RULED SURFACES IN EUCLIDEAN FOUR SPACE

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ABSTRACT OF THESIS

The thesis is concerned with a detailed investigation of the vectors and tensors associated with a ruled surface imbedded in an Euclidean space of four dimensions, which tensors require no more than second order differential equations in their treatment. In other words, this is a study of the simple differential geometry of a ruled surface in R_4. The fundamental forms of and important sets of curves on the surface thus defined are presented, with particular interest centering around the unique curves, the so-called striction curve and the quasi-asymptotic curve. The curvature properties of the surface are investigated with respect to the variation of the normal vectors and curvature conic along a generator of the surface. A few formulae for a ruled V_3 in R_4 are appended. While no striking new results have been obtained, this study seems important in that it asks- and answers- "just what tensors are" for a situation just out of the ordinary.

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RULED SURFACES IN EUCLIDEAN FOUR SPACE

I. INTRODUCTION

The object of this thesis is to study a two dimensional ruled surface in a Euclidean space of four dimensions from the point of view of differential geometry. We use V_2 to designate the surface when the representation is such that the first fundamental form is positive definite (cf. p. 3 and p. 40), and R_4 to designate the surrounding Euclidean Space. The notation used is a combination of ordinary vector and tensor notations. The vector $\overline{x}$, or $x^{\varkappa}$, has components x^1, x^2, x^3, x^4 with respect to some allowable coordinate system.[1] Greek indices refer to the surrounding space, R_4, arabic to the surface V_2 itself. The super-bar is reserved for vectors, indices are necessarily used for tensor quantities of order greater than one. Scalar products are indicated by $\overline{x}\cdot\overline{y}$ or $x^{\varkappa}y_{\varkappa}$ as seems most convenient, etc. The formal manipulation and algebra of tensors used is that developed in Schouten and Struik I.[2] We are mainly concerned with those properties of the surface which depend on the

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[1] O.Veblen 1932,3

[2] J.A.Schouten 1935,3

surrounding space, in the sense that the surface cannot be imbedded in an ordinary space of three dimensions, but not on the particular coordinate system employed, that is:- properties of the surface invariant with respect to transformations of the R_4. Numbers in the footnotes refer to the bibliography at the end of the thesis- which is arranged chronologically.

II. GENERAL DISCUSSION

A ruled surface, V_2, in a Euclidean space of four dimensions, R_4, may be considered as generated by a vector moving along a space curve.[1] If the curve, C, is represented by $\overline{x}(t)$ and the moving vector by $\overline{i}(t)$, where the functions $x^{\varkappa}(t)$, $i^{\varkappa}(t)$ $(\varkappa = 1,2,3,4)$ are real single-valued functions of the parameter t sufficiently regular to permit differentiation as may be required, any point, P, on the surface, with coordinates $y^{\varkappa}$, will be given by

(1) $$\overline{y}(t,u) = \overline{x}(t) + u\,\overline{i}(t)$$

where, if $\overline{i}(t)$ is a unit vector, u is the distance of P from the curve C in the positive direction of $\overline{i}(t)$.

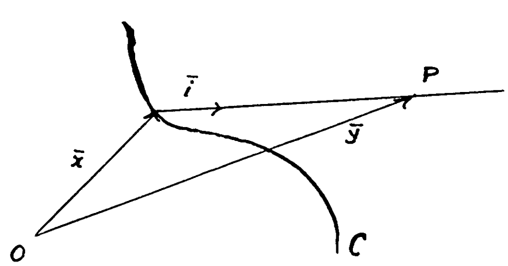

FIG. 1

The curve C is called the directrix of the surface and the vectors $\overline{i}$ the rulings or generators. If all the

[1] W.C.Graustein 1935.1, p.208.

vectors $\overline{i}$ are moved to the same point, they will form a cone which cuts the unit hyper-sphere on the same origin in a curve. This cone is called the director cone of the surface. By unit vector we understand,

(2) $\overline{i}.\overline{i} = 1$

where the . indicates the ordinary scalar product. Distance measured on the surface will be given by

(3) $$ds^2 = d\overline{y}.d\overline{y} = (\overline{x}_t.\overline{x}_t + 2u\overline{i}_t.\overline{x}_t + u^2\overline{i}_t.\overline{i}_t)dt^2 + 2\overline{i}.\overline{x}_t dt\,du + du^2$$

where $\overline{x}_t = \frac{\partial \overline{x}}{\partial t}$ a.s.o.

There is thus defined on the surface a linear element or first fundamental form

(4) $$ds^2 = g_{11}\,dt^2 + 2g_{12}\,dtdu + g_{22}\,du^2$$

or, we may say, a fundamental covariant tensor g_{ij}, $(i,j = 1,2)$ with discriminant $g = g_{11}g_{22} - g_{12}^2$ and a contravariant tensor g^{ij}, where $g^{ij} = G_{ij}/g$, G_{ij} being the cofactor of g_{ij} in its matrix $\|g_{ij}\|$.

If t is the arc length of the curve C

(5) $\overline{x}_t.\overline{x}_t = 1$

and (3) may be written

(6) $$ds^2 = (1 + 2Nu + M^2u^2)dt^2 + 2\cos\vartheta\ dtdu + du^2$$

where $N = \overline{i}_t.\overline{x}_t$, $M^2 = \overline{i}_t.\overline{i}_t$, $\cos\vartheta = \overline{i}.\overline{x}_t$ and ϑ is the angle the rulings make with the curve C.

The orthogonal trajectories of the rulings will be the integral curves of the equation[1]

(7) $\cos\vartheta \, dt + du = 0$

which can be solved by quadratures. Hence we have: <u>A given ruled surface in a Euclidean space of four dimensions can always be transformed so that the directrix is an orthogonal trajectory of the rulings and distance on the surface is measured by</u>

(8) $ds^2 = (1 + 2Nu + M^2u^2)dt^2 + du^2.$

For future reference we note the following formulae calculated from the definitions after (4) and the linear element (8).

(9) Christoffel symbols of the first kind.[2]

$$[11,1] = \tfrac{1}{2}\partial_t g_{11} = \tfrac{1}{2}\partial_t g = u(\bar{x}_t\cdot\bar{i}_{tt} + \bar{x}_{tt}\cdot\bar{i}_t) + u^2\bar{i}_t\cdot\bar{i}_{tt}$$

$$[11,2] = -\tfrac{1}{2}\partial_u g_{11} = -\tfrac{1}{2}\partial_u g = -(\bar{x}_t\cdot\bar{i}_t + u\bar{i}_t\cdot\bar{i}_t)$$

$$[12,1] = \tfrac{1}{2}\partial_u g_{11} = -[11,2]$$

$$[12,2] = 0 \qquad [22,1] = 0 \qquad [22,2] = 0$$

(10) Christoffel symbols of the second kind[3]

$$\Gamma^2_{11} = g^{22}\,[11,2] = [11,2]$$

$$\Gamma^1_{12} = g^{11}\,[12,1] = \frac{1}{2g}\partial_u g = \frac{\bar{x}_t\cdot\bar{i}_t + u\bar{i}_t\cdot\bar{i}_t}{1 + 2u\bar{x}_t\cdot\bar{i}_t + u^2\bar{i}_t\cdot\bar{i}_t}$$

$$\Gamma^2_{12} = 0 \qquad \Gamma^1_{22} = 0 \qquad \Gamma^2_{22} = 0$$

$$\Gamma^1_{11} = g^{11}[11,1] = \frac{1}{2g}\partial_t g = \frac{1}{2}\partial_t(\log g)$$

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[1] L.P.Eisenhart 1909.1

[2] L.P.Eisenhart 1926.1, p.17.

[3] See also J.A.Schouten 1935.3, p.83

(11) The Riemann-Christoffel tensor R_{ijkl} has only one distinct non-zero covariant coordinate.

$$
\begin{aligned}
R_{1212} &= -R_{1221} = R_{2121} \\
&= g_{12}R_{121}^{\cdot\cdot\cdot 1} = R_{121}^{\cdot\cdot\cdot 2} = 2\partial_{[1}\Gamma^{2}_{2]1} + 2\Gamma^{2}_{[1|0|}\Gamma^{0}_{2]1} \\
&= -\partial_2\Gamma^2_{11} + \Gamma^2_{11}\Gamma^1_{21} \\
&= \tfrac{1}{2}\partial^2_{uu}g - \frac{(\partial_u g)^2}{4g} = \bar{i}_t\cdot\bar{i}_t - \frac{(\bar{x}_t\cdot\bar{i}_t + u\bar{i}_t\cdot\bar{i}_t)^2}{g}
\end{aligned}
$$

(12) The Riemann curvature scalar or scalar curvature

$$R = R_{1221}g^{11}g^{22} + R_{2112}g^{11}g^{22} = -(2/g)\,R_{1212}$$

Designating by h^{κ}_{ij} the second covariant derivatives of the functions defining the surface[1], we have

(13) $$
\begin{aligned}
h^{\kappa}_{ij} &= \nabla_i y^{\kappa}_j \\
&= y^{\kappa}_{ij} - \Gamma^{k}_{ij}\, y^{\kappa}_k \quad \text{since} \quad \nabla_j y^{\kappa} = \partial_j y^{\kappa} = y^{\kappa}_j
\end{aligned}
$$

Since $h^{\kappa}_{ji} = y^{\kappa}_{ji} - \Gamma^{k}_{ji}\, y^{\kappa}_k$ and $y^{\kappa}_{ji} = y^{\kappa}_{ij}$ from our assumptions about the nature of the functions y^{κ} and the symmetry of the Christoffel symbols, we see that h^{κ}_{ij} is a symmetric tensor with respect to the covariant indices.

With respect to the contravariant index κ the $h^{\kappa}_{(i)(j)}$ are vectors in the R_4 normal to the surface,[2] and lie in the plane which is completely perpendicular to the tangent plane at any point.[3] The y^{κ}_i are vectors in the

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[1] L.P.Eisenhart 1926.1, p.160.

[2] The (i)(j) indicate that the indices are "dead".

[3] F.S.Woods 1922.2, for perpendicularity in n-dimensional spaces.

tangent plane at any point since they are the tangent vectors to the parametric curves. Then

$y^\kappa_i y^\kappa_i = 1 \qquad y^\kappa_i y^\kappa_j = 0$ from our assumptions (2) and (5)

Differentiating

$$\nabla_j y^\kappa_i y^\kappa_k = 0 \qquad \nabla_j y^\kappa_i y^\kappa_k + y^\kappa_i \nabla_j y^\kappa_k = 0 \qquad (i,j,k = 1,2)$$

and

$$h^\kappa_{ji} y^\kappa_i = 0 \qquad h^\kappa_{ji} y^\kappa_k + y^\kappa_i h^\kappa_{jk} = 0 \qquad (\kappa = 1,2,3,4)$$

for any permutation i, j, k, and the three vectors $h^\kappa_{(i)(j)}$ are normal to the tangent plane at any point of the surface. Since we are working in a four dimensional space, these vectors must lie in the plane which is completely perpendicular to the tangent plane.

There must then be a linear relation among the three vectors. Calculation of their components in terms of the functions defining the surface with the help of the formulae (10) and (11) shows that

$$(14) \qquad \bar{h}_{11} = \frac{1}{g}\begin{vmatrix} \bar{x}_{tt} + u\bar{i}_{tt} & \bar{x}_t + u\bar{i}_t & \bar{i} \\ u(\bar{i}_t.\bar{x}_{tt} + \bar{i}_{tt}.\bar{x}_t) + u^2\bar{i}_t.\bar{i}_{tt} & g & 0 \\ \bar{i}.\bar{x}_{tt} + u\bar{i}.\bar{i}_{tt} & 0 & 1 \end{vmatrix}$$

$$\bar{h}_{12} = \frac{1}{g}\begin{vmatrix} \bar{i}_t & \bar{x}_t + u\bar{i}_t & \bar{i} \\ \bar{i}_t.\bar{x}_t + u\bar{i}_t.\bar{i}_t & g & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$\bar{h}_{22} = 0$ and perforce such a linear relation does exist.

Since $h^{\kappa}_{(i)(j)}$ is a vector perpendicular to the surface

(15) $h^{\kappa}_{(i)(j)} = \overset{1}{h}_{(i)(j)}\ p^{\kappa} + \overset{2}{h}_{(i)(j)}\ q^{\kappa}$, $i,j = 1,2$

where p^{κ} and q^{κ} are unit orthogonal vectors in the normal plane of the surface.[1]

Then, $h^{\kappa}_{ij}\ h^{\kappa}_{ij} = \sum_{a} \overset{a}{h}_{ij}\overset{a}{h}_{ij}$, $(a = 1,2)$

and the Gauss-Codazzi equations of the surface are

(16) $$R_{1212} = \sum_a \overset{a}{h}_{12}\overset{a}{h}_{12}$$
$$= \bar{h}_{12}.\bar{h}_{12}$$

(17) $\nabla_{[i}\overset{1}{h}_{j]k} = \underset{2}{\overset{1}{v}}_{[i}\overset{2}{h}_{j]k}$, where $\underset{2}{\overset{1}{v}}_{i} = -\underset{1}{\overset{2}{v}}_{i} = p^{\kappa}\nabla_i q^{\kappa}$

and $\nabla_{[i}\overset{2}{h}_{j]k} = \underset{1}{\overset{2}{v}}_{[i}\overset{1}{h}_{j]k}$

and the Ricci condition of integrability is

(18) $$\nabla_{[i}\underset{2}{\overset{1}{v}}_{j]} = g^{kl}(\overset{1}{h}_{k[i}\overset{2}{h}_{l]j]})$$

These equations are in the tensor form, but we can also derive 'vector' equations[2]

(19) $$\nabla_1\bar{h}_{12} - \nabla_2\bar{h}_{11} = -R_{121}^{\ \ \ \ 2}\ \bar{y}_2$$
$$-\nabla_2\bar{h}_{21} = R_{212}^{\ \ \ \ 1}\ \bar{y}_1$$

The second fundamental form of the surface is defined by means of these quantities $\bar{h}_{ij}$.

(20) $$\Psi = \bar{h}_{11}dt^2 + 2\bar{h}_{12}dtdu$$

is the second vector form. Its magnitude $\psi = \Psi.\Psi$

(21) $$\psi = \bar{h}_{11}.\bar{h}_{11}dt^4 + 4\bar{h}_{11}.\bar{h}_{12}dt^3du + 4\bar{h}_{12}.\bar{h}_{12}dt^2du^2$$

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[1]This can always be done. Cf. Eisenhart 1926.1, p.143.

[2]For the relation between (16)-(19) see Wilson 1916.2,p.305.

is defined by some authors[1] as the second form, but we shall prefer to use (20).

<u>The Gaussian curvature of a real ruled surface in four dimensional Euclidean space is always negative or zero.</u> The Gaussian curvature of the surface at any point is $K = \frac{1}{2}R$, where R is the scalar curvature defined in (12). From this definition and with the help of formulae (11)

(22) $$K = \frac{-R_{1212}}{g} = \frac{-\bar{h}_{12}\cdot\bar{h}_{12}}{g}$$

If we calculate $\bar{h}_{ij}$ for the surface in the form (6), we have

(14)a.

$$h_{11} = \frac{1}{g}\begin{vmatrix} \bar{x}_{tt} + u\bar{i}_{tt} & \bar{x}_t + u\bar{i}_t & \bar{i} \\ u(\bar{i}_t\cdot\bar{x}_{tt} + \bar{x}_t\cdot\bar{i}_{tt}) + u^2\bar{i}_t\cdot\bar{i}_{tt} & 1 + 2Nu + M^2u^2 & \cos\vartheta \\ \bar{i}\cdot\bar{x}_{tt} + u\bar{i}\cdot\bar{i}_{tt} & \cos\vartheta & 1 \end{vmatrix}$$

$$\bar{h}_{12} = \frac{1}{g}\begin{vmatrix} \bar{i}_t & \bar{x}_t + u\bar{i}_t & \bar{i} \\ N + uM^2 & 1 + 2Nu + M^2u^2 & \cos\vartheta \\ 0 & \cos\vartheta & 1 \end{vmatrix}$$

$$\bar{h}_{22} = 0$$

and

(23) $$\bar{h}_{12}\cdot\bar{h}_{12} = \frac{M^2\sin^2\vartheta - N^2}{g}$$

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[1] L.P. Eisenhart 1926.1, p.166.

and

(24) $$K = \frac{-(M^2\sin^2\vartheta - N^2)}{g^2}$$

Then

(25) $$M^2\sin^2\vartheta - N^2 = 0$$

is the condition that the Gaussian curvature vanish. If $\overline{h}_{12}$ is a non-isotropic vector, it must then be a null vector for K to vanish, and we have

(26) $$\overline{i}_t = a\overline{x}_t + b\overline{i}$$

where $a = \frac{N + uM^2}{Nu + \sin^2\vartheta}$ and $b = \frac{-\cos\vartheta}{Nu + \sin^2\vartheta}$

The tangent plane at any point on the ruled surface is represented by the bivector

(27) $$T^{\kappa\lambda} = (x_t^{[\mu} + ui_t^{[\mu})i^{\nu]} \quad , \quad \kappa,\lambda,\mu,\nu = 1,2,3,4$$

If $K \equiv 0$, $T_0^{\kappa\lambda} = a(1 + u)(x_t^{[\mu} i^{\nu]})$, which is independent of u, and the tangent plane is fixed along a generator. The surface is then the tangent surface of a twisted curve.

Definition: <u>V_2 in R_4 for which $K \equiv 0$ are developable surfaces</u>. Then it follows from above that <u>Ruled surfaces which are developable are twisted curve surfaces</u>. However, in R_4 all surfaces for which $K \equiv 0$ are not necessarily ruled developables.[1] For in general $K = \frac{1}{2}R = -\frac{R_{1212}}{g} = \frac{\overline{h}_{11}\cdot\overline{h}_{22} - \overline{h}_{12}\cdot\overline{h}_{12}}{g}$ and the numerator may vanish

[1] E. B. Wilson 1916.2, p.343.

without $\overline{h}_{12} = 0$ and $\overline{h}_{22} = 0$, as for ruled developables, or without the terms vanishing separately. We shall take (25) as the condition defining ruled developable surfaces when we have represented them in the form (6).

<u>There are no skew ruled surfaces of constant curvature</u>. Substituting for g its value in terms of t and u in (24)

$$K = \frac{-(M^2 \sin^2\vartheta - N^2)}{(1 + 2Nu + M^2u^2)^2}$$

The numerator does not depend on u, and if K is to be a constant

$$N = 0, \ M^2 = 0.$$

But then K = 0, and the surface is developable.

III. THE STRICTION CURVE

We consider now only skew surfaces, that is surfaces for which K is not identically zero. For the infinitely distant points on the rulings the surface behaves like a developable since as $u \to \infty$, $K \to 0(1/u^4)$ from (22). In general K is finite for all finite points of a ruling, but there may be one or more rulings along which K = 0, which rulings may be found by solving (25) for t. For any regular generator K will have a minimum value at the point

(28) $$u_o = \frac{-N}{M^2}$$

found by differentiating expression (24) with respect to u and equating to zero. The value of the Gaussian curvature at the point u_o is

(29) $$K_o = \frac{-M^4}{M^2 \sin^2 \vartheta - N^2}$$

Then at any point

(30) $$K = \frac{K_o}{(1 - K_o(u - u_o)^2)^2}$$

If we ask for the minimum distance between two consecutive generators, t and t + dt, on the surface, we must solve the two following equations[1]

(31) $$\frac{\partial(ds^2)}{\partial(du)} = 2 \cos \vartheta \, dt + 2du = 0$$

$$\frac{\partial(ds^2)}{\partial u} = (2N + 2M^2u)dt^2 = 0$$

whence

(32) $$u = \frac{-N}{M^2} \quad \text{and} \quad \frac{dt}{du} = \frac{-1}{\cos \vartheta}$$

We call the point so defined on the generator the <u>central or striction point</u> of the ruling. The locus of the striction point is the striction curve of the surface. The minimum distance between consecutive generators is then

(33) $$d\sigma^2 = dt^2(1 - N^2/M^2) - \cos^2 \vartheta \, dt^2 = \frac{dt^2 \, (M^2 \sin^2 \vartheta - N^2)}{M^2}$$

The struction curve is in general not an orthogonal

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[1] K. Kommerell 1911.2

trajectory of the rulings since $d\sigma^2$ is not its linear element. If the surface is developable, the $d\sigma^2 = 0$ and and consecutive generators intersect. We might have taken this as a definition of developable surfaces. For such a surface the striction line ccincides with the edge of regression. If we compare (32) and (28) we see that <u>the striction curve is the locus of minimum Gaussian curvature</u>, and moreover <u>this curvature is symmetric about the striction point on any ruling</u>.

We may accept as a theorem of projective geometry in R_4 that two skew lines determine an R_3. Then any two rulings of the V_2 will determine an R_3 in which there will be one line perpendicular to the two given. Thus the argument found in textbooks of ordinary three dimensional differential geometry[1] defining the striction point as the foot of the common perpendicular to two infinitesimally close rulings may also be followed for a ruled surface in R_4.

<u>The tangent planes along a ruling all lie in the same</u> R_3. From (27) this R_3 has covariant components

(34) $$v_\lambda = e_{\lambda\kappa\mu\nu} x_t^\kappa i_t^\mu i^\nu$$

where $e_{\lambda\kappa\mu\nu} = \pm 1$ depending on the permutation of the

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[1] W.C. Graustein 1935.1, p.209

indices[1], and its normal vector is

(35) $$v^{\lambda} = e^{\lambda}_{\cdot\,\varkappa\mu\nu}\, x^{\varkappa}_{t}\, i^{\mu}_{t}\, i^{\nu}$$

Then the angle φ between the tangent plane at a point u and the tangent plane at the striction point may be shown to be a function of the distance between the two points.[2]

(36) $$\tan\varphi = \frac{u - u_0}{\beta}$$

where $\beta^2 = \dfrac{M^2\sin^2\vartheta - N^2}{M^2}$

and the tangent planes rotate around the rulings. From (35) $v^{\lambda}v_{\lambda} = M^2\sin^2\vartheta - N^2$, and the distribution parameter $\beta = \pm\frac{|v|}{M}$. The normal vector $\frac{v^{\lambda}}{M}$ to the surface which lies outside the tangent R_3 along a ruling then plays the part in R_4 that ρ does in arguments in R_3.

Now, the geodesic curvature (as we shall show later) of the directrix is $\frac{\cos\omega}{r_1}$, where ω is the angle between the osculating plane of the curve and the tangent plane of the surface, and r_1 is the radius of first curvature. From the definitions of the quantities involved

(37) $$\frac{\cos\omega}{r_1} = \frac{-N}{\sin\vartheta} - \frac{d\vartheta}{dt}$$

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[1] A.J.McConnell 1936.2, p.8 and p.298 for e systems of special relativity

[2] L.P.Eisenhart 1909.1

If the striction curve is the directrix and also a geodesic, $N = 0$, $\frac{\cos\omega}{r_1} = 0$ and $\cos\vartheta$ is a constant, the rulings cut the directrix under a constant angle. If $N = 0$, $\frac{d\vartheta}{dt} = 0$, the curve must be a geodesic. If $\frac{d\vartheta}{dt} = 0$, $\frac{\cos\omega}{r_1} = 0$ the curve must be the striction curve. So we have Bonnet's theorem[1]: <u>If a curve on a ruled surface possesses two of the following properties, it possesses the third; 1. The curve is the striction curve; 2. The curve is a geodesic; 3. The curve cuts the rulings with a fixed angle</u>.

The properties of the striction curve of the ruled surface in R_4 mentioned above are precisely those of such a surface in R_3. We have been unable to find any which do not carry over, and, conversely, any which do not occur for ruled V_2 in R_3. Further properties are to be found in papers by O. Bonnet, E. Bour, H. Beck and J. Krames.[2]

IV. CURVATURE PROPERTIES OF THE RULED SURFACE

Any non-minimal curve on the surface defined by $y^{\varkappa}(s)$, $(\varkappa = 1,\ldots,4)$ or by $x^i(s)$ $(i=1,2)$, where s is the length of the curve, has an absolute curvature

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[1] O. Bonnet 1867.1

[2] E. Bour 1862.1,2; H. Beck 1928.1; J. Krames 1928.2

vector with respect to the surrounding space defined by[1]

(38) $$n^{\kappa} = \frac{dy^{\lambda}}{ds} \; {}'\nabla_{\lambda} \frac{dy^{\kappa}}{ds}$$

But since $\frac{dy^{\kappa}}{ds} = \frac{\partial y^{\kappa}}{\partial x^{i}} \frac{dx^{i}}{ds}$

(39) $$n^{\kappa} = \frac{\partial y^{\kappa}}{\partial x^{j}} \frac{dx^{i}}{ds} \nabla_{i} \frac{dx^{j}}{ds} + h^{\kappa}_{ij} \frac{dx^{i}}{ds} \frac{dx^{j}}{ds}$$

where

(40) $${}'n^{\kappa} = \frac{\partial y^{\kappa}}{\partial x^{i}} \frac{dx^{j}}{ds} \nabla_{j} \frac{dx^{i}}{ds}$$ is the relative curvature vector of the curve, or the component of the absolute curvature in the surface, and

(41) $${}''n^{\kappa} = h^{\kappa}_{ij} \frac{dx^{i}}{ds} \frac{dx^{j}}{ds}$$ is the component orthogonal to the surface, called the normal curvature vector. Or,

(42) $$n^{\kappa} = {}'n^{\kappa} + {}''n^{\kappa}$$

If r_1 is the radius of curvature of the curve in the given direction with respect to R_4, ρ_g the radius of curvature of the curve with respect to the V_2, and R the radius of curvature of the geodesic in that direction

(43) $$\frac{1}{r_1^2} = \frac{1}{\rho_g^2} + \frac{1}{R^2}$$

(42) expresses the extended Meusnier theorem for a curve on a surface in R_4. All curves through a point P on the surface in the same direction have the same normal curvature. Since we have a definite metric (8), there

[1] J.A. Schouten 1938.1, p.80.

is an angle ω such that

(44) $$\frac{\cos\omega}{r_1} = \frac{1}{\rho_g} \qquad \frac{\sin\omega}{r_1} = \frac{1}{R}.$$

From (41) and (20) we see that the normal curvature vector for a direction at a point is the second vector form of the surface. The normal curvature properties of a surface then may be derived from a study of this quadratic form. Before we begin this study we note that the mean curvature vector at a point is

(45) $$M^{\kappa} = \tfrac{1}{2}g^{ij}h^{\kappa}_{ij} = \frac{1}{2g}\, h^{\kappa}_{11}$$

and the mean curvature is

(46) $$M = \frac{\overline{h}_{11}\cdot\overline{h}_{11}}{4g^2}$$

if the surface is referred to orthogonal parameters (8). If the linear element is (6)

(47) $$M^{\kappa} = \frac{h^{\kappa}_{11} - \cos\vartheta\; h^{\kappa}_{12}}{2g}$$

The ruled surface is a minimal surface, in the sense of least area, if M^{κ} is a null vector.[1] But this requires that

(48) $$g(\overline{x}_{tt} + u\overline{l}_{tt}) + u(\overline{l}_t\cdot\overline{x}_{tt} + \overline{x}_t\cdot\overline{l}_{tt} + u\overline{l}_t\cdot\overline{l}_{tt})(\overline{x}_t + u\overline{l}_t) + g(\overline{l}\cdot\overline{x}_{tt} + u\overline{l}_{tt})\,\overline{l} = 0$$

which is equivalent to the condition that g vanish. The rank of the first fundamental form is then one and the

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[1] E.B. Wilson 1916.2, p.325

surface can be imbedded in an R_3. That is: <u>The only minimal ruled surfaces in R_4 are those of R_3, namely the right helicoid</u>.

Since $''n^{\kappa}$ is perpendicular to the surface at any point for all directions on the surface through that point, P, it lies in the normal plane to the surface at that point and the end point of $''n^{\kappa}$ traces a curve in this plane as the direction changes. From (41) we see that $''n^{\kappa}$ becomes infinite only if $ds^2 = 0$; but these are isotropic directions on the surface, and as long as we deal only with real directions $''n^{\kappa}$ must describe a finite curve. If we consider the surface referred to orthogonal parameters (8) and take as a system of orthogonal unit vectors on the surface those in the directions of the parametric curves,

(49) $\underset{1}{i}^a\!: 0,1 \qquad \underset{2}{i}^a\!: \frac{1}{\sqrt{g}}, 0 \quad (a=1,2)$

any direction dt:du will then be represented by

(50) $i^a = \cos\alpha \, \underset{1}{i}^a + \sin\alpha \, \underset{2}{i}^a$

and $''n^{\kappa}$ for i^a

(51) $''n^{\kappa} = i^a i^b h^{\kappa}_{ab} = M^{\kappa} + \sin 2\alpha \, ''\overset{2}{\underset{1}{n}}{}^{\kappa} - \frac{1}{2}\cos 2\alpha \, ''\underset{2}{n}^{\kappa}$

where $''\underset{2}{n}^{\kappa} = \underset{2}{i}^a \underset{2}{i}^b h^{\kappa}_{ab} = \frac{1}{g} h^{\kappa}_{11}$

$$\overset{2}{\underset{1}{''n}}{}^{\varkappa} = \underset{1}{i}^{a}\underset{2}{i}^{b} h^{\varkappa}_{ab} = \frac{1}{\sqrt{g}}\, h^{\varkappa}_{12}$$

$$\underset{1}{''n}{}^{\varkappa} = \underset{1}{i}^{a}\underset{1}{i}^{b} h^{\varkappa}_{ab} = 0$$

<u>The normal curvature in the direction of the rulings is zero</u>. From (51) we see that <u>the normal curvature vector traces an ellipse in the normal plane with center $M^{\varkappa}$ and a pair of conjugate radii</u> $\frac{1}{2}\underset{2}{''n}{}^{\varkappa}$, $\overset{2}{\underset{1}{''n}}{}^{\varkappa}$. We call this the curvature conic. It is clear that the conic passes through the surface point P since for $\alpha = 0$, $''n^{\varkappa} = 0$.[1] Since $\bar{h}_{11}.\bar{h}_{12}$ is quadratic in terms of the parameter u, we note that there are two points on each ruling where the mean curvature vector is an axis of the curvature conic, when $\bar{h}_{11}$ is perpendicular to $\bar{h}_{12}$.[2]

Conjugate directions at a point are defined as those for which

(52) $\quad \bar{h}_{11}\, dt\, \delta t + \bar{h}_{12}\, (dt\, \delta u + du\, \delta t) = 0.$

At a point for which $\bar{h}_{11} \neq \bar{h}_{12}$ (52) is satisfied by $dt = 0$, $\delta t = 0$. That is: <u>the conjugate directions at a point coincide with and are in the direction of the ruling through the point</u>.

Asymptotic directions by definition are those for which

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[1] The curvature properties may also be studied by means of an extended Dupin indicatrix, the curve in which an R_3 perpendicular to $M^{\varkappa}$ cuts the surface.

[2] See the paragraph on quasi-asymptotic curves on the surface and the diagrams and example at the end of this section.

(53) $\qquad \bar{h}_{11}dt^2 + 2\bar{h}_{12}\,dtdu = 0.$

At any point then for which $\bar{h}_{11} \neq \bar{h}_{12}$, there is only one asymptotic direction, that of the generator through the point. *The conjugate directions and the asymptotic direction at a regular point on a ruled surface in an R_4 coincide with the direction of the ruling through the point.*

We may note, however, that those authors who define the second fundamental form of the surface by (21) obtain in addition to the rulings, two conjugate directions and two definite asymptotic directions which are orthogonal.[1]

If we define those directions for which the normal curvature has maxima and minima as the principal directions at any point, we find four such; one of which is in the direction of the generator, the others are obtained as the roots of the equation[2]

$$(54) \quad \Big\{-4g_{11}h^{\kappa}_{11}h^{\kappa}_{12}\,\lambda^3 + (4h^{\kappa}_{11}h^{\kappa}_{11} - 8g_{11}h^{\kappa}_{12}h^{\kappa}_{12})\,\lambda^2 + 12h^{\kappa}_{11}h^{\kappa}_{12}\,\lambda + 8h^{\kappa}_{12}h^{\kappa}_{12}\Big\}\,\lambda = 0$$

(If we used the more general representation of the surface (6), (54) would be more symmetric in form, but still of the fourth degree in λ.) ($\lambda = dt:du$). The lines of curvature on the surface are the integral curves of (54).

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[1] E.B. Wilson 1916.2.

[2] Differentiate $'n^{\kappa}h^{\kappa}$ (41) and equate to zero.

At least two of the roots of (54) are real since there is at least one change of sign in the coefficients in the expression in brackets, namely the first and the third terms. If we apply the criteria for equal roots of a cubic[1], we find that the directions λ_{α} (α = 1,2,3,4) are in general distinct. If the surface is developable and $a_{11}^2 + 12\, g_{11}\, a_{12}^2 > 0$, two of the directions coincide with that of the generator, the other two are real and distinct. When $\bar{h}_{11}$ is perpendicular to $\bar{h}_{12}$ the parametric directions are two of the principal directions and if $2g_{11}\, a_{22} > a_{11}$ the other two directions are real and make equal angles with the parametric curves. At a regular point the parametric curves cannot both be lines of curvature since $\lambda = \infty$ obviously does not satisfy (54). If we wish to determine whether any two of the other three are orthogonal we must solve

$$\lambda_{\alpha} \lambda_{\beta} + 1 = 0, \qquad \alpha, \beta = 2,3,4 \quad \text{if} \quad \lambda_{1} = \frac{dt}{du} \equiv 0.$$

And it may happen that one pair of principal directions is orthogonal but two pair cannot be so. In fact, Moore and Wilson[2] show that if the directions λ_{α} (α =1,2,3,4) are perpendicular in pairs the curvature ellipse must lie in a plane perpendicular to M^{κ}. Since M^{κ} is a

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[1] F. Cajori 1904.1.

[2] 1916.2, p.350-354.

diameter of the curvature conic for our surface, we have the result: <u>the principal directions</u> λ_{α} <u>cannot be orthogonal in pairs</u>.

The Gaussian curvature is related to the principal curvatures by[1]

$$(55) \qquad K = \frac{1}{6} \sum_{\alpha,\beta} \frac{1}{R_{\alpha} R_{\beta}} \quad , \quad \alpha,\beta = 1,2,3,4$$

where the principal curvatures are the normal curvatures in the principal directions.

Moore and Wilson[2] defining principal directions at any point as those perpendicular directions for which differential changes of normals are perpendicular, obtain

$$(56) \qquad \frac{1}{2g} \overline{h}_{11} \cdot (-g\overline{h}_{12} dt^2 + \overline{h}_{11} dt du + \overline{h}_{12} du^2) = 0$$

and find only two such directions at a point, namely

$$(57) \quad \frac{dt}{du} = \frac{\overline{h}_{11} \cdot \overline{h}_{11} \mp \sqrt{(\overline{h}_{11} \cdot \overline{h}_{11})^2 + 4g(\overline{h}_{11} \cdot \overline{h}_{12})^2}}{g\overline{h}_{11} \cdot \overline{h}_{12}}$$

We believe that our definition agrees better with the more general treatment of the subject found in tensor analysis texts.[3]

In general the tensor h_{ij}^{κ} is of rank two with respect to the index κ [4], since

$$(58) \qquad h_{ij}^{\kappa} v_{\kappa} = 0$$

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[1]J.A. Schouten 1938, 1, p.136.

[2]1916.2, p.350.

[3]See L.P. Eisenhart 1926.1

[4]J.A.Schouten 1938.1.

is a set of two equations in four unknowns which has two linearly independent solutions for $v_\varkappa$. If $h_{11}^\varkappa = \rho h_{12}^\varkappa$, there will be only one solution for (58). A point of the surface for which the rank of $h_{ij}^\varkappa$ as just defined is two is a planar point. A point is called axial if $h_{ij}^\varkappa$ is of rank one at that point. The points of a ruled surface then are planar points in general. But we may ask, are there any points which are axial? The answer is: there is one axial point on each ruling of a ruled surface in R_4. From (14) $h_{11}^\varkappa = \rho h_{12}^\varkappa$, if

$$(59) \qquad x_{tt}^\varkappa + u i_{tt}^\varkappa = a x_t^\varkappa + b i_t^\varkappa + c i^\varkappa$$

whence

$$(60) \qquad u = u_1 = \frac{-x_{tt}^{[\varkappa} x_t^\lambda i_t^\mu i^{\nu]}}{i_{tt}^{[\varkappa} x_t^\lambda i_t^\mu i^{\nu]}}\,, \qquad (\varkappa,\lambda,\mu,\nu = 1,2,3,4)$$

locates a point on the ruling unless the numerator and denominator of (60) both vanish, since each is a fourth order determinant. But this means that $x_{tt}^\varkappa$ and $i_{tt}^\varkappa$ are always linear functions of $x_t^\varkappa, i_t^\varkappa, i^\varkappa$ and the surface lies in an R_3. So we have the result stated.

(60) says that there is one point on each ruling such that the principal normals of all the curves of the surface through that point lie in the R_3 tangent to the surface. The locus of such points is called a quasi-asymptotic curve.[1] Indeed from (53) we see that the

[1] E. Bompiani 1914.1

direction of this curve may be an asymptotic direction.[1]
At u_1 the curvature ellipse becomes a line segment.

$$(61) \qquad {}''n^{\kappa} = \left\{ \frac{dt^2}{ds} + 2\rho \frac{dt}{ds}\frac{du}{ds} \right\} h^{\kappa}_{11}$$

The extremities of the line segment are obtained when

$$\frac{dt}{ds}\left\{ \frac{dt}{ds} + 2\rho \frac{du}{ds} \right\}$$

attains its maximum or minimum. This occurs for

$$\frac{dt}{du} = \frac{-1 \pm \sqrt{1 + 4\rho^2 g_{11}}}{-2g_{11}}$$

These are then the principal directions at the quasi-asymptotic point.

<u>At the quasi-asymptotic point h^{κ}_{11}, and hence M^{κ}, lies in the R_3 tangent to the surface.</u>

From (14)

$$h^{\kappa}_{11} = A(t,u)(x^{\kappa}_{tt} + ui^{\kappa}_{tt}) + B(t,u)(x^{\kappa}_{t} + ui^{\kappa}_{t}) + C(t,u)i^{\kappa}$$

At $u = u_1$, from (69) we have

$$h^{\kappa}_{11} = A(t,u)(ax^{\kappa}_{t} + bi^{\kappa}_{t} + ci^{\kappa}) + B(t,u)(x^{\kappa}_{t}+ui^{\kappa}_{t}) + C(t,u)i^{\kappa}$$
$$= a'(t)x^{\kappa}_{t} + b'(t)i^{\kappa}_{t} + c'(t)i^{\kappa}$$

which from (34) defines the tangent R_3 to the surface.

The locus of quasi-asymptotic points on the surface has also a projective definition. It is the curve of intersection of a given ruled surface and its transversal surface, where the latter is defined as the

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[1] V. Hlavaty 1924.2

locus of straight lines which intersect three consecutive rulings of the given surface. The reader is referred to a series of papers by Ranum[1] and Stouffer[2] dealing with this approach.

Bompiani[3] has also defined quasi-asymptotic curves of the second kind on a surface as those curves whose osculating spaces contain the tangent plane to the surface and which have four point contact with the curve.

In our notation this condition may be expressed by saying: any vector $v^\kappa = \alpha(x_t^\kappa + u i_t^\kappa) + \beta i^\kappa$ must lie in the R_3 : $e_{\kappa\lambda\mu\nu} \frac{dy^\lambda}{ds} \frac{d^2y^\mu}{ds^2} \frac{d^3y^\nu}{ds^3}$, or

$$(62) \qquad e_{\kappa\lambda\mu\nu} v^\kappa \frac{dy^\lambda}{ds} \frac{d^2y^\mu}{ds^2} \frac{d^3y^\nu}{ds^3} = 0$$

Since α and β are arbitrary, (62) may be written as two conditions. If we work with both, i.e. making the determinants into sums of determinants containing single functions as elements, we are led to the same differential equation. This equation is

$$(63) \quad 3(f_{11} + u f_{12}) \frac{d^2u}{dt^2} - 6 f_{12} \left(\frac{du}{dt}\right)^2 + \Big\{3 f_9 + 2 f_{13} + u(3 f_{10} + 2 f_{14})\Big\} \frac{du}{dt} + f_8 u^3 + (f_4 + f_6 + f_7) u^2 + (f_2 + f_3 + f_5) u + f_1 = 0$$

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[1] 1912.1, 1915.1, 1915.2

[2] 1927.1

[3] 1914.1 See also Lane, 1932.1, p.60.

where the f_i are determinants formed from $(x_t^\kappa, i^\kappa, i_t^\kappa, i_{tt}^\kappa, x_{tt}^\kappa, x_{ttt}^\kappa, i_{ttt}^\kappa)$ e.g. $f_1 = |x_t^\kappa\ i^\kappa x_{tt}^\kappa\ x_{ttt}^\kappa|$ a.s.o. This equation is the same as that obtained by Lane(1) p. 60, and we see there are ∞^2 quasi-asymptotic curves of the second kind on a ruled surface in R_4.

A developable ruled surface possesses no quasi-asymptotic curve. For if the surface is developable, $K = 0$, h_{12}^κ is a null vector, h_{11}^κ is not null and there are two real coincident asymptotic directions at a point, namely those of the rulings.

In order to visualize the curvature properties of a ruled surface in R_4, we present the following theorem and accompanying example and diagrams as a summary of such properties.

Theorem: The normal planes to a ruled surface along a generator rotate about a fixed direction in the space R_4 in such a way that the tangent of the angle between the normal plane at a point $P(t,u)$ and the normal plane at the striction point $P_0(t,u_0)$ is a linear function of the distance $|u-u_0|$; and, moreover, the angle is equal to that between the tangent planes at the two points. From (15) the vectors normal to the surface at any point P may be expressed

$$h_{ij}^\kappa = \overset{1}{h}_{ij}\, p^\kappa + \overset{2}{h}_{ij}\, q^\kappa \quad (i,j=1,2)$$

where $\overset{a}{h}_{ij}$ $(a = 1,2)$ are scalar functions depending on the point P and p^κ and q^κ are unit vectors in the normal plane, N, to the surface and mutually orthogonal. In general p^κ and q^κ are invariant functions of (t,u)[1] but are determined only up to a rotation. Let us take p^κ the unit vector in the direction of the normal v^κ (35). p^κ then depends only on the parameter t and not on u and is the same for all normal planes on one ruling. $q^\kappa(t,u)$ will, in some sense, give us, as it varies, with u, the change in the normal planes. $\{q^\kappa, p^\lambda, i^\mu, y_t^\nu\}$ forms at any point P(t,u) a set of four mutually orthogonal vectors. q^κ lies in the tangent R_3 to the surface since it is perpendicular to v^κ $(= p^\kappa)$ and is in fact the intersection of that space with the normal plane. $\{q^\kappa, i^\lambda, y_t^\mu\}$ forms at any point P(t,u) a set of three mutually orthogonal vectors in the tangent R_3. At the directrix $[x_t^\kappa i_t^\lambda]$ is a plane perpendicular to i^κ. q^κ and y_t^κ are parallel to $[x_t^\kappa i_t^\lambda]$ for all u. $[y_t^\kappa i^\lambda]$ is the tangent plane, T, at P and the angle between this plane and the tangent plane T_0 at the striction point of the ruling, P_0, is given by

$$(36) \quad \tan \varphi = \frac{u - u_0}{\beta}$$

$[q^\kappa i^\lambda]$ defines a family of planes, $[S]$, in one-one correspondence with the family of tangent planes $[T]$, and

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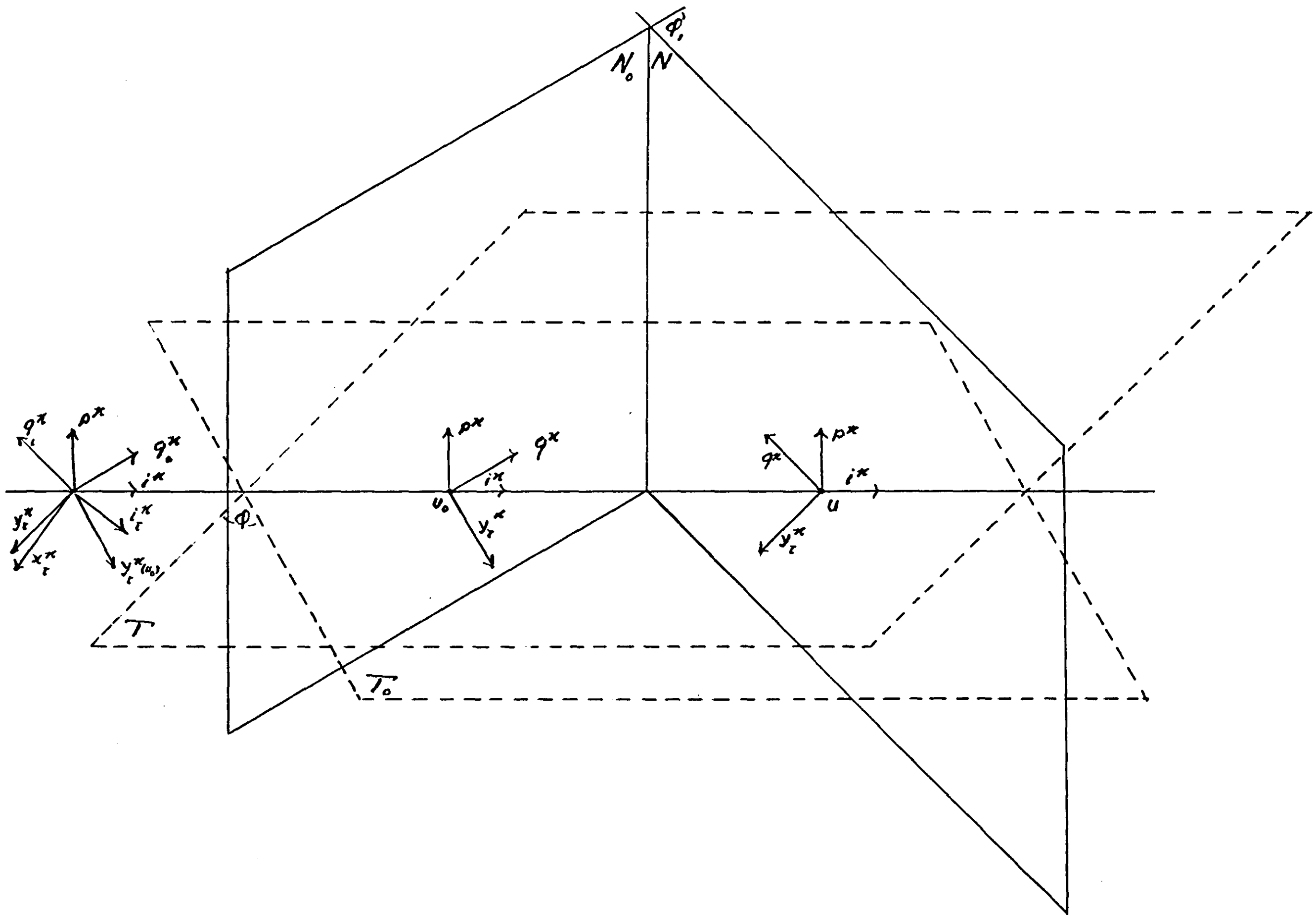
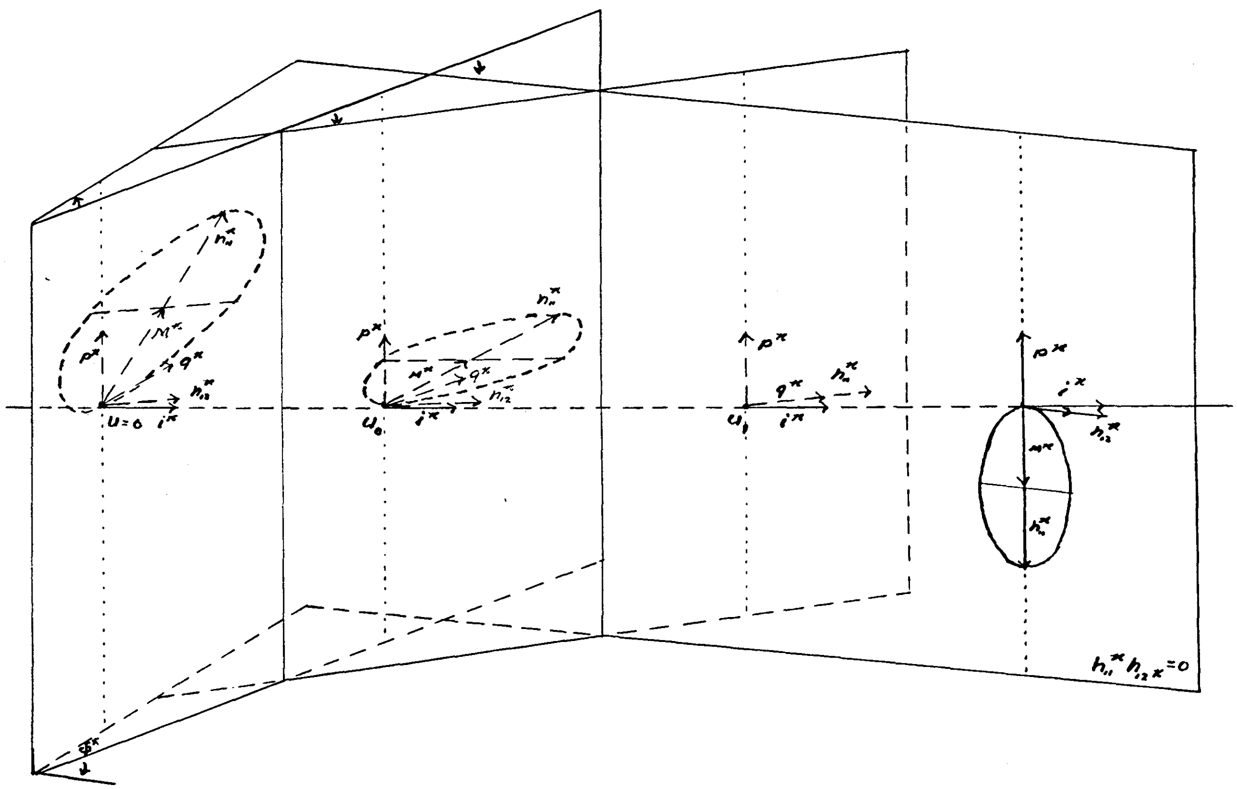
[1] E.B. Wilson 1916.2, p.309.



Normal planes along a generator of a ruled surface in R_4

Figure 2

$$''n^{\varkappa} = M^{\varkappa} + \sin(2\alpha)\frac{1}{\sqrt{g}}h_{12}^{\varkappa} - \frac{1}{2}\cos(2\alpha)\frac{1}{g}h_{11}^{\varkappa}$$



Variation of curvature conic along a generator of a ruled surface in R_4 referred to orthogonal parameters

Figure III

corresponding planes are perpendicular. Then the angle between the plane of $\{S\}$ at P and the plane at P_0 is φ. Or, the angle between $q^{\kappa}(t,u)$ and $q^{\kappa}(t,u_0)$ is φ. $[p^{\kappa}q^{\lambda}]$ defines the family of normal planes, $[N]$, each plane of which contains the common direction $p^{\kappa} = v^{\kappa}$. Then since q^{κ} is parallel to $[x_t^{\kappa} i_t^{\lambda}]$ for all u, as u varies $[p^{\kappa}q^{\lambda}]$ rotates about p^{κ} and the angle between N and N_0 is that between $q^{\kappa}(u)$ and $q^{\kappa}(u_0)$, namely φ.

To describe more concretely the locus generated by the planes $[N]$ if we call the line in the direction i^{κ} the x-axis, that in the direction p^{κ} the y-axis, a plane perpendicular to i^{κ} is represented by

$$x = \lambda$$
$$w = ay + bz$$

This plane to be N, must contain the line

$$x = \lambda$$
$$z = 0$$
$$w = 0$$

Thus, N is given by

$$x = \lambda$$
$$w = bz$$

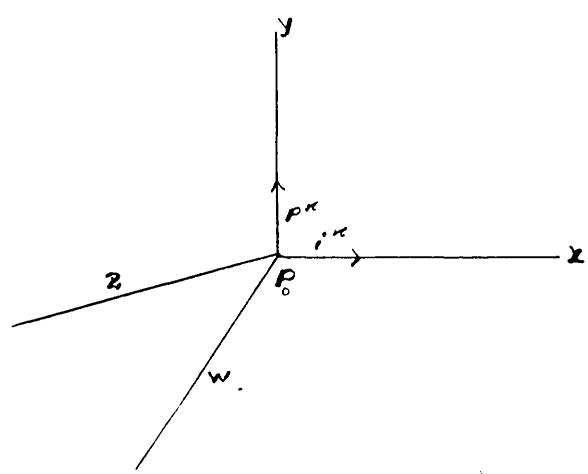


FIG. 4

There is a linear relation between b and λ since the normal planes rotate with the angle φ (36). Say $b = k\lambda$ where k is a constant. (Then the frame of reference has been set up at P_0.) Thus the locus of [N] is:

$$w = kxz,$$

which is a cylinder formed by lines parallel to the y-axis in points of an hyperbolic parabaloid in the wxz-space.

Example: Take a circle in each of two completely perpendicular planes in R_4 defined in terms of the same parameter and consider points as corresponding which have the same value of the parameter. Join corresponding points by straight lines. Then we have the ruled surface defined by

$$\overline{y}(t,u);\ \text{aucost},\quad \text{ausint},\quad \text{bcost} - \text{aucost},\quad \text{bsint} - \text{ausint}$$

where a and b are arbitrary but normalized so $2a^2 = b^2 = 1$. The $\overline{x}(t)$: 0, 0, b cost, b sint b sint can, in our language, be considered the directrix and $\overline{l}(t)$:(a cost, a sint, -a cost, -a sint) the vector in the direction of the rulings for the surface

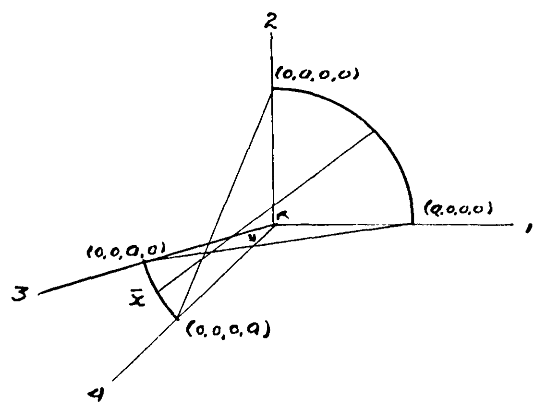


FIG. 5

$$ds^2 = (1 - 2abu + u^2)dt^2 + du^2$$

and t and u are orthogonal parameters. Substitution in the general formulae show that the Gaussian curvature

$$K = \frac{-a^2}{(1 - 2abu + u^2)^2}$$

The quasi-asymptotic point is $u_1 = \infty$, and the striction point $u_o = -ab$. The striction curve $u_o = -ab$ is a particular curve of the family $u = c$ each of which possesses the properties: 1. that the scalar curvatures have ratios which are constants not zero; 2. the curves project orthogonally on to m perpendicular planes into curves so tangents and normals are projections of corresponding tangent and normal spaces of C. These curves have been studied in some detail by M. Syptak[1] and the associated ruled surface by M. Borůvka[2] and are a generalization of helices.

V. THE DEFORMATION OF RULED SURFACES

The question which concerns us now is: can a given ruled surface be transformed into another ruled surface without strain, and if so, in what ways?[3] This is the problem of the applicability of spaces and not that of continuous deformation.[4] Since the necessary

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[1] 1934.2

[2] 1931.1

[3] cf. the argument for surfaces in R_3 based on a study of asymptotic lines, which treatment cannot be carried over into higher spaces. G. Darboux 1896.1 and L. Bianchi 1899.1

[4] cf. L.P. Eisenhart 1926.1. chapter 6.

and sufficient condition that symmetric tensors g_{ij}, h^{μ}_{ij} and the anti-symmetric vector $\overset{a}{\underset{b}{v}}_i$ determine a surface is that they satisfy extended Gauss-Codazzi-Ricci conditions and since the surface is then determined except for a rigid motion, we are not concerned with this type of transformation.

Since we know that any ruled surface may be expressed in terms of orthogonal parameters (8) we ask for the transformation $t'(t,u)$, $u'(t,u)$ which will carry our surface into

(64) $\quad ds^2 = (1 + 2N'u' + M'^2u'^2)dt'^2 + du'^2$

where $N'(t') = N(t)$, $M'(t') = M(t)$.

This gives rise to a set of three partial differential equations of the first order in the two unknown functions $t'(t,u)$, $u'(t,u)$.[3] The question of the integration of this system is a classical problem - once proposed by the French Academie - and, in general, unsolved.

The obvious solution of these equations is the identity transformation, which then gives the same

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[1] $g(t,u) = g(t',u')(\frac{\partial u'}{\partial t})^2 + (\frac{\partial u'}{\partial t})^2$

$$0 = g(t',u')\,\frac{\partial t'}{\partial t}\frac{\partial u'}{\partial u} + \frac{\partial u'}{\partial t}\frac{\partial u'}{\partial u}$$

$$1 = g(t',u')\,(\frac{\partial t'}{\partial u})^2 + (\frac{\partial u'}{\partial u})^2$$

parametrization for two surfaces and a point corresponds to a point with the same coordinates. What conditions are now imposed on the second forms? Or, given (8) determine all surfaces $y^{\kappa}(t,u)$ such that the Gauss-Codazzi-Ricci conditions are satisfied. Or, determine the tensors h^{κ}_{ij}, v_i satisfying these conditions and the necessary symmetry conditions. Then by the integration of

$$g_{ij} = \nabla_i y^{\kappa} \nabla_j y^{\kappa}$$

the surface is determined except for a rigid motion.

If we write out the tensor form of the G-C-R equations, we have five partial differential equation in eight unknown functions. While the actual details of the integration may be difficult a solution does exist and we may say: any ruled surface may be deformed so that the rulings remain rulings and in an ∞^3 ways.

To ascertain whether the surface may be deformed so rulings do not correspond, we might ask the question: can any curve on the surface be transformed into a straight line? Eisenhart in his Differential Geometry[1] treats an analogous question for surfaces in R_3.

<u>Any ruled surface in R_4 can be deformed into a ruled surface in R_3 in an infinity of ways so that rulings correspond.</u> We have to solve the equations:

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[1] 1909.1, p.333.

$$(63)\quad N(t) = \bar{a}_t\cdot\bar{z}_t \qquad 1 = \bar{z}_t\cdot\bar{z}_t \qquad 0 = \bar{a}\cdot\bar{z}_t$$
$$M^2(t) = \bar{a}_t\cdot\bar{a}_t \qquad 1 = \bar{a}\cdot\bar{a}$$

where $\bar{a}$ with three components and $\bar{z}$ with three components determine the surface in the form

$$(64)\quad \bar{w}(t,u) = \bar{z}(t) + u\bar{a}(t)$$

applicable to the surface (1). The linear element of the two surfaces is (8).

Bonnet[1] proved a theorem to the effect that any ruled surface in R_3 which is applicable to a second must have its generators in correspondence with those of the second unless the two surfaces are applicable on a doubly ruled surface in such a way that the rulings of the first surface correspond to one set of the rulings on the quadric while those of the second correspond to the other set of lines on the second degree surface. So we now have the following:

Any two applicable ruled surfaces in R_4 have their generators in correspondence unless they can be deformed into the same doubly ruled surface with one set of generators corresponding to one set of lines of the doubly ruled surface and the generators of the other surface to the second family of lines.[2]

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[1] 1867.1

[2] Any doubly ruled surface in R_4 must lie in an R_3 since all of its points are axial points. Cf. J.A. Schouten 1938.1, p.99.

If we exclude the above case, the problem may be considered that of integrating the equations $g_{ij} = \nabla_i y^{\kappa} \nabla_j y^{\kappa}$, which, if we take the surface in the more general form (6) (i.e. non-orthogonal parameters) means we must solve the five (ordinary) partial differential equations in eight unknowns

(67) $N(t) = \bar{i}_t . \bar{x}_t \qquad 1 = \bar{x}_t . \bar{x}_t \qquad \bar{i} . \bar{x}_t = \cos\vartheta$

$M^2(t) = \bar{i}_t . \bar{i}_t \qquad 1 = \bar{i} . \bar{i}$

If $\bar{i}(t)$ is any curve on the unit hypersphere

(68) $\bar{i}(t)$; $\cos\omega$, $\sin\omega \cos\varphi$, $\sin\omega \sin\varphi \cos\psi$, $\sin\omega \sin\varphi \sin\psi$

the second of conditions (65) determines the parameter t.

(69) $\int_0^t M \, dt$ = arc length of curve $\bar{i}$ on the unit sphere.

(70) $M^2 = (d\omega/dt)^2 + (d\varphi/dt)^2 \sin^2\omega + \sin^2\omega \sin^2\varphi (d\psi/dt)^2$

determines ψ as a function of ω and φ. Then the three last equations of (65) determine $\bar{x}_t$ with one degree of freedom and $\bar{x}$ is obtained by quadratures.

(71) $\bar{x}_t = \cos\vartheta \ \bar{i} + \frac{N}{M^2} \bar{i}_t = \frac{\sqrt{M^2 \sin^2\vartheta - N^2}}{M} \ \bar{p}$ [1]

where $\bar{p}$ is a unit vector orthogonal to $\bar{i}$ and $\bar{i}_t$.

We may picture this as follows: Any curve C on the unit hypersphere determines the parameter t. Then the polar surface of C on the hypersphere, the locus of

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[1] $M^2 \sin^2\vartheta - N^2 = 0$ is the condition for developable ruled surface, p.9 Then $\bar{x}_t = \cos\vartheta \ \bar{i} + N/M^2 \ \bar{i}_t$ and there are ∞^2 surfaces applicable on a given ruled developable.

vectors perpendicular to $\overline{i}$ and $\overline{i}_t$, is determined also. Any curve on the polar surface, C', will be a locus of the vector $\overline{p}$. To any curve C and C' on its polar surface are associated two ruled surfaces, due to the double sign in (69). Any two curves on the unit hypersphere which are polar, determine two ruled surfaces. A ruled surface in R_4 can be bent so that C and C' are arbitrary polar curves on the unit hypersphere. All ruled surfaces with the same C but different C' are parallel in R_4.

Actually to realize the deformation of a surface into another is not so easy. If we ask that form the directrix may take by bending as does Beltrami[1] the problem of determining a curve from its intrinsic equations faces us.[2]

If σ is the angle the tangent plane to the surface at the directrix makes with the osculating plane of the directrix, and ρ is the angle it makes with the plane $[\overline{x}_t \overline{n}_3]$.

$$(72) \quad \overline{i} = \cos\vartheta\, \overline{x}_t + \sin\vartheta \cos\sigma\, \overline{n}_1 + \sin\vartheta \sin\sigma \sin\rho\, \overline{n}_2 + \sin\vartheta \sin\sigma \cos\rho\, \overline{n}_3$$

where $\overline{n}_j$ is the jth normal to the directrix. Then the second of conditions (67) becomes

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[1] 1865.1
[2] A.R. Forsyth 1930.1

$$(73)\quad M^2 = \sin^2\vartheta\,(d\vartheta/dt + \cos\sigma/r_1)^2 + \left\{\frac{d}{dt}(\sin\vartheta\cos\sigma) + \frac{\cos\vartheta}{r_1} - \frac{\sin\vartheta\sin\sigma\sin\rho}{r_2}\right\}^2 + \left\{\frac{d}{dt}(\sin\vartheta\sin\sigma\sin\rho) + \frac{\sin\vartheta\cos\sigma}{r_2} - \frac{\sin\vartheta\sin\sigma\cos\rho}{r_3}\right\}^2 + \left\{\frac{d}{dt}(\sin\vartheta\sin\sigma\cos\rho) + \frac{\sin\vartheta\sin\sigma\sin\rho}{r_3}\right\}^2$$

where r_j is the radius of the jth curvature of the curve In the process of bending the angle ϑ and the relative curvature of the directrix remain fixed, since it depends only on the first form,

$$(74)\quad \frac{d(\bar{i}.\bar{x}_t)}{dt} = \bar{i}.\frac{d}{dt}\,\bar{x}_t + \bar{i}_t.\bar{x}_t$$

$$-\sin\vartheta\,\frac{d\vartheta}{dt} = \frac{\sin\vartheta\,\cos\sigma}{r_1} + N$$

Substitution of (74) in (73) gives a functional relation:

$$(75)\quad F(t, r_1, \frac{dr_1}{dt}, r_2, r_3, \rho, \frac{d\rho}{dt}) = 0.$$

Given any curve, ρ is determined from (75) as a function of one arbitrary parameter, then σ is determined from (74). For any form of directrix the form of the corresponding surface is determined by ρ and σ. But for each form of directrix there are ∞ possible surfaces since (75) is a differential equation in ρ .

As an example: If we ask if the directrix can become a straight line, it is clear that $\frac{\cos\sigma}{r_1} = 0$ for a given curve. That is, it must be a geodesic. Condition (73) then is of the form

$$A(d\rho/dt)^2 + B(\cos\rho)d\rho/dt + C\sin\rho + D = 0.$$

σ is indeterminate.

Suppose the directrix becomes the line $x^1 = t$, $x^2 = x^3 = x^4 = 0$, the conditions are then:

(76) $i^1 = \cos\vartheta \qquad i_t^1 = N$

$$i^{1^2} + i^{2^2} + i^{3^2} + i^{4^2} = 1 \qquad i_t^{1^2} + i_t^{2^2} + i_t^{3^2} + i_t^{4^2} = M^2$$

These equations present essentially two equations in three unknowns, and we have:

1. <u>A ruled surface can be deformed so a curved geodesic becomes a straight line and the director cone has a fixed projection in an R_3 in an ∞^1 number of ways.</u>

2. <u>A ruled surface can be deformed so a geodesic becomes a straight line and the director cone cuts the hypersphere in a great circle in just one way.</u>

3. <u>Any ruled surface can be deformed so the directrix becomes a plane curve and the director cone is arbitrary.</u>

4. <u>Any ruled surface can be deformed so the directrix becomes the quasi-asymptotic curve and the director cone is arbitrary.</u>

VI. ISOTROPIC RULED SURFACES

In the work which has preceded we have been careful to exclude imaginary or isotropic quantities and have dealt only with real functions of real variables representing real geometric entities. We now admit the variables to be complex and turn to a brief study of isotropic manifolds, that is, manifolds with vanishing linear element in some direction or directions. Lense[1] has shown that the only completely ametric manifolds ($ds^2 = 0$ for all directions), X_n, imbedded in a Euclidean R_{2n} are linear spaces.

<u>In R_4 the only completely ametric X_2 is the completely isotropic plane. For if</u>

$$ds^2 = g_{ij}dx^i dx^j = 0 \text{ for all } dx^i\, dx^j, \quad i,j = 1,2$$

$$g_{ij} = 0.$$

But $g_{ij} = y_i^\kappa\, y_j^\kappa$ where $y_i^\kappa = \dfrac{\partial y^\kappa}{\partial x^i}$ and the y^κ define the surface. Differentiating partially with respect to x^k,

$$\frac{\partial g_{ij}}{\partial x^k} = y_i^\kappa\, y_{jk}^\kappa + y_{ik}^\kappa\, y_j^\kappa = 0$$

This relation holds for any i,j,k, so we have also

$$y_j^\kappa\, y_{ki}^\kappa + y_{ji}^\kappa y_k^\kappa = 0$$

$$y_k^\kappa\, y_{ij}^\kappa + y_{kj}^\kappa y_i^\kappa = 0$$

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[1] 1926.2

Adding and subtracting among these three relations

$$y_i^\varkappa \, y_{jk}^\varkappa = 0.$$

But $y_{jk}^\varkappa$ is not a vector, so this must mean that each $y_{jk}^\varkappa = 0$, or $y^\varkappa$ is linear in the x^j, upon integrating.

We may then expect to find some characteristic ruled surfaces containing one or more fields of isotropic vectors, but none with identically vanishing metric.

We consider ruled surfaces whose generators are isotropic. In such a situation we have instead of (2)[1]

(77) $$\bar{j}\cdot\bar{j} = 0$$

whence $g_{22} = 0$ and the metric (which is perfectly well defined) for the surface $y^\varkappa = x^\varkappa + uj^\varkappa$ becomes

(78) $$ds^2 = (\bar{x}_t\cdot\bar{x}_t + 2u\bar{j}_t\cdot\bar{x}_t + u^2\bar{j}_t\cdot\bar{j}_t)dt^2 + 2\bar{j}\cdot\bar{x}_t \, dt\, du$$

and $g = -(\bar{j}\cdot\bar{x}_t)^2$, that is, the first fundamental form is (negative (not positive) definite. Since g_{ij} is well defined we may use the ordinary definitions to calculate that the Gaussian curvature is given by

(79) $$K = \frac{\bar{j}_t\cdot\bar{j}_t}{(\bar{j}\cdot\bar{x}_t)^2} \quad \text{if } \bar{j}\cdot\bar{x}_t \neq 0.$$

This is, in general, not zero. So, in contrast to the situation for a ruled surface in R_3 with isotropic generators, we have: <u>A ruled surface in R_4 with isotropic</u>

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[1] We use $\bar{j}$ for the vector in the direction of the rulings instead of $\bar{i}$ in order to indicate that it is no longer a unit vector.

generators is not necessarily developable. If the surface is developable, then the ds^2 is a perfect square.[1] A developable ruled surface is the tangent surface of a twisted curve (p.9) and may be represented by

$$y^{\kappa} = x^{\kappa} + u x_t^{\kappa}$$

Then if the generators are isotropic $x_t^{\kappa} x_t^{\kappa} = 0$, the directrix is a minimal curve and

$$ds^2 = (\bar{x}_t + u\bar{x}_{tt})^2 dt^2$$

We note from (77) further that the Gaussian curvature of a non-developable ruled surface with isotropic generators is fixed along a ruling. There is, therefore, no "striction point" on a ruling of such a surface in the sense that there is no point of minimum Gaussian curvature, and the argument we used on p. 11 no longer strictly obtains. However, we may argue with Beck[2] as follows, and find that the striction point exists in such a situation and indeed is the ideal point of the ruling. The formula

$$(80) \qquad \omega = \frac{(\bar{x}_t \cdot \bar{j})(\bar{j} \cdot \bar{j}_t) - (\bar{x}_t \cdot \bar{j}_t)(\bar{j} \cdot \bar{j})}{(\bar{j} \cdot \bar{j})(\bar{j}_t \cdot \bar{j}_t) - (\bar{j} \cdot \bar{j}_t)^2}$$

defines the striction point of a ruling in general by

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[1] Cf. J. Eiesland 1911.1.

[2] 1928.1.

the ordinary argument referred to above. (This reduces to our formula (32) when we require $\overline{J}$ and $\overline{x}_t$ to be unit vectors.) Now, if $\overline{J}\cdot\overline{J} = 0$, the denominator of (78) vanishes, and u is infinite if the numerator is finite. We may distinguish two cases: 1. $\overline{x}_t\cdot\overline{J} \neq 0$, the surface is skew and not a tangent surface of a minimal curve. 2. $\overline{x}_t\cdot\overline{J} = 0$, the surface is a minimal cone or the tangent surface of a minimal curve. In which case the striction curve is the edge of regression, Beck classifies such isotropic surfaces by this means. (See paper 1.)

We also note from (77) that the curvature is always positive.

Any non-minimal curve on the surface will have an absolute curvature vector with respect to R_4 defined as on p. 15 and again this is equivalent to the sum of two vectors, the relative and normal curvature vectors, Since the tensors h^{κ}_{ab} and the tangent plane to the surface are defined. If we treat the normal curvature vector by differentiating as before we find that there are only two principal directions at any point, these directions are the roots of the equation

$$(81) \quad (a_{11}g_{11}g_{12} - a_{12}g_{11}^{2})\lambda^{2} + (a_{11}g_{12}^{2} - g_{11}^{2}a_{22})\lambda + 4(a_{12}g_{12}^{2} - a_{22}g_{11}g_{12}) = 0$$

and the principal curvatures in these directions are the same, viz.

$$(82) \quad \mathcal{K} = \frac{1}{R^2} = \frac{a_{12}^2 - a_{11}a_{22}}{2a_{12}g_{11}g_{12} - a_{11}g_{12}^2 - a_{22}g_{11}^2}$$

where $a_{11} = \overline{h}_{11}.\overline{h}_{11}$ etc.

If M^{κ}, the mean curvature vector, is a null vector, the manifold to which it belongs is minimal in the sense of least area. By the use of formulae p. 4 and the definitions of M^{κ} p. 16, we calculate

$$M^{\kappa} = g^{12}\left\{ j_t^{\kappa} - \tfrac{1}{2}g^{12}\,\partial_u g_{11}\, j^{\kappa} \right\}$$

This is null if $g^{12} = 0$, or if $j_t^{\kappa} = \frac{1}{2}g^{12}\,\partial_u g_{11}\, j^{\kappa}$. That is if j_t^{κ} is an isotropic vector, $\overline{j}_t.\overline{j}_t = 0$. But then $K = 0$, and we have: Minimal ruled surfaces in R_4 with isotropic generators are the isotropic developables and conversely.

Any ruled surface containing two families of straight lines can be put in an R_3, so the surface in R_4 containing two sets of isotropic lines is the ordinary sphere.

J. Lense and M. Pinl in a series of papers[1] have treated the subject of isotropic manifolds as the integral surface of $ds^2 = 0$ and have considered various types

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[1] J. Lense 1931.2; 1935.2; 1936.1; 1939.1: M. Pinl 1932.2, 4; 1936.3; 1937.1,2. See also papers by E. Schrenzel 1929.1 and G.F.C. Griss 1934.1.

depending upon the rank of the matrix g_{ij}. They have also considered the surfaces generated by various types curves possessing certain isotropic normal spaces. Since we have obtained no further or new results, we refer to their papers. Note - We may note, however, that since the metric for the particular case we have presented is of rank two there is no difficulty involved in applying the ordinary surface theory, and our results are interesting in that they present another fairly tangible realization of the general tensor theory.

For a classification of possible isotropic manifolds we refer to Schouten and Struik II[1] and to Lense's first paper on the subject.[2]

VII. RULED V_3 IN R_4

The theory of ruled V_3 in R_4 generated by an ∞^1 of planes or a ∞^2 of straight lines can be considered as a generalization of ruled surfaces. Such manifolds are developable in the sense that they possess a singular or focal curve which each generating plane osculates. The important formulae are appended and for more

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[1] 1938.1

[2] 1926.2

information we refer to Schouten and Struik II[1] for the general V_{n-1} in R_n and to Ranum[2] for the projective geometric approach and the theory of related manifolds.

(81) $V_3 : \bar{y}(t,u,v) = \bar{x}(t) + u\bar{x}_t(t) + v\bar{i}(t)$

where $\bar{x}_t \cdot \bar{x}_t = 1$, $\bar{i} \cdot \bar{i} = 1$, $\bar{x}_t \cdot \bar{i} = 0$ and u and v are rectangular cartesian coördinates in a generating plane.

(82) $ds^2 = g_{11}dt^2 + 2g_{12}dtdu + 2g_{13}dtdv + du^2 + dv^2$

where $g_{11} = 1 + 2v\bar{x}_t \cdot \bar{i}_t + 2uv\bar{x}_{tt} \cdot \bar{i}_t + v^2\bar{i}_t \cdot \bar{i}_t + u^2\bar{x}_{tt} \cdot \bar{x}_{tt}$

$g_{12} = 1 + v\bar{x}_t \cdot \bar{i}_t$

$g_{13} = - u\bar{x}_t \cdot \bar{i}_t$

$g_{23} = 0 \qquad g_{22} = 1 \qquad g_{33} = 1$

$g = |g_{ij}| = g_{11} - g_{12}^2 - g_{13}^2$

(85) $\psi = h_{11}dt^2 + 2h_{12}dt\,du + 2h_{13}\,dt\,dv$

where $h_{ij} = (\nabla_{ij}y^\kappa)n^\kappa$, and n^κ is the unit normal to the manifold and $\|h\|$ is of rank two.

(86) $\varkappa = \dfrac{h_{ij}dx^i dx^j}{g_{ij}dx^i dx^j}$ is the normal curvature, or the coordinate of the curvature in the direction n^κ.

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[1] 1938.1, p.62.

[2] 1912.1.

(87) Principal normal curvatures

$$\varkappa = 0,\ \frac{(-h_{11}+2h_{12}g_{12}+2h_{13}g_{13}) \pm \sqrt{(h_{11}-2h_{12}g_{12}-2h_{13}g_{13})^2 - 4(h_{12}^2+h_{13}^2)(-g_{11}+g_{12}^2+g_{13}^2)}}{-2(g_{11}-g_{12}^2-g_{13}^2)}$$

(88) Principal directions: three mutually orthogonal

$$\overset{i}{\underset{1}{\lambda}} : 0, \quad 1, \quad \frac{-h_{12}}{h_{13}} \qquad i = 1,2,3$$

$$\overset{i}{\underset{2}{\lambda}} : \frac{\varkappa_2}{h_{13}-\varkappa_2 g_{13}}, \quad \frac{h_{12}-\varkappa_2 g_{12}}{h_{13}-\varkappa_2 g_{13}}, \quad 1, \qquad \underset{i}{\varkappa} : \text{principal normal curvatures}$$

$$\overset{i}{\underset{3}{\lambda}} : \frac{\underset{3}{\varkappa}}{h_{13}-\underset{3}{\varkappa} g_{13}}, \quad \frac{h_{12}-\underset{3}{\varkappa} g_{12}}{h_{13}-\underset{3}{\varkappa} g_{13}}, \quad 1$$

Principal directions are conjugate directions and orthogonal

(89) $$\varkappa = \underset{2}{\varkappa} \cos^2 \underset{2}{\alpha} + \underset{3}{\varkappa} \cos^2 \underset{3}{\alpha}$$

From which we see that the Dupin indicatrix reduces to a conic and two line segments.

Along any line in a generating plane through the focal point the n^κ are in the same direction.

(90) Asymptotic directions

$$h_{11} dt^2 + 2h_{12}\, dt\, du + 2\, h_{13}\, dt\, dv = 0$$

lie in two planes $\begin{cases} dt = 0 \\ h_{11} dt + 2h_{12}\, du + 2h_{13}\, dv = 0 \end{cases}$

Two points on the same straight line through the focal point of the generating plane in which they lie have asymptotic planes lying in the same R_3, the tangent R_3 to the V_3.

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Martha Hathaway Plass was born on January 25, 1914 in Montclair, New Jersey. She attended the public schools in Montclair and graduated from the High School in 1931 "with honors in Mathematics". She received the the A.B. degree from Wellesley College in 1935, again "with honors" in Mathematics, and the S.M. degree from the Massachusetts Institute of Technology in 1936. At Wellesley she was a Durant Scholar and was elected a member of Phi Beta Kappa. She also is an associate member of Sigma Zi and belongs to the American Mathematical Society.

The second semester of 1937-1938 she was an instructor in Mathematics at Wellesley College. Beginning with the fall of 1938 she has been teaching Mathematics at the University of Maryland.