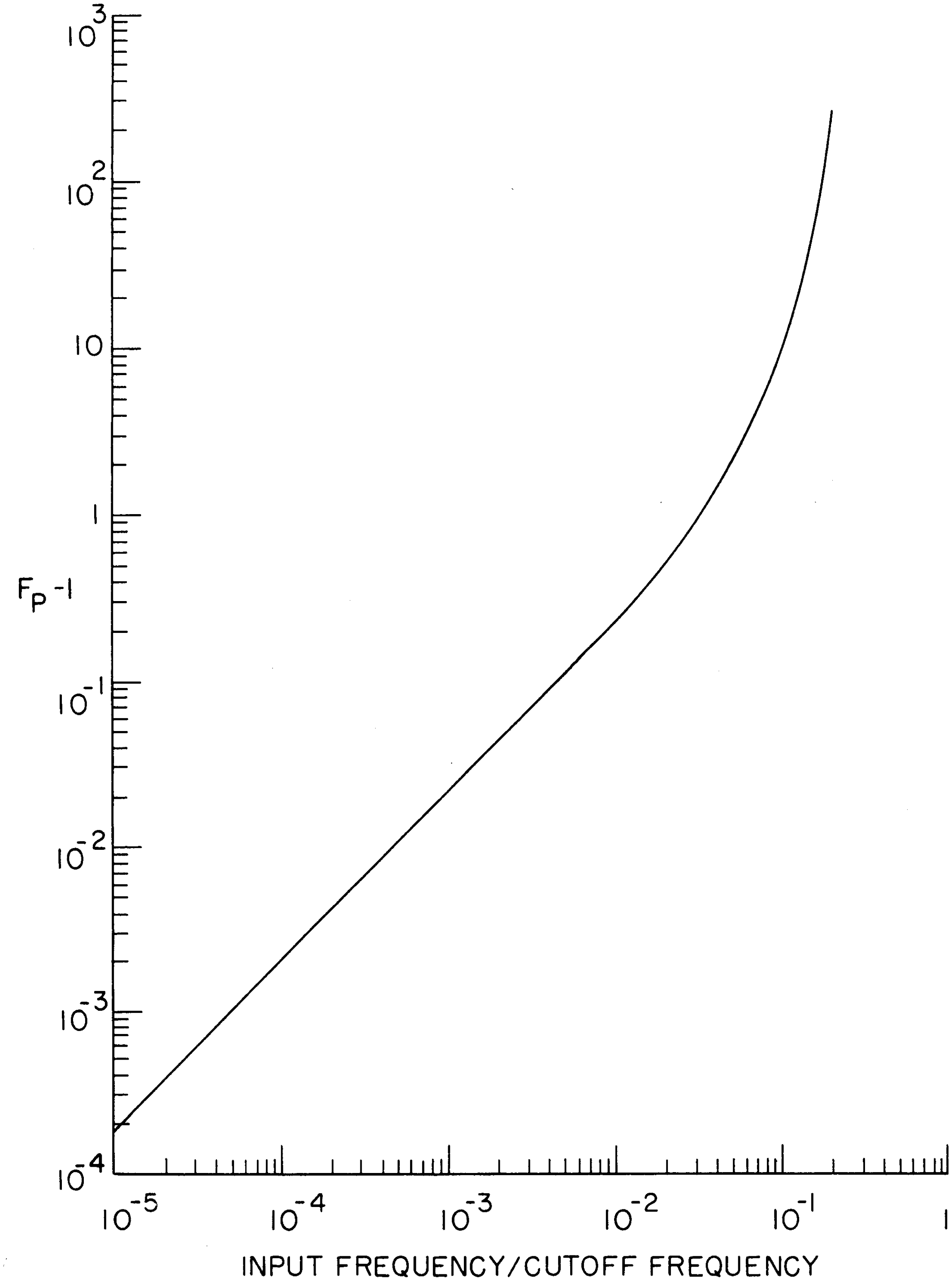

Figure 5.2. Frequency figure of merit for divide-by-two circuit.

frequency ω_s and idler frequency $\omega_o - \omega_s$. Let us also assume that these voltages are large compared to the noise voltages at the corresponding frequencies.

Let $V_{\alpha s}$, $V_{\alpha i}$, $V_{\beta s}$, $V_{\beta i}$ be the α-β voltage components associated with V_s, and V_i. Under our assumptions, it can be shown that

$$\begin{bmatrix} V_{\alpha s} \\ V_{\beta s} \\ V_{\alpha i} \\ V_{\beta i} \end{bmatrix} = \begin{bmatrix} R_s + \frac{S_o}{j(\omega_s+\omega)} & 0 & 0 & \frac{S_1}{j(\omega_o+\omega_s+\omega)} \\ 0 & R_s + \frac{S_o}{j(-\omega_s+\omega)} & \frac{S_1^*}{j(\omega_o-\omega_s+\omega)} & 0 \\ 0 & \frac{S_1}{j(-\omega_s+\omega)} & R_s + \frac{S_o}{j(\omega_o-\omega_s+\omega)} & 0 \\ \frac{S_1^*}{j(\omega_s+\omega)} & 0 & 0 & R_s + \frac{S_o}{j(-\omega_o+\omega_s+\omega)} \end{bmatrix}$$

$$
\left.\begin{array}{c:cccc}
 & 0 & 0 & 0 & \frac{S_\alpha}{j(-\omega_o+\omega)} \\
 & 0 & 0 & \frac{S_\beta}{j(\omega_o-\omega_s)} & 0 \\
 & 0 & \frac{S_\alpha}{j(-\omega_s)} & 0 & 0 \\
 & \frac{S_\beta}{j\omega_s} & 0 & 0 & 0
\end{array}\right]
\begin{bmatrix} I_{\alpha s} \\ I_{\beta s} \\ I_{\alpha i} \\ I_{\beta i} \\ \hdashline I_s \\ I_s^* \\ I_i \\ I_i^* \end{bmatrix}
+
\begin{bmatrix} E_{\alpha s} \\ E_{\beta s} \\ E_{\alpha i} \\ E_{\beta i} \end{bmatrix}
\qquad (5.3)
$$

where $E_{\alpha s}$, $E_{\beta s}$, $E_{\alpha i}$, and $E\beta_i$ are the noise voltage contributions due to the varactor series resistance.

Let us assume that the idler termination is tuned and that this termination is such that

$$\begin{bmatrix} V_{\alpha i} \\ V_{\beta i} \end{bmatrix} = - \begin{bmatrix} R_i & 0 \\ 0 & R_i \end{bmatrix} \begin{bmatrix} I_{\alpha i} \\ E'_{\beta i} \end{bmatrix} + \begin{bmatrix} E'_{\alpha i} \\ E'_{\beta i} \end{bmatrix} . \tag{5.4}$$

$E'_{\alpha i}$ and $E'_{\beta i}$ are the noise voltage sources present in the idler termination.

By using Eqs. (4.67) and (4.68), and assuming that we may choose the time origin so that jS_1 is purely real, we can show that

$$V_{\alpha s} = \left\{ R_s - \frac{|S_1|^2}{\omega_s(\omega_o - \omega_s)(R_i + R_s)} \right\} I_{\alpha s} + \frac{|S_1|}{\omega_s(\omega_o - \omega_s)(R_i + R_s)}$$

$$(jS_\beta - jS_\alpha) I_s + E_{\alpha s} - \frac{|S_1|}{\omega_o - \omega_s} \frac{E_{\beta i} - E'_{\beta i}}{R_i + R_s} \tag{5.5}$$

and

$$V_{\beta s} = \left\{ R_s - \frac{|S_1|^2}{\omega_s(\omega_o - \omega_s)(R_i + R_s)} \right\} I_{\beta s} - \frac{|S_1|}{\omega_s(\omega_o - \omega_s)(R_i + R_s)} (jS_\alpha - jS_\beta)$$

$$I_s^* + E_{\beta s} - \frac{|S_1|}{\omega_o - \omega_s} \frac{E_{\alpha i} - E'_{\alpha i}}{R_i + R_s}. \tag{5.6}$$

It is easy to see that $(jS_\beta - jS_\alpha)$ is the amplitude noise associated with the elastance coefficient S_1.

Equations (5.5) and (5.6), therefore, show that only the amplitude noise associated with the pump affects the noise performance of the varactor parametric amplifier. The phase noise does not. This is an important observation we can make from the analysis of parametric amplifiers given in this section.

REFERENCES

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[2]. P. Penfield, Jr., and R. P. Rafuse, Varactor Applications, The M.I.T. Press, Cambridge, Mass.; 1962.

CHAPTER 6

MULTIFREQUENCY NETWORKS COUPLED WITH LOSSLESS PARAMETRIC DEVICES

Signal and noise voltages may be present at more than one frequency in a system. Systems of this form are noisy oscillators, frequency multipliers, limiters, discriminators, and systems consisting of such devices. In this chapter we shall consider different results that can be achieved by imbedding such a system in a device that obeys the Manley-Rowe relations.

We shall assume that the frequency of the signal present at each of these terminal pairs may be different, but all of these frequencies will be expressible in the form $m\omega_\alpha + n\omega_\beta$ where m and n are integers. It is our purpose in this chapter to develop a noise theory for such multifrequency noisy networks when the latter are coupled with a lossless device that satisfies the Manley-Rowe relations.

We shall begin this chapter by a discussion of the Manley-Rowe relations and the constraints that are thereby imposed upon the coupling network we are considering. We shall then show that if we choose current and voltage variables in a particular way the constraints imposed on these new variables are the same as those imposed upon the currents and voltages present in a linear lossless network. This of course does not mean that the parametric coupling network is lossless. In general we extract more power from this network than we supply to it; the difference in powers is supplied by a pump.

In the remainder of this chapter we extend many of the results obtained for linear noisy networks when they are imbedded in linear lossless networks to multifrequency noisy networks when the latter are imbedded in M-R[1] devices. As in the theory of linear noisy networks, we have found some invariants associated with a multifrequency noisy network when it is imbedded in an M-R device. These invariants have been shown to have the dimensions of energy.

In the last part of this chapter it has been shown how to define the general characteristic-noise matrix when the multifrequency noisy network has been represented in other than impedance formalism.

6.1. MANLEY-ROWE FORMULAS AND CONSTRAINTS

For a linear lossless network the conservation of power requires that the net power delivered to the network at each frequency of interest must be zero. On the other hand, for a nonlinear lossless system power that is supplied to the network at one frequency can be extracted at another frequency. It has been shown [1] that if a nonlinear lossless capacitor is so excited that its current and voltage have components at a

---

[1] A device that obeys the Manley-Rowe relations will be referred to for brevity as an M-R device.

number of frequencies of the form $m\omega_\alpha + n\omega_\beta$, where m and n are integers, then

$$\sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{nP_{mn}}{m\omega_\alpha + n\omega_\beta} = 0 \qquad (6.1)$$

and

$$\sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{mP_{mn}}{m\omega_\alpha + n\omega_\beta} = 0 \qquad (6.2)$$

where P_{mn} is the power input at frequency $m\omega_\alpha + n\omega_\beta$. Equations (6.1) and (6.2) are the Manley-Rowe formulas. They were originally proved for a nonlinear capacitor; but it has been shown [2] that they are also applicable to many types of nonlinear lossless systems.

It is customary to operate such a device by "pumping" it at one of the given frequency or frequencies. We drive it very hard and well into its nonlinear range with a source, known as the pump. In this chapter we will assume that the large pump voltages and currents present at various points within the system are, by design, periodic with some frequency ω_o. Thus the pump voltage or current at a specific point within the network or across one of its terminal pairs is of the form

$$v(t) = \sum_{k=-\infty}^{\infty} V_k e^{jk\omega_o t} \qquad (6.3)$$

and

$$i(t) = \sum_{k=-\infty}^{\infty} I_k e^{jk\omega_o t} . \qquad (6.4)$$

The voltages and currents at other frequencies, known as the sideband frequencies, are assumed to be much smaller than the corresponding pump frequency. In this case the device behaves at the sideband frequencies, as a time-variant linear element instead of a nonlinear element. Consequently, if we represent[2] the device at the sideband frequencies as shown in Fig. 6.1, with power at only one frequency flowing at each terminal pair, we may write

$$\underset{\sim}{V} = \underline{Z_p}\,\underset{\sim}{I} \tag{6.5}$$

where

$$\underset{\sim}{V} = \begin{bmatrix} V_1 \\ \circ \\ \circ \\ \circ \\ V_i \\ \circ \\ \circ \\ \circ \\ V_m \end{bmatrix} \tag{6.6}$$

and

$$\underset{\sim}{I} = \begin{bmatrix} I_1 \\ \circ \\ \circ \\ \circ \\ I_i \\ \circ \\ \circ \\ \circ \\ I_m \end{bmatrix} \tag{6.7}$$

[2]Pump frequency terminals are not shown.

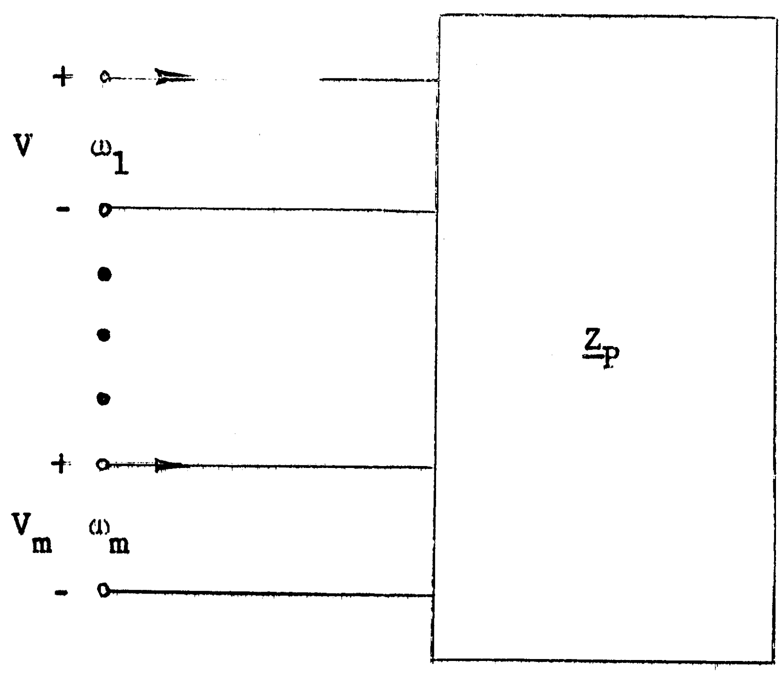

Figure 6.1. M-R device.

The voltage V_i and current I_i are at one of the sideband frequencies $m_i\omega_\alpha + n_i\omega_\beta$.

Let us now assume that we pump the device which obeys the Manley-Rowe relations at frequency ω_o and its harmonics. Usually by design we allow only power to enter and leave the device at a finite number of frequencies. For linearity we may limit ourselves to sidebands with $n_i = 1$.

In this case, we can obtain a relation among the sideband powers that is written

$$\sum_{\substack{i \text{ sideband} \\ \text{frequencies}}} \frac{P_i}{\nu_i} = 0 \qquad (6.8)$$

where P_i is the power into the device at the terminal pair at frequency ω_i, and ν_i[3] is given by

$$\nu_i = \pm k\omega_o + \omega, \quad k \text{ an integer, positive or negative.} \qquad (6.9)$$

The power into the i^{th} port is[4]

$$P_i = [V_i^* I_i + I_i^* V_i]. \qquad (6.10)$$

Let us now define a diagonal matrix $\underline{K}$ whose i^{th} element along the diagonal is ν_i. We may then write Eq. (6.8) as

$$\underset{\sim}{V}^+ \underline{K}^{-1} \underset{\sim}{I} + \underset{\sim}{I}^+ \underline{K}^{-1} \underset{\sim}{V} = 0. \qquad (6.11)$$

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[3]Note here we put $\omega_\beta = \omega$.

[4]Note that we have used half-amplitudes for voltage and current variables.

Let us also define a matrix[5] $\underline{K}^{1/2}$ whose i^{th} element along its diagonal is the positive square root of the corresponding element of the matrix $\underline{K}$ if ν_i is positive. If ν_i is negative, the i^{th} element of $\underline{K}^{1/2}$ will be $j\sqrt{|\nu_i|}$, where j is the square root of -1. We may now define new "current" and "voltage" variables:

$$\underset{\sim}{I}' = \left[\underline{K}^{-1/2}\right] \underset{\sim}{I} \qquad (6.12)$$

$$\underset{\sim}{V}' = \left[\underline{K}^{-1/2}\right]^{+} \underset{\sim}{V}. \qquad (6.13)$$

Let us now write Eq. (6.11) as[6]

$$\underset{\sim}{V}^{+} \left[\underline{K}^{-1/2}\right] \left[\underline{K}^{-1/2}\right] \underset{\sim}{I} + \underset{\sim}{I}^{+} \left[\underline{K}^{-1/2}\right]^{+} \left[\underline{K}^{-1/2}\right]^{+} \underset{\sim}{V} = 0. \qquad (6.14)$$

We can see that by using Eqs. (6.12) and (6.13), we may write Eq. (6.14) as

$$\underset{\sim}{V}'^{+} \underset{\sim}{I}' + \underset{\sim}{I}'^{+} \underset{\sim}{V}' = 0. \qquad (6.15)$$

Equation (6.15) is the constraint imposed by the Manley-Rowe formulas upon the frequency-normalized current and voltage variables $\underset{\sim}{I}'$ and $\underset{\sim}{V}'$ at the terminals of the equivalent circuit of a lossless parametric device. This relation is identical to the constraint that losslessness imposes upon the currents and voltages at the terminals of a linear lossless network.

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[5]The matrix $\underline{K}^{1/2}$ is nonsingular in all but cases of trivial interest.

[6]It is to be pointed out that $\underline{K} = \underline{K}^{+}$ since ν_i's are real.

This does not imply that the network of Fig. 6.1 is lossless.

By transforming properly, we may write Eq. (6.5) as

$$\underset{\sim}{V}' = \left[\underline{K}^{-1/2}\right]^{+} \underline{Z_P} \left[\underline{K}^{1/2}\right] \underset{\sim}{I}'$$

$$= \underline{Z''_P}\, \underset{\sim}{I}' \tag{6.16}$$

where[7]

$$\underline{Z''_P} = \left[\underline{K}^{-1/2}\right]^{+} \underline{Z_P} \left[\underline{K}^{1/2}\right] . \tag{6.17}$$

In case the device is an M-R device it satisfies Eq. (6.16).

Accordingly,

$$\underset{\sim}{I}'^{+} (\underline{Z''_P} + \underline{Z''^{+}_P}) \underset{\sim}{I}' = 0 \tag{6.18}$$

for any arbitrary $\underset{\sim}{I}'$.

Equation (6.18) shows that a frequency-normalized impedance matrix $\underline{Z''_P}$ of an M-R device satisfies the condition

$$\underline{Z''_P} + \underline{Z''^{+}_P} = 0. \tag{6.19}$$

This is analogous to the equation

$$\underline{Z} + \underline{Z}^{+} = 0 \tag{6.20}$$

satisfied by the impedance matrix of a linear lossless network.

6.2. FORMULATION OF THE OPTIMIZATION PROBLEM

In this section we develop the idea of exchangeable frequency-normalized power for a multifrequency noisy network. A characteristic-noise matrix is also derived for such a

[7]We shall indicate a frequency-normalized impedance matrix $\underline{Z}$ by $\underline{Z}''$.

network. The eigenvalues of this matrix may be interpreted as the stationary values of the exchangeable frequency-normalized power.

Exchangeable Frequency-Normalized Power. The terminal voltages of a multifrequency n-port noisy network are related to the currents through the impedance matrix $\underline{Z}$ (see Fig. 6.2). Accordingly,

$$\underset{\sim}{V} = \underline{Z}\, \underset{\sim}{I} + \underset{\sim}{E}. \tag{6.21}$$

Let ν_i be the frequency at the i^{th} port; and let $\underline{K}$ be the matrix as defined in Section 6.1.

We may now express Eq. (6.21) in terms of $\underset{\sim}{V}'$ and $\underset{\sim}{I}'$ by premultiplying it by $\left[\underline{K}^{-1/2}\right]^+$. This gives

$$\left[\underline{K}^{-1/2}\right]^+ \underset{\sim}{V} = \left[\underline{K}^{-1/2}\right]^+ \underline{Z} \left[\underline{K}^{1/2}\right] \left[\underline{K}^{-1/2}\right] \underset{\sim}{I} + \left[\underline{K}^{-1/2}\right]^+ \underset{\sim}{E}. \tag{6.22}$$

Using Eqs. (6.12) and (6.13), we obtain the relation

$$\underset{\sim}{V}' = \underline{Z}''\, \underset{\sim}{I}' + \underset{\sim}{E}' \tag{6.23}$$

where

$$\underline{Z}'' = \left[\underline{K}^{-1/2}\right]^+ \underline{Z} \left[\underline{K}^{1/2}\right] . \tag{6.24}$$

Equation (6.23) is the equation relating the frequency-normalized source variables.

For a one-port linear network we define exchangeable power as the stationary value of the power output obtained by arbitrary variation of the terminal current or voltage. In frequency-normalized variables V' and I' the quantity analogous

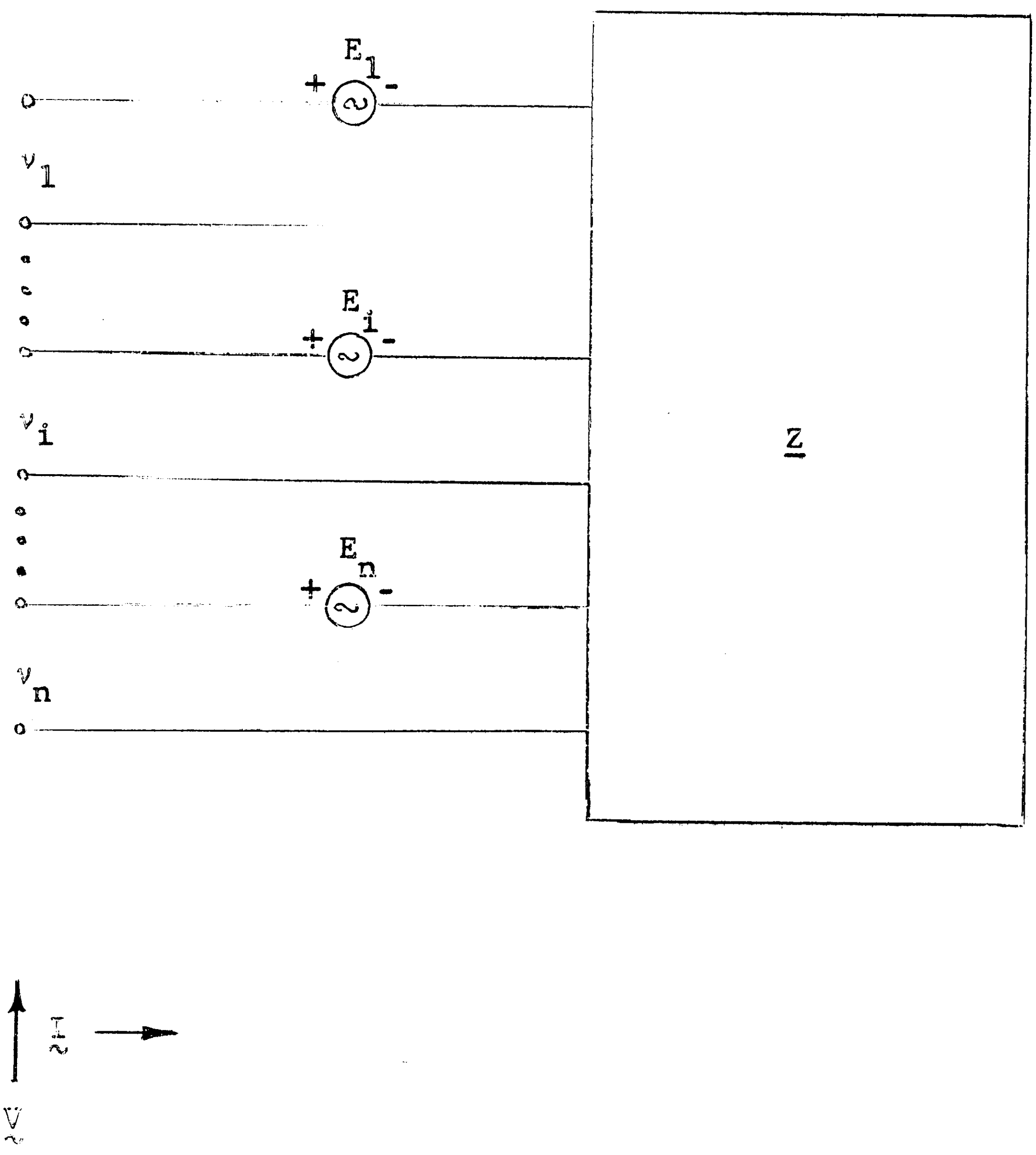

Figure 6.2. Equivalent representation of multifrequency noisy network.

to power is 2 Re $[V'I'^*]$ which is power divided by the frequency ν. This quantity will be called the frequency-normalized power. If V' and I' are the frequency-normalized terminal voltage and current of a one-port network (see Fig. 6.3) whose terminal behavior is given by

$$V = ZI + E \tag{6.25}$$

or

$$V' = Z''I' + E', \tag{6.26}$$

it may be shown very easily that the exchangeable frequency-normalized power output of the network is given by

$$P_e = \frac{\overline{E'E'^*}}{Z'' + Z''^*} \tag{6.27}$$

This exchangeable frequency-normalized power is defined as the stationary value of the frequency-normalized power output of the network obtained by arbitrary variation of the terminal current or voltage.

<u>Parametric Transformations</u>. If the n-port network with the noise column matrix $\underset{\sim}{E}$ and impedance matrix $\underline{Z}$ is connected properly to a 2n-port network, a new n-port network may be obtained. It will have a new noise column matrix $\underset{\sim}{E}_o$ and a new impedance matrix $\underline{Z}_o$. This operation, shown in Fig. 6.4, will be called a transformation or an imbedding of the original network. The analytical relation between the voltages and currents applied to the 2n-port M-R network (the "transformation network") of Fig. 6.4 can be written in the form

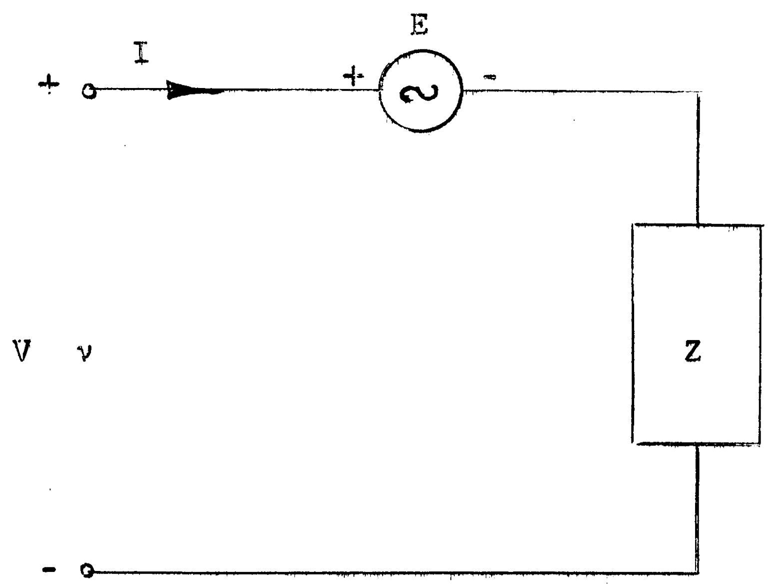

Figure 6.3. One-port noisy network.

$$\underset{\sim}{V}_a = \underline{Z}_{aa}\,\underset{\sim}{I}_a + \underline{Z}_{ab}\,\underset{\sim}{I}_b \tag{6.28}$$

and

$$\underset{\sim}{V}_b = \underline{Z}_{ba}\,\underset{\sim}{I}_a + \underline{Z}_{bb}\,\underset{\sim}{I}_b. \tag{6.29}$$

The column vectors $\underset{\sim}{V}_a$ and $\underset{\sim}{V}_b$ comprise the terminal voltages applied to the transformation network on its two sides, and the column vectors $\underset{\sim}{I}_a$ and $\underset{\sim}{I}_b$ comprise the currents flowing into it. The frequency variables at different parts of the M-R network are shown in Fig. 6.4. The condition that the transformation network is an M-R network can be summarized in the following relations:

$$\left[\underline{K}^{-1/2}\right]^{+} \underline{Z}_T \left[\underline{K}^{1/2}\right] + \left[\underline{K}^{1/2}\right]^{+} \underline{Z}_T^{+} \left[\underline{K}^{-1/2}\right] = 0 \tag{6.30}$$

where

$$\underline{K} = \left[\begin{array}{c|c} \underline{K}_k & 0 \\ \hline 0 & \underline{K}_m \end{array}\right] \tag{6.31}$$

$$\underline{K}_k = \begin{bmatrix} k_1\omega_o+\omega & & & & 0 \\ & \ddots & & & \\ & & k_i\omega_o+\omega & & \\ & & & \ddots & \\ 0 & & & & k_n\omega_o+\omega \end{bmatrix} \tag{6.32}$$

and

$$\underline{K}_m = \begin{bmatrix} m_1\omega_o+\omega & & & & 0 \\ & \ddots & & & \\ & & m_i\omega_o+\omega & & \\ & & & \ddots & \\ 0 & & & & m_n\omega_o+\omega \end{bmatrix}. \tag{6.33}$$

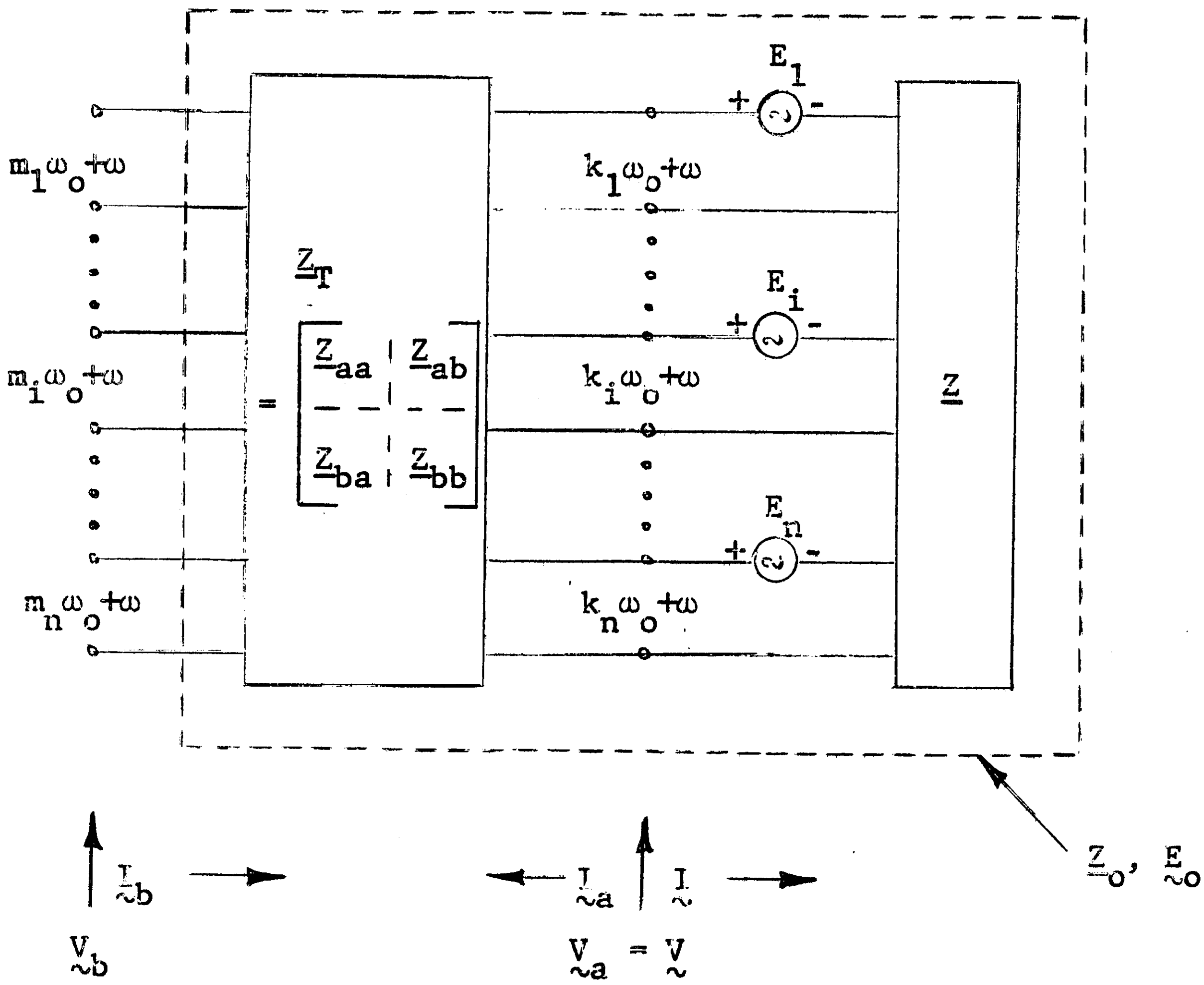

Figure 6.4. General transformation of an n-port multifrequency noisy network.

We may therefore write

$$\left[\underline{K}_k^{-1/2}\right]^{+} \underline{Z}_{aa} \left[\underline{K}_k^{1/2}\right] + \left\{\left[\underline{K}_k^{-1/2}\right]^{+} \underline{Z}_{aa}^{+} \left[\underline{K}_k^{1/2}\right]\right\}^{+} = 0$$

or

$$\underline{Z}''_{aa} + \underline{Z}''^{+}_{aa} = 0 \qquad (6.34)$$

$$\left[\underline{K}_k^{-1/2}\right]^{+} \underline{Z}_{ab} \left[\underline{K}_m^{1/2}\right] + \left\{\left[\underline{K}_m^{-1/2}\right]^{+} \underline{Z}_{ba}^{+} \left[\underline{K}_k^{1/2}\right]\right\}^{+} = 0$$

or

$$\underline{Z}''_{ab} + \underline{Z}''^{+}_{ba} = 0 \qquad (6.35)$$

$$\left[\underline{K}_m^{-1/2}\right]^{+} \underline{Z}_{bb} \left[\underline{K}_m^{1/2}\right] + \left\{\left[\underline{K}_m^{-1/2}\right]^{+} \underline{Z}_{bb}^{+} \left[\underline{K}_m^{1/2}\right]\right\}^{+} = 0$$

or

$$\underline{Z}''_{bb} + \underline{Z}''^{+}_{bb} = 0 \qquad (6.36)$$

and

$$\left[\underline{K}_m^{-1/2}\right]^{+} \underline{Z}_{ba} \left[\underline{K}_k^{1/2}\right] + \left\{\left[\underline{K}_k^{-1/2}\right]^{+} \underline{Z}_{ab}^{+} \left[\underline{K}_m^{1/2}\right]\right\}^{+} = 0$$

or

$$\underline{Z}''_{ba} + \underline{Z}''^{+}_{ab} = 0. \qquad (6.37)$$

The original n-port network, with impedance matrix $\underline{Z}$ and noise column matrix $\underset{\sim}{E}$, imposes the following relation between the column matrices $\underset{\sim}{V}$ and $\underset{\sim}{I}$ of the voltages across, and the currents into, its terminals:

$$\underset{\sim}{V} = \underline{Z}\,\underset{\sim}{I} + \underset{\sim}{E}. \qquad (6.21)$$

The currents $\underset{\sim}{I}$ into the n-port network are, according to Fig. 6.4, equal and opposite to the currents $\underset{\sim}{I}_a$ into one side of the 2n-port network. The voltages $\underset{\sim}{V}$ are equal to the voltages $\underset{\sim}{V}_a$. We thus have

$$\underset{\sim}{V} = V_a; \quad \underset{\sim}{I} = - \underset{\sim}{I}_a. \tag{6.38}$$

Introduction of Eqs. (6.38) into Eq. (6.21) and application of the latter to Eq. (6.28) give

$$\underset{\sim}{I}_a = - (\underline{Z} + \underline{Z}_{aa})^{-1} \underline{Z}_{ab} \underset{\sim}{I}_b + (\underline{Z} + \underline{Z}_{aa})^{-1} \underset{\sim}{E}. \tag{6.39}$$

When this equation is substituted in Eq. (6.29), the final relation between $\underset{\sim}{V}_b$ and $\underset{\sim}{I}_b$ is determined:

$$\underset{\sim}{V}_b = \underline{Z}_o \underset{\sim}{I}_b + \underset{\sim}{E}_o \tag{6.40}$$

where

$$\underline{Z}_o = \underline{Z}_{bb} - \underline{Z}_{ba}(\underline{Z} + \underline{Z}_{aa})^{-1} \underline{Z}_{ab} \tag{6.41}$$

and

$$\underset{\sim}{E}_o = \underline{Z}_{ba}(\underline{Z} + \underline{Z}_{aa})^{-1} \underset{\sim}{E}. \tag{6.42}$$

Equation (6.40) is the matrix relation for the new n-port network obtained from the original one by imbedding it in a 2n-port network. Here $\underline{Z}_o$ is the new impedance matrix, and $\underset{\sim}{E}_o$ is the column matrix of the new open-circuit noise voltages. Conditions 6.34 through 6.37 must be applied to Eqs. (6.41) and (6.42) if the transformation network is to be M-R.

<u>Matrix Formulation of Stationary-Value Problem</u>. We have defined the exchangeable frequency-normalized power for a one-port network as the extremum of frequency-normalized power output obtainable by arbitrary variation of terminal current or voltage. In an obvious generalization, we may extend this

definition to n-port networks by considering the extremum of the frequency-normalized power output of the network obtained by an arbitrary variation of the terminal currents. In this case,[8] we encounter the possibility of the output frequency-normalized power assuming a stationary value rather than an extremum. One may ask whether the stationary value of the frequency-normalized power for the multiport case could be achieved in a simpler way. One method to try is that shown in Fig. 6.5.

The given network is imbedded in a variable (n+1)-port M-R network. For each choice of the variable M-R network, we consider first the frequency-normalized power that can be drawn from the $(n+1)^{th}$ port for various values of the complex current I_{n+1}.

The network operation indicated in Fig. 6.5 is conveniently accomplished by first imbedding the original n-port network in an M-R 2n-port network, as indicated in Fig. 6.4. Open-circuiting all terminal pairs of the resulting n-port network, except the i^{th}, we achieve the n-to-1-port transformation indicated in Fig. 6.5. The exchangeable frequency-normalized power from the i^{th} port of the network can be written in matrix form as

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[8]The case is analogous to that of finding the extremum of power output of an n-port linear noisy network by an arbitrary variation of the terminal currents [3].

$$P'_{e,i} = \frac{\overline{(E'_o)_i (E'_o)_i^*}}{(Z''_o)_i + (Z''_o)_i^*} = \frac{\tilde{\xi}^+ \overline{\tilde{E}'_o \tilde{E}'^+_o} \tilde{\xi}}{\tilde{\xi}^+ (\underline{Z}''_o + \underline{Z}''^+_o) \tilde{\xi}} \tag{6.43}$$

where the (real) column matrix $\tilde{\xi}$ has every element zero except the i^{th}, which is 1:

$$\tilde{\xi} = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_i \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \begin{cases} \xi_j = 0, & j \neq i \\ \xi_i = 1 \end{cases} \tag{6.44}$$

The variation of the M-R network in Fig. 6.5 now corresponds to variation of the transformation network $\underline{Z}_T$ in Fig. 6.4 through all possible forms. We now wish to find the stationary values of $P'_{e,i}$ corresponding to variation of $\underline{Z}_T$.

We can write

$$\begin{aligned} \overline{\tilde{E}'_o \tilde{E}'^+_o} &= \left[\underline{K}_m^{-1/2}\right]^+ \overline{\tilde{E}_o \tilde{E}^+_o} \left[\underline{K}_m^{-1/2}\right] \\ &= \left[\underline{K}_m^{-1/2}\right]^+ \underline{Z}_{ba}(\underline{Z} + \underline{Z}_{aa})^{-1} \overline{\tilde{E}\,\tilde{E}^+} \left[(\underline{Z} + \underline{Z}_{aa})^{-1}\right]^+ \underline{Z}_{ba}^+ \left[\underline{K}_m^{-1/2}\right] \\ &= \underline{Z}''_{ba} \left[\underline{Z}'' + \underline{Z}''_{aa}\right]^{-1} \overline{\tilde{E}'\tilde{E}'^+} \left[(\underline{Z}'' + \underline{Z}''_{aa})^{-1}\right]^+ \underline{Z}''^+_{ba} \\ &= \underline{\tau}^+ \overline{\tilde{E}'\tilde{E}'^+} \underline{\tau} \end{aligned} \tag{6.45}$$

where

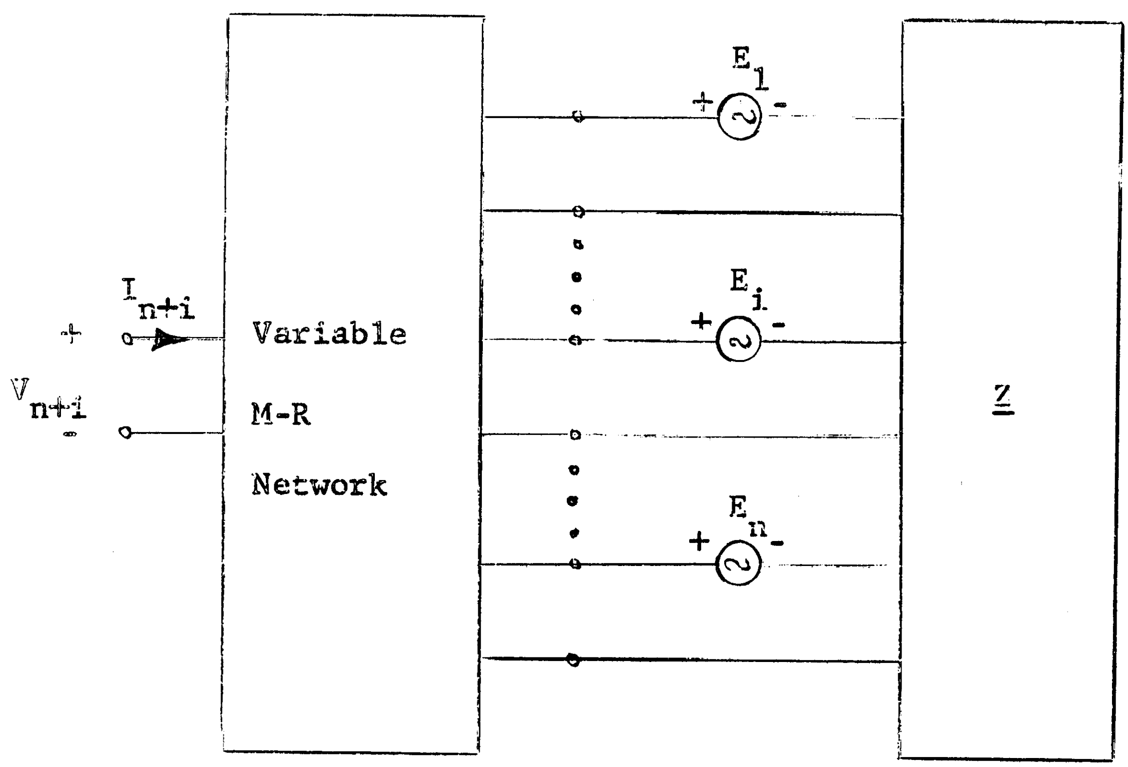

Figure 6.5. Imbedding into an (n+1)-port M-R network.

$$\underline{\tau}^{+} = \underline{Z}''_{ba} \; (\underline{Z}'' + \underline{Z}''_{aa})^{-1}; \tag{6.46}$$

and expressing $\underline{Z}''_{o}$ in terms of $\underline{Z}''$ yields

$$\begin{aligned}
\underline{Z}''_{o} + \underline{Z}''^{+}_{o} &= \left[\underline{K}_{m}^{-1/2}\right]^{+} \underline{Z}_{o} \left[\underline{K}_{m}^{+1/2}\right] + \left\{\left[\underline{K}_{m}^{-1/2}\right]^{+} \underline{Z}_{o} \left[\underline{K}_{m}^{+1/2}\right]\right\}^{+} \\
&= \left[\underline{K}_{m}^{-1/2}\right]^{+} \left\{\underline{Z}_{bb} - \underline{Z}_{ba} \; (\underline{Z} + \underline{Z}_{aa})^{-1} \; \underline{Z}_{ab}\right\} \left[\underline{K}_{m}^{1/2}\right] \\
&\quad + \left\{\left[\underline{K}_{m}^{-1/2}\right]^{+} \left\{\underline{Z}_{bb} - \underline{Z}_{ba} (\underline{Z} + \underline{Z}_{aa})^{-1} \; \underline{Z}_{ab}\right\} \left[\underline{K}_{m}^{1/2}\right]\right\}^{+} \\
&= - \left[\underline{K}_{m}^{-1/2}\right]^{+} \underline{Z}_{ba} (\underline{Z} + \underline{Z}_{aa})^{-1} \; \underline{Z}_{ab} \left[\underline{K}_{m}^{1/2}\right] \\
&\quad - \left[\underline{K}_{m}^{1/2}\right]^{+} \underline{Z}^{+}_{ab} \left[\underline{Z} + \underline{Z}_{aa})^{-1}\right]^{+} \underline{Z}^{+}_{ba} \left[\underline{K}_{m}^{-1/2}\right] \\
&= \underline{\tau}^{+} \; (\underline{Z}'' + \underline{Z}''^{+}) \; \underline{\tau}.
\end{aligned} \tag{6.47}$$

It follows that

$$P'_{e,i} = \frac{(\underset{\sim}{\xi}^{+} \underline{\tau}^{+}) \; \overline{\underset{\sim}{E}' \underset{\sim}{E}'^{+}} \; (\underline{\tau} \; \underset{\sim}{\xi})}{(\underset{\sim}{\xi}^{+} \underline{\tau}^{+}) (\underline{Z}'' + \underline{Z}''^{+}) (\underline{\tau} \; \underset{\sim}{\xi})} \tag{6.48}$$

in which the matrix $\underline{\tau}$ is to be varied through all possible values consistent with the requirement that the transformation network must be M-R.

The significant point now is that $\underline{\tau}$ is actually any square matrix of order n because $\underline{Z}_{ba}$ is entirely unrestricted. Therefore, a new column matrix $\underset{\sim}{x}$ may be defined as

$$\underset{\sim}{x} = \underline{\tau}\,\underset{\sim}{\xi} = \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}, \qquad (6.49)$$

in which the elements take on all possible complex values as the M-R network in Fig. 6.4 is varied through all its allowed forms. Consequently, the stationary values of $P'_{e,i}$ in Eq. (6.48) may be found most conveniently by determining instead the stationary values of the (real) expression

$$P'_{e,i} = \frac{\underset{\sim}{x}^{+}\ \overline{\underset{\sim}{E}'\underset{\sim}{E}'^{+}}\ \underset{\sim}{x}}{\underset{\sim}{x}^{+}\ (\underline{Z}'' + \underline{Z}''^{+})\ \underset{\sim}{x}} \qquad (6.50)$$

as the complex column matrix $\underset{\sim}{x}$ is varied quite arbitrarily.

This is a problem well-known in matrix theory [4]. It may be shown that the stationary values of the exchangeable frequency-normalized power $P'_{e,i}$ are the eigenvalues of the matrix[9]

$$\underline{N} = (\underline{Z}'' + \underline{Z}''^{+})^{-1}\ \overline{\underset{\sim}{E}'\underset{\sim}{E}'^{+}} \qquad (6.51)$$

where

$$\underset{\sim}{E}' = \left[\underline{K}_k^{-1/2}\right]^{+} \underset{\sim}{E} \qquad (6.52)$$

and

[9] It is assumed that the matrix $(\underline{Z}'' + \underline{Z}''^{+})$ is nonsingular.

$$\underline{Z}'' = \left[\underline{K}_k^{-1/2}\right]^{+} \underline{Z} \left[\underline{K}_k^{1/2}\right] . \tag{6.53}$$

We therefore define:

$$\text{Characteristic-noise matrix} = \underline{N} \equiv (\underline{Z}'' + \underline{Z}''^{+})^{-1} \overline{\underset{\sim}{E}'\underset{\sim}{E}'^{+}} \tag{6.54}$$

and conclude that the stationary values of the exchangeable frequency-normalized power $P'_{e,i}$ are the (real) eigenvalues of the characteristic-noise matrix $\underline{N}$.

The physical significance of these eigenvalues will be pointed out in a later section. It may however be noted that $P'_{e,i}$ has the dimensions of energy.

6.3. INVARIANCE OF THE EIGENVALUES OF THE CHARACTERISTIC-NOISE MATRIX

One particular property of the eigenvalues of $\underline{N}$ will be proved in this section. Suppose that the original network with the characteristic-noise matrix $\underline{N}$ is imbedded in a 2n-port M-R network, as shown in Fig. 6.4. A new n-port network results, with the characteristic-noise matrix $\underline{N}'$. The eigenvalues of $\underline{N}'$ are the stationary values of the exchangeable frequency-normalized power obtained in a subsequent imbedding of the type shown in Fig. 6.5.

Theorem 6.1. The eigenvalues of the characteristic-noise matrix $\underline{N}'$ are equal to those of $\underline{N}$. Alternatively, we may state that the eigenvalues of the characteristic-noise matrix $\underline{N}$ are invariant with respect to M-R imbeddings which preserve the number of terminal pairs.

Proof: We showed in Section 6.2 that

$$\underline{Z}''_o + \underline{Z}''^+_o = \underline{\tau}^+ (\underline{Z}'' + \underline{Z}''^+) \underline{\tau} \tag{6.47}$$

and

$$\overline{\underset{\sim}{E}'_o \underset{\sim}{E}'^+_o} = \underline{\tau}^+ \overline{\underset{\sim}{E}' \underset{\sim}{E}'^+} \underline{\tau} \tag{6.45}$$

where

$$\underline{\tau}^+ = \underline{Z}''_{ba} (\underline{Z}'' + \underline{Z}''_{aa})^{-1}. \tag{6.46}$$

We now write[10]

$$\underline{N}' = (\underline{Z}''_o + \underline{Z}''^+_o)^{-1} \overline{\underset{\sim}{E}'_o \underset{\sim}{E}'^+_o}$$

$$= \underline{\tau}^{-1} (\underline{Z}'' + \underline{Z}''^+)^{-1} \overline{\underset{\sim}{E}' \underset{\sim}{E}'^+} \underline{\tau}$$

$$= \underline{\tau}^{-1} \underline{N} \underline{\tau}. \tag{6.55}$$

Equation (6.55) shows that the eigenvalues of $\underline{N}'$ are equal to those of $\underline{N}$.

6.4. CANONICAL FORM OF A MULTIFREQUENCY NOISY NETWORK

Sometimes additional insight into the meaning of the eigenvalues of the characteristic-noise matrix may be obtained [3] from the canonical form of the network. The canonical form of a multifrequency noisy network may be derived from the original network by imbedding it in an M-R network that preserves the number of terminal pairs. This procedure, as shown in Fig. 6.4, led to a new network with an impedance matrix $\underline{Z}_o$, with

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[10]The matrix $\underline{\tau}$ is assumed to be nonsingular.

$$\underline{Z}_o = -\underline{Z}_{ba}(\underline{Z} + \underline{Z}_{aa})^{-1}\,\underline{Z}_{ab} + \underline{Z}_{bb} \tag{6.41}$$

or

$$\underline{Z}''_o = \left[\underline{K}_m^{-1/2}\right]^+ \underline{Z}_{bb} \left[\underline{K}_m^{1/2}\right] - \left\{\left[\underline{K}_m^{-1/2}\right]^+ \underline{Z}_{ba} \left[\underline{K}_k^{1/2}\right]\right\} \left\{\left[\underline{K}_k^{-1/2}\right]^+ \underline{Z} \left[\underline{K}_k^{1/2}\right] + \left[\underline{K}_k^{-1/2}\right]^+ \underline{Z}_{aa} \left[\underline{K}_k^{1/2}\right]\right\}^{-1} \left\{\left[\underline{K}_k^{-1/2}\right]^+ \underline{Z}_{ab} \left[\underline{K}_m^{1/2}\right]\right\} .$$

Using Eq. (6.46) and Eqs. (6.34) through (6.37), we can write

$$\underline{Z}''_o = \underline{\tau}^+ (\underline{Z}''^+ + \underline{Z}''^+_{aa})\,\underline{\tau} + \underline{Z}''_{bb}$$

or

$$\underline{Z}''_o = \frac{1}{2}\,\underline{\tau}^+ (\underline{Z}'' + \underline{Z}''^+)\,\underline{\tau} + \frac{1}{2}\,\underline{\tau}^+ (\underline{Z}''^+ - \underline{Z}'')\,\underline{\tau} + \underline{\tau}^+\,\underline{Z}''_{aa}\,\underline{\tau} + \underline{Z}''_{bb}. \tag{6.56}$$

From Eq. (6.46) we see that $\underline{\tau}$ is independent of $\underline{Z}''_{bb}$ and from Eq. (6.37) we see that the only constraint on $\underline{Z}''_{bb}$ is that it be skew-Hermitian. One possible choice for the matrix $\underline{Z}''_{bb}$, then, is

$$\underline{Z}''_{bb} = \frac{1}{2}\,\underline{\tau}^+ (\underline{Z}'' - \underline{Z}''^+)\,\underline{\tau} - \underline{\tau}^+\,\underline{Z}''^+_{aa}\,\underline{\tau}. \tag{6.57}$$

This choice satisfies the skew-Hermitian constraint, since the Hermitian transpose of this matrix is the negative of the matrix. Introducing Eq. (6.57) into Eq. (6.56), we find that this choice for $\underline{Z}''_{bb}$ gives

$$\underline{Z}''_0 = \frac{1}{2}\,\underline{\tau}^+ (\underline{Z}'' + \underline{Z}''^+)\,\underline{\tau}. \tag{6.58}$$

We have also shown that

$$\overline{\underset{\sim}{E}'_o \, \underset{\sim}{E}'_o} = \underline{\tau}^+ \, \overline{\underset{\sim}{E}' \, \underset{\sim}{E}'^+} \, \underline{\tau}. \tag{6.45}$$

We may write

$$\overline{\underset{\sim}{E}' \, \underset{\sim}{E}'^+} = \left[\underline{K}_k^{-1/2}\right]^+ \overline{\underset{\sim}{E} \, \underset{\sim}{E}^+} \left[\underline{K}_k^{-1/2}\right] . \tag{6.59}$$

The matrix $\overline{\underset{\sim}{E} \, \underset{\sim}{E}^+}$ is positive definite in most of the cases of practical interest [3]. Equation (6.59) shows that the matrix $\overline{\underset{\sim}{E}' \, \underset{\sim}{E}'^+}$ is also positive definite if and only if the matrix $\overline{\underset{\sim}{E} \, \underset{\sim}{E}^+}$ is positive definite, since the matrix $\left[\underline{K}_k^{+1/2}\right]$ is assumed to be a nonsingular matrix. It is always possible to diagonalize simultaneously two Hermitian matrices one of which is positive definite by the same conjunctive transformation [4]. Since $\overline{\underset{\sim}{E}'_o \, \underset{\sim}{E}'^+_o}$ is related to the positive definite matrix $\overline{\underset{\sim}{E}' \, \underset{\sim}{E}'^+}$ by a conjunctive transformation and $\underline{Z}''_o$ in Eq. (6.58) is related to $(\underline{Z}'' + \underline{Z}''^+)$ by the same conjunctive transformation, it is always possible to find an M-R network that will simultaneously diagonalize $\underline{Z}''_o$ and $\overline{\underset{\sim}{E}'_o \, \underset{\sim}{E}'^+_o}$.

We may now write

$$\underline{Z}_o = \left[\underline{K}_m^{1/2}\right]^+ \underline{Z}''_o \left[\underline{K}_m^{-1/2}\right] \tag{6.60}$$

and

$$\overline{\underset{\sim}{E}_o \, \underset{\sim}{E}^+_o} = \left[\underline{K}_m^{1/2}\right]^+ \overline{\underset{\sim}{E}'_o \, \underset{\sim}{E}'^+_o} \left[\underline{K}_m^{1/2}\right] . \tag{6.61}$$

Since $\left[\underline{K}_m^{1/2}\right]$ is a diagonal matrix the same conjunctive transformation will also diagonalize $\underline{Z}_o$ and $\overline{\underset{\sim}{E}_o \, \underset{\sim}{E}^+_o}$. The multifrequency noisy network obtained through this transformation is the canonical network shown in Fig. 6.6.

Since $\overline{\underset{\sim}{E}_o \underset{\sim}{E}_o^+}$ is diagonal, none of the voltages is correlated with any other voltage, and the canonical form consists of a set of independent noisy resistors. The characteristic-noise matrix of the canonical form is diagonal with elements along the diagonal which are the exchangeable frequency-normalized powers of each of these resistors.

$$\underline{N}' = \text{Diag.}\ (\overline{E_1' E_1'^*} / 2R''_{11}, \overline{E_2' E_2'^*} / 2R''_{22}, \cdots, \overline{E_n' E_n'^*} / 2R''_{nn}). \tag{6.62}$$

Since the eigenvalues of the characteristic-noise matrix are invariant under such an M-R imbedding, we see that the elements of $\underline{N}'$ are the eigenvalues of the characteristic-noise matrix of the original network. We may therefore say that the exchangeable frequency-normalized powers of n independently noisy network are the eigenvalues of the characteristic-noise matrix of the original network.

This shows that the following two theorems are true.

<u>Theorem 6.2</u>. Every n-port multifrequency noisy network can be reduced by M-R imbedding to a canonical form consisting of n separate (possibly negative) resistances each in series with an uncorrelated noise voltage generator.

<u>Theorem 6.3</u>. The exchangeable frequency-normalized powers of the n independent sources of the canonical form of any n-port

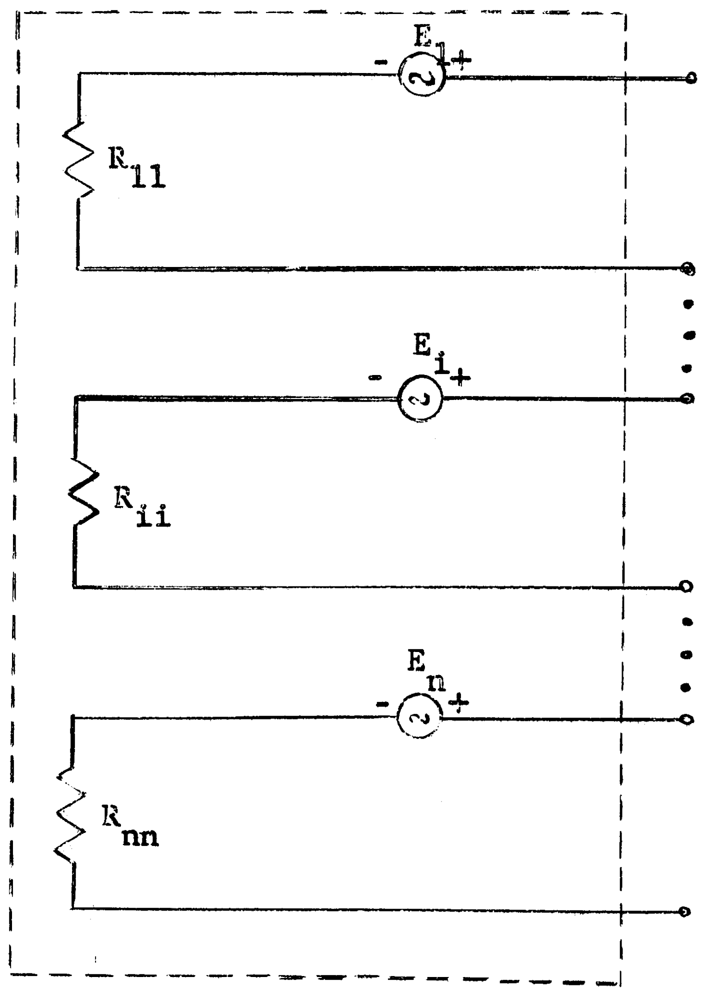

Figure 6.6. Canonical form of a multifrequency noisy network.

multifrequency noisy network are equal to the n eigenvalues of the characteristic-noise matrix $\underline{N}$ of the original network.

6.5. PHYSICAL SIGNIFICANCE OF THE INVARIANTS OF MULTIFREQUENCY NOISY NETWORKS

The terminal relation of a voltage source E in series with a resistor R (see Fig. 6.7) is given by

$$V = R\, I + E. \tag{6.63}$$

Let the frequency of the source be $k\omega_o + \omega$, where k is an integer. Let us now terminate this network in a variable inductor L (see Fig. 6.8).

The value of the average stored energy in the inductor L for arbitrary variation of the terminal current I is given by

$$Q = \frac{1}{2j(k\omega_o + \omega)} \left\{ V\, I^* - V^*\, I \right\} \tag{6.64}$$

$$= \frac{\overline{|E|^2}\, L}{R^2 + L^2\, (k\omega_o + \omega)^2}\, . \tag{6.65}$$

One of the possible stationary values of Q when L is varied over all possible values is given by

$$Q_s = \frac{\overline{|E|^2}}{2R(k\omega_o + \omega)} \tag{6.66}$$

for[11]

[11] According to this interpretation L may be positive or negative. If L is negative, we may terminate the one-port in a capacitor. In that case Q_s will be a stationary value of average stored energy in the capacitor. However, the one-port is terminated either in a single inductor or a capacitor, but not by a combination of both.

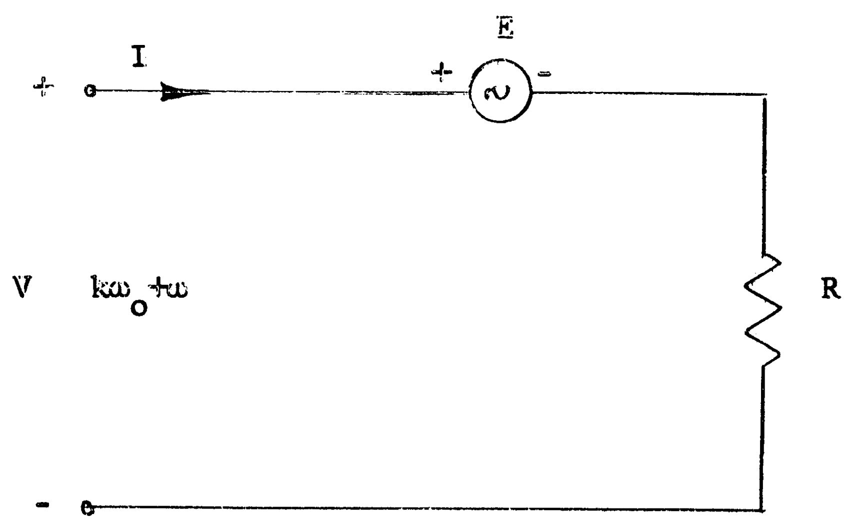

Figure 6.7. One-port noisy network.

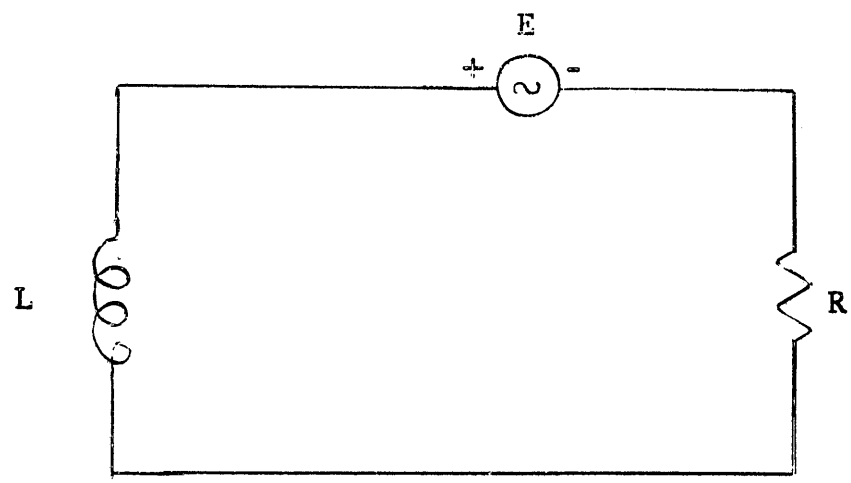

Figure 6.8. One-port-noisy network terminated in an inductance L.

$$L = R/(K\omega_o + \omega). \tag{6.67}$$

It was shown in Section 6.4 that every n-port multifrequency noisy network can be reduced by M-R imbedding to a canonical form consisting of n separate resistances in series with uncorrelated noise voltage generators. Also it was shown that the eigenvalues of the characteristic-noise matrix of the original network and of the canonical form of the network are the same.

We may therefore say that the eigenvalues of the characteristic-noise matrix of a multifrequency noisy network may be interpreted as the stationary values of the average stored energy that can be stored by the canonical form of the network for different terminal constraints. If the first eigenvalue of the characteristic-noise matrix (see Section 6.4) is $\overline{E_1' E_1'^*}/2R_{11}''$, then the terminal constraints on the canonical form of the network is that open-circuit (or short-circuit) all the ports except the 1^{st}; and the 1^{st} port is terminated either in a single inductor or a capacitor.

6.6. MULTIFREQUENCY NETWORKS IN OTHER REPRESENTATIONS

Different matrix representations can be used to describe the terminal noise behavior of multifrequency noisy networks. The investigation of invariants in these different kinds of representations forms the subject matter of this section.

<u>General Matrix Representation</u>. The impedance matrix representation (see Eq. (6.21)) is conveniently rewritten in the form [3]

$$\left[\, \underline{1} \;\vdots\; -\underline{Z} \,\right] \begin{bmatrix} \underset{\sim}{V} \\ --- \\ \underset{\sim}{I} \end{bmatrix} = \underset{\sim}{E} \tag{6.68}$$

where $\underline{1}$ is the identity matrix of the same order as $\underline{Z}$. Any other matrix representation of a multifrequency noisy network can be expressed as

$$\underset{\sim}{v} - \underline{T}\, \underset{\sim}{u} = \underset{\sim}{\delta} \tag{6.69}$$

where $\underset{\sim}{v}$ is a column matrix of the terminal "response", $\underset{\sim}{u}$ is the corresponding column matrix of the terminal "excitation", and $\underset{\sim}{\delta}$ is a column matrix comprising the amplitudes of the internal (noise) sources as seen at the terminals. The square matrix $\underline{T}$ expresses the transformation of the network in the absence of internal sources.

We note now that Eq. (6.69) may also be written in a form similar to Eq. (6.68).

$$\left[\, \underline{1} \;\vdots\; -\underline{T} \,\right] \begin{bmatrix} \underset{\sim}{v} \\ --- \\ \underset{\sim}{u} \end{bmatrix} = \underset{\sim}{\delta} \tag{6.70}$$

<u>Transformation from One Matrix Representation to Another.</u>

The variables $\begin{bmatrix} \underset{\sim}{V} \\ --- \\ \underset{\sim}{I} \end{bmatrix}$ and $\begin{bmatrix} \underset{\sim}{v} \\ --- \\ \underset{\sim}{u} \end{bmatrix}$ can always be related by a linear transformation of the form

$$\underline{R} \begin{bmatrix} \underset{\sim}{v} \\ --- \\ \underset{\sim}{u} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{V} \\ --- \\ \underset{\sim}{I} \end{bmatrix} . \tag{6.71}$$

Let us write

$$\underline{R} = \begin{bmatrix} \underline{R}_{11} & \underline{R}_{12} \\ \underline{R}_{21} & \underline{R}_{22} \end{bmatrix} \tag{6.72}$$

$$\underline{M} = \left[\underline{R}_{11} - \underline{Z}\, \underline{R}_{21} \right]^{-1} . \tag{6.73}$$

We may show [3] that representation 6.68 may be transformed into the representation

$$\left[\underline{1} \;\; \vdots \;\; -\underline{T} \right] \begin{bmatrix} \underset{\sim}{v} \\ --- \\ \underset{\sim}{u} \end{bmatrix} = \underset{\sim}{\delta} \tag{6.70}$$

where

$$\left[\underline{1} \;\; \vdots \;\; -\underline{T} \right] = \underline{M} \left[\underline{1} \;\; \vdots \;\; -\underline{Z} \right] \underline{R} \tag{6.74}$$

and

$$\underset{\sim}{\delta} = \underline{M}\, \underset{\sim}{E} . \tag{6.75}$$

<u>Frequency-Normalized Power Expression and its Transformation.</u> In any matrix representation, the frequency-normalized power P' flowing into the network is a real quadratic form of the excitation response vector $\begin{bmatrix} \underset{\sim}{v} \\ \hline \underset{\sim}{u} \end{bmatrix}$. We have

$$P' = \begin{bmatrix} \underset{\sim}{v} \\ \hline \underset{\sim}{u} \end{bmatrix}^{+} \underline{Q}_T \begin{bmatrix} \underset{\sim}{v} \\ \hline \underset{\sim}{u} \end{bmatrix} \qquad (6.76)$$

where $\underline{Q}_T$ is a Hermitian matrix of order twice that of either $\underset{\sim}{v}$ or $\underset{\sim}{u}$. In the particular case of impedance-matrix representation,

$$P' = \left[\underset{\sim}{V}^{+} \underline{K}^{-1} \underset{\sim}{I} + \underset{\sim}{I}^{+} \underline{K}^{-1} \underset{\sim}{V} \right]$$

$$= \begin{bmatrix} \underset{\sim}{V} \\ \hline \underset{\sim}{I} \end{bmatrix}^{+} \left[\begin{array}{c|c} 0 & \underline{K}^{-1} \\ \hline \underline{K}^{-1} & 0 \end{array} \right] \begin{bmatrix} \underset{\sim}{V} \\ \hline \underset{\sim}{I} \end{bmatrix} \qquad (6.77)$$

Comparing Eqs. (6.76) and (6.77), we find that the $\underline{Q}$ matrix for the impedance representation is

$$\underline{Q}_Z = \left[\begin{array}{c|c} 0 & \underline{K}^{-1} \\ \hline \underline{K}^{-1} & 0 \end{array} \right] . \qquad (6.78)$$

A transformation from one matrix representation into another transforms the $\underline{Q}$ matrix. We have

$$P' = \begin{bmatrix} \tilde{V} \\ --- \\ \tilde{I} \end{bmatrix}^{+} \underline{Q}_Z \begin{bmatrix} \tilde{V} \\ --- \\ \tilde{I} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{v} \\ --- \\ \tilde{u} \end{bmatrix}^{+} \underline{R}^{+}\underline{Q}_Z \, \underline{R} \begin{bmatrix} \tilde{v} \\ --- \\ \tilde{u} \end{bmatrix} \qquad (6.79)$$

Comparison with Eq. (6.76) shows that

$$\underline{Q}_T = \underline{R}^{+}\underline{Q}_Z \, \underline{R} \qquad (6.80)$$

where $\underline{R}$ is the matrix that transforms the general-excitation vector $\begin{bmatrix} \tilde{v} \\ --- \\ \tilde{u} \end{bmatrix}$ into the voltage and current vector $\begin{bmatrix} \tilde{V} \\ --- \\ \tilde{I} \end{bmatrix}$, according to Eq. (6.71).

<u>The General Characteristic-Noise Matrix</u>. Let us define a matrix $\underline{N}_T$ where

$$\underline{N}_T = - \left[\underline{K}^{1/2}\right] \left\{ \left[\underline{1} \;\vdots\; - \underline{T}\right] \underline{Q}_T^{-1} \left[\underline{1} \;\vdots\; - \underline{T}\right]^{+} \right\}^{-1} \overline{\tilde{\delta} \, \tilde{\delta}^{+}} \left[\underline{K}^{-1/2}\right] . \qquad (6.81)$$

For the impedance-matrix representation, we obtain

$$- \left[\underline{1} \;\vdots\; - \underline{Z}\right] \underline{Q}_Z^{-1} \left[\underline{1} \;\vdots\; - \underline{Z}\right]^{+} = \left[\underline{Z}\,\underline{K} + \underline{K}^{+}\underline{Z}^{+}\right]$$

$$= \left[\underline{K}^{1/2}\right]^{+} \left\{ \underline{Z}'' + \underline{Z}''^{+} \right\} \left[\underline{K}^{1/2}\right] . \qquad (6.82)$$

Introducing Eq. (6.82) into Eq. (6.81), we have

$$\underline{N}_Z = \left[\underline{K}^{1/2}\right]\left[\underline{K}^{-1/2}\right]\left\{\underline{Z}'' + \underline{Z}''^{+}\right\}^{-1}\left[\underline{K}^{-1/2}\right]^{+}\overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}}\left[\underline{K}^{-1/2}\right]$$

$$= \left\{\underline{Z}'' + \underline{Z}''^{+}\right\}^{-1}\overline{\underset{\sim}{E}'\underset{\sim}{E}'^{+}}$$

$$= \underline{N}. \tag{6.83}$$

But Eq. (6.83) is identical with the definition Eq. (6.51).

Next, let us relate the general noise matrix $\underline{N}_T$ of Eq. (6.81) to its particular form in the impedance representation. For this purpose, we note that according to Eq. (6.75)

$$\overline{\underset{\sim}{\delta}\,\underset{\sim}{\delta}^{+}} = \underline{M}\,\overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}}\,\underline{M}^{+}. \tag{6.84}$$

Then using Eqs. (6.74), (6.80), and (6.82), we find

$$- \left[\underline{1} \;\vdots\; -\underline{T}\right]\underline{Q}_T^{-1}\left[\underline{1} \;\vdots\; -\underline{T}\right]^{+}$$

$$= -\underline{M}\left[\underline{1} \;\vdots\; -\underline{Z}\right]\underline{R}\,\underline{Q}_T^{-1}\,\underline{R}^{+}\left[\underline{1} \;\vdots\; -\underline{Z}\right]^{+}\underline{M}^{+}$$

$$= -\underline{M}\left[\underline{1} \;\vdots\; -\underline{Z}\right]\underline{Q}_Z^{-1}\left[\underline{1} \;\vdots\; -\underline{Z}\right]^{+}\underline{M}^{+}$$

$$= \underline{M}\left[\underline{K}^{1/2}\right]^{+}\left[\underline{Z}'' + \underline{Z}''^{+}\right]\left[\underline{K}^{1/2}\right]^{+}\underline{M}^{+}.$$

We may then write

$$\underline{N}_T = -\left[\underline{K}^{1/2}\right]\left\{\left[\underline{1} \;\vdots\; -\underline{T}\right]\underline{Q}_T^{-1}\left[\underline{1} \;\vdots\; -\underline{T}\right]^{+}\right\}^{-1}\overline{\underset{\sim}{\delta}\,\underset{\sim}{\delta}^{+}}\left[\underline{K}^{-1/2}\right]$$

$$= \left[\underline{K}^{1/2}\right]\underline{M}^{+-1}\left[\underline{K}^{-1/2}\right]\left[\underline{Z}'' + \underline{Z}''^{+}\right]^{-1}\left[\underline{K}^{-1/2}\right]^{+}\underline{M}^{-1}\underline{M}\,\overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}}\,\underline{M}^{+}\left[\underline{K}^{-1/2}\right]$$

$$= \left\{ \left[\underline{K}^{1/2} \right] \underline{M}^{+-1} \left[\underline{K}^{-1/2} \right] \right\} \left\{ (\underline{Z}'' + \underline{Z}''^{+})^{-1} \overline{\underset{\sim}{E}' \underset{\sim}{E}'^{+}} \right\} \left\{ \left[\underline{K}^{1/2} \right] \underline{M}^{+-1} \left[\underline{K}^{-1/2} \right] \right\}^{-1}$$

$$= \underline{M}' \, \underline{N}_Z \, \underline{M}'^{-1} \qquad (6.85)$$

where

$$\underline{M}' = \left[\underline{K}^{1/2} \right] \underline{M}^{+-1} \left[\underline{K}^{-1/2} \right] . \qquad (6.86)$$

According to Eq. (6.85), the characteristic-noise matrix $\underline{N}_T$ of the general matrix representation of a multifrequency noisy network is related by a similarity transformation to the characteristic-noise matrix $\underline{N}_Z$ of the impedance matrix representation of the same network. Therefore, $\underline{N}_T$ and $\underline{N}_Z$ have the same eigenvalues.

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CHAPTER 7

INVARIANTS OF MULTIFREQUENCY NOISY NETWORKS

It may be shown [1] that a nonlinear system driven by a strong periodic signal and containing internal noise sources may be considered as a device exchanging power at a number of frequencies. In general, power at a number of frequencies flows through each physical port of this device, either by design or accidentally. However, for theoretical considerations such a device may be represented [2] as a multiport network, with each port exchanging power at only one frequency. The study of terminal-noise behavior of pumped nonlinear systems may, therefore, be considered as the study of noise performance of multifrequency noisy networks.

The invariants of linear time-invariant n-ports to nonsingular linear lossless imbeddings of various kinds have been investigated by Haus and Adler [3], Mason [4], Schaug-Pettersen and Tonning [5], and Youla [6]. The investigation of invariants of this type to different kinds of imbeddings of multifrequency noisy networks forms the subject of this chapter.

7.1. IMPEDANCE-MATRIX REPRESENTATION OF MULTIFREQUENCY NOISY NETWORKS

At any "frequency deviation" ω, the terminal-noise behavior of a periodically-driven nonlinear system is specified completely by an impedance matrix $\underline{Z}$, and the complex Fourier amplitudes of

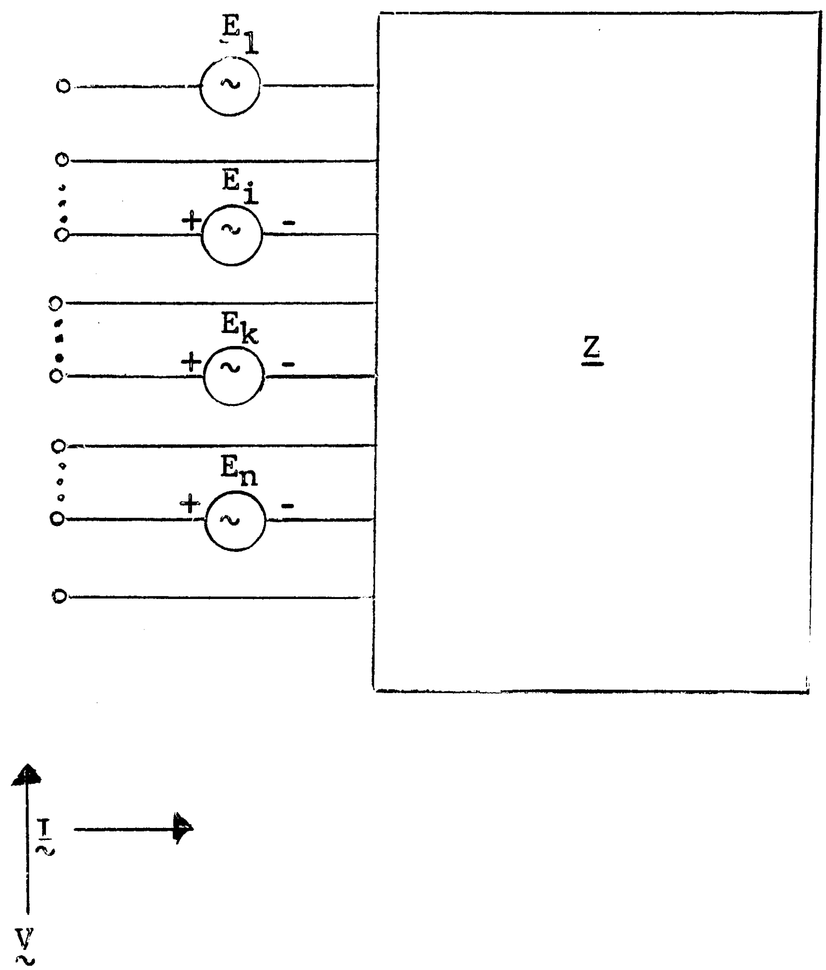

Figure 7.1. Equivalent representation of multifrequency noisy network.

its open-circuit terminal voltages $E_1, E_2, \cdots, E_n$[1] (see Fig. 7.1). In matrix form, $\underline{Z}$ denotes a square n-by-n array

$$\underline{Z} = \begin{bmatrix} Z_{11} & \cdots & Z_{1i} & \cdots & Z_{1k} & \cdots & Z_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ Z_{i1} & \cdots & Z_{ii} & \cdots & Z_{ik} & \cdots & Z_{in} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ Z_{k1} & \cdots & Z_{ki} & \cdots & Z_{kk} & \cdots & Z_{kn} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ Z_{n1} & \cdots & Z_{ni} & \cdots & Z_{nk} & \cdots & Z_{nn} \end{bmatrix} \tag{7.1}$$

The complex amplitudes of the open-circuit terminal voltages are represented by a column matrix $\underset{\sim}{E}$:

$$E = \begin{bmatrix} E_1 \\ \vdots \\ E_i \\ \vdots \\ E_k \\ \vdots \\ E_n \end{bmatrix} \tag{7.2}$$

The Fourier amplitudes $E_1, E_2, \cdots, E_n$ are complex random variables, the physical significance of which usually appears in their self- and cross-power spectral densities $\overline{E_i E_k^*}$.[2]

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[1]In the study of terminal-noise behavior of pumped nonlinear systems n is an even number. However, the results given in this paper are true for any n, an integer.

[2]The bar indicates an average over an ensemble of noise processes with identical statistical properties.

As in the rest of this work, only frequencies with positive values of frequency deviation ω are retained.

A convenient summary of the power spectral densities is the matrix

$$\overline{\underset{\sim}{E}\,\underset{\sim}{E}^{\dagger}}{}^{3} = \begin{bmatrix} \overline{E_1E_1^*} & \cdots & \overline{E_1E_i^*} & \cdots & \overline{E_1E_k^*} & \cdots & \overline{E_1E_n^*} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \overline{E_iE_1^*} & \cdots & \overline{E_iE_i^*} & \cdots & \overline{E_iE_k^*} & \cdots & \overline{E_iE_n^*} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \overline{E_kE_1^*} & \cdots & \overline{E_kE_i^*} & \cdots & \overline{E_kE_k^*} & \cdots & \overline{E_kE_n^*} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \overline{E_nE_1^*} & \cdots & \overline{E_nE_k^*} & \cdots & \overline{E_nE_k^*} & \cdots & \overline{E_nE_n^*} \end{bmatrix} \tag{7.3}$$

By examining the quadratic form associated with the matrix $\overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}}$ it may be shown that the matrix $\overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}}$ is a positive definite or positive semi-definite matrix [3]. In general, if the voltages of $\overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}}$ are noise voltages, it can usually be argued from physical grounds (barring trivial degeneracies and noiseless positive or negative resistances) that the matrix is positive definite.

7.2. TRANSFORMATIONS

If the n-port network with the generator column matrix $\underset{\sim}{E}$ and the impedance matrix $\underline{Z}$ is connected properly to a 2n-port

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[3] The superscript dagger indicates the two-step operation composed of forming the complex conjugate of and transposing the matrix to which it refers. Briefly, $\underline{A}^{+}$ is called the Hermitian conjugate of any matrix $\underline{A}$.

network, a new n-port network may be obtained. It will have a new noise column matrix $\tilde{E}'$ and a new impedance matrix $\underline{Z}'$. This operation, shown in Fig. 7.2, is called a transformation.

The analytical relation between the voltages and currents applied to the 2n-port network (the "transformation network") of Fig. 7.2 can be written in the form

$$\tilde{V}_o = \underline{Z}_{oo}\, \tilde{I}_o + \underline{Z}_{oi}\, \tilde{I}_i \tag{7.4}$$

$$\tilde{V}_i = \underline{Z}_{io}\, \tilde{I}_o + \underline{Z}_{ii}\, \tilde{I}_i. \tag{7.5}$$

The column vectors $\tilde{V}_o$ and $\tilde{V}_i$ comprise the terminal voltages applied to the transformation network on its two sides, and the column vectors $\tilde{I}_o$ and $\tilde{I}_i$ are the corresponding terminal currents.

The original n-port network, with impedance matrix $\underline{Z}$ and noise column matrix $\tilde{E}$, impose the following relation between the column matrices $\tilde{V}$ and $\tilde{I}$ of the voltages across, and currents into, its terminals:

$$\tilde{V} = \underline{Z}\, \tilde{I} + \tilde{E}. \tag{7.6}$$

The currents $\tilde{I}$ into the n-port network are, according to Fig. 7.2, equal and opposite to the currents $\tilde{I}_i$ into one side of the 2n-port network. The voltages $\tilde{V}$ are equal to the voltages $\tilde{V}_i$. We thus have

$$\tilde{V} = \tilde{V}_i; \qquad \tilde{I} = -\tilde{I}_i. \tag{7.7}$$

When these relations are used, the final relation between $\tilde{V}_o$ and $\tilde{I}_o$ is determined:

$$\tilde{V}_o = \underline{Z}'\, \tilde{I}_o + \tilde{E}' \tag{7.8}$$

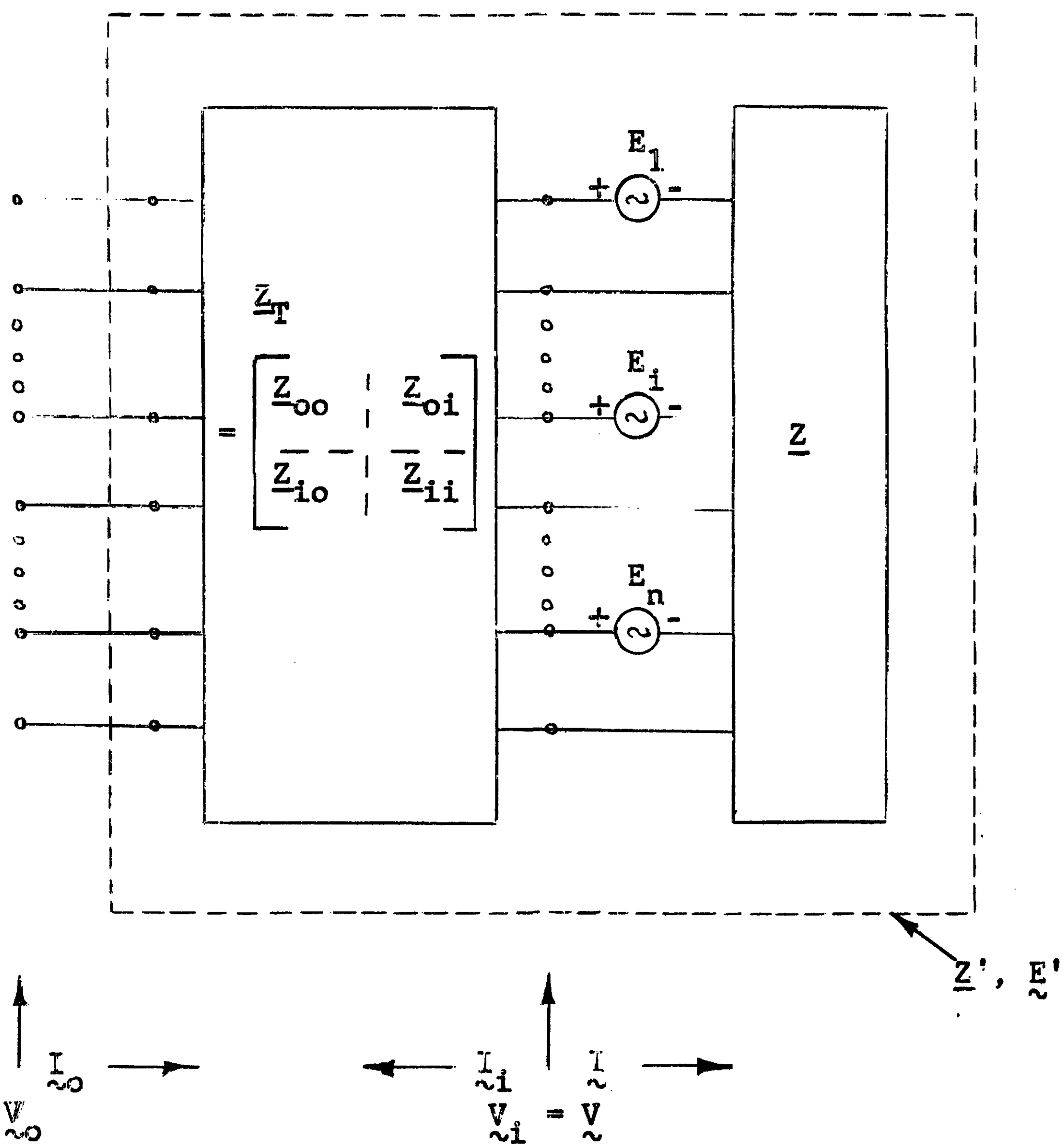

Figure 7.2. General transformation of an n-port noisy network.

where

$$\underline{Z}' = \underline{Z}_{oo} - \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1}\, \underline{Z}_{io} \tag{7.9}$$

and

$$\underset{\sim}{E}' = \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1}\, \underset{\sim}{E}. \tag{7.10}$$

Here $\underline{Z}'$ is the new impedance matrix, and $\underset{\sim}{E}'$ is the column matrix of the new open-circuit noise voltages.

7.3. LINEAR TRANSFORMATIONS

In case the transformation network[4] is linear, the matrices $\underline{Z}_{oo}$, $\underline{Z}_{oi}$, $\underline{Z}_{ii}$ and $\underline{Z}_{io}$ are all diagonal. This is a basic property of linear networks.

Diagonal Condition. It may be shown (See Appendix A) that an n-by-n matrix $\underline{A}$ is diagonal if and only if

$$\underline{A}\,\underline{\mu} - \underline{\mu}\,\underline{A} = \underline{0} \tag{7.11}$$

where $\underline{\mu}$ is represented as

$$\underline{\mu} = \begin{bmatrix} 1 & & & & & \\ & 2 & & & 0 & \\ & & \cdot & & & \\ & & & k & & \\ & 0 & & & \cdot & \\ & & & & & n \end{bmatrix}. \tag{7.12}$$

Theorem 7.1. If the transformation network is linear and does not contain any internal signal/noise generators, its impedance matrix (see Fig. 7.2)

[4]It is assumed that all the transformation networks considered in this chapter do not contain any internal signal/noise generators.

$$\underline{Z}_T = \begin{bmatrix} \underline{Z}_{oo} & \underline{Z}_{oi} \\ \underline{Z}_{io} & \underline{Z}_{ii} \end{bmatrix} \tag{7.13}$$

satisfies the condition

$$\underline{Z}_T \begin{bmatrix} \underline{\mu} & 0 \\ 0 & \underline{\mu} \end{bmatrix} - \begin{bmatrix} \underline{\mu} & 0 \\ 0 & \underline{\mu} \end{bmatrix} \underline{Z}_T = \underline{0}. \tag{7.14}$$

Proof. Since the transformation network is linear, the matrices $\underline{Z}_{oo}$, $\underline{Z}_{oi}$, $\underline{Z}_{ii}$ and $\underline{Z}_{io}$ are all diagonal.

These matrices, therefore, satisfy the equations

$$\underline{Z}_{oo}\underline{\mu} - \underline{\mu}\underline{Z}_{oo} = \underline{0} \tag{7.15}$$

$$\underline{Z}_{oi}\underline{\mu} - \underline{\mu}\underline{Z}_{oi} = \underline{0} \tag{7.16}$$

$$\underline{Z}_{ii}\underline{\mu} - \underline{\mu}\underline{Z}_{ii} = \underline{0} \tag{7.17}$$

and

$$\underline{Z}_{io}\underline{\mu} - \underline{\mu}\underline{Z}_{io} = \underline{0}. \tag{7.18}$$

We can write

$$\underline{Z}_T \begin{bmatrix} \underline{\mu} & 0 \\ 0 & \underline{\mu} \end{bmatrix} - \begin{bmatrix} \underline{\mu} & 0 \\ 0 & \underline{\mu} \end{bmatrix} \underline{Z}_T$$

$$= \begin{bmatrix} \underline{Z}_{oo} & \underline{Z}_{oi} \\ \underline{Z}_{io} & \underline{Z}_{ii} \end{bmatrix} \begin{bmatrix} \underline{\mu} & 0 \\ 0 & \underline{\mu} \end{bmatrix} - \begin{bmatrix} \underline{\mu} & 0 \\ 0 & \underline{\mu} \end{bmatrix} \begin{bmatrix} \underline{Z}_{oo} & \underline{Z}_{oi} \\ \underline{Z}_{io} & \underline{Z}_{ii} \end{bmatrix}$$

$$= \begin{bmatrix} \underline{Z}_{oo}\underline{\mu} - \underline{\mu}\underline{Z}_{oo} & \underline{Z}_{oi}\underline{\mu} - \underline{\mu}\underline{Z}_{oi} \\ \underline{Z}_{io}\underline{\mu} - \underline{\mu}\underline{Z}_{io} & \underline{Z}_{ii}\underline{\mu} - \underline{\mu}\underline{Z}_{ii} \end{bmatrix}$$

by virtue of Eqs. (7.15) through (7.18).

This proves Theorem 7.1.

<u>Theorem 7.2</u>. Cascading of a noiseless linear reciprocal network with a multifrequency noisy network, described by its impedance matrix $\underline{Z}$ and a noise voltage column matrix $\underset{\sim}{E}$ does not change the value of the quantity[5]

$$Q = \triangle(\underline{Z}\,\underline{\mu} - \underline{\mu}\,\underline{Z})\overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}}^{-1}\;\overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}}^{-1}\;(\underline{\mu}\,\underline{Z}^{+} - \underline{Z}^{+}\underline{\mu}) \tag{7.19}$$

provided the matrix $\overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}}$[6] is nonsingular.

<u>Proof</u>. Let a multifrequency noisy network have a terminal relation

$$\underset{\sim}{V} = \underline{Z}\,\underset{\sim}{I} + \underset{\sim}{E}. \tag{7.6}$$

If this multifrequency noisy network is cascaded with a linear noiseless network (see Fig. 7.2), the new terminal relations are given by

$$\underset{\sim}{V}_o = \underline{Z}'\,\underset{\sim}{I}_o + \underset{\sim}{E}' \tag{7.8}$$

where

$$\underline{Z}' = \underline{Z}_{oo} - \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1}\underline{Z}_{io} \tag{7.9}$$

and

$$\underset{\sim}{E}' = \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1}\,\underset{\sim}{E}. \tag{7.10}$$

The cascading network is linear. It therefore satisfies the Eqs. (7.15) through (7.18).

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[5] $\triangle\underline{A}$ is the determinant of matrix $\underline{A}$.

[6] If the matrix $\overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}}$ is singular, but the matrix $(\underline{Z\mu} - \underline{\mu Z})$ is nonsingular, we can show that the quantity

$$\triangle(\underline{Z\mu} - \underline{\mu Z})^{-1}\,\overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}}\;\overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}}\;(\underline{\mu Z}^{+} - \underline{Z}^{+}\underline{\mu})^{-1}$$

remains invariant. However, in most of the cases of interest, the matrix $\overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}}$ is positive definite and hence nonsingular.

$$\underline{Z}'\underline{\mu} - \underline{\mu}\underline{Z}' = - \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1}\underline{Z}_{io}\underline{\mu} + \underline{\mu}\underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1}\underline{Z}_{io}$$

$$= - \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1}\underline{\mu}\underline{Z}_{io} + \underline{Z}_{oi}\underline{\mu}(\underline{Z} + \underline{Z}_{ii})^{-1}\underline{Z}_{io}$$

$$= \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1}\left\{- \underline{\mu}(\underline{Z} + \underline{Z}_{ii}) + (\underline{Z} + \underline{Z}_{ii})\underline{\mu}\right\}(\underline{Z} + \underline{Z}_{ii})^{-1}\underline{Z}_{io}$$

$$= \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1}(\underline{Z}\underline{\mu} - \underline{\mu}\underline{Z})(\underline{Z} + \underline{Z}_{ii})^{-1}\underline{Z}_{io}.$$

We now have

$$Q' = \triangle(\underline{Z}'\underline{\mu} - \underline{\mu}\underline{Z}')\ \overline{\underset{\sim}{E}'\underset{\sim}{E}'^{+}}^{\,-1}\ \overline{\underset{\sim}{E}'\underset{\sim}{E}'^{+}}^{\,-1}\ (\underline{Z}'\underline{\mu} - \underline{\mu}\underline{Z}')^{+}$$

$$= \triangle\left\{\underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1}\ (\underline{Z}\underline{\mu} - \underline{\mu}\underline{Z})(\underline{Z} + \underline{Z}_{ii})^{-1}\ \underline{Z}_{io}\right\}$$

$$\left\{\underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1}\ \overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}}\ (\underline{Z} + \underline{Z}_{ii})^{+\,-1}\ \underline{Z}^{+}_{oi}\right\}^{-1}$$

$$\left\{\underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1}\ \overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}}\ (\underline{Z} + \underline{Z}_{ii})^{+\,-1}\ \underline{Z}^{+}_{oi}\right\}^{-1}$$

$$\left\{\underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1}\ (\underline{Z}\underline{\mu} - \underline{\mu}\underline{Z})(\underline{Z} + \underline{Z}_{ii})^{-1}\ \underline{Z}_{io}\right\}^{+}.$$

The cascading network is assumed to be reciprocal. The impedance matrix $\underline{Z}_T$ is, therefore, symmetrical.

$$\underline{Z}_T - \underline{Z}_T^t = \underline{0} \tag{7.20}$$

or

$$\underline{Z}_{oo} - \underline{Z}_{oo}^t = \underline{0} \tag{7.21}$$

$$\underline{Z}_{oi} - \underline{Z}_{io}^t = \underline{0} \tag{7.22}$$

and

$$\underline{Z}_{ii} - \underline{Z}_{ii}^t = \underline{0}. \tag{7.23}$$

By using Eq. (7.22), we can show that

$$Q' = \triangle (\underline{Z}\underline{\mu} - \underline{\mu}\underline{Z}) \overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}}^{-1} \overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}}^{-1} (\underline{Z}\underline{\mu} - \underline{\mu}\underline{Z})^{+} = Q.$$

This shows that $Q' = Q$.

In this section we have proved the invariance of a quantity associated with a multifrequency noisy network when the latter is cascaded with a noiseless linear reciprocal network. The quantity Q has the same physical dimensions as those of $1/(\text{power})^2$.

7.4. LINEAR LOSSLESS TRANSFORMATIONS

In Section 7.3, the transformation network was assumed to be only linear. Let us now consider the case when the transformation network is not only linear but also lossless.

Condition of Losslessness and Linearity. In case the transformation network is lossless, it can be shown [3] that the impedance matrix $\underline{Z}_T$ satsifies the condition

$$\underline{Z}_T + \underline{Z}_T^{+} = \underline{0} \tag{7.24}$$

or

$$\underline{Z}_{oo} + \underline{Z}_{oo}^{+} = \underline{0} \tag{7.25}$$

$$\underline{Z}_{oi} + \underline{Z}_{io}^{+} = \underline{0} \tag{7.26}$$

and

$$\underline{Z}_{ii} + \underline{Z}_{ii}^{+} = \underline{0}. \tag{7.27}$$

Since the transformation network is also linear, it also satisfies Eqs. (7.15) through (7.18). By using these relations, Eqs. (7.25) through (7.27) may be written as

$$\underline{Z}_{oo}\underline{\mu} + \underline{\mu}\underline{Z}_{oo}^{+} = \underline{0} \tag{7.28}$$

$$\underline{Z}_{oi}\underline{\mu} + \underline{\mu}\underline{Z}_{io}^{+} = \underline{0} \tag{7.29}$$

and

$$\underline{Z}_{ii}\underline{\mu} + \underline{\mu}\underline{Z}_{ii}^{+} = \underline{0}. \tag{7.30}$$

<u>Part I</u>.

Theorem 7.3. The eigenvalues of the matrix[7]

$$\underline{S}_1 = (\underline{Z} + \underline{Z}^{+})^{-1} \overline{\tilde{E}\,\tilde{E}^{+}} \tag{7.31}$$

associated with a multifrequency noisy network described by an impedance matrix $\underline{Z}$ and a noise voltage column matrix $\tilde{E}$ are invariant to a linear lossless transformation.

<u>Proof</u>. Let a multifrequency noisy network be described by an impedance matrix $\underline{Z}$ and a noise voltage column matrix $\tilde{E}$.

In case this multifrequency noisy network is cascaded with a linear lossless network, the new impedance and the noise voltage column matrices of the resulting network are given by

$$\underline{Z}' = \underline{Z}_{oo} - \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1}\,\underline{Z}_{io} \tag{7.9}$$

$$\tilde{E}' = \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1}\,\tilde{E}. \tag{7.10}$$

Since the transformation network is lossless, it satisfies Eqs. (7.25) through (7.27).

The $\underline{S}_1'$ matrix for the resulting network is given by

$$\underline{S}_1' = (\underline{Z}' + \underline{Z}'^{+})^{-1}\,\overline{\tilde{E}'\tilde{E}'^{+}}. \tag{7.32}$$

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[7] It is assumed that the matrix $(\underline{Z} + \underline{Z}^{+})$ is nonsingular.

We can write

$$\underline{Z}' + \underline{Z}'^{+} = \underline{Z}_{oo} - \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1}\,\underline{Z}_{io} + \underline{Z}_{oo}^{+} - \underline{Z}_{io}^{+}(\underline{Z} + \underline{Z}_{ii})^{+\,-1}\,\underline{Z}_{oi}^{+}$$

$$= \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1}(\underline{Z} + \underline{Z}^{+})(\underline{Z} + \underline{Z}_{ii})^{+\,-1}\,\underline{Z}_{oi}^{+}.$$

Let us write

$$\underline{A}^{+} = \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1}. \qquad (7.33)$$

Accordingly

$$\underline{Z}' + \underline{Z}'^{+} = \underline{A}^{+}(\underline{Z} + \underline{Z}^{+})\underline{A} \qquad (7.34)$$

$$\overline{\tilde{E}'\tilde{E}'^{+}} = \underline{A}^{+}\,\overline{\tilde{E}\,\tilde{E}^{+}}\,\underline{A}. \qquad (7.35)$$

From Eq. (7.32)

$$\underline{S}_1' = \underline{A}^{-1}(\underline{Z} + \underline{Z}^{+})^{-1}\,\overline{\tilde{E}\,\tilde{E}^{+}}\,\underline{A}. \qquad (7.36)$$

Hence, the matrices $\underline{S}_1$ and $\underline{S}_1'$ are related by a similarity transformation. The eigenvalues of $\underline{S}_1'$ and $\underline{S}_1$ are therefore the same.[8]

This proves the theorem.

Physical Significance of these Invariants. Let us cascade the n-port multifrequency noisy network with a linear lossless 2n-port network (see Fig. 7.2). Open-circuiting all terminal pairs of the resulting n-port network except the i^{th}, we achieve an n-to-1-port lossless transformation, as indicated in Fig. 7.3. The exchangeable power from the i^{th} port can be written in matrix form as

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[8]The matrix $\underline{A}$ is nonsingular in all but degenerate cases.

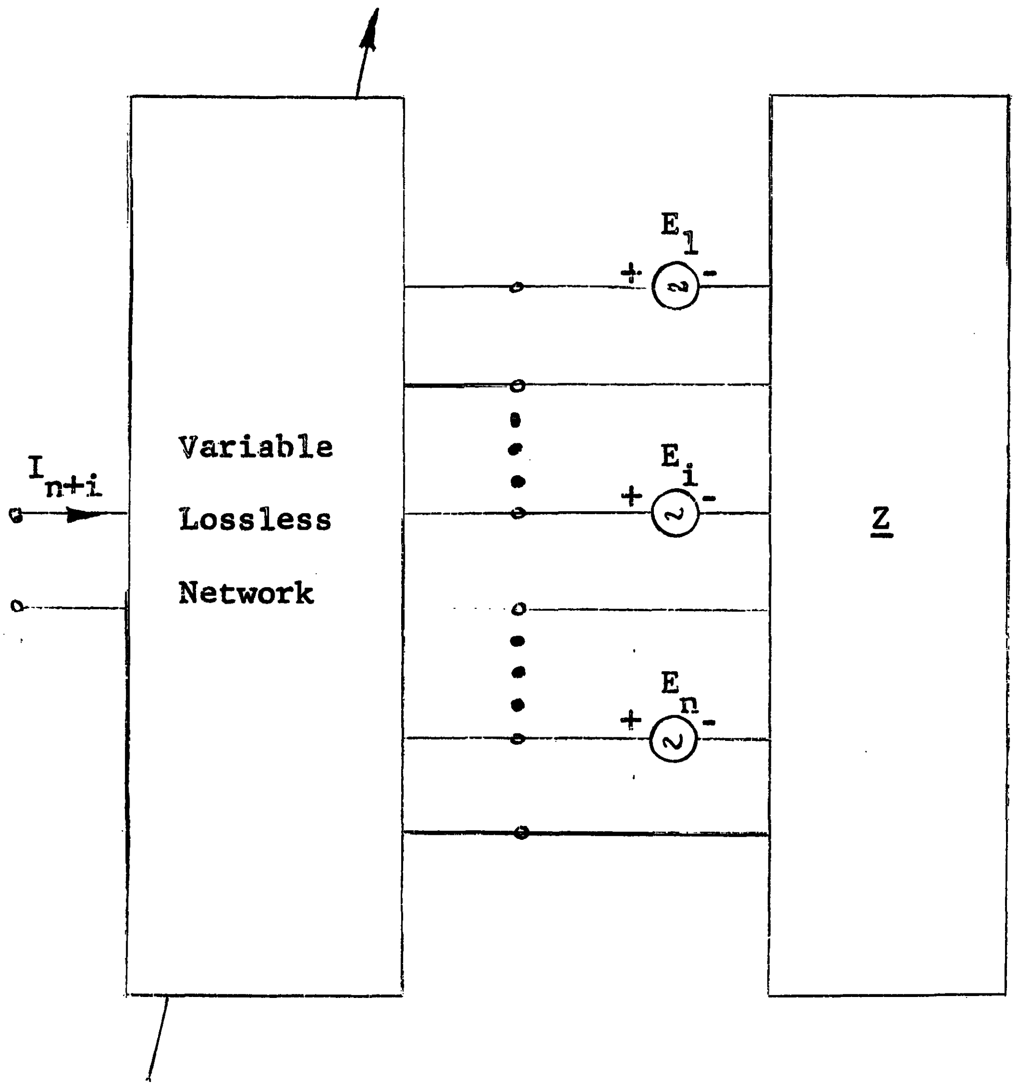

Figure 7.3. Cascading of a variable lossless network with a multifrequency noisy network.

$$P_{e,i} = \frac{\overline{E_i' E_i'^{*}}}{Z_{ii}' + Z_{ii}'^{*}} = \frac{\underset{\sim}{\xi}^{+} \overline{\underset{\sim}{E}' \underset{\sim}{E}'^{+}}\, \underset{\sim}{\xi}}{\underset{\sim}{\xi}^{+} (\underline{Z}' + \underline{Z}'^{+}) \underset{\sim}{\xi}} \tag{7.37}$$

where the (real) column matrix $\underset{\sim}{\xi}$ has every element zero except the i^{th}, which is 1:

$$\underset{\sim}{\xi} = \begin{bmatrix} \xi_1 \\ \cdot \\ \cdot \\ \cdot \\ \xi_i \\ \cdot \\ \cdot \\ \cdot \\ \xi_n \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} . \qquad \begin{aligned} \xi_j &= 0, \; j \neq i \\ \xi_i &= 1 \end{aligned} \tag{7.38}$$

Using Eqs. (7.34) and (7.35), we can now write

$$P_{e,i} = \frac{\underset{\sim}{\xi}^{+} \underline{A}^{+} \overline{\underset{\sim}{E}\, \underset{\sim}{E}^{+}}\, \underline{A}\, \underset{\sim}{\xi}}{\underset{\sim}{\xi}^{+} \underline{A}^{+} (\underline{Z} + \underline{Z}^{+}) \underline{A}\, \underset{\sim}{\xi}} \tag{7.39}$$

A new column matrix $\underset{\sim}{x}$ may be defined as

$$\underset{\sim}{x} = \underline{A}\, \underset{\sim}{\xi} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_i \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} . \tag{7.40}$$

It may be worth pointing out that $\underset{\sim}{x}$ is not an arbitrary n-dimensional vector if the transformation network is restricted to be linear and lossless.

Equation (7.39) may now be written as

$$P_{e,i} = \frac{\underset{\sim}{x}^{+} \overline{\underset{\sim}{E} \underset{\sim}{E}^{+}} \underset{\sim}{x}}{\underset{\sim}{x}^{+}(\underline{Z} + \underline{Z}^{+}) \underset{\sim}{x}} \tag{7.41}$$

It may be shown [3] that the stationary values of the quantity $P_{e,i}$ when the vector $\underset{\sim}{x}$ is arbitrarily varied are given by the eigenvalues of the matrix

$$\underline{S}_1 = (\underline{Z} + \underline{Z}^{+})^{-1} \overline{\underset{\sim}{E} \underset{\sim}{E}^{+}} . \tag{7.31}$$

It is therefore evident that the eigenvalues of the matrix $\underline{S}_1$ may be interpreted as the possible stationary values of the exchangeable power that can be obtained from the multi-frequency noisy network by cascading the latter with a linear lossless network.

There is another possible interpretation to these invariants if the multifrequency noisy network describes only the terminal-noise behavior of a pumped nonlinear system. In this case the value of n is equal to two and the two frequencies of interest are $\omega_o + \omega$ and $-\omega_o + \omega$. ω_o is the frequency of the pump and ω is the frequency deviation.

It may be shown [7] that the eigenvalues of the matrix

$$\underline{N} = (\underline{Z}'' + \underline{Z}''^{+})^{-1} \overline{\underset{\sim}{E}'' \underset{\sim}{E}''^{+}} \tag{7.42}$$

may be interpreted as the stationary values of exchangeable amplitude/phase noise power that we can get from the system when the linear lossless network cascaded with the system is arbitrarily varied. The impedance matrix $\underline{Z}''$ and the noise voltage column matrix $\underset{\sim}{E}''$ are given in the amplitude-phase representation. The transformations between amplitude-phase and α - β representations are given by

$$\underline{Z}'' = \underline{\lambda_v} \underline{Z}\, \underline{\lambda_i}^{-1} \tag{7.43}$$

and

$$\underset{\sim}{E}'' = \underline{\lambda_v}\, \underset{\sim}{E} \tag{7.44}$$

where

$$\underline{\lambda_v} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} e^{-j\phi v} & 0 \\ 0 & e^{j\phi v} \end{bmatrix} \tag{7.45}$$

and

$$\underline{\lambda_i} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} e^{-j\phi i} & 0 \\ 0 & e^{j\phi i} \end{bmatrix} . \tag{7.46}$$

ϕv and ϕi are the phase angles of carrier voltage and carrier current at the frequency ω_o. $\underline{Z}$ and $\underset{\sim}{E}$ in Eqs. (7.43) and (7.44) are given in the α - β representation. Let us assume that the carrier current and carrier voltage are in phase. Accordingly,

$$\underline{\lambda_v} = \underline{\lambda_i} . \tag{7.47}$$

It may be verified that

$$\underline{\lambda}_v^{+} = \frac{1}{2}\,\underline{\lambda}_v^{-1}. \tag{7.48}$$

We can now write

$$\begin{aligned} \underline{N} &= (\underline{Z}'' + \underline{Z}''^{+})^{-1}\,\overline{\tilde{E}''\tilde{E}''^{+}} \\ &= \frac{1}{\sqrt{2}}\,\underline{\lambda}_v^{+\,-1}\,(\underline{Z} + \underline{Z}^{+})^{-1}\,\overline{\tilde{E}\,\tilde{E}^{+}}\,\frac{1}{\sqrt{2}}\,\underline{\lambda}_v^{+} \\ &= \frac{1}{\sqrt{2}}\,\underline{\lambda}_v^{+\,-1}\,\underline{S}_1\,\frac{1}{\sqrt{2}}\,\underline{\lambda}_v^{+}. \end{aligned} \tag{7.49}$$

The eigenvalues of $\underline{N}$ and $\underline{S}_1$ are, therefore, the same.

The eigenvalues of $\underline{S}_1$ may hence be interpreted as the stationary values of the exchangeable amplitude/phase noise power that we can get by a linear lossless transformation from a nonlinear system pumped at frequency ω_o when the frequency deviation ω is not arbitrarily small.

Part II

Theorem 7.4. The characteristic values of the matrix[9]

$$\underline{S}_2 = (\underline{Z}\underline{\mu} + \underline{\mu}\underline{Z}^{+})^{-1}\,\overline{\tilde{E}\,\tilde{E}^{+}} \tag{7.50}$$

remain invariant to a linear lossless transformation.

Matrix $\underline{\mu}$ is given by Eq. (7.12).

Proof. The $\underline{S}_2'$ matrix for the multifrequency noisy network obtained by cascading a linear lossless 2n-port network with an n-port multifrequency noisy network is given by

[9]It is assumed that the matrix $(\underline{Z}\underline{\mu} + \underline{\mu}\underline{Z}^{+})$ is nonsingular.

$$\underline{S}_2' = (\underline{Z}'\underline{\mu} + \underline{\mu}\underline{Z}'^{+})^{-1} \; \overline{\tilde{E}'\tilde{E}'^{+}} \; . \tag{7.51}$$

We then have

$$\underline{Z}'\underline{\mu}+\underline{\mu}\underline{Z}'^{+} = \underline{Z}_{oo}\underline{\mu} - \underline{Z}_{oi}(\underline{Z}+\underline{Z}_{ii})^{-1}\underline{Z}_{io}\underline{\mu}+\underline{\mu}\underline{Z}^{+}_{oo} - \underline{\mu}\underline{Z}^{+}_{io}(\underline{Z} + \underline{Z}_{ii})^{+\,-1}\,\underline{Z}^{+}_{oi}$$

Because of Eqs. (7.28) through (7.30)

$$\begin{aligned}\underline{Z}'\underline{\mu}+\underline{\mu}\underline{Z}'^{+} &= \underline{Z}_{oi}(\underline{Z}+\underline{Z}_{ii})^{-1}\,\underline{\mu}\underline{Z}^{+}_{oi} + \underline{Z}_{oi}\underline{\mu}(\underline{Z} + \underline{Z}_{ii})^{+\,-1}\,\underline{Z}^{+}_{oi} \\ &= \underline{Z}_{oi}(\underline{Z}+\underline{Z}_{ii})^{-1}\left\{\underline{\mu}(\underline{Z}^{+} + \underline{Z}^{+}_{ii}) + (\underline{Z}+\underline{Z}_{ii})\underline{\mu}\right\}(\underline{Z} + \underline{Z}_{ii})^{+\,-1}\,\underline{Z}^{+}_{oi} \\ &= \underline{A}^{+}(\underline{Z}\underline{\mu} + \underline{\mu}\underline{Z}^{+})\,\underline{A}\end{aligned} \tag{7.52}$$

where

$$\underline{A}^{+} = \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1}. \tag{7.33}$$

We also have

$$\overline{\tilde{E}'\tilde{E}'^{+}} = \underline{A}^{+} \; \overline{\tilde{E}\,\tilde{E}^{+}} \; \underline{A}. \tag{7.35}$$

From Eq. (7.50)

$$\begin{aligned}\underline{S}_2' &= \underline{A}^{-1}(\underline{Z}\underline{\mu} + \underline{\mu}\underline{Z}^{+})^{-1} \; \overline{\tilde{E}\,\tilde{E}^{+}} \; \underline{A} \\ &= \underline{A}^{-1}\,\underline{S}_2\,\underline{A}.\end{aligned} \tag{7.53}$$

Again, $\underline{S}_2'$ and $\underline{S}_2$ are related by a similarity transformation. They therefore have the same eigenvalues.

Part III.

Theorem 7.5. The matrix[10]

[10] It is again assumed that $(\underline{Z}\underline{\mu} + \underline{Z}\underline{\mu}^{+})$ is nonsingular.

$$\underline{S}_3 = (\underline{Z}\underline{\mu} + \underline{\mu}\underline{Z}^+)^{-1} \quad (\underline{Z} + \underline{Z}^+) \tag{7.54}$$

goes through a similarity transformation when the network is subjected to a linear lossless transformation. The eigenvalues of $\underline{S}_3$, therefore, remain invariant.

Proof. According to Eqs. (7.34) and (7.52)

$$\underline{Z}' + \underline{Z}'^+ = \underline{A}^+(\underline{Z} + \underline{Z}^+) \ \underline{A} \tag{7.34}$$

$$\underline{Z}'\underline{\mu} + \underline{\mu}\underline{Z}'^+ = \underline{A}^+(\underline{Z}\underline{\mu} + \underline{\mu}\underline{Z}^+) \ \underline{A}. \tag{7.52}$$

We can then write

$$\begin{aligned} \underline{S}_3' &= (\underline{Z}'\underline{\mu} + \underline{\mu}\underline{Z}'^+)^{-1} \ (\underline{Z}' + \underline{Z}'^+) \\ &= \underline{A}^{-1}(\underline{Z}\underline{\mu} + \underline{\mu}\underline{Z}^+)^{-1} \ (\underline{Z} + \underline{Z}^+) \ \underline{A} \\ &= \underline{A}^{-1} \ \underline{S}_3 \ \underline{A}. \end{aligned} \tag{7.55}$$

This shows that the matrices $\underline{S}_3'$ and $\underline{S}_3$ possess the same eigenvalues.

We would like to point out that the eigenvalues of $\underline{S}_2$ have the dimensions of power, and those of $\underline{S}_3$ are dimensionless numbers.

7.5. LINEAR LOSSLESS RECIPROCAL TRANSFORMATIONS

In this section we assume that the transformation network is also reciprocal.

Condition of Reciprocity. If the transformation network is linear, lossless and reciprocal, its impedance matrix $\underline{Z}_T$ satisfies Eqs. (7.25) through (7.30) and also it satisfies the conditions:

$$\underline{Z}_{oo} - \underline{Z}_{oo}^{t} = \underline{0} \tag{7.21}$$

$$\underline{Z}_{oi} - \underline{Z}_{io}^{t} = \underline{0} \tag{7.22}$$

and

$$\underline{Z}_{ii} - \underline{Z}_{ii}^{t} = \underline{0}. \tag{7.23}$$

Theorem 7.6. The eigenvalues of the matrices[11]

$$\underline{T}_1 = (\underline{Z} - \underline{Z}^t)^{-1} (\underline{Z} + \underline{Z}^+)(\underline{Z} - \underline{Z}^t)^{*-1} (\underline{Z} + \underline{Z}^+)^* \tag{7.56}$$

$$\underline{T}_2 = (\underline{Z} - \underline{Z}^t)^{-1} \overline{\underset{\sim}{E}\,\underset{\sim}{E}^+} (\underline{Z} - \underline{Z}^t)^{*-1} \overline{\underset{\sim}{E}\,\underset{\sim}{E}^+}^* \tag{7.57}$$

$$\underline{T}_3 = (\underline{Z} - \underline{Z}^t)^{-1} (\underline{Z}\underline{\mu} + \underline{\mu}\underline{Z}^+)(\underline{Z} - \underline{Z}^t)^{*-1} (\underline{Z}\underline{\mu} + \underline{\mu}\underline{Z}^+)^* \tag{7.58}$$

$$\underline{T}_4 = (\underline{Z} - \underline{Z}^t)^{-1} (\underline{Z} + \underline{Z}^+)(\underline{Z} - \underline{Z}^t)^{*-1} (\underline{Z}\underline{\mu} + \underline{\mu}\underline{Z}^+)^* \tag{7.59}$$

$$\underline{T}_5 = (\underline{Z} - \underline{Z}^t)^{-1} (\underline{Z}\underline{\mu} + \underline{\mu}\underline{Z}^+)(\underline{Z} - \underline{Z}^t)^{*-1} (\underline{Z} + \underline{Z}^+)^* \tag{7.60}$$

$$\underline{T}_6 = (\underline{Z} - \underline{Z}^t)^{-1} \overline{\underset{\sim}{E}\,\underset{\sim}{E}^+} (\underline{Z} - \underline{Z}^t)^{*-1} (\underline{Z} + \underline{Z}^+)^* \tag{7.61}$$

$$\underline{T}_7 = (\underline{Z} - \underline{Z}^t)^{-1} \overline{\underset{\sim}{E}\,\underset{\sim}{E}^+} (\underline{Z} - \underline{Z}^t)^{*-1} (\underline{Z}\underline{\mu} + \underline{\mu}\underline{Z}^+)^* \tag{7.62}$$

$$\underline{T}_8 = (\underline{Z} - \underline{Z}^t)^{-1} (\underline{Z} + \underline{Z}^+)(\underline{Z} - \underline{Z}^t)^{*-1} \overline{\underset{\sim}{E}\,\underset{\sim}{E}^+}^* \tag{7.63}$$

and

$$\underline{T}_9 = (\underline{Z} - \underline{Z}^t)^{-1} (\underline{Z}\underline{\mu} + \underline{\mu}\underline{Z}^+)(\underline{Z} - \underline{Z}^t)^{*-1} \overline{\underset{\sim}{E}\,\underset{\sim}{E}^+}^* \tag{7.64}$$

associated with a multifrequency noisy network remain invariant when the network is subjected to a linear, lossless, and reciprocal transformation.

Proof. We showed in Section 7.4 that

$$\underline{Z}' + \underline{Z}'^+ = \underline{A}^+(\underline{Z} + \underline{Z}^+)\, \underline{A} \tag{7.34}$$

$$\underline{Z}'\underline{\mu} + \underline{\mu}\underline{Z}'^+ = \underline{A}^+(\underline{Z}\underline{\mu} + \underline{\mu}\underline{Z}^+)\, \underline{A} \tag{7.52}$$

[11] We make the assumption that $(\underline{Z} - \underline{Z}^t)$ is nonsingular.

and

$$\overline{\underset{\sim}{E}'\underset{\sim}{E}'^{+}} = \underline{A}^{+} \overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}} \underline{A} \tag{7.35}$$

where

$$\underline{A}^{+} = \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1}. \tag{7.33}$$

We can now write

$$\underline{Z}' - \underline{Z}'^{t} = \underline{Z}_{oo} - \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1} \underline{Z}_{io} - \underline{Z}^{t}_{oo} + \underline{Z}^{t}_{io}(\underline{Z} + \underline{Z}_{ii})^{t-1}\underline{Z}^{t}_{oi}.$$

By virtue of Eqs. (7.21) through (7.23),

$$\begin{aligned} \underline{Z}'-\underline{Z}'^{t} &= - \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1} \underline{Z}^{t}_{oi} + \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{t-1} \underline{Z}^{t}_{oi} \\ &= \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1}\left\{- (\underline{Z}^{t} + \underline{Z}^{t}_{ii}) + (\underline{Z} + \underline{Z}_{ii})\right\}(\underline{Z} + \underline{Z}_{ii})^{t-1} \underline{Z}^{t}_{oi} \\ &= \underline{A}^{+} (\underline{Z} - \underline{Z}^{t}) \underline{A}^{*}. \end{aligned} \tag{7.65}$$

<u>Case 1</u>. The $\underline{T}_1'$ matrix for the new multifrequency noisy network is given by

$$\begin{aligned} \underline{T}_1' &= (\underline{Z}' - \underline{Z}'^{t})^{-1} (\underline{Z}' + \underline{Z}'^{+})(\underline{Z}' - \underline{Z}'^{t})^{*-1} (\underline{Z}' + \underline{Z}'^{+})^{*} \\ &= \underline{A}^{*-1} (\underline{Z} - \underline{Z}^{t})^{-1} (\underline{Z} + \underline{Z}^{+})(\underline{Z} - \underline{Z}^{t})^{*-1} (\underline{Z} + \underline{Z}^{+})^{*} \underline{A}^{*} \\ &= \underline{A}^{*-1} \underline{T}_1 \underline{A}^{*}. \end{aligned} \tag{7.66}$$

This proves that the eigenvalues of $\underline{T}_1'$ are equal to the eigenvalues of $\underline{T}_1$.

<u>Case 2</u>. We also can write

$$\begin{aligned} \underline{T}_2' &= (\underline{Z}' - \underline{Z}'^{t})^{-1} \overline{\underset{\sim}{E}'\underset{\sim}{E}'^{+}} (\underline{Z}' - \underline{Z}'^{t})^{*-1} \overline{\underset{\sim}{E}'\underset{\sim}{E}'^{+}}^{*} \\ &= \underline{A}^{*-1} (\underline{Z} - \underline{Z}^{t})^{-1} \overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}} (\underline{Z} - \underline{Z}^{t})^{*-1} \overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}}^{*} \underline{A}^{*} \\ &= \underline{A}^{*-1} \underline{T}_2 \underline{A}^{*}. \end{aligned} \tag{7.67}$$

This shows the equality of eigenvalues of $\underline{T}_2'$ and $\underline{T}_2$.

The proofs of other cases mentioned in Theorem 7.6 are very similar and they therefore are not given.

7.6. LINEAR PURELY LOSSY TRANSFORMATION

A transformation network is said to be purely lossy[12] if the value of the reactive power stored in the network is identically zero.

Condition of being Purely Lossy. It is known from circuit theory that the transformation network is purely lossy if and only if

$$\underline{Z}_T - \underline{Z}_T^+ = \underline{0} \tag{7.68}$$

or

$$\underline{Z}_{oo} - \underline{Z}_{oo}^+ = \underline{0} \tag{7.69}$$

$$\underline{Z}_{oi} - \underline{Z}_{io}^+ = \underline{0} \tag{7.70}$$

and

$$\underline{Z}_{ii} - \underline{Z}_{ii}^+ = \underline{0}. \tag{7.71}$$

Since it is also linear, it also satisfies Eqs. (7.15) through (7.18).

Theorem 7.7. The matrices[13]

$$\underline{U}_1 = (\underline{Z} - \underline{Z}^+)^{-1} \overline{\tilde{E}\,\tilde{E}^+} \tag{7.72}$$

$$\underline{U}_2 = (\underline{Z}\underline{\mu} - \underline{\mu}\underline{Z}^+)^{-1} \overline{\tilde{E}\,\tilde{E}^+} \tag{7.73}$$

---

[12] As in the rest of this chapter, it is assumed that the transformation network does not contain any internal signal/noise generators.

[13] We make the assumption that the matrices $(\underline{Z} - \underline{Z}^+)$ and $(\underline{Z}\underline{\mu} - \underline{\mu}\underline{Z}^+)$ are nonsingular.

and

$$\underline{U}_3 = (\underline{Z}\underline{\mu} - \underline{\mu}\underline{Z}^+)^{-1} (\underline{Z} - \underline{Z}^+) \tag{7.74}$$

where $\underline{Z}$ and $\underset{\sim}{E}$ are the impedance and noise voltage column matrices of a multifrequency noisy network go through a similarity transformation whenever the network is subjected to a linear purely lossy transformation. The eigenvalues of the matrices, therefore, remain invariant.

Proof. The matrix $\underline{U}_1'$ is given by

$$\underline{U}_1' = (\underline{Z}' - \underline{Z}'^+)^{-1} \overline{\underset{\sim}{E}'\underset{\sim}{E}'^+} . \tag{7.75}$$

We can write

$$\underline{Z}' - \underline{Z}'^+ = \underline{Z}_{oo} - \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1} \underline{Z}_{io} - \underline{Z}_{oo}^+ + \underline{Z}_{io}^+(\underline{Z} + \underline{Z}_{ii})^{+-1} \underline{Z}_{oi}^+$$

$$= - \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1} \underline{Z}_{oi}^+ + \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{+-1} \underline{Z}_{oi}^+$$

$$= \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1}\left\{- (\underline{Z}^+ + \underline{Z}_{ii}^+) + (\underline{Z} + \underline{Z}_{ii})\right\} (\underline{Z} + \underline{Z}_{ii})^{+-1} \underline{Z}_{oi}^+$$

$$= \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1} (\underline{Z} - \underline{Z}^+)(\underline{Z} + \underline{Z}_{ii})^{+-1} \underline{Z}_{oi}^+$$

$$= \underline{A}^+(\underline{Z} - \underline{Z}^+) \underline{A}. \tag{7.76}$$

Let us now write

$$\underline{U}_1' = (\underline{Z}' - \underline{Z}'^+)^{-1} \overline{\underset{\sim}{E}'\underset{\sim}{E}'^+} = \underline{A}^{-1} (\underline{Z} - \underline{Z}^+)^{-1} \overline{\underset{\sim}{E}\,\underset{\sim}{E}^+} \underline{A}$$

$$= \underline{A}^{-1} \underline{U}_1 \underline{A}. \tag{7.77}$$

This again shows that the eigenvalues of $\underline{U}_1'$ remain invariant. It may also be shown that

$$\underline{U}_2' = \underline{A}^{-1} \underline{U}_2 \underline{A} \tag{7.78}$$

and

$$\underline{U}_3' = \underline{A}^{-1}\, \underline{U}_3 \underline{A}. \tag{7.79}$$

Equations (7.78) and (7.79) prove the validity of the rest of the statements made in Theorem 7.7.

7.7. LINEAR LOSSLESS TRANSFORMATIONS OF TWO-FREQUENCY TWO-PORT NETWORKS

For two-frequency two-port noisy networks, we have been able to find two matrices the characteristic values of which remain invariant when the two-port network is subjected to a linear lossless transformation. It has also been shown that the eigenvalues of these matrices may be interpreted as stationary values of exchangeable noise power.

Let the two frequencies present at the two ports of the network (see Fig. 7.4) be $\omega_o + \omega$ and $- \omega_o + \omega$. Let also the terminal-noise behavior of the network be described by[14]

$$\underset{\sim}{V} = \underline{Z}\, \underset{\sim}{I} + \underset{\sim}{E} \tag{7.80}$$

or

$$\begin{bmatrix} V_\alpha \\ V_\beta \end{bmatrix} = \begin{bmatrix} Z_{\alpha\alpha} & Z_{\alpha\beta} \\ Z_{\beta\alpha} & Z_{\beta\beta} \end{bmatrix} \begin{bmatrix} I_\alpha \\ I_\beta \end{bmatrix} + \begin{bmatrix} E_\alpha \\ E_\beta \end{bmatrix}. \qquad (7.81$$

Let this network be cascaded with a linear lossless network as shown in Fig. 7.5. Let us open-circuit the port at frequency $- \omega_o + \omega$ (β - port).

[14] It is assumed that these representations are given in the α - β representation.

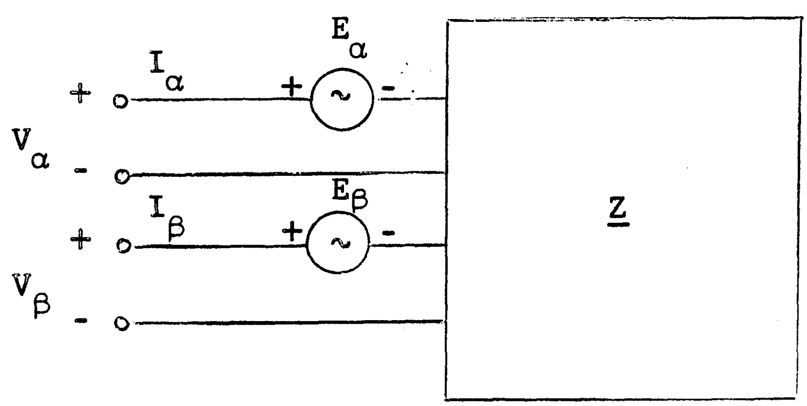

Figure 7.4. Two-frequency two-port noisy network.

<u>Theorem 7.8</u>. The stationary values of the exchangeable noise power that we can get at frequency $\omega_o + \omega$ from a two-frequency two-port noisy network by a linear lossless transformation are given by eigenvalues of the matrix[15]

$$\underline{N}_\alpha = \left\{ \underline{P}(\underline{Z} + \underline{Z}^+)\underline{P}^+ + [\underline{P}(\underline{Z} + \underline{Z}^+)\underline{P}^+]^* \right\}^{-1} \left\{ \underline{P}\,\overline{\underset{\sim}{E}\,\underset{\sim}{E}^+}\,\underline{P}^+ + [\underline{P}\,\overline{\underset{\sim}{E}\,\underset{\sim}{E}^+}\,\underline{P}^+]^* \right\} \tag{7.82}$$

where

$$\underline{P} = \begin{bmatrix} Z_{\beta\beta} & -Z_{\alpha\beta} \\ j & 0 \end{bmatrix}. \tag{7.83}$$

<u>Proof</u>. Since the transformation network is linear and lossless, it satisfies Eqs. (7.25) through (7.29). It may also be shown easily that for a two-port network

$$\underline{Z}_{oo} = \begin{bmatrix} jx_1 & 0 \\ 0 & jx_2 \end{bmatrix} \tag{7.84}$$

$$\underline{Z}_{ii} = \begin{bmatrix} jx_1' & 0 \\ 0 & jx_2' \end{bmatrix} \tag{7.85}$$

---

[15] It is, of course, assumed that the matrix $\left\{ \underline{P}(\underline{Z} + \underline{Z}^+)\underline{P}^+ + [\underline{P}(\underline{Z} + \underline{Z}^+)\underline{P}^+]^* \right\}$ is nonsingular.

$$\underline{Z}_{oi} = \begin{bmatrix} m_1 + jn_1 & 0 \\ 0 & m_2 + jn_2 \end{bmatrix} \qquad (7.86)$$

where x_1, x_2, x_1', x_2', m_1, n_1, m_2, and n_2 are real numbers.

The exchangeable noise power from the α-port can be written in matrix form as

$$P_{e,\alpha} = \frac{\overline{E_\alpha' E_\alpha'^*}}{Z_{\alpha\alpha}' + Z_{\alpha\alpha}'^*} = \frac{\tilde{\xi}^+ \overline{\tilde{E}' \tilde{E}'^+} \tilde{\xi}}{\tilde{\xi}^+ (\underline{Z}' + \underline{Z}'^+) \tilde{\xi}} \qquad (7.87)$$

where the matrix $\tilde{\xi}$ is given by

$$\tilde{\xi} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} . \qquad (7.88)$$

The variation of the lossless network is Fig. 7.5 now corresponds to variation of the transformation network $\underline{Z}_T$ through all possible lossless forms. We wish to find the stationary values of $P_{e,\alpha}$ corresponding to variation of $\underline{Z}_T$.

We may write

$$\overline{\tilde{E}' \tilde{E}'^+} = \underline{Z}_{oi} (\underline{Z} + \underline{Z}_{ii})^{-1} \overline{\tilde{E} \tilde{E}^+} [(\underline{Z} + \underline{Z}_{ii})^{-1}]^+ \underline{Z}_{oi}^+ \qquad (7.89)$$

$$\underline{Z}' + \underline{Z}'^+ = \underline{Z}_{oi} (\underline{Z} + \underline{Z}_{ii})^{-1} [(\underline{Z} + \underline{Z}^+)] [(\underline{Z} + \underline{Z}_{ii})^{-1}]^+ \underline{Z}_{oi}^+ . \qquad (7.90)$$

By means of Eqs. (7.83) through (7.85),

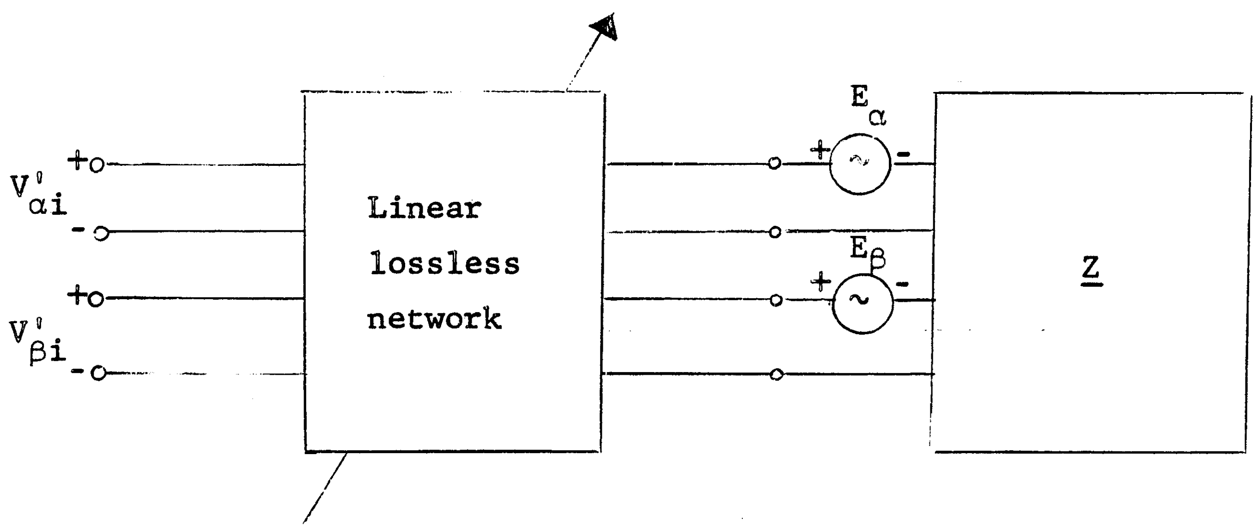

Figure 7.5. Two-frequency two-port noisy network in cascade with a variable lossless network.

$$\underline{Z}_{oi}(\underline{Z}+\underline{Z}_{ii})^{-1} = \begin{bmatrix} m_1 + jn_1 & 0 \\ 0 & m_2 + jn_2 \end{bmatrix} \begin{bmatrix} Z_{\alpha\alpha} + jx_1' & Z_{\alpha\beta} \\ Z_{\beta\alpha} & Z_{\beta\beta} + jx_2' \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} m_1 + jn_1 & 0 \\ 0 & m_2 + jn_2 \end{bmatrix} \begin{bmatrix} \dfrac{Z_{\beta\beta} + jx_2'}{\triangle} & -\dfrac{Z_{\alpha\beta}}{\triangle} \\ -\dfrac{Z_{\beta\alpha}}{\triangle} & \dfrac{Z_{\alpha\alpha} + jx_1'}{\triangle} \end{bmatrix} \qquad (7.91)$$

where[16]

$$\triangle = \begin{vmatrix} Z_{\alpha\alpha} + jx_1' & Z_{\alpha\beta} \\ Z_{\beta\alpha} & Z_{\beta\beta} + jx_2' \end{vmatrix} . \qquad (7.92)$$

Let us write

$$\underline{A}^+ = \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1}. \qquad (7.33)$$

From Eq. (7.87),

$$P_{e,\alpha} = \frac{\underset{\sim}{\xi}^+ \underline{A}^+ \overline{\underset{\sim}{E}\,\underset{\sim}{E}^+} \underline{A}\,\underset{\sim}{\xi}}{\underset{\sim}{\xi}^+ \underline{A}^+(\underline{Z} + \underline{Z}^+)\underline{A}\,\underset{\sim}{\xi}}$$

$$= \frac{\underset{\sim}{x}^+ \overline{\underset{\sim}{E}\,\underset{\sim}{E}^+}\,\underset{\sim}{x}}{\underset{\sim}{x}^+ (\underline{Z} + \underline{Z}^+)\,\underset{\sim}{x}} \qquad (7.93)$$

where

$$\underset{\sim}{x} = \underline{A}\,\underset{\sim}{\xi}. \qquad (7.94)$$

---

[16] It is assumed that $\triangle$ is nonzero.

We can write

$$\underset{\sim}{x}^+ = \underset{\sim}{\xi}^+ \underline{A}^+ = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} m_1 + jn_1 & 0 \\ 0 & m_2 + jn_2 \end{bmatrix} \begin{bmatrix} \frac{Z_{\beta\beta} + jx_2'}{\triangle} & -\frac{Z_{\alpha\beta}}{\triangle} \\ -\frac{Z_{\beta\alpha}}{\triangle} & \frac{Z_{\alpha\alpha} + jx_1'}{\triangle} \end{bmatrix}$$

$$= \frac{m_1 + jn_1}{\triangle} \begin{bmatrix} Z_{\beta\beta} + jx_2' & -Z_{\alpha\beta} \end{bmatrix} \quad (7.95)$$

$$= \frac{m_1 + jn_1}{\triangle} \begin{bmatrix} 1 & x_2' \end{bmatrix} \begin{bmatrix} Z_{\beta\beta} & -Z_{\alpha\beta} \\ j & 0 \end{bmatrix} \quad (7.96)$$

$$= \frac{m_1 + jn_1}{\triangle} \begin{bmatrix} 1 & x_2' \end{bmatrix} \underline{P} \quad (7.97)$$

$$= \frac{m_1 + jn_1}{\triangle} \begin{bmatrix} y_1 & y_2 \end{bmatrix} \underline{P} \quad (7.98)$$

where y_1 and y_2 are real numbers, positive or negative.

Let us write

$$\underset{\sim}{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} . \quad (7.99)$$

Accordingly

$$P_{e,\alpha} = \frac{\underset{\sim}{y}^t \, \underline{P} \, \overline{\underset{\sim}{E} \, \underset{\sim}{E}^+} \, \underline{P}^+ \, \underset{\sim}{y}}{\underset{\sim}{y}^+ \, \underline{P}(\underline{Z} + \underline{Z}^+)\underline{P}^+ \, \underset{\sim}{y}} . \quad (7.100)$$

It may be shown that $\underline{P}\,\overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}}\,\underline{P}^{+}$ and $\underline{P}(\underline{Z} + \underline{Z}^{+})\,\underline{P}^{+}$ are Hermitian matrices. We may therefore write

$$P_{e,\alpha} = \frac{\underset{\sim}{y}^{t}\left\{\underline{P}\,\overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}}\,\underline{P}^{+} + (\underline{P}\,\overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}}\,\underline{P}^{+})^{*}\right\}\underset{\sim}{y}}{\underset{\sim}{y}^{t}\left\{\underline{P}(\underline{Z} + \underline{Z}^{+})\underline{P}^{+} + [\underline{P}(\underline{Z} + \underline{Z}^{+})\underline{P}^{+}]^{*}\right\}\underset{\sim}{y}} \tag{7.101}$$

It must be pointed out here that the matrices

$$\underline{G} = \underline{P}\,\overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}}\,\underline{P}^{+} + (\underline{P}\,\overline{\underset{\sim}{E}\,\underset{\sim}{E}^{+}}\,\underline{P}^{+})^{*} \tag{7.102}$$

and

$$\underline{H} = \underline{P}(\underline{Z} + \underline{Z}^{+})\underline{P}^{+} + [\underline{P}(\underline{Z} + \underline{Z}^{+})\underline{P}^{+}]^{*} \tag{7.103}$$

are the real symmetric matrices.

As the transformation network is varied through all possible values, the elements of real column matrix $\underset{\sim}{y}$ take on all possible real values. Consequently, the stationary values of $P_{e,\alpha}$ in Eq. (7.101) may be found most conveniently by determining instead the stationary values of the (real) expression

$$P_{e,\alpha} = \frac{\underset{\sim}{y}^{t}\,\underline{G}\,\underset{\sim}{y}}{\underset{\sim}{y}^{t}\,\underline{H}\,\underset{\sim}{y}} \tag{7.104}$$

as the real column matrix $\underset{\sim}{y}$ is varied quite arbitrarily.

This is a well-known problem in matrix theory [8]. The stationary values of $P_{e,\alpha}$ are, therefore, given by eigenvalues of the matrix

$$\underline{N}_{\alpha} = \underline{H}^{-1}\underline{G}. \tag{7.105}$$

This proves Theorem 7.8.

Theorem 7.9. The stationary values of the exchangeable noise power that we can get at frequency $-\omega_o + \omega$ from a two-frequency, two-port noisy network by a linear lossless transformation are given by characteristic values of the matrix[17]

$$\underline{N}_\beta = \left\{\underline{Q}(\underline{Z} + \underline{Z}^+)\underline{Q}^+ + [\underline{Q}(\underline{Z} + \underline{Z}^+)\underline{Q}^+]^*\right\}^{-1} \left\{\underline{Q}\,\overline{\underset{\sim}{E}\,\underset{\sim}{E}^+}\,\underline{Q}^+ + (\underline{Q}\,\overline{\underset{\sim}{E}\,\underset{\sim}{E}^+}\,\underline{Q}^+)^*\right\} \tag{7.106}$$

where

$$\underline{Q} = \begin{bmatrix} -Z_{\beta\alpha} & Z_{\alpha\alpha} \\ 0 & j \end{bmatrix}.$$

Proof. The proof of this is very similar to that of Theorem 7.8 and is not given.

Let us call $\underline{N}_\alpha$ and $\underline{N}_\beta$ the characteristic noise matrices of the two-frequency, two-port noisy network.

Theorem 7.10. The eigenvalues of the matrices $\underline{N}_\alpha$ and $\underline{N}_\beta$ are invariant to a linear lossless transformation that preserves the number of terminal pairs.

Proof. Suppose that the original two-frequency, two-port network has characteristic noise matrices $\underline{N}_\alpha$ and $\underline{N}_\beta$.

Suppose this network is cascaded with a 4-port linear lossless network, as shown in Fig. 7.6. A new 2-port network results, with the characteristic noise matrices $\underline{N}_\alpha'$ and $\underline{N}_\beta'$. The

[17] Nonsingularity of the matrix $\left\{\underline{Q}(\underline{Z} + \underline{Z}^+)\underline{Q}^+ + [\underline{Q}(\underline{Z}+\underline{Z}^+)\underline{Q}^+]^*\right\}$ is assumed.

eigenvalues of $\underline{N}'_\alpha$ and $\underline{N}'_\beta$ are the stationary values of the exchangeable noise power obtained by a further linear lossless transformation. The second transformation network is completely variable. One possible variation removes the first 4-port transformation network. In cascade with this, we can use, if we like, any other linear lossless transformation network. Accordingly, the stationary values of the exchangeable noise powers at the two frequencies $\omega_o + \omega$ and $-\omega_o + \omega$ do not change when the two-port noisy network is subjected to a linear lossless transformation so as to get a new two-frequency, two-port noisy network.

This proves Theorem 7.10.

7.8. CANONICAL FORM OF TWO-FREQUENCY, TWO-PORT NOISY NETWORKS

Lossless network transformations performed on a two-frequency, two-port noisy network in such a way that the number of terminal pairs remain unchanged, change the impedance matrix as well as the noise spectra.

It may be shown [3] that at any particular frequency, every linear n-port noisy network can be reduced by linear lossless transformation into a canonical form consisting of n separate resistances in series with uncorrelated noise voltage generators.

Investigation of a simple form of $\underline{Z}$ and $\underset{\sim}{E}$ for a two-frequency, two-port noisy network by a linear lossless transformation forms the subject of this section.

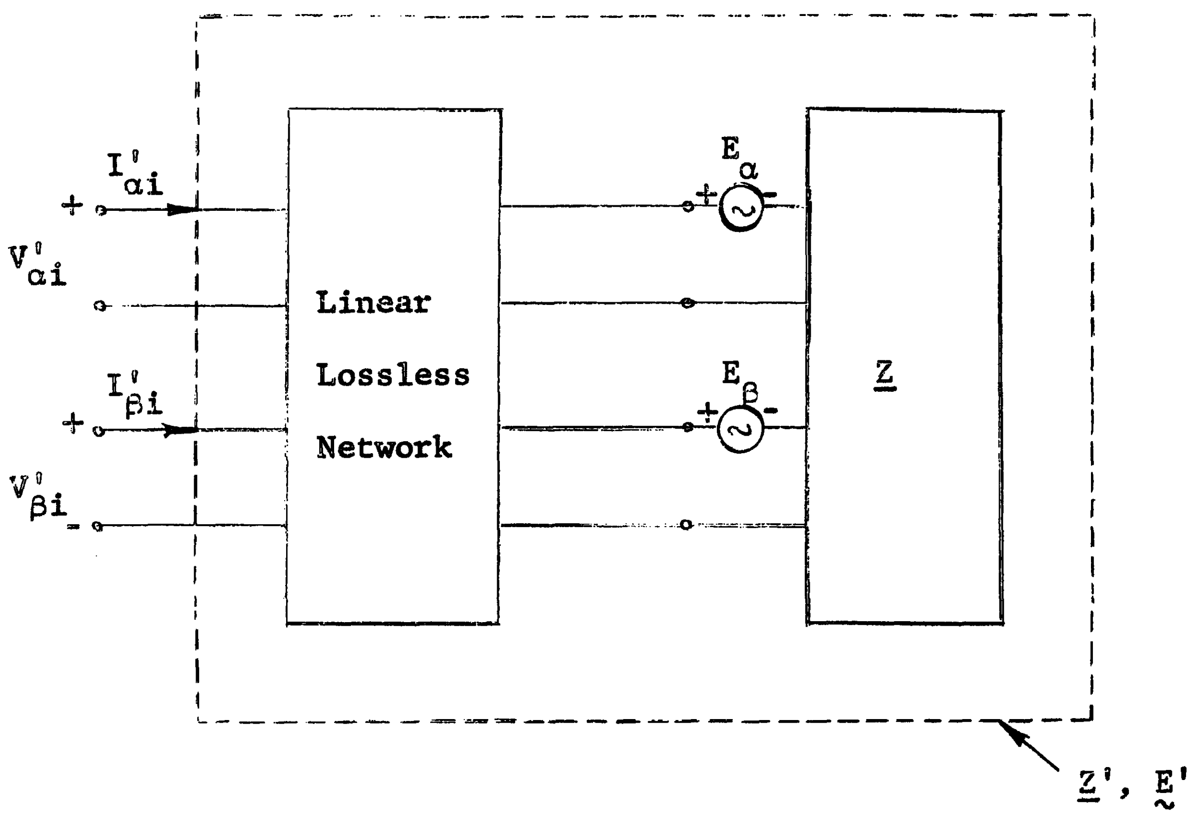

Figure 7.6. Two-frequency, two-port noisy network in cascade with a linear lossless network.

<u>Theorem 7.11</u>. One of the simplest forms that we can get for the impedance and noise voltage column matrices of a two-frequency, two-port noisy network by linear lossless transformation is given by

$$\underline{Z}' = \begin{bmatrix} 1 & k_1 \\ k_2 & 1 \end{bmatrix} \tag{7.107}$$

and

$$\overline{\underset{\sim}{E}'\underset{\sim}{E}'^{+}} = \begin{bmatrix} \overline{|E'_\alpha|^2} & \overline{E'_\alpha E'^{*}_\beta} \\ \overline{E'_\beta E'^{*}_\alpha} & \overline{|E'_\beta|^2} \end{bmatrix} \tag{7.108}$$

where k_1/k_2 = a fixed number determined by the impedance matrix of the original two-frequency, two-port noisy network.

<u>Proof</u>. Let the terminal-noise behavior of a two-frequency, two-port noisy network be described by

$$\underset{\sim}{V} = \underline{Z}\,\underset{\sim}{I} + \underset{\sim}{E} \tag{7.80}$$

or

$$\begin{bmatrix} V_\alpha \\ V_\beta \end{bmatrix} = \begin{bmatrix} Z_{\alpha\alpha} & Z_{\alpha\beta} \\ Z_{\beta\alpha} & Z_{\beta\beta} \end{bmatrix} \begin{bmatrix} I_\alpha \\ I_\beta \end{bmatrix} + \begin{bmatrix} E_\alpha \\ E_\beta \end{bmatrix} . \tag{7.81}$$

Let us get a new two-frequency, two-port noisy network by a linear lossless transformation, as shown in Fig. 7.5.

The terminal-noise behavior of the resulting network is given by

$$\underset{\sim}{V}_o = \underline{Z}' \underset{\sim}{I}_o + \underset{\sim}{E}' \qquad (7.8)$$

where

$$\underline{Z}' = \underline{Z}_{oo} + \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1} \underline{Z}_{oi}^{+} \qquad (7.109)$$

and

$$\underset{\sim}{E}' = \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1} \underset{\sim}{E}. \qquad (7.10)$$

Since the transformation network is linear and lossless, it satisfies Eqs.(7.84) through (7.86).

We can write

$$\underline{Z}' = \begin{bmatrix} jx_1 & 0 \\ 0 & jx_2 \end{bmatrix} + \begin{bmatrix} m_1 + jn_1 & 0 \\ 0 & m_2 + jn_2 \end{bmatrix} \begin{bmatrix} Z_{\alpha\alpha} + jx_1' & Z_{\alpha\beta} \\ Z_{\beta\alpha} & Z_{\beta\beta} + jx_2' \end{bmatrix}^{-1} \begin{bmatrix} m_1 - jn_1 & 0 \\ 0 & m_2 - jn_2 \end{bmatrix}$$

$$= \begin{bmatrix} (m_1^2 + n_1^2)\dfrac{Z_{\beta\beta} + jx_2'}{\triangle} + jx_1 & -\dfrac{Z_{\alpha\beta}}{\triangle}(m_1 + jn_1)(m_2 - jn_2) \\ -\dfrac{Z_{\beta\alpha}}{\triangle}(m_1 - jn_1)(m_2 + jn_2) & (m_2^2 + n_2^2)\dfrac{Z_{\alpha\alpha} + jx_1'}{\triangle} + jx_2 \end{bmatrix} \qquad (7.110)$$

where $\triangle$ is given by Eq. (7.92).

x_1, x_2, x_1', x_2', m_1, n_1, m_2, and n_2 are arbitrary real numbers. We can therefore choose these numbers in such a way that

$$\frac{m_1^2 + n_1^2}{\triangle} (Z_{\beta\beta} + jx_2') + jx_1 = 1 \tag{7.111}$$

$$\frac{m_2^2 + n_2^2}{\triangle} (Z_{\alpha\alpha} + jx_1') + jx_2 = 1. \tag{7.112}$$

By looking at Eq. (7.92) we can see that we can make $\triangle$ equal to any arbitrary complex number.

Let us write

$$\triangle = |D| e^{j\phi_1} \tag{7.113}$$

$$Z_{\alpha\beta} = |Z_{\alpha\beta}| e^{j\phi_{\alpha\beta}} \tag{7.114}$$

$$Z_{\beta\alpha} = |Z_{\beta\alpha}| e^{j\phi_{\beta\alpha}} \tag{7.115}$$

and

$$(m_1 + jn_1)(m_2 - jn_2) = |C| e^{j\phi_2}. \tag{7.116}$$

Accordingly,

$$-\frac{Z_{\alpha\beta}}{\triangle} (m_1 + jn_1)(m_2 - jn_2) = -\frac{|Z_{\alpha\beta}||C|}{|D|} e^{j(\phi_{\alpha\beta} + \phi_2 - \phi_1)} \tag{7.117}$$

$$-\frac{Z_{\beta\alpha}}{\triangle} (m_1 - jn_1)(m_2 + jn_2) = -\frac{|Z_{\beta\alpha}||C|}{|D|} e^{j(\phi_{\beta\alpha} - \phi_2 - \phi_1)}. \tag{7.118}$$

Let us choose ϕ_1 and ϕ_2 in such a way that

$$\phi_{\alpha\beta} + \phi_2 - \phi_1 = 0 \tag{7.119}$$

and

$$\phi_{\beta\alpha} - \phi_2 - \phi_1 = 0. \tag{7.120}$$

Let us also write

$$-\frac{|Z_{\alpha\beta}||C|}{|D|} = k_1 \tag{7.121}$$

and

$$-\frac{|Z_{\beta\alpha}||C|}{|D|} = k_2. \tag{7.122}$$

We can therefore write

$$\underline{Z}' = \begin{bmatrix} 1 & k_1 \\ k_2 & 1 \end{bmatrix} \tag{7.107}$$

where

$$k_1/k_2 = |Z_{\alpha\beta}| / |Z_{\beta\alpha}|. \tag{7.123}$$

From Eqs. (7.119) and (7.120)

$$\phi_2 = \frac{\phi_{\beta\alpha} - \phi_{\alpha\beta}}{2} \tag{7.124}$$

and

$$\phi_1 = \frac{\phi_{\beta\alpha} + \phi_{\alpha\beta}}{2} . \tag{7.125}$$

In this case, of course, the noise power matrix $\overline{\underset{\sim}{E}'\underset{\sim}{E}'^{+}}$ is given by

$$\overline{\underset{\sim}{E}'\underset{\sim}{E}'^{+}} = \begin{bmatrix} \overline{|E'_\alpha|^2} & \overline{E'_\alpha E'^{*}_\beta} \\ \overline{E'_\beta E'^{*}_\alpha} & \overline{|E'_\beta|^2} \end{bmatrix} \tag{7.126}$$

where

$$\underset{\sim}{E}' = \underline{Z}_{oi}(\underline{Z} + \underline{Z}_{ii})^{-1} \underset{\sim}{E}. \tag{7.10}$$

This shows the validity of Theorem 7.11.

7.9. MULTIFREQUENCY NOISY NETWORKS IN OTHER REPRESENTATIONS

In the foregoing analysis we have found some invariants of a multifrequency noisy network to different kinds of transformations that preserve the number of terminal pairs when the network is described in terms of its impedance representation. Other kinds of representations can also be used; and for each new kind of representation results analogous to those obtained with the aid of impedance formalism can be derived. The invariants obtained in these different kinds of representations are, of course, the same.

7.10. SEPARATE IMBEDDINGS OF LINEAR NOISY NETWORKS

At any frequency, a linear n-port containing internal signal or noise generators is specified completely with respect to its terminal pairs by its impedance matrix $\underline{Z}$ and the complex Fourier amplitudes of its open-circuit terminal voltages $E_1, E_2, \cdots, E_n$ (see Fig. 7.7). In case this n-port network is connected as shown in Fig. 7.8 to another 2n-port network, a new n-port network may be obtained. This operation will be called separate imbedding of the original network. The transformation network, in this case, is said to be separate.

The results we have obtained in Sections 7.3, 7.4, 7.5, 7.6, and 7.7 may be easily extended to linear noisy networks when an additional constraint is put on the transformation network. This constraint is that the transformation network be separate.

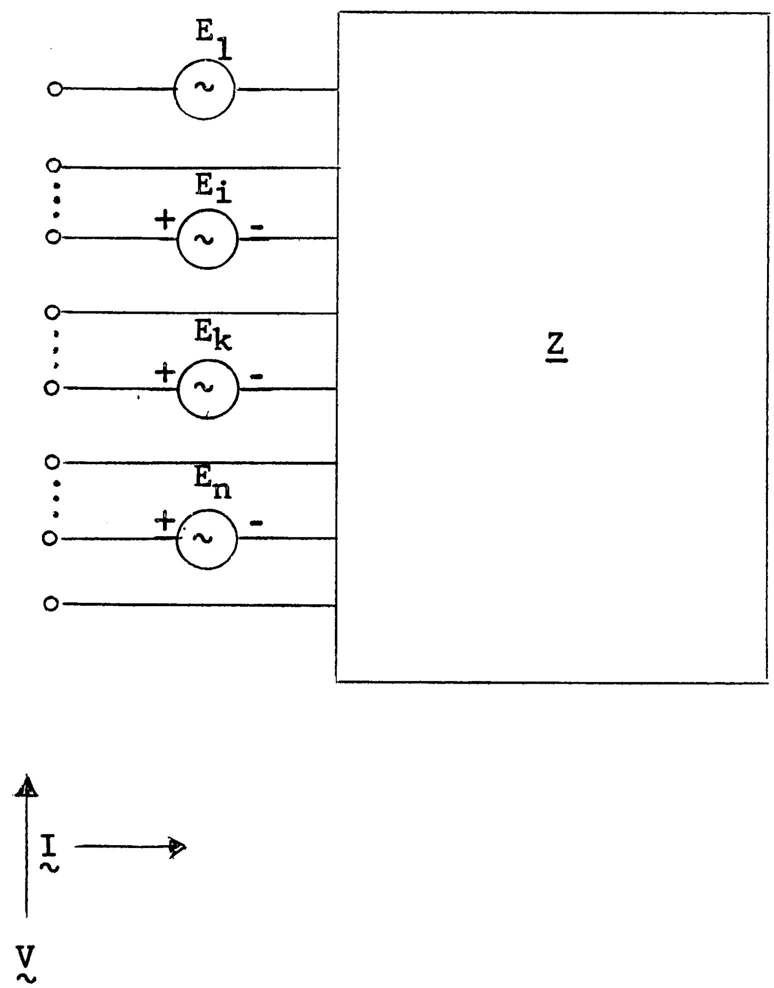

Figure 7.7. Equivalent representation of linear network with internal noise sources.

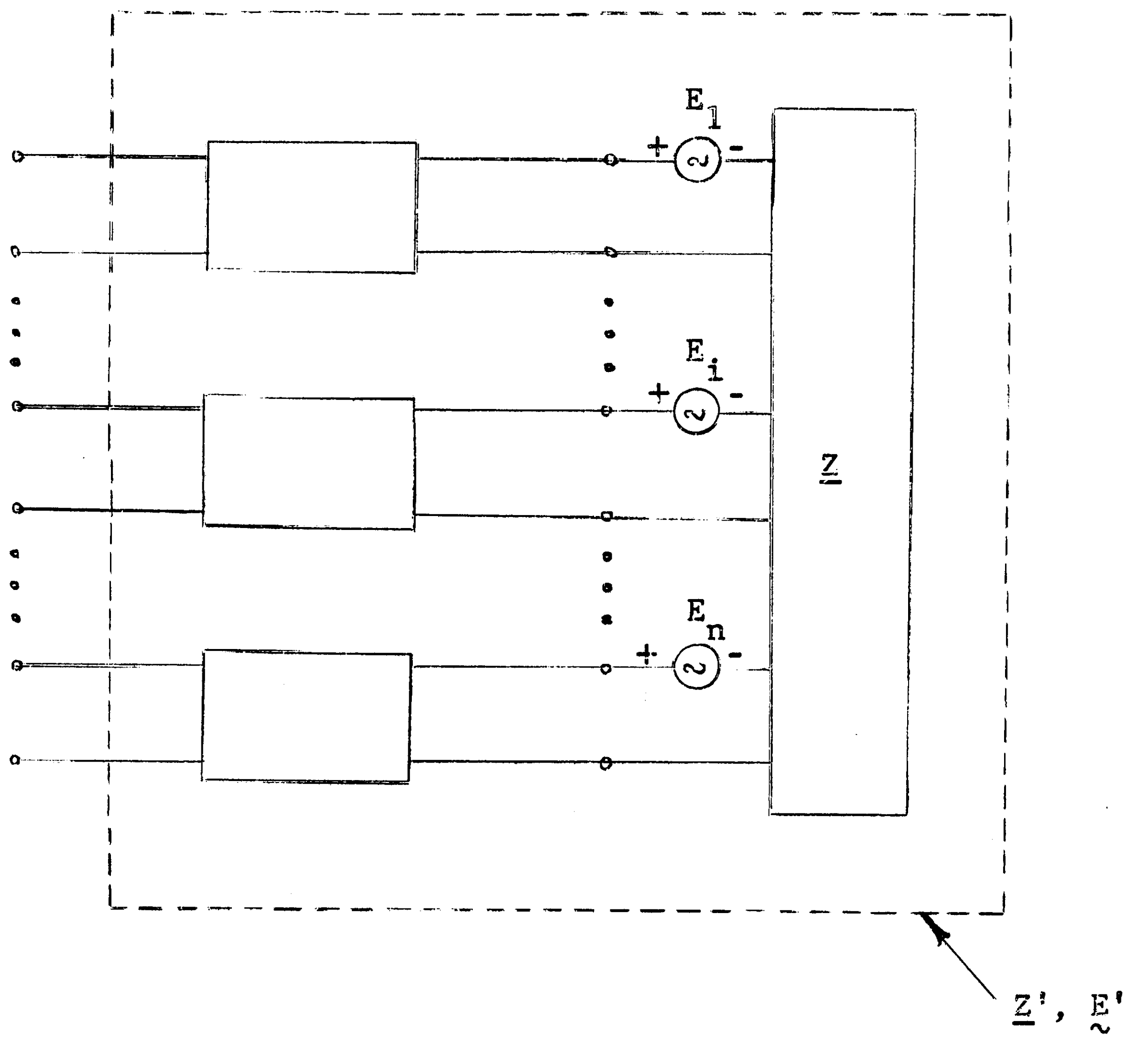

Figure 7.8. General separate transformation of an n-port network.

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CHAPTER 8

CONCLUSIONS

A great deal of material has been covered in the last seven chapters.

It was the purpose of this thesis to develop a general theory applicable to the analysis of noise performance of pumped nonlinear systems.

In Chapter 2, several forms of characterizations were given for the representation of internal noise sources in pumped nonlinear systems. Some of these characterizations are very similar to those that can be obtained for linear noisy networks. The difference is that at each frequency of interest two sources rather than one needed in linear noisy networks is essential for the representation of noise sources in pumped nonlinear systems.

Several ways of characterizing the noise performance of pumped nonlinear systems have been given in Chapter 3. One of these methods leads to a set of figures of merits which remain invariant when the system is cascaded with a linear lossless network. This method of defining a set of figures of merit is based on the concepts of exchangeable amplitude and phase noise powers. These concepts have also been developed in this chapter. However, this method does not enable us to get a cascade formula for a combination of two or more transducers; in other words, if a set of pumped nonlinear systems are connected in cascade, and the combination is driven by a single source, we have not been able to express the set of figures of merit for the combination in terms of the sets of figures of merit for each of the pumped nonlinear systems when they are individually driven by the same source. In Chapter 3, an attempt is also made to define a set of figures of merit for the pumped nonlinear system in terms of the open-circuit noise voltage matrices and a gain matrix. It has also been shown that we can get a cascade formula by using this definition of set of figures of merit. Finally, in Chapter 3, a third set of figures of merit have been proposed. This set of figures of merit have been defined in terms of the variances of input and output parameters. This set of figures of merit seems to have the greatest physical significance. The values of these figures of merit, however, depend not only on the value of the source impedance but also on the value of the load impedance. This is not a very desirable feature.

The detailed analysis of noise performance of abrupt-junction varactor frequency multipliers has been given in Chapter 4. It has been shown that it is possible to express output amplitude and phase noise currents in terms of input amplitude and phase noise sources and in terms of physical sources of noise that may be present at several locations in the circuit. The set of figures of merit defined for these multipliers in terms of variances of input and output parameters have been expressed in terms of the modulation ratios of the varactor. They have also been evaluated for some of the multipliers by using a digital computer. The expressions for the set of figures of merit can be used to find the point of optimum noise performance, and to find the direction in which improved noise performance can be obtained. This has not been done in our thesis.

The analysis of noise performance of divide-by-two circuit using an abrupt-junction varactor diode is taken up next in the first part of Chapter 5. The values of set of figures of merit have been obtained for this device, and the values of these figures of merit have been illustrated in a set of plots. Optimization techniques can also be used for this circuit to find out the point of optimum noise performance. This has not been our aim in this thesis, and it has not been attempted.

It is certainly possible that the pump used to drive the nonlinear system can itself be noisy. The techniques developed in the preceding chapters have been used in Chapter 5 to analyze the noise performance of parametric amplifiers in which the pump may be noisy. It has been shown for such amplifiers that only the amplitude noise present in the pump affects the amplifier noise performance, and the phase noise does not. The case of analysis of noise performance of parametric amplifiers using a varactor diode has been taken as a typical case. The same methods can be easily applied to the analysis of noise performance of devices driven by noisy pumps. Examples of such systems are discriminators, degenerate amplifiers, and systems consisting of such devices. Detailed investigation of spot noise performance of such devices is possible using the techniques developed in the earlier chapters. This has not been attempted in this thesis.

In Chapter 6 investigation of imbedding of multifrequency noisy networks in lossless parametric devices has been done. A characteristic noise matrix has been defined for these multifrequency noisy networks. The eigenvalues of this characteristic noise matrix remain invariant when the multi-frequency noisy network is subjected to a lossless parametric imbedding. An attempt is also made to give physical significance to these invariants with some success. A canonic form is also obtained for these multifrequency noisy networks when they are imbedded in lossless parametric devices. Most of the results obtained for these multifrequency noisy networks are analogous to those obtained for linear noisy networks when the latter are imbedded in linear lossless networks.

Chapter 7 deals with the invariants of multifrequency noisy networks when they are subjected to linear transformations of different kinds. Physical significance has been given to some of the invariants that we have obtained. It is anticipated that these invariants may become useful to characterize the performance of these devices by comparing output parameters with input parameters. The concept of separate imbeddings for linear noisy networks has been introduced in this chapter; and it has been pointed out that most of the results obtained for multifrequency noisy networks when they are imbedded in linear networks of different kinds may be shown to be valid for linear noisy networks when the latter are subjected to separate linear transformations of the same kind. It seems possible to develop a new theory of noise performance of linear noisy networks by using this idea of separate imbedding.

In summary, a general framework has been laid for the analysis of noise performance of pumped nonlinear systems. Techniques have been developed for the representation of noise sources in such devices, and for their characterization of noise performance. Specific problems are not treated in detail, even though modest attempts have been made to get an insight into the noise performance of harmonic generators and dividers.

8.1. SUGGESTIONS FOR FUTURE WORK

A number of problems of interest have been suggested in the preceding paragraphs which deserve further investigation.

Even though we have proposed three ways of characterizing the noise performance of pumped nonlinear systems, each one of these methods lacks one feature or another possessed by the noise figure defined for linear noisy networks. Future research may be directed in finding such a set of figures of merit for the pumped nonlinear systems.

As pointed out earlier, specific problems are not treated in detail in this thesis. Research may also be directed in analyzing the noise performance of specific devices like discriminators, parametric amplifiers, limiters, modulators, and systems consisting of such devices.

Our attempts to give physical significance to some of the invariants obtained in Chapter 7 have only met with partial success. This is another area in which further investigation is suggested.

The idea of separate imbeddings has been introduced in Chapter 7. This concept also needs further refinements and investigation. It is possible that a whole new theory of linear noisy networks can be developed by using this idea of separate imbeddings.

These are only some of the areas which needs further investigation and research.

APPENDIX A

DIAGONAL CHARACTER OF A MATRIX

A square n x n matrix $[a_{ij}]$ is defined to be a diagonal matrix if

$$a_{ij} = 0 \qquad i \neq j. \tag{A-1}$$

A diagonal matrix $\underline{A}$ with diagonal elements $a_1, a_2, \cdots, a_n$ is written as $\underline{A} = \text{diag}\,[a_1, a_2, \cdots, a_n]$.

Theorem A-1. A square n x n matrix $\underline{A}$ is a diagonal matrix if and only if

$$\underline{A}\,\underline{\mu} - \underline{\mu}\,\underline{A} = \underline{0} \tag{A-2}$$

where the matrix $\underline{\mu}$ is represented as

$$\underline{\mu} = \begin{bmatrix} \mu_1 & & & 0 \\ & \mu_2 & & \\ & & \ddots & \\ 0 & & & \mu_n \end{bmatrix} \tag{A-3}$$

$\mu_1, \mu_2, \cdots, \mu_n$ are any nonzero real or complex numbers which satisfy the following condition:

$$\mu_i - \mu_j \neq 0 \qquad \text{unless } i = j \tag{A-4}$$

Proof. Let a square n x n matrix $\underline{A}$ be represented as

$$\underline{A} = \begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ii} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{k1} & \cdots & a_{ki} & \cdots & a_{kk} & \cdots & a_{kn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nk} & \cdots & a_{nn} \end{bmatrix} \qquad \text{(A-5)}$$

We can write

$$\underline{A}\underline{\mu}-\underline{\mu}\underline{A} = \begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ii} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{k1} & \cdots & a_{ki} & \cdots & a_{kk} & \cdots & a_{kn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nk} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \mu_1 & & & & \\ & \ddots & & & 0 \\ & & \mu_i & & \\ & & & \ddots & \\ & 0 & & \mu_k & \\ & & & & \ddots \\ & & & & & \mu_n \end{bmatrix}$$

$$- \begin{bmatrix} \mu_1 & & & & \\ & \ddots & & & 0 \\ & & \mu_i & & \\ & & & \ddots & \\ & & & \mu_k & \\ & 0 & & & \ddots \\ & & & & & \mu_n \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ii} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{k1} & \cdots & a_{ki} & \cdots & a_{kk} & \cdots & a_{kn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nk} & \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \cdots & (\mu_i - \mu_1)a_{1i} & \cdots & (\mu_k - \mu_1)a_{1k} & \cdots & (\mu_n - \mu_1)a_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ (\mu_1 - \mu_i)a_{i1} & \cdots & 0 & \cdots & (\mu_k - \mu_i)a_{ik} & \cdots & (\mu_n - \mu_i)a_{in} \\ \vdots & & \vdots & & \vdots & & \vdots \\ (\mu_1 - \mu_k)a_{k1} & \cdots & (\mu_i - \mu_k)a_{ki} & \cdots & 0 & \cdots & (\mu_n - \mu_k)a_{kn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ (\mu_1 - \mu_n)a_{n1} & \cdots & (\mu_i - \mu_n)a_{ni} & \cdots & (\mu_k - \mu_n)a_{nk} & \cdots & 0 \end{bmatrix} \qquad \text{(A-6)}$$

Let $\underline{A}$ be a diagonal matrix. The property of a diagonal matrix is that

$$a_{ij} = 0 \qquad i \neq j \qquad \text{(A-1)}$$

From Eqs. (A-1), (A-5), and (A-6), we have

$$\underline{A}\,\underline{\mu} - \underline{\mu}\,\underline{A} = \underline{0}. \qquad \text{(A-2)}$$

Let us now assume that Eq. (A-2) is satisfied.

From Eq. (A-6) we can then write

$$(\mu_i - \mu_j)a_{ij} = 0 \qquad i \neq j \qquad \text{(A-7)}$$

By virtue of Eq. (A-4), it follows that

$$a_{ij} = 0 \qquad i \neq j. \qquad \text{(A-1)}$$

This proves Theorem A-1.

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BIOGRAPHY

The author was born in Kumta, India on May 13, 1940. He attended primary and secondary schools in Kumta, India; and graduated from Gibb High School ranking first for the state. He was awarded a Government of India postmatriculation scholarship for his study at Karnatak College, Dharwar. He received his B.Sc. (Hons.) degree from Karnatak University at Dharwar in 1958. He was first for the University. He obtained his B.E. (Dist.) degree from Indian Institute of Science, Bangalore in 1962. He is a winner of the Alfred Hay Gold medal for being the best student of the Institute in the Department of Electrical Communication Engineering. He received his S.M. degree from Massachusetts Institute of Technology in 1963.

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