Chapter 5

Overlapping Generations Models

5.1 OLG and Life-Cycle Savings

5.1.1 Households

- Consider a household born in period t, living in periods t and $t+1$. We denote by c_t^y his consumption when young and c_{t+1}^o his consumption when old.

- Preferences are given by

$$u(c_t^y) + \beta u(c_{t+1}^o)$$

  where β denotes a discount factor and u is a neoclassical utility function.

- The household is born with zero initial wealth, saves only for life-cycle consumption smoothing, and dies leaving no bequests to future generations. The household receives labor income possibly in both periods of life. We denote by l^y and l^o the endowments of effective labor when young and when old, respectively. The budget constraint during the first period of life is thus

$$c_t^y + a_t \leq w_t l^y,$$

whereas the budget constraint during the second period of life is

$$c_{t+1}^o \leq w_{t+1} l^o + (1 + R_{t+1}) a_t.$$

Adding up the two constraints (and assuming that the household can freely borrow and lend when young, so that a_t can be either negative or positive), we derive the intertemporal budget constraint of the household:

$$c_t^y + \frac{c_{t+1}^o}{1 + R_{t+1}} \leq h_t \equiv w_t l^y + \frac{w_{t+1} l^o}{1 + R_{t+1}}$$

- The household choose consumption and savings so as to maximize life utility subject to his intertemporal budget:

$$\max \left[u(c_t^y) + \beta u(c_{t+1}^o) \right]$$

$$s.t. \quad c_t^y + \frac{c_{t+1}^o}{1 + R_{t+1}} \leq h_t.$$

The Euler condition gives:

$$u'(c_t^y) = \beta(1 + r_{t+1}) u'(c_{t+1}^o).$$

In words, the household chooses savings so as to smooth (the marginal utility of) consumption over his life-cycle.

- With CEIS preferences, $u(c) = c^{1-1/\theta}/(1 - 1/\theta)$, the Euler condition reduces to

$$\frac{c_{t+1}^o}{c_t^y} = [\beta(1 + R_{t+1})]^\theta.$$

Life-cycle consumption growth is thus an increasing function of the return on savings and the discount factor. Combining with the intertemporal budget, we infer

$$h_t = c_t^y + \frac{c_{t+1}^o}{1+R_{t+1}} = c_t^y + \beta^\theta (1+R_{t+1})^{\theta-1} c_t^y$$

and therefore optimal consumption during youth is given by

$$c_t^y = m(r_{t+1}) \cdot h_t$$

where

$$m(R) \equiv \frac{1}{1+\beta^\theta (1+R)^{\theta-1}}.$$

Finally, using the period-1 budget, we infer that optimal life-cycle saving are given by

$$a_t = w_t l^y - m(R_{t+1}) h_t = [1-m(R_{t+1})] w_t l^y - m(R_{t+1}) \frac{w_{t+1} l^o}{1+R_{t+1}}$$

5.1.2 Population Growth

- We denote by N_t the size of generation t and assume that population grows at constant rate n :

$$N_{t+1} = (1+n) N_t$$

- It follows that the size of the labor force in period t is

$$L_t = N_t l^y + N_{t-1} l^o = N_t \left[l^y + \frac{l^o}{1+n} \right]$$

  We henceforth normalize $l^y + l^o/(1+n) = 1$, so that $L_t = N_t$.

- *Remark:* As always, we can reinterpret N_t as effective labor and n as the growth rate of exogenous technological change.

5.1.3 Firms and Market Clearing

- Let $k_t = K_t/L_t = K_t/N_t$. The FOCs for competitive firms imply:

$$\begin{aligned} r_t &= f'(k_t) \equiv r(k_t) \\ w_t &= f(k_t) - f'(k_t)k_t \equiv w(k_t) \end{aligned}$$

  On the other hand, the arbitrage condition between capital and bonds implies $1+R_t = 1 + r_t - \delta$, and therefore

$$R_t = f'(k_t) - \delta \equiv r(k_t) - \delta$$

- Total capital is given by the total supply of savings:

$$K_{t+1} = a_t N_t$$

  Equivalently,

$$(1 + n)k_{t+1} = a_t.$$

5.1.4 General Equilibrium

- Combining $(1 + n)k_{t+1} = a_t$ with the optimal rule for savings, and substituting $r_t = r(k_t)$ and $w_t = w(k_t)$, we infer the following general-equilibrium relation between savings and capital in the economy:

$$(1 + n)k_{t+1} = [1 - m(r(k_{t+1}) - \delta)]w(k_t)l^y - m(r(k_{t+1}) - \delta)\frac{w(k_{t+1})l^o}{1 + r(k_{t+1}) - \delta}.$$

- We rewrite this as an implicit relation between k_{t+1} and k_t :

$$\Phi(k_{t+1}, k_t) = 0.$$

Note that

$$\Phi_1 = (1+n) + h\frac{\partial m}{\partial R}\frac{\partial r}{\partial k} + ml^o \frac{\partial}{\partial k}\left(\frac{w}{1+r}\right),$$
$$\Phi_2 = -(1-m)\frac{\partial w}{\partial k} l^y.$$

Recall that $\frac{\partial m}{\partial R} \lesseqgtr 0 \Leftrightarrow \theta \gtreqless 1$, whereas $\frac{\partial r}{\partial k} = F_{KK} < 0$, $\frac{\partial w}{\partial k} = F_{LK} > 0$, and $\frac{\partial}{\partial k}\left(\frac{w}{1+r}\right) > 0$. It follows that Φ_2 is necessarily negative, but Φ_1 may be of either sign:

$$\Phi_2 < 0 \quad \text{but} \quad \Phi_1 \lesseqgtr 0.$$

We can thus always write k_t as a function of k_{t+1}, but to write k_{t+1} as a function of k_t,we need Φ to be monotonic in k_{t+1}.

- A sufficient condition for the latter to be the case is that savings are non-decreasing in real returns:

$$\theta \geq 1 \Rightarrow \frac{\partial m}{\partial r} \geq 0 \Rightarrow \Phi_1 > 0$$

In that case, we can indeed express k_{t+1} as a function of k_t :

$$k_{t+1} = G(k_t).$$

Moreover, $G' = -\frac{\Phi_2}{\Phi_1} > 0$, and therefore k_{t+1} increases monotonically with k_t. However, there is no guarantee that $G' < 1$. Therefore, in general there can be multiple steady states (and poverty traps). See **Figure 1**.

- On the other hand, if θ is sufficiently lower than 1, the equation $\Phi(k_{t+1}, k_t) = 0$ may have multiple solutions in k_{t+1} for given k_t. That is, it is possible to get *equilibrium indeterminacy*. Multiple equilibria indeed take the form of *self-fulfilling prophesies*. The anticipation of a high capital stock in the future leads agents to expect a low return on savings, which in turn motivates high savings (since $\theta < 1$) and results to a high capital stock in the future. Similarly, the expectation of low k in period $t+1$ leads to high returns and low savings in the period t, which again vindicates initial expectations. See **Figure 2**.

5.2 Some Examples

5.2.1 Log Utility and Cobb-Douglas Technology

- Assume that the elasticity of intertemporal substitution is unit, that the production technology is Cobb-Douglas, and that capital fully depreciates over the length of a generation:

$$u(c) = \ln c, \quad f(k) = k^{\alpha}, \quad \text{and} \quad \delta = 1.$$

- It follows that the MPC is constant,

$$m = \frac{1}{1+\beta}$$

and one plus the interest rate equals the marginal product of capital,

$$1 + R = 1 + r(k) - \delta = r(k)$$

where

$$\begin{aligned} r(k) &= f'(k) = \alpha k^{\alpha-1} \\ w(k) &= f(k) - f'(k)k = (1-\alpha)k^{\alpha}. \end{aligned}$$

- Substituting into the formula for G, we conclude that the law of motion for capital reduces to

$$k_{t+1} = G(k_t) = \frac{f'(k_t)k_t}{\zeta(1+n)} = \frac{\alpha k_t^{\alpha}}{\zeta(1+n)}$$

where the scalar $\zeta > 0$ is given by

$$\zeta \equiv \frac{(1+\beta)\alpha + (1-\alpha)l^o/(1+n)}{\beta(1-\alpha)l^y}$$

Note that ζ is increasing in l^o, decreasing in l^y, decreasing in β, and increasing in α (decreasing in $1-\alpha$). Therefore, G (savings) decreases with an increase in l^o and a decrease in l^y, with an decrease in β, or with an increase in α.

5.2.2 Steady State

- The steady state is any fixed point of the G mapping:

$$k_{olg} = G(k_{olg})$$

Using the formula for G, we infer

$$f'(k_{olg}) = \zeta(1+n)$$

and thus $k_{olg} = (f')^{-1}(\zeta(1+n))$.

- Recall that the golden rule is given by

$$f'(k_{gold}) = \delta + n,$$

and here $\delta = 1$. That is, $k_{gold} = (f')^{-1}(1+n)$.

- Pareto optimality requires

$$k_{olg} < k_{gold} \Leftrightarrow r > \delta + n \Leftrightarrow \zeta > 1,$$

while Dynamic Inefficiency occurs when

$$k_{olg} > k_{gold} \Leftrightarrow r < \delta + n \Leftrightarrow \zeta < 1.$$

Note that

$$\zeta = \frac{(1+\beta)\alpha + (1-\alpha)l^o/(1+n)}{\beta(1-\alpha)l^y}$$

is increasing in l^o, decreasing in l^y, decreasing in β, and increasing in α (decreasing in $1-\alpha$). Therefore, inefficiency is less likely the higher l^o, the lower l^y, the lower is β, and the higher α.

- Provide intuition...

- In general, ζ can be either higher or lower than 1. There is thus no guarantee that there will be no dynamic inefficiency. But, Abel et al argue that the empirical evidence suggests $r > \delta + n$, and therefore no evidence of dynamic inefficiency.

5.2.3 No Labor Income When Old: The Diamond Model

- Assume $l^o = 0$ and therefore $l^y = 1$. That is, household work only when young. This case corresponds to Diamond's OLG model.

- In this case, ζ reduces to

$$\zeta = \frac{(1+\beta)\alpha}{\beta(1-\alpha)}.$$

ζ is increasing in α and $\zeta = 1 \Leftrightarrow \alpha = \frac{1}{2+1/\beta}$. Therefore,

$$r \gtrless n + \delta \Leftrightarrow \zeta \gtrless 1 \Leftrightarrow \alpha \gtrless (2+1/\beta)^{-1}$$

Note that, if $\beta \in (0,1)$, then $(2+1/\beta)^{-1} \in (0,1/3)$ and therefore dynamic inefficiency is possible only if α is sufficiently lower than $1/3$. This suggests that dynamic inefficiency is rather unlikely. However, in an OLG model β can be higher than 1, and the higher β the more likely to get dynamic inefficiency in the Diamond model.

- Finally, note that dynamic inefficiency becomes *less* likely as we increase l^o, that is, as we increase income when old (hint: retirement benefits).

5.2.4 Perpetual Youth: The Blanchard Model

- We now reinterpret n as the rate of exogenous technological growth. We assume that household work the same amount of time in every period, meaning that in effective terms $l^o = (1+n)l^y$. Under the normalization $l^y + l^o/(1+n) = 1$, we infer $l^y = l^o/(1+n) = 1/2$.

- The scalar ζ reduces to

$$\zeta = \frac{2(1+\beta)\alpha + (1-\alpha)}{\beta(1-\alpha)}$$

Note that ζ is increasing in α, and since $\alpha > 0$, we have

$$\zeta > \frac{2(1+\beta)0 + (1-0)}{\beta(1-0)} = \frac{1}{\beta}.$$

- If $\beta \in (0,1)$, it is necessarily the case that $\zeta > 1$. It follows that necessarily $r > n + \delta$ and thus

$$k_{blanchard} < k_{gold},$$

meaning that it is impossible to get dynamic inefficiency.

- Moreover, recall that the steady state in the Ramsey model is given by

$$\beta[1 + f'(k_{ramsey}) - \delta] = 1 + n \Leftrightarrow$$
$$f'(k_{ramsey}) = (1+n)/\beta \Leftrightarrow$$
$$k_{ramsey} = (f')^{-1}((1+n)/\beta)$$

while the OLG model has

$$f'(k_{blanchard}) = \zeta(1+n) \Leftrightarrow$$
$$k_{blanchard} = (f')^{-1}(\zeta(1+n))$$

Since $\zeta > 1/\beta$, we conclude that the steady state in Blanchard's model is necessarily lower than in the Ramsey model. We conclude

$$k_{blanchard} < k_{ramsey} < k_{gold}.$$

- Discuss the role of "perpetual youth" and "new-comers".

5.3 Ramsey Meets Diamond: The Blanchard Model

topic covered in recitation

notes to be completed

5.4 Various Implications

- Dynamic inefficiency and the role of government
- Ricardian equivalence breaks, public debt crowds out investment.
- Fully-funded social security versus pay-as-you-go.
- Bubbles

notes to be completed