14.451: Macroeconomics
Spring 2003

Problem set3

1 OLG and Social Security

Consider the OLG economy studied in class. Individuals live for two periods. They start their life with no financial assets and they leave no bequests to future generations. They are concerned only with consumption in the two periods of their life. An individual thus accumulates assets only over his life cycle (between the first and second period of his life). An individual works $l^y > 0$ when he is young and $l^o \geq 0$ when he is old.

To simplify, throughout we assume the following: There is no exogenous technological change; popululation is constant at N in all periods; utility is logarithmic utility; technology is Cobb-Douglas; and there is full depreciation:

$$u(c) = \log c,$$

$$F(K,N) = K^{\alpha}N^{1-\alpha} \Rightarrow f(k) \equiv F(k,1) = k^{\alpha}.$$

$$\delta = 1, \quad \gamma = 0.$$

1.1 Part I: The Decentralized Equilibrium

Please consult the notes of Marios for thi exercise.

Households: The problem faced by a typical member of generation t is

$$\begin{aligned} & Max[u(c_t^y) + \beta u(c_{t+1}^o)] \\ s.t.\ c_t^y + a_t \quad \leq \quad & w_t l^y \qquad\qquad bc\ when\ young \\ and \quad c_{t+1}^o \quad \leq \quad & w_{t+1} l^o + R_{t+1} a_t \quad bc\ when\ old \end{aligned}$$

where c_t^y denotes consumption when young, c_{t+1}^o denotes consumption when old, a_t denotes savings when young, R_{t+1} is the interest reat between t and $t+1$, and w_t (w_{t+1}) is the wage in period t $(t+1)$.

Firms: The firms maximize their profits under the technological constraint given by the production function, thus they solve:

$$Max\ \pi_t = F(K,N) - (r_t + \delta)K - w_t N.$$

Note that $R_t \equiv 1 + r_t$ by definition.

Market clearing: To close the model we use the market clearing condition

$$\begin{aligned} K_{t+1} &= a_t N \Leftrightarrow \\ k_{t+1} &= a_t \end{aligned}$$

where $k_{t+1} \equiv K_{t+1}/N$.

a) Find the individual agent's optimal consumption and optimal savings.

b) Solve the firms' problem and characterize the equilibrium R_t and w_t in terms of $k_t = K_t/N$.

c) Find the law of motion of the capital stock; that is, solve for $k_{t+1} = G(k_t)$. Verify that $G(0) = 0$, $G' > 0$, $G'' < 0$, $G'(0) = \infty$, $G'(\infty) = 0$.

d) Derive the steady-state capital stock and the steady-state interest rate.

e) Assume $l^y = 1$, $l^o = 0$, to capture the idea of retirement. Discuss the dynamic efficiency or inefficiency.

f) Assume $l^y = 1/2$, $l^o = 1/2$, to capture the idea of perpetual youth. Discuss the dynamic efficiency or inefficiency.

1.2 Part II: Social Security

We now introduce social security taxes for young agents and retirement benefits for the elderly. Let τ_t^y denote the tax on young people and τ_t^o the tax on old people during period t ($\tau_t^y > 0 > \tau_t^o$).

Government: The government budget is

$$B_{t+1} = R_t B_t - \tau_t^y N - \tau_t^o N$$

Define $b_t \equiv B_t/N$ and rewrite the budget as

$$b_{t+1} = R_t b_t - \tau_t^y - \tau_t^o.$$

Households: Because of the presence of taxes, the household's problem now becomes

$$\begin{array}{rcll} & & Max[u(c_t^y) + \beta u(c_{t+1}^o)] & \\ s.t.\ c_t^y + a_t & \leq & w_t l^y - \tau_t^y & bc\ when\ young \\ and\quad c_{t+1}^o & \leq & w_{t+1} l^o - \tau_t^o + R_{t+1} a_t & bc\ when\ old \end{array}$$

Firms: The firms' problem is exactly as before.

Market clearing: Total savings are now invested either in capital or in government debt. Therefore, the market clearing condition becomes

$$\begin{aligned} K_{t+1} + B_{t+1} &= a_t N \Leftrightarrow \\ k_{t+1} + b_{t+1} &= a_t. \end{aligned}$$

Consider the following two fiscal institutions:

i) Fully Funded Social Security System.

Suppose the government imposes a tax equal to $d > 0$ on the young generation in period t, invests that at the competitive interest rate R_{t+1}, and distributes the return back to the old generation in period $t+1$:

$$\begin{aligned} \tau_t^y &= d > 0 \\ b_t &= -d > 0 \\ \tau_{t+1}^o &= -R_{t+1} d < 0 \end{aligned}$$

Verify that the above scheme satisfies the government budget. Find the law of motion of the capital and the steady state. How the presence of a fully-funded social security system affects capital accumulation?

ii) Pay-As-You-Go Social Security:

Suppose the government imposes a tax equal to $d > 0$ on the young generation in period t, distributes the tax proceeds immediately to the old generation in the same period, and rolls over no debt (or assets) from period to period:

$$\begin{aligned} b_t &= 0 \\ \tau_t^y &= d > 0 \\ \tau_t^o &= -d < 0 \end{aligned}$$

Verify that the above scheme satisfies the government budget. Find the law of motion of the capital and the steady state. How the presence of a pay-as-you-go social security system affects the capital accumulation? If the economy is initially dynamically efficient, how does a marginal increase in d affect the welfare of current a future generations? What happens if initially the economy is dynamically inefficient?

2 Endogenous Growth

2.1 Part I: Technological Change

Barro and Sala-Y-Martin, Exercise 4.7

2.2 Part II: Human Capital

Consider a model of endogenous growth where the accumulation of human capital requires time and no other resources. In particular, suppose that:

$$\begin{aligned} y_t &= A k_t^\alpha [h_t l_t]^\beta \\ \dot{h}_t &= B h_t (1 - l_t) \\ \dot{k}_t &= y_t - c_t \end{aligned}$$

Where k_t denotes physical capital, h_t denotes human capital, l_t is the fraction of time spent in production, and $1 - l_t$ is the fraction of time spent in the accumulation of human capital. We assume $0 < \alpha < 1,\ 0 < \beta < 1$, and $\alpha + \beta \lesseqqgtr 1$.

a) Assume that there are constants s and l such that $c_t = (1-s)y_t$ and $l_t = l$ in all t. What is the growth rate of h?

b) Does the economy converge to a steady state or to a ballanced growth path? How does your answer depend on whether $\alpha + \beta$ is $<$, $=$, or $>$ than 1? What is the growth rate of k, c, and y when $\alpha + \beta = 1$?

Now we will endogenize s and l, using the Ramsey model with CEIS utility and letting $\alpha + \beta = 1$. The representative agent maximizes

$$\int e^{-\rho t} \left[\frac{c_t^{1-\theta}}{1-\theta} \right] dt$$

subject to

$$\begin{aligned} \dot{k}_t &= A k_t^{\alpha} [h_t l_t]^{1-\alpha} - c_t \\ \dot{h}_t &= B h_t (1 - l_t) \end{aligned}$$

c) Set up the Hamiltonian and derive the optimality conditions. Get rid of the Lagrange multipliers.

d) The two laws of motion for k and h (see above) and the optimality conditions (from part a) characterize the equilibrium path. This is a system in c_t, $\dot{c}_t$, k_t, $\dot{k}_t$, h_t, $\dot{h}_t$, and l_t. Define $\omega_t = k_t/h_t$, $\chi_t = c_t/k_t$, and $\gamma_t = \dot{c}_t/c_t$. Observe that $\dot{\omega}_t/\omega_t = \dot{k}_t/k_t - \dot{h}_t/h_t$ and $\dot{\chi}_t/\chi_t = \dot{c}_t/c_t - \dot{k}_t/k_t$. Rewrite the equilibrium system in terms of γ_t, ω_t, $\dot{\omega}_t$, χ_t, $\dot{\chi}_t$, and l_t.

e) Suppose there is a steady state point such $(\gamma_t, \omega_t, \chi_t, l_t) = (\gamma^*, \omega^*, \chi^*, l^*)$ and $\dot{\omega}_t = \dot{\chi}_t = 0$. Characterize the steady state values $(\gamma^*, \omega^*, \chi^*, l^*)$. What are the corresponding growth rates of y, c, k, and h? How does an increase in B or in A affect γ^* and l^*? Interpret.

2.3 Part III: Romer

Barro and Sala-Y-Martin, Exercise 6.3