

D-branes and String Field Theory

by

Ilya Sigalov

Submitted to the Department of Physics
in partial fulfillment of the requirements for the degree of
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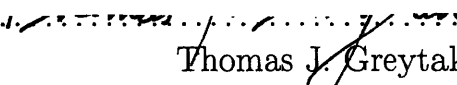
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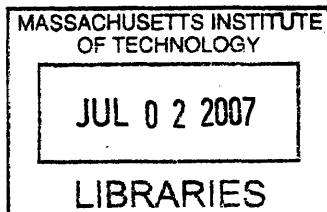
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Abstract

In this thesis we study the D-brane physics in the context of Witten's cubic string field theory. We compute first few terms the low energy effective action for the non-abelian gauge field A_μ from Witten's action. We show that after the appropriate field redefinition which relates the string field theory variables to the worldsheet variables one obtains the correct Born-Infeld terms. We then compute the rolling tachyon solution in the context of string field theory. We show that after the appropriate field redefinition we obtain the rolling tachyon solution of Sen.

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Chapter 1

Introduction

Modern physics classifies the observed forces of nature into four interactions: electromagnetic, weak, strong and gravitational. Three of them: electromagnetic, weak and strong have fully consistent quantum-mechanical description as specific quantum field theory with $SU(3) \times SU(2) \times U(1)$ gauge symmetry. This quantum field theory, describes all three interactions within one framework. Together with specific data for particle spectra and fundamental constants it describes all observed accelerator particle physics.

The fourth interaction: gravitation has only classical description. It is also a gauge theory where the role of the gauge transformations is played by the reparametrizations of the space time. However, the quantization of gravity has proven to be extremely difficult problem. In the core of it is the issue of renormalizability of the quantum theory. When a classical field theory is quantized, say, via path integral methods the perturbative expansion of the observable quantities, such as scattering amplitudes, around the non-interacting theory is obtained. In this perturbative expansion individual terms are divergent, however for a class of QFT's via a careful and systematic procedure called renormalization, it was shown that the divergencies can be canceled out between the individual terms, contributing to each order in perturbation theory. While weak, strong and electromagnetic interactions have been shown to be renormalizable, it was also found that the gravity, beyond the second order in perturbation theory, is not.

One more attractive physical idea which has been extensively explored is that of unification. It is known that electromagnetic, weak and strong interactions appear in the low energy limit of a class of larger quantum field theories with bigger gauge symmetry group. When going to the low energy limit, this large gauge symmetry is spontaneously broken and some of the gauge bosons receive masses, with the remaining symmetry being $SU(3) \times SU(2) \times U(1)$ of strong, weak and electromagnetic forces. The unified description is attractive because it uses less fundamental constants to specify the theory, and explains the family structure of the particle spectrum.

Probably one of the most challenging physics problems today is that of finding a unifying theory that would describe all four fundamental forces as one interaction, where the splitting into apparently different interactions would appear only in the low energy limit. Although gravitation is not renormalizable, in this context it would appear in the special regime of a bigger and self consistent theory. In the last 15 years string theory has become the most prominent candidate for such a theory.

The string theory's basic assumption is that instead of being pointlike objects, the elementary particles are extended objects - strings. Strings can be open or closed. While moving in spacetime a string spans a surface called the string worldsheet. Elementary particles with different quantum numbers appear as various string excitations as we look at strings from large distances. One of the very attractive features of this point of view is that different particles appear as different excitations of one fundamental string.

In mid-eighties it was found that in flat spacetime there are five consistent supersymmetric string theories, which must live in 10-dimensional Minkowski space. All of these theories are related by various duality transformations, and it is generally assumed, that these are different limits of unique fundamental theory. The connection to our 4-dimensional world is made via procedure called compactification, where 6 of the 10 dimensions are assumed to span some compact manifold. Excitation of the string in the compact direction would require large energies, leaving strings to move freely in the four spacetime dimensions.

These five string theories are defined in a perturbative fashion, where the spectra

of excited states in noninteracting theory are well defined, and scattering amplitudes are computed order by order in perturbation theory. All of the five formulations are related by so called duality transformations - which state that perturbative definition of one string theory appear in a specific regime of another. It was then assumed that these five apparently different string theories are perturbative expansions around different backgrounds of some fundamental underlying theory.

With the research that followed the seminal paper of Polchinski [2] the importance of D-branes in string theory has been fully realized. D-branes in string theory appear as localized $D + 1$ dimensional objects, whose perturbations are described by the open string degrees of freedom. Open strings are allowed to end on the D-brane only. Different open string backgrounds correspond to D-branes with different geometry. From the space time point of view, D-branes are classical solutions to the supergravity equations of motion.

An important question is to find a theory, which would possess a so called background independence. Such theory should allow a description of different string models as different backgrounds specified in terms of some fundamental variable. Perturbative expansions around these backgrounds would give us the five string theories and their compactifications. Different D-brane configurations would also be described as open string backgrounds in the fundamental background independent theory. The best known candidate for the background independent formulation of string theory is String Field Theory (SFT). While there are SFT formulations for some of the 10-dimensional supersymmetric string theories, most significant progress in understanding different string backgrounds using string field theory was achieved in Witten's cubic string field theory (CSFT) which describes the 26-dimensional open bosonic string. The lowest state of 26-dimensional bosonic string has negative mass squared, which is the indication that the theory is unstable. Before the remarkable paper of Sen [1] it was assumed that the presence of the tachyon presents fundamental problems with open bosonic string. Witten's cubic string field theory (CSFT) provided the action which describes the interaction of the tachyonic and all the excited states. From this action one can compute the effective potential for the tachyonic

state. Ashoke Sen in [1] has made three conjectures about the relationship between open string vacua with dramatically different geometrical properties:

1. The effective potential for the tachyon has a minimum, furthermore, the energy difference between the perturbative vacuum and the minimum of the tachyon potential is exactly the energy of the space-filling 25 dimensional D-brane.
2. In the perturbative expansion around the tachyonic vacuum there are no open string excitations. The minimum of the tachyonic potential corresponds to the closed string vacuum where the D-brane has decayed and only closed string excitations are remaining.
3. The lower dimensional branes should be realized as the soliton configurations of the tachyon and other string excitations.

These conjectures have been verified by numerical computations in CSFT in [3, 4, 5, 6, 7, 8, 9, 10] as well as in subsequent research (see [11, 12, 13, 14] for reviews of this work).

Sen's conjectures have made CSFT into an important tool for studying various string backgrounds. In my work with Washington Taylor and Erasmo Coletti we have concentrated on two aspects of string field theory. In Chapter 2, following [15] we describe the calculation of the low energy effective action of the 25-dimensional D-brane from CSFT. In the low energy limit there are two excitations of the open bosonic string remaining: the tachyon ϕ , which has negative mass squared and the massless vector field A_μ which describes the perturbations of the D-brane. The gauge field A_μ takes values from $U(n)$ group. From earlier work analyzing string scattering amplitudes and string partition function [16, 17, 18] it was known, that the low energy effective action for the $U(1)$ vector field A_μ is so called Born-Infeld action

$$S_{BI} = -\frac{1}{(2\pi g_{YM})^2} \int dx \sqrt{-\det(\eta_{\mu\nu} + 2\pi g_{YM} F_{\mu\nu})} \quad (1.1)$$

The lowest term in the power expansion of (1.1) is the usual electromagnetic action. It has also been assumed that for the $U(n)$ models with $n \geq 2$ the resulting action

must be a non-abelian generalization of (1.1). The leading terms in the expansion of such action were computed by analyzing string scattering amplitudes and, in the supersymmetric case by the analysis of restrictions imposed by supersymmetry. The full form of the non-abelian vector field effective action is not known.

The development of SFT provided us with an alternative method of computation of effective action. In the work of Taylor [19], the first calculation of the vector field effective action from SFT was done, and it was shown, that in the abelian case, up to terms of order A^2 the action is simply the standard electromagnetic term. Further progress in the computation of effective action was made with [20] where the method of computation of Feynman diagrams in string field theory was proposed. Using this method in [15] we have computed the effective action for abelian and non-abelian cases as the low energy effective action of string field theory up to terms of the order A^4 . We have also checked by numerical computation that non gauge invariant terms A^n cancel for $n > 4$.

Probably the most important realization of [15] was that the variables in the world sheet and string field theory formulations of string theory are not the same, but rather related by a complicated field redefinition. In the worldsheet formulation of string theory the vector A_μ has standard gauge transformation from the start, and the resulting action is a function of the gauge invariant stress tensor $F_{\mu\nu}$ and its derivatives.

In SFT, gauge invariance for A_μ is dictated by the gauge invariance of the string field theory. After the higher energy excitations are eliminated using the equations of motion, the gauge invariance for A_μ becomes rather complicated non-linear transformation. The field redefinition is required to bring the gauge transformation into the standard form. Only after this field redefinition is made, do we obtain the Born-Infeld action in the $U(1)$ case and in the $U(n)$ case its non-abelian generalization.

Another aspect of D-brane physics we study from string field theory point of view is the rolling tachyon solution of Sen [22]. The rolling tachyon solution is the string background which was proven by Sen to be exact solution of string field theory equations of motion. In Chapter 3 following [21] we solve the CSFT equations of

motion for the rolling tachyon initial condition. The initial condition for the tachyon $\phi(x, t)$ is specified by the asymptotics $\phi(x, t) \sim e^t$ as $t \rightarrow -\infty$. The tachyon field is assumed to stay constant along the spatial coordinates. The motivation for this work comes from the works of Sen [22, 23, 24] on the rolling tachyon in worldsheet formulation of string theory and the work of Moeller and Zwiebach [25] where the rolling tachyon solution in CSFT was calculated in the level zero truncation (which basically means that the theory is truncated to include the tachyon field only, all other fields are forcibly set to zero). In [25] authors have found that the rolling tachyon solution in the level 0 truncation develops evergrowing oscillations. This picture is dramatically different from the worldsheet picture, where the worldsheet tachyon $T(t) \sim e^t$ was shown by Sen to be exact solution. Moeller and Zwiebach conjectured that the situation might get amended by including massive string modes into consideration. We show in [21] that this is not the case, and in fact the growing oscillations are present in CSFT rolling tachyon solution even when we include the massive string modes. The apparent contradiction is resolved by the field redefinition $T(\phi)$ which relates the worldsheet tachyon into the CSFT one. We compute the field redefinition to the leading order in power expansion in ϕ and show that it indeed maps the rolling solution of Sen into the CSFT rolling solution.

Chapter 2

Abelian and nonabelian vector field effective actions from string field theory

2.1 Introduction

Despite major advances in our understanding of nonperturbative features of string theory and M-theory over the last eight years, we still lack a fundamental nonperturbative and background-independent definition of string theory. String field theory seems to incorporate some features of background independence which are missing in other approaches to string theory. Recent work, following the conjectures of Sen [1], has shown that Witten's open bosonic string field theory successfully describes multiple distinct open string vacua with dramatically different geometrical properties, in terms of the degrees of freedom of a single theory (see [11, 12, 13, 14] for reviews of this work). An important feature of string field theory, which allows it to transcend the usual limitations of local quantum field theories, is its essential nonlocality. String field theory is a theory which can be defined with reference to a particular background in terms of an infinite number of space-time fields, with highly nonlocal interactions. The nonlocality of string field theory is similar in spirit to that of noncommutative

field theories which have been the subject of much recent work [26], but in string field theory the nonlocality is much more extreme. In order to understand how string theory encodes a quantum theory of gravity at short distance scales, where geometry becomes poorly defined, it is clearly essential to achieve a better understanding of the nonlocal features of string theory.

While string field theory involves an infinite number of space-time fields, most of these fields have masses on the order of the Planck scale. By integrating out the massive fields, we arrive at an effective action for a finite number of massless fields. In the case of a closed string field theory, performing such an integration would give an effective action for the usual multiplet of gravity/supergravity fields. This action will, however, have a complicated nonlocal structure which will appear through an infinite family of higher-derivative terms in the effective action. In the case of the open string, integrating out the massive fields leads to an action for the massless gauge field. Again, this action is highly nonlocal and contains an infinite number of higher-derivative terms. This nonlocal action for the massless gauge field in the bosonic open string theory is the subject of this chapter. By explicitly integrating out all massive fields in Witten's open string field theory (including the tachyon), we arrive at an effective action for the massless open string vector field. We compute this effective action term-by-term using the level-truncation approximation in string field theory, which gives us a very accurate approximation to each term in the action.

It is natural to expect that the effective action we compute for the massless vector field will take the form of the Born-Infeld action, including higher-derivative terms. Indeed, we show that this is the case, although some care must be taken in making this connection. Early work deriving the Born-Infeld action from string theory [16, 27] used world-sheet methods [17, 18, 18]. More recently, in the context of the supersymmetric nonabelian gauge field action, other approaches, such as κ -symmetry and the existence of supersymmetric solutions, have been used to constrain the form of the action (see [28] for a recent discussion and further references). In this work we take a different approach. We start with string field theory, which is a manifestly off-shell formalism. Our resulting effective action is therefore also an off-shell action.

This action has a gauge invariance which agrees with the usual Yang-Mills gauge invariance to leading order, but which has higher-order corrections arising from the string field star product. A field redefinition analogous to the Seiberg-Witten map [29, 30] is necessary to get a field which transforms in the usual fashion [31, 32]. We identify the leading terms in this transformation and show that after performing the field redefinition our action indeed takes the Born-Infeld form in the abelian theory. In the nonabelian theory, there is an additional subtlety, which was previously encountered in related contexts in [31, 32]. Extra terms appear in the form of the gauge transformation which cannot be removed by a field redefinition. These additional terms, however, are trivial and can be dropped, after which the standard form of gauge invariance can be restored by a field redefinition. This leads to an effective action in the nonabelian theory which takes the form of the nonabelian Born-Infeld action plus derivative correction terms.

It may seem surprising that we integrate out the tachyon as well as the fields in the theory with positive mass squared. This is, however, what is implicitly done in previous work such as [16, 27] where the Born-Infeld action is derived from bosonic string theory. The abelian Born-Infeld action can similarly be derived from recent proposals for the coupled tachyon-vector field action [33, 34, 35, 36] by solving the equation of motion for the tachyon at the top of the hill. In the supersymmetric theory, of course, there is no tachyon on a BPS brane, so the supersymmetric Born-Infeld action should be derivable from a supersymmetric open string field theory by only integrating out massive fields. Physically, integrating out the tachyon corresponds to considering fluctuations of the D-brane in stable directions, while the tachyon stays balanced at the top of its potential hill. While open string loops may give rise to problems in the effective theory [37], at the classical level the resulting action is well-defined and provides us with an interesting model in which to understand the nonlocality of the Born-Infeld action. The classical effective action we derive here must reproduce all on-shell tree-level scattering amplitudes of massless vector fields in bosonic open string theory. To find a sensible action which includes quantum corrections, it is probably necessary to consider the analogue of the calculation in

this chapter in the supersymmetric theory, where there is no closed string tachyon.

The structure of this chapter is as follows: In Section 2 we review the formalism of string field theory, set notation and make some brief comments regarding the Born-Infeld action. In Section 3 we introduce the tools needed to calculate terms in the effective action of the massless fields. Section 4 contains a calculation of the effective action for all terms in the Yang-Mills action. Section 5 extends the analysis to include the next terms in the Born-Infeld action in the abelian case and Section 6 does the same for the nonabelian analogue of the Born-Infeld action. Section 7 contains concluding remarks. Some useful properties of the Neumann matrices appearing in the 3-string vertex of Witten's string field theory are included in the Appendix.

2.2 Review of formalism

Subsection 2.2.1 summarizes our notation and the basics of string field theory. In subsection 2.2.2 we review the method of [20] for computing terms in the effective action. The last subsection, 2.2.3, contains a brief discussion of the Born-Infeld action.

2.2.1 Basics of string field theory

In this subsection we review the basics of Witten's open string field theory [38]. For further background information see the reviews [39, 40, 41, 13]. The degrees of freedom of string field theory (SFT) are functionals $\Phi[x(\sigma); c(\sigma), b(\sigma)]$ of the string configuration $x^\mu(\sigma)$ and the ghost and antighost fields $c(\sigma)$ and $b(\sigma)$ on the string at a fixed time. String functionals can be expressed in terms of string Fock space states, just as functions in $\mathcal{L}^2(\mathbb{R})$ can be expressed as linear combinations of harmonic oscillator eigenstates. The Fock module of a single string of momentum p is obtained by the action of the matter, ghost and antighost oscillators on the (ghost number one) highest weight vector $|p\rangle$. The action of the raising and lowering oscillators on

$|p\rangle$ is defined by the creation/annihilation conditions and commutation relations

$$\begin{aligned}
a_{n \geq 1}^\mu |p\rangle &= 0, & [a_m^\mu, a_{-n}^\nu] &= \eta^{\mu\nu} \delta_{m,n}, \\
p^\mu |k\rangle &= k^\mu |k\rangle, & & \\
b_{n \geq 0} |p\rangle &= 0, & \{b_m, c_{-n}\} &= \delta_{m,n}, \\
c_{n \geq 1} |p\rangle &= 0. & &
\end{aligned} \tag{2.1}$$

Hermitian conjugation is defined by $a_n^{\mu\dagger} = a_{-n}^\mu$, $b_n^\dagger = b_{-n}$, $c_n^\dagger = c_{-n}$. The single-string Fock space is then spanned by the set of all vectors $|\chi\rangle = \cdots a_{n_2} a_{n_1} \cdots b_{k_2} b_{k_1} \cdots c_{l_2} c_{l_1} |p\rangle$ with $n_i, k_i < 0$ and $l_i \leq 0$. String fields of ghost number 1 can be expressed as linear combinations of such states $|\chi\rangle$ with equal number of b 's and c 's, integrated over momentum.

$$|\Phi\rangle = \int d^{26}p \left(\phi(p) + A_\mu(p) a_{-1}^\mu - i\alpha(p) b_{-1} c_0 + B_{\mu\nu}(p) a_{-1}^\mu a_{-1}^\nu + \cdots \right) |p\rangle. \tag{2.2}$$

The Fock space vacuum $|0\rangle$ that we use is related to the $SL(2, \mathbb{R})$ invariant vacuum $|1\rangle$ by $|0\rangle = c_1 |1\rangle$. Note that $|0\rangle$ is a Grassmann odd object, so that we should change the sign of our expression whenever we interchange $|0\rangle$ with a Grassmann odd variable. The bilinear inner product between the states in the Fock space is defined by the commutation relations and

$$\langle k | c_0 | p \rangle = (2\pi)^{26} \delta(k + p). \tag{2.3}$$

The SFT action can be written as

$$S = -\frac{1}{2} \langle V_2 | \Phi, Q_B \Phi \rangle - \frac{g}{3} \langle V_3 | \Phi, \Phi, \Phi \rangle \tag{2.4}$$

where $|V_n\rangle \in \mathcal{H}^n$. This action is invariant under the gauge transformation

$$\delta|\Phi\rangle = Q_B|\Lambda\rangle + g(\langle \Phi, \Lambda | V_3 \rangle - \langle \Lambda, \Phi | V_3 \rangle) \tag{2.5}$$

with Λ a string field gauge parameter at ghost number 0. Explicit oscillator representations of $\langle V_2|$ and $\langle V_3|$ are given by [42, 43, 44, 45]

$$\langle V_2| = \int d^{26}p \langle p|^{(1)} \otimes \langle -p|^{(2)} (c_0^{(1)} + c_0^{(2)}) \exp(a^{(1)} \cdot C \cdot a^{(2)} - b^{(1)} \cdot C \cdot c^{(2)} - b^{(1)} \cdot C \cdot c^{(2)}) \quad (2.6)$$

and

$$\begin{aligned} \langle V_3| = & \mathcal{N} \int \prod_{i=1}^3 \left(d^{26}p_i \langle p_i|^{(i)} c_0^{(i)} \right) \delta\left(\sum p_j\right) \\ & \times \exp\left(\frac{1}{2}a^{(r)} \cdot V^{rs} \cdot a^{(s)} - p^{(r)} V_0^{rs} \cdot a^{(s)} + \frac{1}{2}p^{(r)} V_{00}^{rr} p^{(r)} - b^{(r)} \cdot X^{rs} \cdot c^{(s)}\right) \end{aligned} \quad (2.7)$$

where all inner products denoted by \cdot indicate summation from 1 to ∞ except in $b \cdot X$, where the summation includes the index 0. The contracted Lorentz indices in a_n^μ and p_μ are omitted. $C_{mn} = (-1)^n \delta_{mn}$ is the BPZ conjugation matrix. The matrix elements V_{mn}^{rs} and X_{mn}^{rs} are called Neumann coefficients. Explicit expressions for the Neumann coefficients and some relevant properties of these coefficients are summarized in the Appendix. The normalization constant \mathcal{N} is defined by

$$\mathcal{N} = \exp\left(-\frac{1}{2} \sum_r V_{00}^{rr}\right) = \frac{3^{9/2}}{2^6}, \quad (2.8)$$

so that the on-shell three-tachyon amplitude is given by $2g$. We use units where $\alpha' = 1$.

2.2.2 Calculation of effective action

String field theory can be thought of as a (nonlocal) field theory of the infinite number of fields that appear as coefficients in the oscillator expansion (2.2). In this chapter, we are interested in integrating out all massive fields at tree level. This can be done using standard perturbative field theory methods. Recently an efficient method of performing sums over intermediate particles in Feynman graphs was proposed in [20]. We briefly review this approach here; an alternative approach to such computations

has been studied recently in [46].

In this chapter, while we include the massless auxiliary field α appearing in the expansion (2.2) as an external state in Feynman diagrams, all the massive fields we integrate out are contained in the Feynman-Siegel gauge string field satisfying

$$b_0|\Phi\rangle = 0, \quad (2.9)$$

This means that intermediate states in the tree diagrams we consider do not have a c_0 in their oscillator expansion. For such states, the propagator can be written in terms of a Schwinger parameter τ as

$$\frac{b_0}{L_0} = b_0 \int_0^\infty d\tau e^{-\tau L_0}, \quad (2.10)$$

In string field theory, the Schwinger parameters can be interpreted as moduli for the Riemann surface associated with a given diagram [47, 48, 49, 41, 50].

In field theory one computes amplitudes by contracting vertices with external states and propagators. Using the quadratic and cubic vertices (2.6), (2.7) and the propagator (2.10) we can do same in string field theory. To write down the contribution to the effective action arising from a particular Feynman graph we include a vertex $\langle V_3| \in \mathcal{H}^{*3}$ for each vertex of the graph and a vertex $|V_2\rangle$ for each internal edge. The propagator (2.10) can be incorporated into the quadratic vertex through ¹

$$|P\rangle = - \int_0^\infty d\tau e^{\tau(1-p^2)} |\tilde{V}_2\rangle. \quad (2.13)$$

¹Consider the tachyon propagator as an example. We contract $c_0|p_1\rangle$ and $c_0|p_2\rangle$ with $\langle P|$ to get

$$\langle P| c_0|p_1\rangle c_0|p_2\rangle = - \int_0^\infty d\tau e^{\tau(1-p_1^2)} \delta(p_1 + p_2) = - \frac{\delta(p_1 + p_2)}{p_1^2 - 1}. \quad (2.11)$$

This formula assumes that both momenta are incoming. Setting $p_1 = -p_2 = p$ and using the metric with $(-, +, +, \dots, +)$ signature we have

$$- \frac{1}{p^2 + m^2} = \frac{1}{p_0^2 - \vec{p}^2 - m^2} \quad (2.12)$$

thus (2.11) is indeed the correct propagator for the scalar particle of mass $m^2 = -1$.

where in the modified vertex $|\tilde{V}_2(\tau)\rangle$ the ghost zero modes c_0 are canceled by the b_0 in (2.10) and the matrix C_{mn} is replaced by

$$\tilde{C}_{mn}(\tau) = e^{-m\tau}(-1)^m\delta_{mn}. \quad (2.14)$$

With these conventions, any term in the effective action can be computed by contracting the three-vertices from the corresponding Feynman diagram on the left with factors of $|P\rangle$ and low-energy fields on the right (or vice-versa, with $|V_3\rangle$'s on the right and $\langle P|$'s on the left). Because the resulting expression integrates out *all* Feynman-Siegel gauge fields along interior edges, we must remove the contribution from the intermediate massless vector field by hand when we are computing the effective action for the massless fields. Note that in [20], a slightly different method was used from that just described; there the propagator was incorporated into the three-vertex rather than the two-vertex. Both methods are equivalent; we use the method just described for convenience.

States of the form

$$\exp\left(\lambda \cdot a^\dagger + \frac{1}{2}a^\dagger \cdot S \cdot a^\dagger\right) |p\rangle \quad (2.15)$$

are called squeezed states. The vertex $|V_3\rangle$ and the propagator $|P\rangle$ are (linear combinations of) squeezed states and thus are readily amenable to computations. The inner product of two squeezed states is given by [51]

$$\begin{aligned} & \langle 0 | \exp(\lambda \cdot a + \frac{1}{2}a \cdot S \cdot a) \exp(\mu \cdot a^\dagger + \frac{1}{2}a^\dagger \cdot V \cdot a^\dagger) | 0 \rangle \\ &= \text{Det}(1 - S \cdot V)^{-1/2} \exp[\lambda \cdot (1 - V \cdot S)^{-1} \cdot \mu \\ & \quad + \frac{1}{2}\lambda \cdot (1 - V \cdot S)^{-1} \cdot V \cdot \lambda + \frac{1}{2}\mu \cdot S \cdot (1 - V \cdot S)^{-1} \cdot \mu] \quad (2.16) \end{aligned}$$

and (neglecting ghost zero-modes)

$$\begin{aligned}
& \langle 0 | \exp(b \cdot \lambda_b - \lambda_c \cdot c - b \cdot S \cdot c) \exp(b^\dagger \cdot \mu_b + \mu_c \cdot c^\dagger + b^\dagger \cdot V \cdot c^\dagger) | 0 \rangle \\
&= \text{Det}(1 - S \cdot V) \exp[-\lambda_c \cdot (1 - V \cdot S)^{-1} \cdot \mu_b - \mu_c \cdot (1 - S \cdot V)^{-1} \cdot \lambda_b \\
&\quad + \lambda_c \cdot (1 - V \cdot S)^{-1} \cdot V \cdot \lambda_b + \mu_c \cdot S \cdot (1 - V \cdot S)^{-1} \cdot \mu_b]. \quad (2.17)
\end{aligned}$$

Using these expressions, the combination of three-vertices and propagators associated with any Feynman diagram can be simply rewritten as an integral over modular (Schwinger) parameters of a closed form expression in terms of the infinite matrices $V_{nm}, X_{nm}, \tilde{C}_{nm}(\tau)$. The schematic form of these integrals is

$$\begin{aligned}
& (\langle V_3 |)^v (|P\rangle)^i \sim \left(\prod_{j=1}^i \int d\tau^j \right) \frac{\text{Det}(1 - \hat{C}\hat{X})}{\text{Det}(1 - \hat{C}\hat{V})^{13}} \\
& \times (\langle 0 |)^{3v-2i} \exp\left(\frac{1}{2} a^\dagger \cdot S \cdot a^\dagger + \mu \cdot a^\dagger + b^\dagger \cdot U \cdot c^\dagger + \mu_c \cdot c^\dagger + b^\dagger \cdot \mu_b\right) \quad (2.18)
\end{aligned}$$

where $\hat{C}, \hat{X}, \hat{V}$ are matrices with blocks of the form \tilde{C}, X, V arranged according to the combinatorial structure of the diagram. The matrix \hat{C} and the squeezed state coefficients S, U, μ, μ_b, μ_c depend implicitly on the modular parameters τ^i .

2.2.3 The effective vector field action and Born-Infeld

In this subsection we describe how the effective action for the vector field is determined from SFT, and we discuss the Born-Infeld action [52] which describes the leading terms in this effective action. For a more detailed review of the Born-Infeld action, see [53]

As discussed in subsection 2.1, the string field theory action is a space-time action for an infinite set of fields, including the massless fields $A_\mu(x)$ and $\alpha(x)$. This action has a very large gauge symmetry, given by (2.5). We wish to compute an effective action for $A_\mu(x)$ which has a single gauge invariance, corresponding at leading order to the usual Yang-Mills gauge invariance. We compute this effective action in several steps. First, we use Feynman-Siegel gauge (2.9) for all *massive* fields in the theory. This leaves a single gauge invariance, under which A_μ and α have linear components

in their gauge transformation rules. This partial gauge fixing is described more precisely in section 2.5.2. Following this partial gauge fixing, all massive fields in the theory, including the tachyon, can be integrated out using the method described in the previous subsection, giving an effective action

$$\tilde{S}[A_\mu(x), \alpha(x)] \tag{2.19}$$

depending on A_μ and α . We can then further integrate out the field α , which has no kinetic term, to derive the desired effective action

$$S[A_\mu(x)]. \tag{2.20}$$

The action (2.20) still has a gauge invariance, which at leading order agrees with the Yang-Mills gauge invariance

$$\delta A_\mu(x) = \partial_\mu \lambda(x) - ig_{\text{YM}}[A_\mu(x), \lambda(x)] + \dots \tag{2.21}$$

The problem of computing the effective action for the massless gauge field in open string theory is an old problem, and has been addressed in many other ways in past literature. Most methods used in the past for calculating the effective vector field action have used world-sheet methods. While the string field theory approach we use here has the advantage that it is a completely off-shell formalism, as just discussed the resulting action has a nonstandard gauge invariance [32]. In world-sheet approaches to this computation, the vector field has the standard gauge transformation rule (2.21) with no further corrections. A general theorem [54] states that there are no deformations of the Yang-Mills gauge invariance which cannot be taken to the usual Yang-Mills gauge invariance by a field redefinition. In accord with this theorem, we identify in this chapter field redefinitions which take the massless vector field A_μ in the SFT effective action (2.20) to a gauge field \hat{A}_μ with the usual gauge invariance. We write the resulting action as

$$\hat{S}[\hat{A}_\mu(x)]. \tag{2.22}$$

This action, written in terms of a conventional gauge field, can be compared to previous results on the effective action for the open string massless vector field.

Because the mass-shell condition for the vector field $A_\mu(p)$ in Fourier space is $p^2 = 0$, we can perform a sensible expansion of the action (2.20) as a double expansion in p and A . We write this expansion as

$$S[A_\mu] = \sum_{n=2}^{\infty} \sum_{k=0}^{\infty} S_{A^n}^{[k]} \quad (2.23)$$

where $S_{A^n}^{[k]}$ contains the contribution from all terms of the form $\partial^k A^n$. A similar expansion can be done for \hat{S} , and we similarly denote by $\check{S}_{\alpha^m A^n}^{[k]}$ the sum of the terms in \check{S} of the form $\partial^k \alpha^m A^n$.

Because the action $\hat{S}[\hat{A}]$ is a function of a gauge field with conventional gauge transformation rules, this action can be written in a gauge invariant fashion; *i.e.* in terms of the gauge covariant derivative $\hat{D}_\mu = \partial_\mu - ig_{\text{YM}}[\hat{A}, \cdot]$ and the field strength $\hat{F}_{\mu\nu}$. For the abelian theory, \hat{D}_μ is just ∂_μ , and there is a natural double expansion of \hat{S} in terms of p and F . It was shown in [16, 27] that in the abelian theory the set of terms in \hat{S} which depend only on \hat{F} , with no additional factors of p (*i.e.*, the terms in $\hat{S}_{\hat{A}^n}^{[n]}$) take the Born-Infeld form (dropping hats)

$$S_{BI} = -\frac{1}{(2\pi g_{\text{YM}})^2} \int dx \sqrt{-\det(\eta_{\mu\nu} + 2\pi g_{\text{YM}} F_{\mu\nu})} \quad (2.24)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (2.25)$$

is the gauge-invariant field strength. Using $\log(\det M) = \text{tr}(\log(M))$ we can expand in F to get

$$S_{BI} = -\frac{1}{(2\pi g_{\text{YM}})^2} \int dx \left(1 + \frac{(2\pi g_{\text{YM}})^2}{4} F_{\mu\nu} F^{\mu\nu} - \frac{(2\pi g_{\text{YM}})^4}{8} \left(F_{\mu\nu} F^\nu{}_\lambda F^\lambda{}_\sigma F^{\sigma\mu} - \frac{1}{4} (F_{\mu\nu} F^{\mu\nu})^2 \right) + \dots \right). \quad (2.26)$$

We expect that after the appropriate field redefinition, the result we calculate from string field theory for the effective vector field action (2.20) should contain as a leading part at each power of \hat{A} terms of the form (2.26), as well as higher-derivative terms of the form $\partial^{n+k}A^n$ with $k > 0$. We show in section 5 that this is indeed the case.

The nonabelian theory is more complicated. In the nonabelian theory we must include covariant derivatives, whose commutators mix with field strengths through relations such as

$$[D_\mu, D_\nu]F_{\lambda\sigma} = [F_{\mu\nu}, F_{\lambda\sigma}]. \quad (2.27)$$

In this case, there is no systematic double expansion in powers of D and F . It was pointed out by Tseytlin in [56] that when F is taken to be constant, and both commutators $[F, F]$ and covariant derivatives of field strengths DF are taken to be negligible, the nonabelian structure of the theory is irrelevant. In this case, the action reduces to the Born-Infeld form (2.24), where the ordering ambiguity arising from the matrix nature of the field strength F is resolved by the symmetrized trace (STr) prescription whereby all possible orderings of the F 's are averaged over. While this observation is correct, it seems that the symmetrized trace formulation of the nonabelian Born-Infeld action misses much of the important physics of the full vector field effective action. In particular, this simplification of the action gives the wrong spectrum around certain background fields, including those which are T-dual to simple intersecting brane configurations [57, 58, 59, 60]. It seems that the only systematic way to deal with the nonabelian vector field action is to include all terms of order F^n at once, counting D at order $F^{1/2}$. The first few terms in the nonabelian vector field action for the bosonic theory were computed in [61, 62, 63]. The terms in the action up to F^4 are given by

$$S_{\text{nonabelian}} = \int -\frac{1}{4}\text{Tr} F^2 + \frac{2ig_{YM}}{3}\text{Tr}(F^3) + \frac{(2\pi g_{YM})^2}{8}\text{STr}\left(F^4 - \frac{1}{4}(F^2)^2\right) + \dots \quad (2.28)$$

In section 6, we show that the effective action we derive from string field theory agrees with (2.28) up to order F^3 after the appropriate field redefinition.

2.3 Computing the effective action

In this section we develop some tools for calculating low-order terms in the effective action for the massless fields by integrating out all massive fields. Section 2.3.1 describes a general approach to computing the generating functions for terms in the effective action and gives explicit expressions for the generating functions of cubic and quartic terms. Section 2.3.2 contains a general derivation of the quartic terms in the effective action for the massless fields. Section 2.3.3 describes the method we use to numerically approximate the coefficients in the action.

2.3.1 Generating functions for terms in the effective action

A convenient way of calculating SFT diagrams is to first compute the off-shell amplitude with generic external coherent states

$$|G\rangle = \exp(J_{m\mu}a_{-m}^\mu - b_{-m}\mathcal{J}_{bm} + \mathcal{J}_{cm}c_{-m})|p\rangle \quad (2.29)$$

where the index m runs from 1 to ∞ in $J_{m\mu}^\mu$ and \mathcal{J}_{bm} and from 0 to ∞ in \mathcal{J}_{cm} .

Let $\Omega_M(p^i, J^i, \mathcal{J}_b^i, \mathcal{J}_c^i; 1 \leq i \leq M)$ be the sum of all connected tree-level diagrams with M external states $|G^i\rangle$. Ω_M is a generating function for all tree-level off-shell M -point amplitudes and can be used to calculate all terms we are interested in in the effective action. Suppose that we are interested in a term in the effective action whose j 'th field $\psi_{\mu_1, \dots, \mu_N}^{(j)}(p)$ is associated with the Fock space state

$$\prod_{m,n,q} a_{i_m}^{\mu_m} b_{k_n} c_{l_q} |p\rangle. \quad (2.30)$$

We can obtain the associated off-shell amplitude by acting on Ω_M with the corresponding differential operator for each j

$$\int dp \psi_{\mu_1, \dots, \mu_N}^{(j)}(p) \prod_{m,n,q} \frac{\partial}{\partial J_{i_m \mu_m}^j} \frac{\partial}{\partial \mathcal{J}_{b k_n}^j} \frac{\partial}{\partial \mathcal{J}_{c l_q}^j} \quad (2.31)$$

and setting J^j , \mathcal{J}_b^j , and \mathcal{J}_c^j to 0. Thus, all the terms in the effective action which we

are interested in can be obtained from Ω_M .

When we calculate a certain diagram with external states $|G^i\rangle$ by applying formulae (2.16) and (2.17) for inner products of coherent and squeezed states the result has the general form

$$\Omega_M = \delta\left(\sum p^r\right) \int \prod_{\ell=1}^{N_{\text{prop}}} d\tau_\ell \mathcal{F}(p, \tau) \times \exp\left(\frac{1}{2} J_m^i \Delta_{mn}^{ij}(\tau) J_n^j - p^i \Delta_{0m}^{ij}(\tau) J_m^j + p_i \Delta_{00}^{ij}(\tau) p_j + \text{ghosts}\right). \quad (2.32)$$

A remarkable feature is that (2.32) depends on the sources $J^j, \mathcal{J}_b^j, \mathcal{J}_c^j$ only through the exponent of a quadratic form. Wick's theorem is helpful in writing the derivatives of the exponential in an efficient way. Indeed, the theorem basically reads

$$\prod_{i=1}^M \frac{\partial}{\partial J_n^i} \exp\left(\frac{1}{2} J_m^j \Delta_{mn}^{jk} J_n^k\right) \Bigg|_{J_n^i=0} = \text{Sum over all contraction products} \quad (2.33)$$

where the sum is taken over all pairwise contractions, with the contraction between (n, i) and (m, j) carrying the factor Δ_{nm}^{ij} .

Note that Ω_M includes contributions from all the intermediate fields in Feynman-Siegel gauge. To compute the effective action for A_μ we must project out the contribution from intermediate A_μ 's.

Three-point generating function

Here we illustrate the idea sketched above with the simple example of the three-point generating function. This generating function provides us with an efficient method of computing the coefficients of the SFT action and the SFT gauge transformation. Plugging $|G^i\rangle$, $1 \leq i \leq 3$ into the cubic vertex (2.7) and using (2.16), (2.17) to evaluate the inner products we find

$$\Omega_3 = -\frac{\mathcal{N}g}{3} \delta\left(\sum_r p^r\right) \exp\left(\frac{1}{2} p^r V_{00}^{rs} p^s - p^r V_{0n}^{rs} J_n^s + \frac{1}{2} J_m^r V_{mn}^{rs} J_n^s - \mathcal{J}_{cm}^r X_{mn}^{rs} \mathcal{J}_{bn}^s\right). \quad (2.34)$$

As an illustration of how this generating function can be used consider the three-tachyon term in the effective action. The external tachyon state is $\int dp \phi(p)|p\rangle$. The three-tachyon vertex is obtained from (2.34) by simple integration over momenta and setting the sources to 0. No differentiations are necessary in this case. The three-tachyon term in the action is then

$$-\frac{g}{3} \langle V_3 | \phi, \phi, \phi \rangle = -\frac{\mathcal{N}g}{3} \int \delta(\sum_s p^s) \prod_r dp^r \phi(p_r) \exp\left(\frac{1}{2} p^r V_{00}^{rs} p^s\right) = -\frac{\mathcal{N}g}{3} \int dx \tilde{\phi}(x)^3 \quad (2.35)$$

where

$$\tilde{\phi}(x) = \exp\left(-\frac{1}{2} V_{00}^{11} \partial^2\right) \phi(x). \quad (2.36)$$

For on-shell tachyons, $\partial^2 \phi(x) = -\phi(x)$, so that we have

$$-\frac{g}{3} \langle V_3 | \phi, \phi, \phi \rangle = -\frac{g}{3} \mathcal{N} e^{\frac{3}{2} V_{00}^{11}} \int dx \phi(x)^3 = -\frac{g}{3} \int dx \phi(x)^3. \quad (2.37)$$

The normalization constant cancels so that the on-shell three-tachyon amplitude is just $2g$, in agreement with conventions used here and in [64].

Four-point generating function

Now let us consider the generating function for all quartic off-shell amplitudes (see Figure 2-1). The amplitude Ω_4 after contracting all indices can be written as

$$\Omega_4 = \frac{\mathcal{N}^2 g^2}{2} \int_0^\infty d\tau e^{\tau(1-(p_1+p_2)^2)} \langle \tilde{V}_2 | |R(1, 2)\rangle |R(3, 4)\rangle \quad (2.38)$$

where

$$|R(i, j)\rangle^{(k)} = \langle G^i | \langle G^j | |V_3\rangle^{(ijk)}. \quad (2.39)$$

Applying (2.16), (2.17) to the inner products in (2.39) we get

$$\begin{aligned}
|R(1, 2)\rangle = \exp\left(\frac{1}{2}p_\mu^\alpha U_{00}^{\alpha\beta} p^{\mu\beta} - p_\mu^\alpha U_{0n}^{\alpha\beta} J_n^{\alpha\mu} + \frac{1}{2}J_{m\mu}^\alpha U_{mn}^{\alpha\beta} J_n^{\beta\mu}\right. \\
+ a_{-m\mu}^{(3)} U_{mn}^{33} a_{-n}^{\mu(3)} + (J_{m\mu}^\alpha U_{mn}^{\alpha 3} - p_\mu^\alpha U_{0n}^{\alpha 3}) a_{-n}^{\mu(3)} - \mathcal{J}_{cm}^\alpha X_{mn}^{\alpha\beta} \mathcal{J}_{bn}^\beta \\
\left. + b_{-m}^{(3)} X_{mn}^{3\alpha} \mathcal{J}_{bn}^\alpha - \mathcal{J}_{cm}^\alpha X_{mn}^{\alpha 3} c_{-n}^{(3)} - b_{-m}^{(3)} X_{mn}^{33} c_{-n}^{(3)}\right) c_0 | -p^1 - p^2 \rangle. \quad (2.40)
\end{aligned}$$

Here $\alpha, \beta \in 1, 2$ and

$$U^{rs} = \begin{pmatrix} V_{00}^{rs} - V_{00}^{r3} - V_{00}^{3s} + V_{00}^{33} & V_{0n}^{rs} - V_{0n}^{3s} \\ V_{n0}^{rs} - V_{n0}^{r3} & V_{mn}^{rs} \end{pmatrix}. \quad (2.41)$$

Using (2.16), (2.17) one more time to evaluate the inner products in (2.38) we obtain

$$\begin{aligned}
\Omega_4 = \frac{\mathcal{N}^2 g^2}{2} \delta\left(\sum_i p^i\right) \int_0^\infty d\tau e^\tau \text{Det} \left(\frac{1 - \tilde{X}^2}{(1 - \tilde{V}^2)^{13}} \right) \\
\times \exp \left(\frac{1}{2} p_\mu^i Q_{00}^{ij} p^{j\mu} - p_\mu^i Q_{0n}^{ij} J_n^{j\mu} + \frac{1}{2} J_{m\mu}^i Q_{mn}^{ij} J_n^{j\mu} - \mathcal{J}_{cm}^i Q_{mn}^{ij} \mathcal{J}_{bn}^j \right). \quad (2.42)
\end{aligned}$$

Here $i, j \in 1, 2, 3, 4$. the matrices \tilde{V} and \tilde{X} are defined by

$$\tilde{V}_{mn} = e^{-\frac{m}{2}\tau} V_{mn} e^{-\frac{n}{2}\tau}, \quad \tilde{X}_{mn} = e^{-\frac{m}{2}\tau} X_{mn} e^{-\frac{n}{2}\tau}. \quad (2.43)$$

The matrices Q^{ij} and \tilde{Q}^{ij} are defined through the tilded matrices \tilde{Q}^{ij} and \tilde{Q}^{ij}

$$\tilde{Q}_{mn}^{ij} = e^{-\frac{m}{2}\tau} Q_{mn}^{ij} e^{-\frac{n}{2}\tau}, \quad \tilde{Q}_{mn}^{ij} = e^{-\frac{m}{2}\tau} \tilde{Q}_{mn}^{ij} e^{-\frac{n}{2}\tau} \quad (2.44)$$

where the tilded matrices \tilde{Q} and \tilde{Q} are defined through \tilde{V} , \tilde{U} , \tilde{X}

$$\begin{aligned}
\tilde{Q}^{\alpha\beta} = \tilde{U}^{\alpha 3} \frac{1}{1 - \tilde{V}^2} \tilde{V} \tilde{U}^{3\beta} + \tilde{U}^{\alpha\beta}, \quad \tilde{Q}^{\alpha\beta} = \tilde{X}^{\alpha 3} \frac{1}{1 - \tilde{X}^2} \tilde{X} \tilde{X}^{3\beta} + \tilde{X}^{\alpha\beta}, \\
\tilde{Q}_{mn}^{\alpha\alpha'} = - \left(\tilde{U}^{\alpha 3} \frac{1}{1 - \tilde{V}^2} C \tilde{U}^{3\alpha'} \right)_{mn} + \delta_{0m} \delta_{0n} \tau, \quad \tilde{Q}^{\alpha\alpha'} = - \tilde{X}^{\alpha 3} \frac{1}{1 - \tilde{X}^2} C \tilde{X}^{3\alpha'} \quad (2.45)
\end{aligned}$$

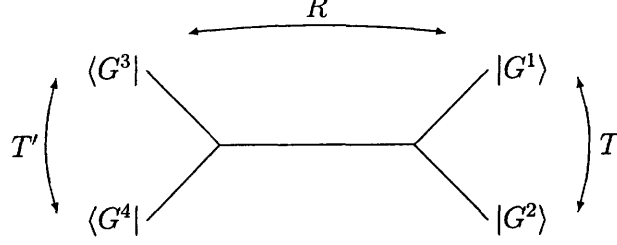


Figure 2-1: Twists T , T' and reflection R are symmetries of the amplitude.

with $\alpha, \beta \in 1, 2$; $\alpha', \beta' \in 3, 4$. The matrix \tilde{U} includes zero modes while \tilde{V} does not, so one has to understand $\tilde{U}\tilde{V}$ in (2.45) as a product of \tilde{U} , where the first column is dropped, and \tilde{V} . Similarly $\tilde{V}\tilde{U}$ is the product of \tilde{V} and \tilde{U} with the first row of \tilde{U} omitted.

The matrices Q^{ij} are not all independent for different i and j . The four-point amplitude is invariant under the twist transformation of either of the two vertices as well as under the interchange of the two (see Figure 2-1). In addition the whole block matrix Q_{mn}^{ij} has been defined in such a way that it is symmetric under the simultaneous exchange of i with j and m with n . Algebraically, we can use properties (A.7a, A.7b, A.7c) of Neumann coefficients to show that the matrices Q^{ij} satisfy

$$\begin{aligned}
(Q^{\alpha\beta})^T &= Q^{\beta\alpha}, & CQ^{\alpha\beta}C &= Q^{3-\alpha\ 3-\beta}, & Q^{\alpha\beta} &= Q^{\alpha+2\ \beta+2}, \\
(Q^{\alpha'\beta'})^T &= Q^{\beta'\alpha'}, & CQ^{\alpha'\beta'}C &= Q^{7-\alpha'\ 7-\beta'}, & Q^{\alpha'\beta'} &= Q^{\alpha'-2\ \beta'-2}, \\
(Q^{\alpha\alpha'})^T &= Q^{\alpha'\alpha}, & CQ^{\alpha\alpha'}C &= Q^{3-\alpha\ 7-\alpha'}, & Q^{\alpha\alpha'} &= Q^{\alpha+2\ \alpha'-2}.
\end{aligned} \tag{2.46}$$

The analogous relations are satisfied by ghost matrices \mathcal{Q} .

Note that we still have some freedom in the definition of the zero modes of the matter matrices Q . Due to the momentum conserving delta function we can add to the exponent in the integrand of (2.42) any expression proportional to $\sum p_i$. To fix this freedom we require that after the addition of such a term the new matrices \bar{Q} satisfy $\bar{Q}_{00}^{ii} = \bar{Q}_{0n}^{ii} = 0$. This gives

$$\bar{Q}_{00}^{ij} = Q_{00}^{ij} - \frac{1}{2}Q_{00}^{jj} - \frac{1}{2}Q_{00}^{ii}, \quad \bar{Q}_{0n}^{ij} = Q_{0n}^{ij} - Q_{0n}^{jj}. \tag{2.47}$$

and $\bar{Q}_{mn}^{ij} = Q_{mn}^{ij}$ for $m, n > 0$. The addition of any term proportional to $\sum p_i$ corresponds in coordinate space to the addition of a total derivative. In coordinate space we have essentially integrated by parts the terms $\partial_\sigma \partial^\sigma \psi_{\mu_1 \dots \mu_n}(x)$ and $\partial^{\mu_j} \psi_{\mu_1 \dots \mu_j \dots \mu_n}(x)$ thus fixing the freedom of integration by parts.

To summarize, we have rewritten Ω_4 in terms of \bar{Q} 's as

$$\Omega_4 = \frac{\mathcal{N}^2 g^2}{2} \delta\left(\sum_i p^i\right) \int_0^\infty d\tau e^\tau \text{Det} \left(\frac{1 - \tilde{X}^2}{(1 - \tilde{V}^2)^{13}} \right) \times \exp \left(\frac{1}{2} p_\mu^i \bar{Q}_{00}^{ij} p^{j\mu} - p_\mu^i \bar{Q}_{0n}^{ij} J_n^{j\mu} + \frac{1}{2} J_{m\mu}^i \bar{Q}_{mn}^{ij} J_n^{j\mu} - \mathcal{J}_{cm}^i Q_{mn}^{ij} \mathcal{J}_{bn}^j \right). \quad (2.48)$$

There are only three independent matrices \bar{Q} . For later use we find it convenient to denote the independent \bar{Q} 's by $A = \bar{Q}^{12}$, $B = \bar{Q}^{13}$, $C = \bar{Q}^{14}$. Then the matrix \bar{Q}_{mn}^{ij} can be written as

$$\bar{Q}_{mn}^{ij} = \begin{pmatrix} 0 & A_{mn} & B_{mn} & C_{mn} \\ (-1)^{m+n} A_{mn} & 0 & (-1)^{m+n} C_{mn} & (-1)^{m+n} B_{mn} \\ B_{mn} & C_{mn} & 0 & A_{mn} \\ (-1)^{m+n} C_{mn} & (-1)^{m+n} B_{mn} & (-1)^{m+n} A_{mn} & 0 \end{pmatrix}. \quad (2.49)$$

In the next section we derive off-shell amplitudes for the massless fields by differentiating Ω_4 . The generating function Ω_4 defined in (2.48) and supplemented with the definition of the matrices \tilde{V} , \tilde{X} , \bar{Q} , Q given in (2.41), (2.43), (2.44), (2.45), (2.47) and (2.49) provides us with all information about the four-point tree-level off-shell amplitudes.

2.3.2 Effective action for massless fields

In this subsection we compute explicit expressions for the general quartic off-shell amplitudes of the massless fields, including derivatives to all orders. Our notation for the massless fields is, as in (2.2),

$$|\Phi_{\text{massless}}\rangle = \int d^d p (A_\mu(p) a_{-1}^\mu - i\alpha(p) b_{-1} c_0) |p\rangle. \quad (2.50)$$

External states with A_μ and α in the k 'th Fock space are inserted using

$$D_\mu^{A,k} = \int dp A_\mu(p) \frac{\partial}{\partial J_{1\mu}^k} \Bigg|_{J^k = \mathcal{J}_{b,c}^k = 0} \quad \text{and} \quad D^{\alpha,k} = -i \int dp \alpha(p) \frac{\partial}{\partial \mathcal{J}_{b1}^k} \frac{\partial}{\partial \mathcal{J}_{c0}^k} \Bigg|_{J^k = \mathcal{J}_{b,c}^k = 0}. \quad (2.51)$$

We can compute all quartic terms in the effective action $\check{S}[A_\mu, \alpha]$ by computing quartic off-shell amplitudes for the massless fields by acting on Ω_4 with D^A and D^α . First consider the quartic term with four external A 's. The relevant off-shell amplitude is given by $\prod_{i=1}^4 D_{\mu_i}^{A,i} \Omega_4$ where Ω_4 is given in (2.48) and $D_{\mu_i}^{A,i}$ is given in (2.51). Performing the differentiations we get

$$S_{A^4} = \frac{1}{2} \mathcal{N}^2 g^2 \int \prod_i dp^i \delta(p^1 + p^2 + p^3 + p^4) A^{\mu_1}(p_1) A^{\mu_2}(p_2) A^{\mu_3}(p_3) A^{\mu_4}(p_4) \\ \times \int_0^\infty d\tau e^\tau \text{Det} \left(\frac{1 - \tilde{X}^2}{(1 - \tilde{V}^2)^{13}} \right) \left(\mathcal{I}_{A^4}^0 + \mathcal{I}_{A^4}^2 + \mathcal{I}_{A^4}^4 \right) \exp \left(\frac{1}{2} p_\mu^i \bar{Q}_{00}^{ij} p^{j\mu} \right). \quad (2.52)$$

Here $\mathcal{I}_{A^4}^0, \mathcal{I}_{A^4}^2, \mathcal{I}_{A^4}^4$ are defined by

$$\mathcal{I}_{A^4}^0 = \frac{1}{8} \sum_{i_i \neq i_j} \bar{Q}_{11}^{i_1 i_2} \bar{Q}_{11}^{i_3 i_4} \eta_{\mu_{i_1} \mu_{i_2}} \eta_{\mu_{i_3} \mu_{i_4}}, \\ \mathcal{I}_{A^4}^2 = \frac{1}{4} \sum_{i_i \neq i_j} \bar{Q}_{11}^{i_1 i_2} \bar{Q}_{10}^{i_3 j_1} \bar{Q}_{10}^{i_4 j_2} p_{\mu_{i_3}}^{j_1} p_{\mu_{i_4}}^{j_2} \eta_{\mu_{i_1} \mu_{i_2}}, \\ \mathcal{I}_{A^4}^4 = \bar{Q}_{10}^{i_1} \bar{Q}_{10}^{j_2} \bar{Q}_{10}^{k_3} \bar{Q}_{10}^{l_4} p_{\mu_1}^i p_{\mu_2}^j p_{\mu_3}^k p_{\mu_4}^l. \quad (2.53)$$

Other amplitudes with α 's and A 's all have the same pattern as (2.52). The amplitude with one α and three A 's is obtained by replacing $A_{\mu_{i_1}}(p^{i_1})$ in formula (2.52) with $i\alpha(p^{i_1})$ and the sum of $\mathcal{I}_{A^4}^{0,2,4}$ with the sum of

$$\mathcal{I}_{\alpha A^3}^1 = \frac{1}{2} \sum_{i_i \neq i_j} \mathcal{Q}_{01}^{i_1 i_1} \bar{Q}_{11}^{i_2 i_3} \bar{Q}_{10}^{i_4 k} p_{\mu_{i_4}}^k \eta_{\mu_{i_2} \mu_{i_3}}, \\ \mathcal{I}_{\alpha A^3}^3 = \frac{1}{6} \sum_{i_i \neq i_j} \mathcal{Q}_{01}^{i_1 i_1} \bar{Q}_{10}^{i_2 j} \bar{Q}_{10}^{i_3 k} \bar{Q}_{10}^{i_4 l} p_{\mu_{i_2}}^j p_{\mu_3}^k p_{\mu_4}^l. \quad (2.54)$$

The amplitude with two A 's and two α 's is obtained by replacing $A_{\mu_1}(p^{i_1})A_{\mu_2}(p^{i_2})$ with $-\alpha(p^{i_1})\alpha(p^{i_2})$ and the sum of $\mathcal{I}_{A^4}^{0,2,4}$ with the sum of

$$\begin{aligned}\mathcal{I}_{\alpha^2 A^2}^0 &= \frac{1}{4} \sum_{i_i \neq i_j} (\mathcal{Q}_{01}^{i_1 i_1} \mathcal{Q}_{01}^{i_2 i_2} - \mathcal{Q}_{01}^{i_1 i_2} \mathcal{Q}_{01}^{i_2 i_1}) \bar{Q}_{11}^{i_3 i_4} \eta_{\mu_3 \mu_4}, \\ \mathcal{I}_{\alpha^2 A^2}^2 &= \frac{1}{4} \sum_{i_i \neq i_j} (\mathcal{Q}_{01}^{i_1 i_1} \mathcal{Q}_{01}^{i_2 i_2} - \mathcal{Q}_{01}^{i_1 i_2} \mathcal{Q}_{01}^{i_2 i_1}) \bar{Q}_{10}^{i_3 k} \bar{Q}_{10}^{i_4 l} p_{\mu_3}^k p_{\mu_4}^l.\end{aligned}\quad (2.55)$$

It is straightforward to write down the analogous expressions for the terms of order $\alpha^3 A$ and α^4 . However, as we shall see later, it is possible to extract all the information about the coefficients in the expansion of the effective action for A_μ in powers of field strength up to F^4 from the terms of order A^4 , $A^3 \alpha$, and $A^2 \alpha^2$.

The off-shell amplitudes (2.52), (2.53), (2.54) and (2.55) include contributions from the intermediate gauge field. To compute the quartic terms in the effective action we must subtract, if nonzero, the amplitude with intermediate A_μ . In the case of the abelian theory this amplitude vanishes due to the twist symmetry. In the nonabelian case, however, the amplitude with intermediate A_μ is nonzero. The level truncation method in the next section makes it easy to subtract this contribution at the stage of numerical computation.

As in (2.23), we expand the effective action in powers of p . As an example of a particular term appearing in this expansion, let us consider the space-time independent (zero-derivative) term of (2.52). In the abelian case there is only one such term: $A_\mu A^\mu A_\nu A^\nu$. The coefficient of this term is

$$\gamma = \frac{1}{2} \mathcal{N}^2 g^2 \int_0^\infty d\tau e^\tau \text{Det} \left(\frac{1 - \tilde{X}^2}{(1 - \tilde{V}^2)^{13}} \right) (A_{11}^2 + B_{11}^2 + C_{11}^2) \quad (2.56)$$

where the matrices A, B and C are those in (2.49). In the nonabelian case there are two terms, $\text{Tr}(A_\mu A^\mu A_\nu A^\nu)$ and $\text{Tr}(A_\mu A_\nu A^\mu A^\nu)$, which differ in the order of gauge fields. The coefficients of these terms are obtained by keeping $A_{11}^2 + C_{11}^2$ and B_{11}^2 terms in (2.56) respectively.

2.3.3 Level truncation

Formula (2.56) and analogous formulae for the coefficients of other terms in the effective action contain integrals over complicated functions of infinite-dimensional matrices. Even after truncating the matrices to finite size, these integrals are rather difficult to compute. To get numerical values for the terms in the effective action, we need a good method for approximately evaluating integrals of the form (2.56). In this subsection we describe the method we use to approximate these integrals. For the four-point functions, which are the main focus of the computations in this chapter, the method we use is equivalent to truncating the summation over intermediate fields at finite field level. Because the computation is carried out in the oscillator formalism, however, the complexity of the computation only grows polynomially in the field level cutoff.

Tree diagrams with four external fields have a single internal propagator with Schwinger parameter τ . It is convenient to do a change of variables

$$\sigma = e^{-\tau}. \tag{2.57}$$

We then truncate all matrices to size $L \times L$ and expand the integrand in powers of σ up to σ^{M-2} , dropping all terms of higher order in σ . We denote this approximation scheme by $\{L, M\}$. The σ^n term of the series contains the contribution from all intermediate fields at level $k = n + 2$, so in this approximation scheme we are keeping all oscillators $a_{k \leq L}^\mu$ in the string field expansion, and all intermediate particles in the diagram of mass $m^2 \leq M - 1$. We will use the approximation scheme $\{L, L\}$ throughout this chapter. This approximation really imposes only one restriction—the limit on the mass of the intermediate particle. It is perhaps useful to compare the approximation scheme we are using here with those used in previous work on related problems. In [20] analogous integrals were computed by numerical integration. This corresponds to $\{L, \infty\}$ truncation. In earlier papers on level truncation in string field theory, such as [3, 4, 5] and many others, the (L, M) truncation scheme was used, in which fields of mass up to $L - 1$ and interaction vertices with total mass of fields in

the vertex up to $M - 3$ are kept. Our $\{L, L\}$ truncation scheme is equivalent to the $(L, L + 2)$ truncation scheme by that definition.

To explicitly see how the σ expansion works let us write the expansion in σ of a generic integrand and take the integral term by term

$$\int_0^1 \frac{d\sigma}{\sigma^2} \sigma^{p^2} \sum_{n=0}^{\infty} c_n(p^i) \sigma^n = \sum_{n=0}^{\infty} \frac{c_n(p^i)}{p^2 + n - 1}. \quad (2.58)$$

Here $p = p_1 + p_2 = p_3 + p_4$ is the intermediate momentum. This is the expansion of the amplitude into poles corresponding to the contributions of (open string) intermediate particles of fixed level. We can clearly see that dropping higher powers of σ in the expansion means dropping the contribution of very massive particles. We also see that to subtract the contribution from the intermediate fields A_μ and α we can simply omit the term $c_1(p)\sigma^{p^2-1}$ in (2.58).

While the Taylor expansion of the integrand might seem difficult, it is in fact quite straightforward. We notice that \tilde{V}^{rs} , and \tilde{X}^{rs} are both of order σ . Therefore we can simply expand the integrand in powers of matrices \tilde{V} and \tilde{X} . For example, the determinant of the matter Neumann coefficients is

$$\text{Det}(1 - \tilde{V}^2)^{-13} = \exp\left(-13 \text{Tr} \text{Log}(1 - \tilde{V}^2)\right). \quad (2.59)$$

Looking again at (2.52) we notice that the only matrix series' that we will need are $\text{Log}(1 - \tilde{V}^2)$ for the determinant (and the analogue for \tilde{X}) and $1/(1 - \tilde{V}^2)$ for \tilde{Q}^{ij} . Computation of these series is straightforward.

It is also easy to estimate how computation time grows with L and M . The most time consuming part of the Taylor expansion in σ is the matrix multiplication. Recall that \tilde{V} is an $L \times L$ matrix whose coefficients are proportional to σ^n at leading order. Elements of \tilde{V}^k are polynomials in σ with M terms. To construct a series $a_0 + a_1 \tilde{V} + \dots + a_M \tilde{V}^M + \mathcal{O}(\sigma^{M+1})$ we need M matrix multiplications $\tilde{V}^k \cdot \tilde{V}$. Each matrix multiplication consists of L^3 multiplications of its elements. Each multiplication of the elements has on the average $M/2$ multiplications of monomials. The total complexity

therefore grows as $L^3 M^2$.

The method just described allows us to compute approximate coefficients in the effective action at any particular finite level of truncation. In [20], it was found empirically that the level truncation calculation gives approximate results for finite on-shell and off-shell amplitudes with errors which go as a power series in $1/L$. Based on this observation, we can perform a least-squares fit on a finite set of level truncation data for a particular term in the effective action to attain a highly accurate estimate of the coefficient of that term. We use this method to compute coefficients of terms in the effective action which are quartic in A throughout the remainder of this chapter.

2.4 The Yang-Mills action

In this section we assemble the Yang-Mills action, picking the appropriate terms from the two, three and four-point Green functions. We write the Yang-Mills action as

$$S_{YM} = \int d^d x \text{Tr} \left(-\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2} \partial_\mu A_\nu \partial^\nu A^\mu + ig_{YM} \partial_\mu A_\nu [A^\mu, A^\nu] + \frac{1}{4} g_{YM}^2 [A_\mu, A_\nu] [A^\mu, A^\nu] \right). \quad (2.60)$$

In section 2.4.1 we consider the quadratic terms of the Yang-Mills action. In section 2.4.2 consider the cubic terms and identify the Yang-Mills coupling constant g_{YM} in terms of the SFT (three tachyon) coupling constant g . This provides us with the expected value for the quartic term. In section 2.4.3 we present the results of a numerical calculation of the (space-time independent) quartic terms and verify that we indeed get the Yang-Mills action.

2.4.1 Quadratic terms

The quadratic term in the action for massless fields, calculated from (2.4), and (2.6) is

$$\check{S}_{A^2} = \int d^d x \text{Tr} \left(-\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu - \alpha^2 + \sqrt{2} \alpha \partial_\mu A^\mu \right). \quad (2.61)$$

Completing the square in α and integrating the term $(\partial A)^2$ by parts we obtain

$$\check{S}_{A_2} = \int d^d x \text{Tr} \left(-\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2} \partial_\mu A_\nu \partial^\nu A^\mu - B^2 \right) \quad (2.62)$$

where we denote

$$B = \alpha - \frac{1}{\sqrt{2}} \partial_\mu A^\mu. \quad (2.63)$$

Eliminating α using the leading-order equation of motion, $B = 0$, leads to the quadratic terms in (2.60). Subleading terms in the equation of motion for α lead to higher-order terms in the effective action, to which we return in the following sections.

2.4.2 Cubic terms

The cubic terms in the action for the massless fields are obtained by differentiating (2.34). The terms cubic in A are given by

$$\begin{aligned} \check{S}_{A^3} = & \frac{\mathcal{N}g}{3} \int \prod_i dp_i \delta \left(\sum_j p_j \right) \text{Tr} \left(A_\mu(p_1) A_\nu(p_2) A_\lambda(p_3) \right) \exp \left(\frac{1}{2} p^r V_{00}^{rr} p^r \right) \\ & \times \left(\eta^{\nu\lambda} p^{r\mu} V_{01}^{r1} V_{11}^{32} + \eta^{\mu\lambda} p^{r\nu} V_{01}^{r2} V_{11}^{13} + \eta^{\mu\nu} p^{r\lambda} V_{01}^{r3} V_{11}^{12} \right) \\ & + p^{r\mu} V_{01}^{r1} p^{s\nu} V_{01}^{s2} p^{t\lambda} V_{01}^{t3}. \end{aligned} \quad (2.64)$$

To compare with the Yang-Mills action we perform a Fourier transform and use the properties of the Neumann coefficients to combine similar terms. We then get

$$\begin{aligned} \check{S}_{A^3} = & -i\mathcal{N}g \int dx \text{Tr} \left(V_{11}^{12} V_{01}^{12} (\partial_\mu \tilde{A}_\nu [\tilde{A}^\mu, \tilde{A}^\nu]) \right. \\ & \left. + \frac{1}{3} (V_{01}^{12})^3 (\partial_\lambda \tilde{A}^\mu \partial_\mu \tilde{A}^\nu \partial_\nu \tilde{A}^\lambda - \partial_\nu \tilde{A}^\mu \partial_\lambda \tilde{A}^\nu \partial_\mu \tilde{A}^\lambda) + (V_{01}^{12})^3 [\tilde{A}_\nu, \partial^\lambda \tilde{A}_\mu] \partial^\mu \partial^\nu \tilde{A}_\lambda \right) \end{aligned} \quad (2.65)$$

where, following the notation introduced in (2.36), we have

$$\tilde{A}_\mu = \exp \left(-\frac{1}{2} V_{00}^{11} \partial^2 \right) A_\mu. \quad (2.66)$$

To reproduce the cubic terms in the Yang-Mills action, we are interested in the terms in (2.65) of order ∂A^3 . The remaining terms and the terms coming from the expansion of the exponential of derivatives contribute to higher-order terms in the effective action, which we discuss later. The cubic terms in the action involving the α field are

$$\begin{aligned}\check{S}_{A\alpha^2} &= -i\mathcal{N}gV_{01}^{12} (X_{01}^{12})^2 \int dx \text{Tr}(\bar{A}^\mu [\partial_\mu \bar{\alpha}, \bar{\alpha}]), \\ \check{S}_{A^2\alpha} &= \check{S}_{\alpha^3} = 0.\end{aligned}\tag{2.67}$$

$\check{S}_{A^2\alpha}$ vanishes because $X_{01}^{11} = 0$, and \check{S}_{α^3} is zero because $[\alpha, \alpha] = 0$. After α is eliminated using its equation of motion, (2.67) first contributes terms at order $\partial^3 A^3$.

The first line of (2.65) contributes to the cubic piece of the F^2 term. Substituting the explicit values of the Neumann coefficients:

$$\begin{aligned}V_{00}^{11} &= -\log(27/16), & V_{11}^{12} &= 16/27, \\ V_{01}^{12} &= -2\sqrt{2}/3\sqrt{3}, & X_{01}^{12} &= 4/(3\sqrt{3}).\end{aligned}\tag{2.68}$$

we write the lowest-derivative term of (2.65) as

$$S_{A^3}^{[1]} = i\frac{g}{\sqrt{2}} \int d^d x \text{Tr}(\partial_\mu A_\nu [A^\mu, A^\nu]).\tag{2.69}$$

We can now predict the value of the quartic amplitude at zero momentum. From (2.60) and (2.69) we see that the Yang-Mills constant is related to the SFT coupling constant by

$$g_{YM} = \frac{1}{\sqrt{2}}g.\tag{2.70}$$

This is the same relation between the gauge boson and tachyon couplings as the one given in formula (6.5.14) of Polchinski [64]. We expect the nonderivative part of the quartic term in the effective action to add to the quadratic and cubic terms to form

the full Yang-Mills action, so that

$$S_{A_4}^{[0]} = \frac{1}{4} g_{YM}^2 [A_\mu, A_\nu]^2. \quad (2.71)$$

2.4.3 Quartic terms

As we have just seen, to get the full Yang-Mills action the quartic terms in the effective action at $p = 0$ must take the form (2.71). We write the nonderivative part of the SFT quartic effective action as

$$S_{A_4}^{[0]} = g^2 \int dx \left(\gamma_+ \text{Tr}(A_\mu A^\mu)^2 + \frac{1}{4} \gamma_- \text{Tr}[A_\mu, A^\nu]^2 \right). \quad (2.72)$$

We can use the method described in section 2.3.3 to numerically approximate the coefficients γ_+ and γ_- in level truncation. In the limit $L \rightarrow \infty$ we expect that $\gamma_+ \rightarrow 0$ and that $\gamma_- \rightarrow g_{YM}^2/g^2 = 1/2$. As follows from formula (2.56) and the comment below it γ_\pm are given by:

$$\begin{aligned} \gamma_+ &= \frac{1}{2} \mathcal{N}^2 \int_0^\infty e^\tau d\tau \text{Det} \left(\frac{1 - \tilde{X}^2}{(1 - \tilde{V}^2)^{13}} \right) (A_{11}^2 + B_{11}^2 + C_{11}^2), \\ \gamma_- &= \mathcal{N}^2 \int_0^\infty e^\tau d\tau \text{Det} \left(\frac{1 - \tilde{X}^2}{(1 - \tilde{V}^2)^{13}} \right) B_{11}^2. \end{aligned} \quad (2.73)$$

We have calculated these integrals including contributions from the first 100 levels. We have found that as the level L increases the coefficients γ_+ and γ_- indeed converge to their expected values ². The leading term in the deviation decays as $1/L$ as expected. Figure 2-2 shows the graphs of $\gamma_\pm(L)$ vs L . Table 2.1 explicitly lists the results from the first 10 levels. At level 100 we get $\gamma_+ = 0.0037$, $\gamma_- = 0.4992$ which is within 0.5% of the expected values. One can improve precision even more by doing a least-squares fit of $\gamma_\pm(L)$ with an expansion in powers of $1/L$ with indeterminate coefficients. The contributions to γ_\pm from the even and odd level fields are oscillatory. Thus, the fit for only even or only odd levels works much better. The least-squares

²We were recently informed of an analytic proof of this result in SFT, which will appear in [65]

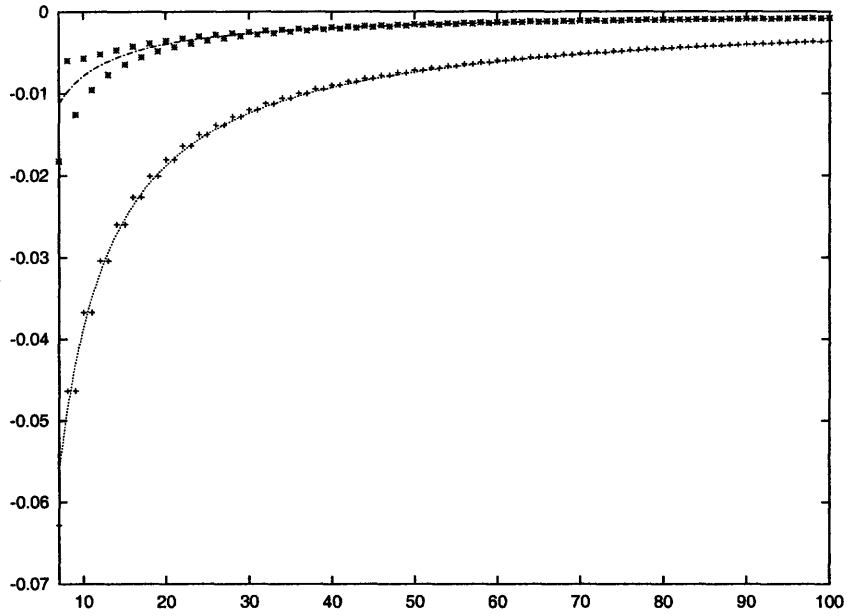


Figure 2-2: Deviation of the coefficients of quartic terms in the effective action from the expected values, as a function of the level of truncation L . The coefficient γ_+ is shown with crosses and $\gamma_- - 1/2$ is shown with stars. The curves given by fitting with a power series in $1/L$ are graphed in both cases.

Level	$\gamma_+(n)$	$\gamma_-(n)$	$\gamma_-(n) - \frac{1}{2}$
0	-0.844	0	-0.500
2	-0.200	0.592	0.092
3	-0.200	0.417	-0.083
4	-0.097	0.504	0.004
5	-0.097	0.468	-0.032
6	-0.063	0.495	-0.005
7	-0.063	0.483	-0.017
8	-0.047	0.494	-0.006
9	-0.047	0.487	-0.013
10	-0.037	0.494	-0.006

Table 2.1: Coefficients of the constant quartic terms in the action for the first 10 levels.

fit for the last 25 even levels gives

$$\begin{aligned}\gamma_+(L) &\approx -5 \cdot 10^{-8} - \frac{0.35807}{L} - \frac{0.0091}{L^2} - \frac{1.6}{L^3} + \frac{15}{L^4} + \dots \\ \gamma_-(L) &\approx \frac{1}{2} - 2 \cdot 10^{-8} - \frac{0.0795838}{L} + \frac{0.1212}{L^2} + \frac{1.02}{L^3} - \frac{1.24}{L^4} + \dots\end{aligned}\quad (2.74)$$

We see that when $L \rightarrow \infty$ the fitted values of γ_{\pm} are in agreement with the Yang-Mills quartic term to 7 digits of precision ³.

The calculations we have described so far provide convincing evidence that the SFT effective action for A_{μ} reproduces the nonabelian Yang-Mills action. This is encouraging in several respects. First, it shows that our method of computing Feynman diagrams in SFT is working well. Second, the agreement with on-shell calculations is another direct confirmation that cubic SFT provides a correct off-shell generalization of bosonic string theory. Third, it encourages us to extend these calculations further to get more information about the full effective action of A_{μ} .

2.5 The abelian Born-Infeld action

In this section we consider the abelian theory, and compute terms in the effective action which go beyond the leading Yang-Mills action computed in the previous section. As discussed in Section 2.3, we expect that the effective vector field theory computed from string field theory should be equivalent under a field redefinition to a theory whose leading terms at each order in A take the Born-Infeld form (2.26). In this section we give evidence that this is indeed the case. In the abelian theory, the terms at order A^3 vanish identically, so the quartic terms are the first ones of interest beyond the quadratic Yang-Mills action. In subsection 2.5.1 we use our results on the general quartic term from 2.3.2 to explicitly compute the terms in the effective action at order $\partial^2 A^4$. We find that these terms are nonvanishing. We find, however,

³Note that in [19], an earlier attempt was made to calculate the coefficients γ_{\pm} from SFT. The results in that paper are incorrect; the error made there was that odd-level fields, which do not contribute in the abelian action due to twist symmetry, were neglected. As these fields do contribute in the nonabelian theory, the result for γ_- obtained in [19] had the wrong numerical value. Our calculation here automatically includes odd-level fields, and reproduces correctly the expected value.

that the gauge invariance of the effective action constrains the terms at this order to live on a one-parameter family of terms related through field redefinitions, and that the terms we find are generated from the Yang-Mills terms \hat{F}^2 with an appropriate field redefinition. We discuss general issues of field redefinition and gauge invariance in subsection 2.5.2; this discussion gives us a framework with which to analyze more complicated terms in the effective action. In subsection 2.5.3 we analyze terms of the form $\partial^4 A^4$, and show that these terms indeed take the form predicted by the Born-Infeld action after the appropriate field redefinition. In subsection 2.5.4 we consider higher-order terms with no derivatives, and give evidence that terms of order $(A \cdot A)^n$ vanish up to $n = 5$ in the string field theory effective action.

2.5.1 Terms of the form $\partial^2 A^4$

In the abelian theory, all terms in the Born-Infeld action have the same number of fields and derivatives. If we assume that the effective action for A_μ calculated in SFT directly matches the Born-Infeld action (plus higher-order derivative corrections) we would expect the $\partial^2 A^4$ terms in the expansion of the effective action to vanish. The most general form of the quartic terms with two derivatives is parameterized as ⁴

$$S_{A^4}^2 = g^2 \int d^{26}x (c_1 A_\mu A^\mu \partial_\sigma A_\nu \partial^\sigma A^\nu + c_2 A_\mu A_\nu \partial_\sigma A^\mu \partial^\sigma A^\nu + c_3 A_\mu A_\nu \partial^\mu A_\sigma \partial^\nu A^\sigma + c_4 A_\mu A_\nu \partial^\nu A_\sigma \partial^\sigma A^\mu + c_5 A_\mu A_\nu A_\sigma \partial^\mu \partial^\nu A^\sigma + c_6 A_\mu A^\mu \partial_\sigma A^\nu \partial_\nu A^\sigma). \quad (2.75)$$

When α is eliminated from the massless effective action \check{S} using the equation of motion, we might then expect that all coefficients c_n in the resulting action (2.75) should vanish. Let us now compute these terms explicitly. From (2.62) and (2.67) we see that the equation of motion for α in the effective theory of the massless fields

⁴Recall that in section 2.3.1 we fixed the integration by parts freedom by integrating by parts all terms with $\partial^2 A_\lambda$ and $\partial \cdot A$. Formula (2.75) gives the most general combination of terms with four A 's and two derivatives that do not have $\partial^2 A_\lambda$ and $\partial \cdot A$.

reads (in the abelian theory)

$$\alpha = \frac{1}{\sqrt{2}} \partial^\mu A_\mu + \mathcal{O}((A, \alpha)^3). \quad (2.76)$$

The coefficients c_1, \dots, c_6 thus get contributions from the two-derivative term of (2.52), the one-derivative term of (2.54) and the zero-derivative term of (2.55). We first consider the contribution from the four-gauge boson amplitude (2.52). All the expressions for these contributions, which we denote $(\delta c_i)_{A^4}$, are of the form

$$(\delta c_i)_{A^4} = \frac{1}{2} \mathcal{N}^2 \int_0^\infty d\tau e^\tau \text{Det} \left(\frac{1 - \tilde{X}^2}{(1 - \tilde{V}^2)^{13}} \right) P_{\partial^2 A^4, i}(A, B, C). \quad (2.77)$$

Here $P_{\partial^2 A^4, i}$ are polynomials in the elements of the matrices A , B and C which were defined in (2.49). It is straightforward to derive expressions for the polynomials $P_{\partial^2 A^4, i}$ from (2.52) and (2.53), so we just give the result here

$$\begin{aligned} P_{\partial^2 A^4, 1} &= -2(A_{11}^2 A_{00} + B_{11}^2 B_{00} + C_{11}^2 C_{00}), \\ P_{\partial^2 A^4, 2} &= -2(A_{11}^2 (B_{00} + C_{00}) + B_{11}^2 (A_{00} + C_{00}) + C_{11}^2 (A_{00} + B_{00})), \\ P_{\partial^2 A^4, 3} &= 2(A_{11}(B_{10}^2 + C_{10}^2) - B_{11}(A_{10}^2 + C_{10}^2) + C_{11}(A_{10}^2 + B_{10}^2)), \\ P_{\partial^2 A^4, 4} &= 4(A_{11}A_{10}(B_{10} + C_{10}) - B_{11}B_{10}(A_{10} + C_{10}) + C_{11}C_{10}(A_{10} + B_{10})), \\ P_{\partial^2 A^4, 5} &= 4(A_{11}B_{10}C_{10} - B_{11}A_{10}C_{10} + C_{11}A_{10}B_{10}), \\ P_{\partial^2 A^4, 6} &= 2(A_{11}A_{10}^2 - B_{11}B_{10}^2 + C_{11}C_{10}^2). \end{aligned} \quad (2.78)$$

The terms in the effective action \check{S} which contain α 's and contribute to $S[A]$ at order $\partial^2 A^4$ can similarly be computed from (2.54) and are given by (2.55)

$$\check{S}_{\alpha A^3}^{[1]} + \check{S}_{\alpha^2 A^2}^{[0]} = g^2 \int d^{26}x (\sigma_1 \alpha A_\mu A_\nu \partial^\mu A^\nu + \sigma_2 \partial^\mu \alpha A_\mu A_\nu A^\nu + \sigma_3 \alpha^2 A_\mu A^\mu) \quad (2.79)$$

where the coefficients σ_i are given by

$$\begin{aligned}
P_{\partial\alpha A^3,1} &= 4Q_{01}^{11}(A_{11}(B_{10} + C_{10}) - B_{11}(A_{10} + C_{10}) + C_{11}(B_{10} + A_{10})), \\
P_{\partial\alpha A^3,2} &= 4Q_{01}^{11}(A_{11}A_{10} - B_{11}B_{10} + C_{11}C_{10}) \\
P_{\alpha^2 A^2} &= 2((Q_{01}^{11})^2 - (Q_{01}^{12})^2)A_{11} - ((Q_{01}^{11})^2 - (Q_{01}^{13})^2)B_{11} + ((Q_{01}^{11})^2 - (Q_{01}^{14})^2)C_{11}.
\end{aligned} \tag{2.80}$$

Computation of the integrals up to level 100 and using a least-squares fit gives us

$$\begin{aligned}
(\delta c_1)_{A^4} &\approx -2.1513026, & (\delta c_4)_{A^4} &\approx 0.9132288, & \sigma_1 &\approx -0.4673613, \\
(\delta c_2)_{A^4} &\approx -4.3026050, & (\delta c_5)_{A^4} &\approx -2.0134501, & \sigma_2 &\approx 0.2171165, \\
(\delta c_3)_{A^4} &\approx -2.0134501, & (\delta c_6)_{A^4} &\approx 1.4633393, & \sigma_3 &\approx 1.6829758.
\end{aligned} \tag{2.81}$$

Elimination of α with (2.76) gives

$$\begin{aligned}
c_1 &\approx -2.1513026, & c_4 &\approx 4.302605, \\
c_2 &\approx -4.302605, & c_5 &\approx 0, \\
c_3 &\approx 0, & c_6 &\approx 2.1513026.
\end{aligned} \tag{2.82}$$

These coefficients are not zero, so that the SFT effective action does not reproduce the abelian Born-Infeld action in a straightforward manner. Thus, we need to consider a field redefinition to put the effective action into the usual Born-Infeld form. To understand how this field redefinition works, it is useful to study the gauge transformation in the effective theory. Without directly computing this gauge transformation, we can write the general form that the transformation must take; the leading terms can be parameterized as

$$\begin{aligned}
\delta A_\mu &= \partial_\mu \lambda + g_{YM}^2 (\varsigma_1 A^2 \partial_\mu \lambda + \varsigma_2 A_\nu \partial_\mu A^\nu \lambda \\
&\quad + \varsigma_3 A_\nu \partial^\nu A_\mu \lambda + \varsigma_4 A_\mu \partial \cdot A \lambda + \varsigma_5 A_\mu A_\nu \partial^\nu \lambda) + \mathcal{O}(\partial^3 A^2 \lambda).
\end{aligned} \tag{2.83}$$

The action (2.75) must be invariant under this gauge transformation. This gauge

invariance imposes a number of *a priori* restrictions on the coefficients c_i, ς_i . When we vary the F^2 term in the effective action (2.60) the nonlinear part of (2.83) generates $\partial^3 A^3 \lambda$ terms. Gauge invariance requires that these terms cancel the terms arising from the linear gauge transformation of the $\partial^2 A^4$ terms in (2.75). This cancellation gives homogeneous linear equations for the parameters c_i and ς_i . The general solution of these equations depends on one free parameter γ :

$$\begin{aligned}
c_1 = -c_6 = -\gamma, & & \varsigma_1 = -\gamma, \\
c_2 = -c_4 = -2\gamma, & & \varsigma_5 = -2\gamma, \\
c_3 = c_5 = 0, & & \varsigma_2 = \varsigma_3 = \varsigma_4 = 0.
\end{aligned} \tag{2.84}$$

The coefficients c_i calculated above satisfy these relations to 7 digits of precision. From the numerical values of the c_i 's, we find

$$\gamma \approx 2.1513026 \pm 0.0000005. \tag{2.85}$$

We have thus found that the $\partial^2 A^4$ terms in the effective vector field action derived from SFT lie on a one-parameter family of possible combinations of terms which have a gauge invariance of the desired form. We can identify the degree of freedom associated with this parameter as arising from the existence of a family of field transformations with nontrivial terms at order A^3

$$\begin{aligned}
\hat{A}_\mu &= A_\mu + g^2 \gamma A^2 A_\mu, \\
\hat{\lambda} &= \lambda.
\end{aligned} \tag{2.86}$$

We can use this field redefinition to relate a field \hat{A} with the standard gauge transformation $\delta \hat{A}_\mu = \partial_\mu \hat{\lambda}$ to a field A transforming under (2.83) with ς_i and γ satisfying

(2.84). Indeed, plugging this change of variables into

$$\begin{aligned}\delta\hat{A}_\mu &= \partial\hat{\lambda}, \\ S_{BI} &= -\frac{1}{4}\int dx\hat{F}^2 + \mathcal{O}(\hat{F}^3).\end{aligned}\tag{2.87}$$

gives (2.83) and (2.75) with c_i, ς_i satisfying (2.84).

We have thus found that nonvanishing $\partial^2 A^4$ terms arise in the vector field effective action derived from SFT, but that these terms can be removed by a field redefinition. We would like to emphasize that the logic of this subsection relies upon using the fact that the effective vector field theory has a gauge invariance. The existence of this invariance constrains the action sufficiently that we can identify a field redefinition that puts the gauge transformation into standard form, without knowing in advance the explicit form of the gauge invariance in the effective theory. Knowing the field redefinition, however, in turn allows us to identify this gauge invariance explicitly. This interplay between field redefinitions and gauge invariance plays a key role in understanding higher-order terms in the effective action, which we explore further in the following subsection.

2.5.2 Gauge invariance and field redefinitions

In this subsection we discuss some aspects of the ideas of gauge invariance and field redefinitions in more detail. In the previous subsection, we determined a piece of the field redefinition relating the vector field A in the effective action derived from string field theory to the gauge field \hat{A} in the Born-Infeld action by using the existence of a gauge invariance in the effective theory. The rationale for the existence of the field transformation from A to \hat{A} can be understood based on the general theorem of the rigidity of the Yang-Mills gauge transformation [54, 55]. This theorem states that any deformation of the Yang-Mills gauge invariance can be mapped to the standard gauge invariance through a field redefinition. At the classical level this field redefinition can

be expressed as

$$\begin{aligned}\hat{A}_\mu &= \hat{A}_\mu(A), \\ \hat{\lambda} &= \hat{\lambda}(A, \lambda).\end{aligned}\tag{2.88}$$

This theorem explains, for example, why noncommutative Yang-Mills theory, which has a complicated gauge invariance involving the noncommutative star product, can be mapped through the Seiberg-Witten map (field redefinition) to a gauge theory written in terms of a gauge field with standard transformation rules [29, 66]. Since in string field theory the parameter α' (which we have set to unity) parameterizes the deformation of the standard gauge transformation of A_μ , the theorem states that some field redefinition exists which takes the effective vector field theory arising from SFT to a theory which can be written in terms of the field strength $\hat{F}_{\mu\nu}$ and covariant derivative \hat{D}_μ of a gauge field \hat{A}_μ with the standard transformation rule⁵.

There are two ways in which we can make use of this theorem. Given the explicit expression for the effective action from SFT, one can assume that such a transformation exists, write the most general covariant action at the order of interest, and find a field redefinition which takes this to the effective action computed in SFT. Applying this approach, for example, to the $\partial^2 A^4$ terms discussed in the previous subsection, we would start with the covariant action \hat{F}^2 , multiplied by an unknown overall coefficient ζ , write the field redefinition (2.86) in terms of the unknown γ , plug in the field redefinition, and match with the effective action (2.75), which would allow us to fix γ and $\zeta = -1/4$.

A more direct approach can be used when we have an explicit expression for the gauge invariance of the effective theory. In this case we can simply try to construct a field redefinition which relates this invariance to the usual Yang-Mills gauge invariance. When finding the field redefinition relating the deformed and undeformed theories, however, a further subtlety arises, which was previously encountered in related situations [31, 32]. Namely, there exists for any theory a class of trivial gauge

⁵In odd dimensions there would also be a possibility of Chern-Simons terms

invariances. Consider a theory with fields ϕ_i and action $S(\phi_i)$. This theory has trivial gauge transformations of the form

$$\delta\phi_i = \mu_{ij} \frac{\delta S}{\delta\phi_j} \quad (2.89)$$

where $\mu_{ij} = -\mu_{ji}$. Indeed, the variation of the action under this transformation is $\delta S = \mu_{ij} \frac{\delta S}{\delta\phi_i} \frac{\delta S}{\delta\phi_j} = 0$. These transformations are called trivial because they do not correspond to a constraint in the Hamiltonian picture. The conserved charges associated with trivial transformations are identically zero. In comparing the gauge invariance of the effective action $S[A]$ to that of the Born-Infeld action, we need to keep in mind the possibility that the gauge invariances are not necessarily simply related by a field redefinition, but that the invariance of the effective theory may include additional terms of the form (2.89). In considering this possibility, we can make use of a theorem (theorem 3.1 of [67]), which states that under suitable regularity assumptions on the functions $\frac{\delta S}{\delta\phi_i}$ any gauge transformation that vanishes on shell can be written in the form (2.89). Thus, when identifying the field redefinition transforming the effective vector field A to the gauge field \hat{A} , we allow for the possible addition of trivial terms.

The benefit of the first method described above for determining the field redefinition is that we do not need to know the explicit form of the gauge transformation. Once the field redefinition is known we can find the gauge transformation law in the effective theory of A_μ up to trivial terms by plugging the field redefinition into the standard gauge transformation law of \hat{A}_μ . In the explicit example of $\partial^2 A^4$ terms considered in the previous subsection we determined that the gauge transformation of the vector field A_μ is given by

$$\delta A_\mu = \partial_\mu \lambda - g_{\text{YM}}^2 \gamma (A^2 \partial_\mu \lambda - 2A_\mu A_\nu \partial^\nu \lambda) \quad (2.90)$$

plus possible trivial terms which we did not consider. We have found the numerical value of γ in (2.85). If we had been able to directly compute this gauge transformation law, finding the field redefinition (2.86) would have been trivial. Unfortunately, as

we shall see in a moment, the procedure for computing the higher-order terms in the gauge invariance of the effective theory is complicated to implement, which makes the second method less practical in general for determining the field redefinition. We can, however, at least compute the terms in the gauge invariance which are of order $A\lambda$ directly from the definition (2.5). Thus, for these terms the second method just outlined for computing the field redefinition can be used. We use this method in section 2.6.1 to compute the field redefinition including terms at order ∂A^2 and $\partial^2 A$ in the nonabelian theory.

Let us note that the field redefinition that makes the gauge transformation standard is not unique. There is a class of field redefinitions that preserves the gauge structure and mass-shell condition

$$\begin{aligned}\hat{A}'_\mu &= \hat{A}_\mu + T_\mu(\hat{F}) + \hat{D}_\mu \xi(\hat{A}), \\ \hat{\lambda}' &= \hat{\lambda} + \delta_{\hat{\lambda}} \xi(\hat{A}_\mu).\end{aligned}\tag{2.91}$$

In this field redefinition $T_\mu(\hat{F})$ depends on \hat{A}_μ only through the covariant field strength and its covariant derivatives. The term ξ is a trivial (pure gauge) field redefinition, which is essentially a gauge transformation with parameter $\xi(A)$. The resulting ambiguity in the effective Lagrangian has a field theory interpretation based on the equivalence theorem [68]. According to this theorem, different Lagrangians give the same S-matrix elements if they are related by a change of variables in which both fields have the same gauge variation and satisfy the same mass-shell condition.

Let us now describe briefly how the different forms of gauge invariance arise in the world-sheet and string field theory approaches to computing the vector field action. We primarily carry out this discussion in the context of the abelian theory, although similar arguments can be made in the nonabelian case. In a world-sheet sigma model calculation one introduces the boundary interaction term

$$\oint_{\gamma} A_\mu \frac{dX^\mu}{d\tau} d\tau.\tag{2.92}$$

This term is explicitly invariant under

$$A_\mu \rightarrow A_\mu + \partial_\mu \lambda. \quad (2.93)$$

Provided that one can find a systematic method of calculation that respects this gauge invariance, the resulting effective action will possess this gauge invariance as well. This is the reason calculations such as those in [16, 27] give an effective action with the usual gauge invariance.

In the cubic SFT calculation, on the other hand, the gauge invariance is much more complicated. The original theory has an infinite number of gauge invariances, given by (2.5). We have fixed all but one of these gauge symmetries; the remaining symmetry comes from a gauge transformation that may change the field α , but which keeps all other auxiliary fields at zero. A direct construction of this gauge transformation in the effective theory of A_μ is rather complicated, but can be described perturbatively in three steps:

1. Make an SFT gauge transformation (in the full theory with an infinite number of fields) with the parameter

$$|\Lambda'\rangle = \frac{i}{\sqrt{2}} \lambda(x) b_{-1} |0\rangle. \quad (2.94)$$

This gauge transformation transforms α and A_μ as

$$\begin{aligned} \delta A_\mu &= \partial_\mu \lambda + i g_{YM}(\dots), \\ \delta \alpha &= \sqrt{2} \partial^2 \lambda + i g_{YM}(\dots), \end{aligned} \quad (2.95)$$

and transforms all fields in the theory in a computable fashion.

2. The gauge transformation $|\Lambda'\rangle$ takes us away from the gauge slice we have fixed by generating fields associated with states containing c_0 at all higher levels. We now have to make a second gauge transformation with a parameter $|\Lambda''(\lambda)\rangle$ that will restore our gauge of choice. The order of magnitude of the auxiliary

fields we have generated at higher levels is $\mathcal{O}(g\lambda\Phi)$. Therefore $|\Lambda''(\lambda)\rangle$ is of order $g\lambda\Phi$. Since we already used the gauge parameter at level zero, we will choose $|\Lambda''\rangle$ to have nonvanishing components only for massive modes. Then this gauge transformation does not change the massless fields linearly, so the contribution to the gauge transformation at the massless level will be of order $\mathcal{O}(g^2\lambda\Phi^2)$. The gauge transformation generated by $|\Lambda''(\lambda)\rangle$ can be computed as a perturbative expansion in g . Combining this with our original gauge transformation generated by $|\Lambda'\rangle$ gives us a new gauge transformation which transforms the massless fields linearly according to (2.95), but which also keeps us in our chosen gauge slice.

3. In the third step we eliminate all the fields besides A_μ using the classical equations of motion. The SFT equations of motion are

$$Q_B|\Phi\rangle = -g\langle\Phi, \Phi|V_3\rangle. \quad (2.96)$$

The BRST operator preserves the level of fields; therefore, the solutions for massive fields and α in terms of A_μ will be of the form

$$\psi_{\mu_1, \dots, \mu_n} = \mathcal{O}(gA^2), \quad (2.97)$$

$$\alpha = \frac{1}{\sqrt{2}}\partial \cdot A + \mathcal{O}(gA^2) \quad (2.98)$$

where $\psi_{\mu_1, \dots, \mu_n}$ is a generic massive field. Using these EOM to eliminate the massive fields and α in the gauge transformation of A_μ will give terms of order $\mathcal{O}(g^2A^2)$.

To summarize, the gauge transformation in the effective theory for A_μ is of the form

$$\delta A_\mu = \partial_\mu \lambda + R_\mu(A, \lambda), \quad (2.99)$$

where R_μ is a specific function of A and λ at order $g^2A^2\lambda$, which can in principle be

computed using the method just described. In the nonabelian theory, there will also be terms at order $gA\lambda$ arising directly from the original gauge transformation $|\Lambda\rangle$; these terms are less complicated and can be computed directly from the cubic string field vertex.

In this subsection, we have discussed two approaches to computing the field redefinition which takes us from the effective action $S[A]$ to a covariant action written in terms of the gauge field \hat{A} , which should have the form of the Born-Infeld action plus derivative corrections. In the following sections we use these two approaches to check that various higher-order terms in the SFT effective action indeed agree with known terms in the Born-Infeld action, in both the abelian and nonabelian theories.

2.5.3 Terms of the form $\partial^4 A^4$

The goal of this subsection is to verify that after an appropriate field redefinition the $\partial^4 A^4$ terms in the abelian effective action derived from SFT transform into the $\hat{F}^4 - \frac{1}{4}(\hat{F}^2)^2$ terms of the Born-Infeld action (including the correct constant factor of $(2\pi g_{YM})^2/8$). To demonstrate this, we use the first method discussed in the previous subsection. Since the total number of $\partial^4 A^4$ terms is large we restrict attention to a subset of terms: namely those terms where indices on derivatives are all contracted together. These terms are independent from other terms at the same order in the effective action. By virtue of the equations of motion (2.76) the diagrams with α do not contribute to these terms. This significant simplification is the reason why we choose to concentrate on these terms. Although we only compute a subset of the possible terms in the effective action, however, we find that these terms are sufficient to fix both coefficients in the Born-Infeld action at order F^4 .

The terms we are interested in have the general form

$$\begin{aligned}
S_{(\partial,\partial)A^4}^4 = g^2 \int d^{26}x & (d_1(\partial_\mu A_\lambda \partial^\mu A^\lambda)^2 + d_2 \partial_\mu A_\lambda \partial_\nu A^\lambda \partial^\mu A_\sigma \partial^\nu A^\sigma \\
& + d_3 A_\lambda \partial_\nu A^\lambda \partial_\mu A_\sigma \partial^\mu \partial^\nu A^\sigma + d_4 \partial_\mu A_\lambda \partial_\nu A^\lambda A_\sigma \partial^\mu \partial^\nu A^\sigma \\
& + d_5 A_\lambda A^\lambda \partial_\mu \partial_\nu A_\sigma \partial^\mu \partial^\nu A^\sigma + d_6 A_\lambda \partial_\mu \partial_\nu A^\lambda A_\sigma \partial^\mu \partial^\nu A^\sigma). \quad (2.100)
\end{aligned}$$

The coefficients for these terms in the effective action are given by

$$d_i = \frac{1}{2} \mathcal{N}^2 \int_0^\infty d\tau e^\tau \text{Det} \left(\frac{1 - \tilde{X}^2}{(1 - \tilde{V}^2)^{13}} \right) P_i^{(4)}(A, B, C) \quad (2.101)$$

with

$$\begin{aligned} P_1^{(4)} &= P_5^{(4)} = A_{11}^2 A_{00}^2 + B_{11}^2 B_{00}^2 + C_{11}^2 C_{00}^2, \\ P_2^{(4)} &= P_6^{(4)} = A_{11}^2 (B_{00}^2 + C_{00}^2) + B_{11}^2 (A_{00}^2 + C_{00}^2) + C_{11}^2 (A_{00}^2 + B_{00}^2), \\ P_3^{(4)} &= 4A_{11}^2 A_{00} (B_{00} + C_{00}) + 4B_{11}^2 B_{00} (A_{00} + C_{00}) + 4C_{11}^2 C_{00} (A_{00} + B_{00}), \\ P_4^{(4)} &= 4A_{11}^2 B_{00} C_{00} + 4B_{11}^2 A_{00} C_{00} + 4C_{11}^2 A_{00} B_{00}. \end{aligned} \quad (2.102)$$

Computation of the integrals gives us

$$\begin{aligned} d_1 &= d_5 \approx 3.14707539, & d_3 &\approx 18.51562023, \\ d_2 &= d_6 \approx 2.96365920, & d_4 &\approx 0.99251621. \end{aligned} \quad (2.103)$$

To match these coefficients with the BI action we need to write the general field redefinition to order $\partial^2 A^3$ (again, keeping only terms with all derivatives contracted)

$$\begin{aligned} \hat{A}_\mu &= A_\mu + g^2 (\gamma A^2 A_\mu + \alpha_1 A_\mu A_\sigma \partial^2 A^\sigma + \alpha_2 A^2 \partial^2 A_\mu \\ &\quad + \alpha_3 A_\mu \partial_\lambda A_\sigma \partial^\lambda A^\sigma + \alpha_4 A_\sigma \partial_\lambda A_\mu \partial^\lambda A^\sigma). \end{aligned} \quad (2.104)$$

Using the general theorem quoted in the previous subsection, we know that there is a field redefinition relating the action containing the terms (2.100) to a covariant action written in terms of a conventional field strength \hat{F} . The coefficients of \hat{F}^2 and \hat{F}^3 are already fixed, so the most generic action up to \hat{F}^4 is

$$\text{Tr} \int dx \left(-\frac{1}{4} \hat{F}^2 + g^2 \left(a \hat{F}^4 + b (\hat{F}^2)^2 \right) + O(\hat{F}^6) \right). \quad (2.105)$$

We plug the change of variables (2.104) into this equation and collect $\partial^4 A^4$ terms

with derivatives contracted together:

$$\begin{aligned}
g^2 \int d^{26}x & \left((\alpha_1 - \alpha_3 + 4b)(\partial_\mu A_\lambda \partial^\mu A^\lambda)^2 \right. \\
& + (\alpha_1 + 2\alpha_2 - \alpha_4 + 2a)\partial_\mu A_\lambda \partial_\nu A^\lambda \partial^\mu A_\sigma \partial^\nu A^\sigma \\
& + (4\alpha_1 + 4\alpha_2 - 2\alpha_3 - \alpha_4)A_\lambda \partial_\nu A^\lambda \partial_\mu A_\sigma \partial^\mu \partial^\nu A^\sigma \\
& + (2\alpha_1 + 2\alpha_2 - \alpha_4)\partial_\mu A_\lambda \partial_\nu A^\lambda A_\sigma \partial^\mu \partial^\nu A^\sigma \\
& \left. + \alpha_2 A_\lambda A^\lambda \partial_\mu \partial_\nu A_\sigma \partial^\mu \partial^\nu A^\sigma + \alpha_1 A_\lambda \partial_\mu \partial_\nu A^\lambda A_\sigma \partial^\mu \partial^\nu A^\sigma \right).
\end{aligned} \tag{2.106}$$

The assumption that (2.100) can be written as (2.106) translates into a system of linear equations for a , b and $\alpha_1, \dots, \alpha_4$ with the right hand side given by d_1, \dots, d_6 .

This system is non-degenerate and has a unique solution

$$\begin{aligned}
\alpha_1 &= d_6 \approx 2.9636592, \\
\alpha_2 &= d_5 \approx 3.1470754, \\
\alpha_3 &= \frac{1}{2}(-d_3 + d_4 + 2d_5 + 2d_6) \approx -2.6508174, \\
\alpha_4 &= -d_4 + 2d_5 + 2d_6 \approx 11.2289530, \\
a &= \frac{1}{2}(d_2 - d_4 + d_6) \approx 2.4674011, \\
b &= \frac{1}{8}(2d_1 - d_3 + d_4 + 2d_5) \approx -0.6168503.
\end{aligned} \tag{2.107}$$

This determines the coefficients a and b in the effective action (2.105) to 8 digits of precision. These values agree precisely with those that we expect from the Born-Infeld action, which are given by

$$\begin{aligned}
a &= \frac{\pi^2}{4} \approx 2.4674011, \\
b &= -\frac{\pi^2}{16} \approx -0.6168502.
\end{aligned} \tag{2.108}$$

Thus, we see that after a field redefinition, the effective vector theory derived from string field theory agrees with Born-Infeld to order F^4 , and correctly fixes the coeffi-

icients of both terms at that order. This calculation could in principle be continued to compute higher-derivative corrections to the Born-Infeld action of the form $\partial^6 A^4$ and higher, but we do not pursue such calculations further here.

Note that, assuming we know that the Born-Infeld action takes the form

$$S_{BI} = -T \int dx \sqrt{-\det \left(\eta^{\mu\nu} + T^{-\frac{1}{2}} F^{\mu\nu} \right)}. \quad (2.109)$$

with undetermined D-brane tension, we can fix $T = 1/(2\pi\alpha'g_{YM})^2$ from the coefficients at F^2 and F^4 . We may thus think of the calculations done so far as providing another way to determine the D-brane tension from SFT.

2.5.4 Terms of the form A^{2n}

In the preceding discussion we have focused on terms in the effective action which are at most quartic in the vector field A_μ . It is clearly of interest to extend this discussion to terms of higher order in A . A complete analysis of higher-order terms, including all momentum dependence, involves considerable additional computation. We have initiated analysis of higher-order terms by considering the simplest class of such terms: those with no momentum dependence. As for the quartic terms of the form $(A^\mu A_\mu)^2$ discussed in Section 4.2, we expect that all terms in the effective action of the form

$$(A^\mu A_\mu)^n \quad (2.110)$$

should vanish identically when all diagrams are considered. In this subsection we consider terms of the form (2.110). We find good numerical evidence that these terms indeed vanish, up to terms of the form A^{10} .

In Section 4.2 we found strong numerical evidence that the term (2.110) vanishes for $n = 2$ by showing that the coefficient γ_+ in (2.72) approaches 0 in the level-truncation approximation. This A^4 term involves only one possible diagram. As n increases, the number of diagrams involved in computing A^{2n} increases exponentially, and the complexity of each diagram also increases, so that the primary method used

in this chapter becomes difficult to implement. To study the terms (2.110) we have used a somewhat different method, in which we directly truncate the theory by only including fields up to a fixed total oscillator level, and then computing the cubic terms for each of the fields below the desired level. This was the original method of level truncation used in [3] to compute the tachyon 4-point function, and in later work [4, 5] on level truncation on the problem of tachyon condensation. As discussed in Section 3.3, the method we are using for explicitly calculating the quartic terms in the action involves truncating on the level of the intermediate state in the 4-point diagram, so that the two methods give the same answers. While level truncation on oscillators is very efficient for computing low-order diagrams at high level, however, level truncation on fields is more efficient for computing high-order diagrams at low level.

In [5], a recursive approach was used to calculate coefficients of ϕ^n in the effective tachyon potential from string field theory using level truncation on fields. Given a cubic potential

$$V = \sum_{i,j} d_{ij} \psi_i \psi_j + \sum_{i,j,k} g t_{ijk} \psi_i \psi_j \psi_k \quad (2.111)$$

for a finite number of fields $\psi_i, i = 1, \dots, N$ at $p = 0$, the effective action for $a = \psi_1$ when all other fields are integrated out is given by

$$V_{\text{eff}}(a) = \sum_{n=2}^{\infty} \frac{1}{n} v_{n-1}^1 a^n g^n \quad (2.112)$$

where v_n^i represents the summation over all graphs with n external a edges and a single external ψ^i , with no internal a 's. The v 's satisfy the recursion relations

$$\begin{aligned} v_1^i &= \delta_1^i \\ v_n^i &= \frac{3}{2} \sum_{m=1}^{n-1} d^{ij} t_{jkl} \hat{v}_m^k \hat{v}_{n-m}^l \end{aligned} \quad (2.113)$$

where d^{ij} is the inverse matrix to d_{ij} and

$$\hat{v}_n^i = \begin{cases} 0, & i = 1 \text{ and } n > 1 \\ v_n^i, & \text{otherwise} \end{cases} \quad (2.114)$$

has been defined to project out internal a edges.

We have used these relations to compute the effective action for A_μ at $p = 0$. We computed all quadratic and cubic interactions between fields up to level 8 associated with states which are scalars in 25 of the space-time dimensions and which include an arbitrary number of matter oscillators a_{-n}^{25} . Plugging the resulting quadratic and cubic coefficients into the recursion relations (2.113) allows us to compute the coefficients $c_{2n} = v_{2n-1}^1/2n$ in the effective action for the gauge field A_μ

$$\sum_{n=1}^{\infty} -c_{2n} g^n (A^\mu A_\mu)^n \quad (2.115)$$

for small values of n . We have computed these coefficients up to $n = 7$ at different levels of field truncation up to $L = 8$. The results of this computation are given in Table 2.2 up to $n = 5$, including the predicted value at $L = \infty$ from a $1/L$ fit to the data at levels 2, 4, 6 and 8. The results in Table 2.2 indicate that, as expected, all

Level	c_4	c_6	c_8	c_{10}
2	0.200	1.883	6.954	28.65
4	0.097	1.029	6.542	37.49
6	0.063	0.689	5.287	37.62
8	0.046	0.517	4.325	34.18
∞	0.001	0.014	-0.229	1.959

Table 2.2: Coefficients of A^{2n} at various levels of truncation

coefficients c_{2n} will vanish when the level is taken to infinity. The initial contribution at level 2 is canceled to within 0.6% for terms A^4 , within 0.8% for terms A^6 , within 4% for terms A^8 , and within 7% for terms A^{10} . It is an impressive success of the level-truncation method that for c_{10} , the cancellation predicted by the $1/L$ expansion is so good, given that the coefficients computed in level truncation *increase* until level

$L = 8$. We have also computed the coefficients for larger values of n , but for $n > 5$ the numerics are less compelling. Indeed, the approximations to the coefficients c_{12} and beyond continue to grow up to level 8. We expect that a good prediction of the cancellation of these higher-order terms would require going to higher level.

The results found here indicate that the method of level truncation in string field theory seems robust enough to correctly compute higher-order terms in the vector field effective action. Computing terms with derivatives at order A^6 and beyond would require some additional work, but it seems that a reasonably efficient computer program should be able to do quite well at computing these terms, even to fairly high powers of A .

2.6 The nonabelian Born-Infeld action

We now consider the theory with a nonabelian gauge group. As we discussed in section 2.2.3, the first term beyond the Yang-Mills action in the nonabelian analogue of the Born-Infeld action has the form $\text{Tr } \hat{F}^3$. As in the previous section, we expect that a field redefinition is necessary to get this term from the effective nonabelian vector field theory derived from SFT. In this section we compute the terms in the effective vector field theory to orders $\partial^3 A^3$ and $\partial^2 A^4$, and we verify that after a field redefinition these terms reproduce the corresponding pieces of the \hat{F}^3 term, with the correct coefficients. In section 2.6.1 we analyze $\partial^3 A^3$ terms, and in subsection 2.6.2 we consider the $\partial^2 A^4$ terms.

2.6.1 $\partial^3 A^3$ terms

In section 2.4.2 we showed that the terms of the form ∂A^3 in the nonabelian SFT effective action for A contribute to the \hat{F}^2 term after a field redefinition. We now consider terms at order $\partial^3 A^3$. Recall from (2.65) and (2.67) that the full effective

action for α and A at this order is given by

$$\begin{aligned} \check{S}_{A^3}[A, \alpha] = ig_{YM} \int dx \operatorname{Tr} & \left(\frac{1}{6} (\partial_\lambda \tilde{A}^\mu \partial_\mu \tilde{A}^\nu \partial_\nu \tilde{A}^\lambda - \partial_\nu \tilde{A}^\mu \partial_\lambda \tilde{A}^\nu \partial_\mu \tilde{A}^\lambda) \right. \\ & \left. - \partial_\mu \tilde{A}_\nu [\tilde{A}^\mu, \tilde{A}^\nu] + \frac{1}{2} [\tilde{A}_\nu, \partial^\lambda \tilde{A}_\mu] \partial^\mu \partial^\nu \tilde{A}_\lambda + \tilde{A}^\mu [\partial_\mu \tilde{\alpha}; \tilde{\alpha}] \right) \end{aligned} \quad (2.116)$$

where $\tilde{A}_\mu = \exp(-\frac{1}{2}V_{00}^{11}\partial^2)A_\mu$, and similarly for $\tilde{\alpha}$. After eliminating α using (2.61) and (2.116) and integrating by parts to remove terms containing ∂A , we find that the complete set of terms at order $\partial^3 A^3$ is given by

$$\begin{aligned} S^{[3]}[A]_{A^3} = ig_{YM} \int dx \operatorname{Tr} & \left(\frac{2}{3} (\partial_\lambda A^\mu \partial_\mu A^\nu \partial_\nu A^\lambda - \partial_\nu A^\mu \partial_\lambda A^\nu \partial_\mu A^\lambda) \right. \\ & \left. + \frac{1}{2} V_{00}^{11} (\partial_\mu \partial^2 A_\nu [A^\mu, A^\nu] + \partial_\mu A_\nu [\partial^2 A^\mu, A^\nu] + \partial_\mu A_\nu [A^\mu, \partial^2 A^\nu]) \right). \end{aligned} \quad (2.117)$$

Note that unlike the quartic terms in A , our expressions for these terms are exact.

Let us now consider the possible terms that we can get after the field redefinition to the field \hat{A} with standard gauge transformation rules. Following the analysis of [63], we write the most general covariant action to order \hat{F}^3 (keeping D at order $F^{1/2}$ as discussed above)

$$-\frac{1}{4}\hat{F}^2 + ig_{YM}a\hat{F}^3 + \chi\hat{D}_\sigma\hat{F}^{\sigma\mu}\hat{D}^\nu\hat{F}_{\nu\mu} + \mathcal{O}(\hat{F}^4), \quad (2.118)$$

where

$$\hat{D}_\mu = \partial_\mu - ig_{YM}[\hat{A}_\mu, \cdot]. \quad (2.119)$$

The action (2.118) is not invariant under field redefinitions which keep the gauge invariance unchanged. Under the field redefinition

$$\hat{A}'_\mu = \hat{A}_\mu + v\hat{D}_\sigma\hat{F}_\mu^\sigma. \quad (2.120)$$

we have

$$\begin{aligned} a' &= a, \\ \chi' &= \chi - v. \end{aligned} \tag{2.121}$$

Thus, the coefficient a is defined unambiguously, while χ can be set to any chosen value by a field redefinition.

Just as we have an exact formula for the cubic terms in the SFT action, we can also compute the gauge transformation rule exactly to quadratic order in A using (2.5). After some calculation, we find that the gauge variation for A_μ to order $A^2\lambda$ is given by (before integrating out α)

$$\begin{aligned} \delta A_\mu &= \partial_\mu \lambda - ig_{YM} \left([A_\mu, \lambda]_\star - [\partial_\mu A_\nu, \partial^\nu \lambda]_\star + [A^\nu, \partial_\mu \partial_\nu \lambda]_\star + \right. \\ &\quad \left. \frac{1}{\sqrt{2}} [\partial_\mu B, \lambda]_\star - \frac{1}{\sqrt{2}} [B, \partial_\mu \lambda]_\star \right). \end{aligned} \tag{2.122}$$

where $B = \alpha - \frac{1}{\sqrt{2}} \partial_\mu A^\mu$ as in section (2.4.1). The commutators are taken with respect to the product

$$f(x) \star g(x) = f(x) e^{-V_{00}^{11} (\vec{\partial}^2 + \vec{\partial} \cdot \vec{\partial} + \vec{\partial}^2)} g(x). \tag{2.123}$$

The equation of motion for α at leading order is simply $B = 0$. Eliminating α we therefore have

$$\delta A_\mu = \partial_\mu \lambda - ig_{YM} \left([A_\mu, \lambda]_\star + [\partial^\nu \lambda, \partial_\mu A_\nu]_\star + [A^\nu, \partial_\mu \partial_\nu \lambda]_\star \right). \tag{2.124}$$

We are interested in considering the terms at order $\partial^2 A \lambda$ in this gauge variation. Recall that in section 2.5.2 we observed that the gauge transformation may include extra trivial terms which vanish on shell. Since the leading term in the equation of motion for A arises at order $\partial^2 A$, it is possible that (2.124) may contain a term of the form

$$\delta A_\mu = \rho [\lambda, \partial^2 A_\mu - \partial_\mu \partial \cdot A] + \mathcal{O}(\lambda A^2) \tag{2.125}$$

in addition to a part which can be transformed into the standard nonabelian gauge variation through a field redefinition. Thus, we wish to consider the one-parameter family of gauge transformations

$$\begin{aligned} \delta A &= \partial_\mu \lambda - ig_{YM}([A_\mu, \lambda] - V_{00}^{11}[\partial^2 A_\mu, \lambda] \\ &- V_{00}^{11}[\partial_\nu A_\mu, \partial^\nu \lambda] - V_{00}^{11}[A_\mu, \partial^2 \lambda] + \rho[\lambda, \partial^2 A_\mu - \partial_\mu \partial \cdot A] + \mathcal{O}(\lambda A^2, \lambda \partial^4 A)), \end{aligned} \quad (2.126)$$

where ρ is an as-yet undetermined constant. We now need to show, following the second method discussed in subsection 2.5.2, that there exists a field redefinition which takes a field A with action (2.117) and a gauge transformation of the form (2.126) to a gauge field \hat{A} with an action of the form (2.118) and the standard nonabelian gauge transformation rule.

The leading terms in the field redefinition can be parameterized as

$$\begin{aligned} \hat{A}_\mu &= A_\mu + v_1 \partial_\mu \partial \cdot A + v_2 \partial^2 A_\mu + ig_{YM}(v_3[A_\sigma, \partial_\mu A^\sigma] + v_4[A_\mu, \partial \cdot A] + v_5[\partial_\sigma A_\mu, A^\sigma]), \\ \hat{\lambda} &= \lambda + v_6 \partial^2 \lambda + ig_{YM}(v_7[\partial \cdot A, \lambda] + v_8[A_\sigma, \partial^\sigma \lambda]). \end{aligned} \quad (2.127)$$

The coefficient v_1 can be chosen arbitrarily through a gauge transformation, so we simply choose $v_1 = -v_2$. The requirement that the RHS of (2.127) varied with (2.126) and rewritten in terms of \hat{A} , $\hat{\lambda}$ gives the standard transformation law for \hat{A} , $\hat{\lambda}$ up to terms of order $\mathcal{O}(\hat{\lambda}\hat{A}^2)$ gives a system of linear equations with solutions depending on one free parameter v .

$$\begin{aligned} v_2 &= -v_1 = v, & \rho &= V_{00}^{11}, \\ v_3 &= 1 - \frac{1}{2}V_{00}^{11} + v, & v_6 &= 0, \\ v_4 &= -V_{00}^{11} + v, & v_7 &= V_{00}^{11}, \\ v_5 &= -V_{00}^{11} + 2v, & v_8 &= \frac{1}{2}V_{00}^{11}. \end{aligned} \quad (2.128)$$

It is easy to see that the parameter v generates the field redefinition (2.120). For

simplicity, we set $v = 0$. The field redefinition is then given by

$$\hat{A}_\mu = A_\mu - ig_{YM} \left(\left(\frac{1}{2} V_{00}^{11} - 1 \right) [A_\sigma, \partial_\mu A^\sigma] + V_{00}^{11} [A_\mu, \partial \cdot A] + V_{00}^{11} [\partial_\sigma A_\mu, A^\sigma] \right). \quad (2.129)$$

We can now plug in the field redefinition (2.129) into the action (2.118) and compare with the $\partial^3 A^3$ term in the SFT effective action (2.117). We find agreement when the coefficients in (2.118) are given by

$$a = \frac{2}{3}, \quad \chi = 0. \quad (2.130)$$

Thus, we have shown that the terms of order $\partial^3 A^3$ in the effective nonabelian vector field action derived from SFT are in complete agreement with the first nontrivial term in the nonabelian analogue of the Born-Infeld theory, including the overall constant. Note that while the coefficient of a agrees with that in (2.28), the condition $\chi = 0$ followed directly from our choice $v = 0$; other choices of v would lead to other values of χ , which would be equivalent under the field redefinition (2.120).

2.6.2 $\partial^2 A^4$ terms

In the abelian theory, the $\partial^2 A^4$ terms disappear after the field redefinition. In the nonabelian case, however, the term proportional to \hat{F}^3 contains terms of the form $\partial^2 \hat{A}^4$. In this subsection, we show that these terms are correctly reproduced by string field theory after the appropriate field redefinition. Just as in section 2.5.3, for simplicity we shall concentrate on the $\partial^2 A^4$ terms where the Lorentz indices on derivatives are contracted together.

The terms of interest in the effective nonabelian vector field action can be written in the form

$$S_{A^4}^{[2]} = g_{YM}^2 \int d^{26}x \left(f_1 \partial_\sigma A_\mu A^\mu \partial^\sigma A_\nu A^\nu + f_2 \partial_\sigma A_\mu A^\mu A^\nu \partial^\sigma A_\nu + f_3 A^\mu \partial_\sigma A_\mu A_\nu \partial^\sigma A^\nu \right. \\ \left. + f_4 \partial_\sigma A_\mu \partial^\sigma A^\mu A_\nu A^\nu + f_5 \partial_\sigma A_\mu \partial^\sigma A_\nu A^\mu A^\nu + f_6 \partial_\sigma A_\mu A^\nu \partial^\sigma A_\mu A^\nu \right) \quad (2.131)$$

where the coefficients f_i will be determined below. The coefficients of the terms in the field redefinition which are linear and quadratic in A were fixed in the previous subsection. The relevant terms in the field redefinition for computing the terms we are interested in here are generic terms of order A^3 with no derivatives, as well as those from (2.129) that do not have ∂_μ 's contracted with A_μ 's. Keeping only these terms we can parametrize the field redefinition as

$$\hat{A}_\mu = A_\mu + ig_{YM} \left(1 - \frac{V_{00}^{11}}{2}\right) [A_\sigma, \partial_\mu A^\sigma] + g_{YM}^2 (\rho_1 A_\sigma A_\mu A^\sigma + \rho_2 A^2 A_\mu + \rho_3 A_\mu A^2). \quad (2.132)$$

In the abelian case this formula reduces to (2.86) with $\rho_1 + \rho_2 + \rho_3 = 2\gamma$. Plugging this field redefinition into the action

$$\hat{S}[\hat{A}_\mu] = \int \text{Tr} \left(-\frac{1}{4} \hat{F}^2 + \frac{2i}{3} g_{YM} \hat{F}^3 + \mathcal{O}(\hat{F}^4) \right). \quad (2.133)$$

and collecting $\partial^2 A^4$ terms with indices on derivatives contracted together we get

$$\begin{aligned} g_{YM}^2 \int dx & \left[\left(\frac{1}{2} V_{00}^{11} - 1 - \rho_3 \right) \partial_\sigma A_\mu A^\mu \partial^\sigma A_\nu A^\nu - (\rho_2 + \rho_3 + V_{00}^{11}) \partial_\sigma A_\mu A^\mu A_\nu \partial^\sigma A^\nu \right. \\ & + \left(\frac{1}{2} V_{00}^{11} - 1 - \rho_2 \right) A_\mu \partial^\sigma A^\mu A_\nu \partial^\sigma A^\nu - (\rho_2 + \rho_3) \partial_\sigma A_\mu \partial^\sigma A^\mu A_\nu A^\nu \\ & \left. + (2 - 2\rho_1) \partial_\sigma A_\mu \partial^\sigma A_\nu A^\mu A^\nu - \rho_1 \partial_\sigma A_\mu A_\nu \partial^\sigma A^\mu A^\nu \right]. \quad (2.134) \end{aligned}$$

Comparing (2.134) and (2.131) we can write the unknown coefficients in the field redefinition in terms of the f_i 's through

$$\rho_1 = -f_6, \quad \rho_2 = \rho_3 = -\frac{1}{2} f_4. \quad (2.135)$$

We also find a set of constraints on the f_i 's which we expect the values computed from the SFT calculation to satisfy, namely

$$f_1 - \frac{1}{2} f_4 = -1 + \frac{1}{2} V_{00}^{11}, \quad f_2 - f_4 = -V_{00}^{11}, \quad f_5 - 2f_6 = 2. \quad (2.136)$$

On the string field theory side the coefficients f_i are given by

$$f_i = \frac{1}{2} \mathcal{N}^2 \int_0^\infty d\tau e^\tau \text{Det} \left(\frac{1 - \tilde{X}^2}{(1 - \tilde{V}^2)^{13}} \right) P_{\partial^2 A^4, i}(A, B, C) \quad (2.137)$$

where, in complete analogy with the previous examples, the polynomials $P_{\partial^2 A^4, i}$ derived from (2.52) and (2.53) have the form

$$\begin{aligned} P_{\partial^2 A^4, 1} &= -2(A_{11}^2 B_{00} + C_{11}^2 B_{00}), & P_{\partial^2 A^4, 4} &= -4(A_{11}^2 A_{00} + C_{11}^2 C_{00}), \\ P_{\partial^2 A^4, 2} &= -4(A_{11}^2 C_{00} + C_{11}^2 A_{00}), & P_{\partial^2 A^4, 5} &= -4B_{11}^2 (A_{00} + C_{00}), \\ P_{\partial^2 A^4, 3} &= -2(A_{11}^2 B_{00} + C_{11}^2 B_{00}), & P_{\partial^2 A^4, 6} &= -4B_{11}^2 B_{00}. \end{aligned} \quad (2.138)$$

Numerical computation of the integrals gives

$$\begin{aligned} f_1 &\approx -2.2827697, & f_4 &\approx -2.0422916, \\ f_2 &\approx -1.5190433, & f_5 &\approx -2.5206270, \\ f_3 &\approx -2.2827697, & f_6 &\approx -2.2603135. \end{aligned} \quad (2.139)$$

As one can easily check, the relations (2.136) are satisfied with high accuracy. This verifies that the $\partial^2 A^4$ terms we have computed in the effective vector field action are in agreement with the \hat{F}^3 term in the nonabelian analogue of the Born-Infeld action.

2.7 Conclusions

In this chapter we have computed the effective action for the massless open string vector field by integrating out all other fields in Witten's cubic open bosonic string field theory. We have calculated the leading terms in the off-shell action $S[A]$ for the massless vector field A_μ , which we have transformed using a field redefinition into an action $\hat{S}[\hat{A}]$ for a gauge field \hat{A} which transforms under the standard gauge transformation rules. For the abelian theory, we have shown that the resulting action agrees with the Born-Infeld action to order \hat{F}^4 , and that zero-momentum terms vanish to

order A^{10} . For the nonabelian theory, we have shown agreement with the nonabelian effective vector field action previously computed by world-sheet methods to order \hat{F}^3 . These results demonstrate that string field theory provides a systematic approach to computing the effective action for massless string fields. In principle, the calculation in this chapter could be continued to determine higher-derivative corrections to the abelian Born-Infeld action and higher-order terms in the nonabelian theory.

As we have seen in this chapter, comparing the string field theory effective action to the effective gauge theory action computed using world-sheet methods is complicated by the fact that the fields defined in SFT are related through a nontrivial field redefinition to the fields defined through world-sheet methods. In particular, the massless vector field in SFT has a nonstandard gauge invariance, which is only related to the usual Yang-Mills gauge invariance through a complicated field redefinition. This is a similar situation to that encountered in noncommutative gauge theories, where the gauge field in the noncommutative theory—whose gauge transformation rule is nonstandard and involves the noncommutative star product—is related to a gauge field with conventional transformation rules through the Seiberg-Witten map. In the case of noncommutative Yang-Mills theories, the structure of the field redefinition is closely related to the structure of the gauge-invariant observables of the theory, which in that case are given by open Wilson lines [69]. A related construction recently appeared in [70], where a field redefinition was used to construct matrix objects transforming naturally under the D4-brane gauge field in a matrix theory of D0-branes and D4-branes. An important outstanding problem in string field theory is to attain a better understanding of the observables of the theory (some progress in this direction was made in [71, 72]). It seems likely that the problem of finding the field redefinition between SFT and world-sheet fields is related to the problem of understanding the proper observables for open string field theory.

While we have focused in this chapter on calculations in the bosonic theory, it would be even more interesting to carry out analogous calculations in the supersymmetric theory. There are currently several candidates for an open superstring field theory, including the Berkovits approach [73] and the (modified) cubic Witten

approach [74, 75, 76]. (See [77] for further references and a comparison of these approaches.) In the abelian case, a superstring calculation should again reproduce the Born-Infeld action, including all higher-derivative terms. In the nonabelian case, it should be possible to compute all the terms in the nonabelian effective action. Much recent work has focused on this nonabelian action, and at this point the action is constrained up to order F^6 [28]. It would be very interesting if some systematic insight into the form of this action could be gained from SFT.

The analysis in this chapter also has an interesting analogue in the closed string context. Just as the Yang-Mills theory describing a massless gauge field can be extended to a full stringy effective action involving the Born-Infeld action plus derivative corrections, in the closed string context the Einstein theory of gravity becomes extended to a stringy effective action containing higher order terms in the curvature. Some terms in this action have been computed, but they are not yet understood in the same systematic sense as the abelian Born-Infeld theory. A tree-level computation in closed string field theory would give an effective action for the multiplet of massless closed string fields, which should in principle be mapped by a field redefinition to the Einstein action plus higher-curvature terms [31]. Lessons learned about the nonlocal structure of the effective vector field theory discussed in this chapter may have interesting generalizations to these nonlocal extensions of standard gravity theories.

Another direction in which it would be interesting to extend this work is to carry out an explicit computation of the effective action for the tachyon in an unstable brane background, or for the combined tachyon-vector field effective action. Some progress on the latter problem was made in [32]. Because the mass-shell condition for the tachyon is $p^2 = 1$, it does not seem to make any sense to consider an effective action for the tachyon field, analogous to the Born-Infeld action, where terms of higher order in p are dropped. Indeed, it can be shown that when higher-derivative terms are dropped, any two actions for the tachyon which keep only terms $\partial^k \phi^{m+k}$, $m \geq 0$, can be made perturbatively equivalent under a field redefinition (which may, however, have a finite radius of convergence in p). Nonetheless, a proposal for an effective

tachyon + vector field action of the form

$$S = V(\phi) \sqrt{-\det(\eta_{\mu\nu} + F_{\mu\nu} + \partial_\mu \phi \partial_\nu \phi)} \quad (2.140)$$

was given in [34, 35, 36] (see also [33]). Quite a bit of recent work has focused on this form of effective action (see [78] for a recent summary with further references), and there seem to be many special properties for this action with particular forms of the potential function $V(\phi)$. It would be very interesting to explicitly construct the tachyon-vector action using the methods of this chapter. A particularly compelling question related to this action is that of closed string radiation during the tachyon decay process. In order to understand this radiation process, it is necessary to understand back-reaction on the decaying D-brane [79], which in the open string limit corresponds to the computation of loop diagrams. Recent work [37] indicates that for the superstring, SFT loop diagrams on an unstable Dp -brane with $p < 7$ should be finite, so that it should be possible to include loop corrections in the effective tachyon action in such a theory. The resulting effective theory should shed light on the question of closed string radiation from a decaying D-brane.

Ultimately, however, it seems that the most important questions which may be addressed using the type of effective field theory computed in this chapter have to do with the nonlocal nature of string theory. The full effective action for the massless fields on a D-brane, given by the Born-Infeld action plus derivative corrections, or by the nonabelian vector theory on multiple D-branes, has a highly nonlocal structure. Such nonlocal actions are very difficult to make sense of from the point of view of conventional quantum field theory. Nonetheless, there is important structure hidden in the nonlocality of open string theory. For example, the instability associated with contact interactions between two parts of a D-brane world-volume which are separated on the D-brane but coincident in space-time is very difficult to understand from the point of view of the nonlocal theory on the D-brane, but is implicitly contained in the classical nonlocal D-brane action. At a more abstract level, we expect that in any truly background-independent description of quantum gravity, space-time geometry

and topology will be an emergent phenomenon, not manifest in any fundamental formulation of the theory. A nongeometric formulation of the theory is probably necessary for addressing questions of cosmology and for understanding very early universe physics before the Planck time. It seems very important to develop new tools for grappling with such issues, and it may be that string field theory may play an important role in developments in this direction. In particular, the way in which conventional gauge theory and the nonlocal structure of the D-brane action is encoded in the less geometric variables of open string field theory may serve as a useful analogue for theories in which space-time geometry and topology emerge from a nongeometric underlying theory.

Chapter 3

Taming the Tachyon in Cubic String Field Theory

3.1 Introduction

The tachyon of the open bosonic string has played an important role in recent years in the development of string field theory as a background-independent formulation of string theory. Following Sen's conjectures regarding this tachyon [1], significant progress has been made towards demonstrating that both the unstable vacuum containing the tachyon and the "true" vacuum where the tachyon has condensed are well-defined states in Witten's cubic open string field theory (CSFT) [38]. This is important evidence that string field theory is capable of describing multiple distinct vacuum configurations using a single set of degrees of freedom, as one would expect for a background-independent formulation of the theory. Some of the work in this area is reviewed in [80, 81].

An important aspect of the open string tachyon which is not yet fully understood, however, is the dynamical process through which the tachyon rolls from the unstable vacuum to the stable vacuum. A review of previous work on this problem is given in [81]. Computations using CFT, boundary states, RG flow analysis and boundary string field theory (BSFT) [82, 83, 84, 85] show that the tachyon should monotonically roll towards the true vacuum, but should not arrive at the true vacuum in finite

time [22]-[90]. In BSFT variables, where the tachyon T goes to $T \rightarrow \infty$ in the stable vacuum, the time-dependence of the tachyon field goes as $T(t) = e^t$. This dynamics is intuitively fairly transparent, and follows from the fact that e^t is a marginal boundary operator [91, 92, 93, 22, 90]. Other approaches to understanding the rolling tachyon from a variety of viewpoints including DBI-type actions [94]-[96], S-branes and time-like Liouville theory [97]-[101], matrix models [102]-[107], and fermionic boundary CFT [108] lead to a similar picture of the time dynamics of the tachyon.

In CSFT, on the other hand, the rolling tachyon dynamics appears much more complicated. In [25], Moeller and Zwiebach used level truncation to analyze the time dependence of the tachyon. They found that at low levels of truncation, the tachyon rolls well past the minimum of the potential, then turns around and begins to oscillate with ever increasing amplitude. It was further argued by Fujita and Hata in [109] that such oscillations are a natural consequence of the form of the CSFT equations of motion, which include an exponential of time derivatives acting on the tachyon field.

These two apparently completely different pictures of the tachyon dynamics raise an obvious puzzle. Which picture is correct? Does the tachyon converge monotonically to the true vacuum, or does it undergo wild oscillations? Is there a problem with the BSFT approach? Does the CSFT analysis break down for some reason such as a branch point singularity at a finite value of t ? Does the dynamics in CSFT behave better when higher-level states are included? Is CSFT an incomplete formulation of the theory?

In this chapter we resolve this puzzle. We carry out a systematic level-truncation analysis of the tachyon dynamics for a particular solution in CSFT. We compute the trajectory $\phi(t)$ as a power series in e^t at various levels of truncation. We show that indeed the dynamics in CSFT has wild oscillations. We find, however, that the trajectory $\phi(t)$ is well-defined in the sense that increasing both the level of truncation in CSFT and the number of terms retained in the power series in e^t leads to a convergent value of $\phi(t)$ for any fixed t , at least below an upper bound $t < t_b$ associated with the limit of our computational ability.

We reconcile this apparent discrepancy with the results of BSFT by demonstrating

that a field redefinition which takes the CSFT action to the BSFT action also maps the wildly oscillating CSFT solution to the well-behaved BSFT exponential solution. This qualitative change in behavior through the field redefinition is possible because the field redefinition relating the tachyon in the two formulations is nonlocal and includes terms with arbitrarily many time derivatives. Such field redefinitions are generically expected to be necessary when relating the background-independent CSFT degrees of freedom to variables appropriate for a particular background [31]. A similar field redefinition involving higher derivatives was shown in [43] to be necessary to relate the massless vector field \hat{A}_μ of CSFT on a D-brane with the usual gauge field A_μ appearing in the Yang-Mills and Born-Infeld actions. Other approaches to the rolling tachyon using CSFT appear in [110]-[113]; related approaches which have been studied include p -adic SFT [114, 115], open-closed SFT [116], and vacuum string field theory [117, 118]. Closed string production during the rolling process is described in [119, 120, 121].

The chapter is organized as follows. Section 3.2 describes the general approach that we use to find the rolling tachyon solution and gives the leading order terms in the solution explicitly. Section 3.3 describes the results of numerically solving the equations of motion in level-truncated CSFT. Section 3.4 is dedicated to finding the leading terms in the field redefinition that relates the effective tachyon actions in Boundary and Cubic String Field Theory. Section 3.5 contains conclusions and a discussion of our results. Some technical details regarding our methods of calculation are relegated to Appendices.

As this the research described in this chapter was being completed the paper [122] appeared, which treats the same system, although without considering massive fields. The analysis of [122] is carried out using analytic methods which give an approximate rolling tachyon solution when all fields other than the tachyon are neglected. The solution in their paper shares some qualitative features with our results—in particular, they find a solution which has similar behavior for negative time, and their solution also rolls past the naive minimum of the tachyon potential. Their solution has a cusp at $t = 0$ where the solution has a discontinuous first derivative; we believe that their

solution breaks down at this point, but that their solution is good for $t < 0$ and that the analytic methods they use in deriving their results are of interest and may help in understanding the dynamics of the system.

3.2 Solving the CSFT equations of motion

We are interested in finding a solution to the complete open string field theory equations of motion. The full CSFT action contains an infinite number of fields, coupled through cubic terms which contain exponentials of derivatives (see [80] for a detailed review). Thus, we have a nonlocal action in which it is difficult to make sense of an initial value problem (see [123, 124, 125, 126] for some discussion of such equations with infinite time derivatives).

Nonetheless, we can systematically develop a solution valid for all times by assuming that as $t \rightarrow -\infty$ the solution approaches the perturbative vacuum at $\phi = 0$. In this limit the equation of motion is the free equation for the tachyon field $\ddot{\phi}(t) = \phi(t)$, with solution $\phi(t) = ce^t$. For $t \ll 0$, we can perform a perturbative expansion in the small parameter e^t . The computations carried out in this chapter indicate that this power series indeed seems convergent for all t . A related approach was taken in [25, 109], where an expansion in $\cosh t$ was proposed. This allows a one-parameter family of solutions with $\dot{\phi}(0) = 0$, but is more technically involved due to the more complicated structure of $\cosh nt$ compared with e^{nt} . We restrict attention here to the simplest case of solutions which can be expanded in e^t , but we expect that a more general class of solutions can be constructed using this approach. Note that in most previous work on this problem, solutions have been constructed using Wick rotation of periodic solutions; in this chapter we work directly with the real solution which is a sum of exponentials.

The infinite number of fields of CSFT represents an additional complication. We can, however, systematically integrate out any finite set of fields to arrive at an effective action for the tachyon field which we can then solve using the method just described. We do this using the level-truncation approximation to CSFT including

fields up to a fixed level. We find that the resulting trajectory $\phi(t)$ converges well for fixed t as the level of truncation is increased.

We thus compute the solution $\phi(t)$ with the desired behavior e^t as $t \rightarrow -\infty$ in two steps. In the first step, described in subsection 3.2.1, we compute the tachyon effective action, eliminating all the other modes using equations of motion. Some technical details of this calculation are relegated to Appendix A. In the second step, described in subsection 3.2.2, we write down the equation of motion for the effective theory and solve it perturbatively in powers of e^t .

3.2.1 Computing the effective action

We are interested in a spatially homogeneous rolling tachyon solution. One can compute such a solution by solving the equations of motion for the infinite family of string fields with all the spatial derivatives set to 0. Labeling string fields ψ_i , the cubic string field theory equations of motion (in the Feynman-Siegel gauge) take the schematic form

$$(\partial_t^2 - m_i^2)\psi_i(t) = g e^{V_{00}^{11}(\partial_s^2 + \partial_u^2 + \partial_s \partial_u)} C_i^{jk}(\partial_u, \partial_s) \psi_j(s)\psi_k(u)|_{s=u=t} \quad (3.1)$$

where all possible pairs of fields appear on the RHS. The coefficients C_{ijk} multiplying each term may contain a finite order polynomial in the derivatives ∂_s, ∂_u . Plugging in the Ansatz $\phi(t) = \psi_0(t) = e^t + \dots$ with all other fields vanishing at order e^t it is clear that we can systematically solve the equations for all fields order by order in e^t . This is one way of systematically solving order by order for $\phi(t)$.

We will find it convenient to think of the perturbative solution for $\phi(t)$ in terms of an effective action $S[\phi]$ which arises by integrating out all the massive string fields at tree level. Perturbatively, we can solve the equations of motion (3.1) for all fields except $\phi = \psi_0$ as power series in ϕ , by recursively plugging in the equations of motion for all fields except ϕ on the RHS until all that remains is a perturbative expansion in terms of $\phi(t)$ and its derivatives. We have used two approaches to compute the effective action $S[\phi]$. One approach is to explicitly use the equations (3.1) for all

fields up to a fixed level. This approach is useful for generating terms to high powers in g but becomes unwieldy for fields at high levels. The second approach we use is to compute the effective action as a diagrammatic sum using the level truncation on oscillator method developed in [20]. This approach is useful for calculating low-order terms in the effective potential where high-level fields are included. Some details of the oscillator approach are described in Appendix A.

The leading terms in the tachyon action are the quadratic and cubic terms coming directly from the CSFT action

$$S[\phi] = \frac{1}{2} \int dt \phi(t) (-\partial_t^2 + 1) \phi(t) - \frac{g}{3} \left(e^{\frac{1}{2} V_{00}^{11} (\partial_t^2 - 1)} \phi(t) \right)^3 + \dots \quad (3.2)$$

where

$$V_{00}^{11} = -\log \left(\frac{27}{16} \right) \quad (3.3)$$

is the Neumann coefficient for the three tachyon vertex.

Integrating out the massive fields at tree level gives rise to higher-order terms $g^2 \phi^4, \dots$ with even more complicated derivative structures. The resulting effective action can be written in terms of the (temporal) Fourier modes $\phi(w)$ of $\phi(t)$ as

$$S[\phi] = \sum_n \frac{g^{n-2}}{n!} \int \prod_{i=1}^n dw_i (2\pi)^n \delta \left(\sum_i w_i \right) \Xi_n^{\text{CSFT}}(w_1, \dots, w_n) \phi(w_1) \dots \phi(w_n) \quad (3.4)$$

where the functions $\Xi_n^{\text{CSFT}}(w_1, \dots, w_n)$ determine the derivative structure of the terms at order $g^{n-2} \phi^n$. The quadratic and cubic terms following from (3.2) are

$$\Xi_2^{\text{CSFT}}(w_1, w_2) = (1 - w_1 w_2), \quad (3.5)$$

$$\Xi_3^{\text{CSFT}}(w_1, w_2, w_3) = -2 e^{-\frac{1}{2} V_{00}^{11} (w_1^2 + w_2^2 + w_3^2 + 3)}. \quad (3.6)$$

One way to obtain the approximate classical effective action for the tachyon field is to use the equations of motion for a few low level massive fields to eliminate these fields explicitly from the action. The higher level massive fields are set to zero (level truncation).

As an example, we now explicitly compute the quartic term in the effective action (3.4) in the level 2 truncation. In the case of CSFT for a single D-brane the combined level of fields coupled by a cubic interaction must be even. For example, there is no vertex coupling two tachyons (level zero) with the gauge boson (level 1). It follows that there are no tree level Feynman diagrams with all external tachyons and internal fields of odd level. Thus, in calculating the tachyonic effective action we may set odd level fields to 0. Fixing Feynman-Siegel gauge, the only fields involved are the tachyon ϕ and three level 2 massive fields with $m^2 = 1$: β , B_μ and $B_{\mu\nu}$. The terms in the action contributing to the four-tachyon term in the effective action are

$$\begin{aligned} & \frac{1}{2} \int dt \beta (\partial_t^2 + 1) \beta - B_{\mu\nu} (\partial_t^2 + 1) B^{\mu\nu} - B_\mu (\partial_t^2 + 1) B^\mu + \\ & g \int dt a_1 \tilde{\phi}^2 \tilde{B}_\mu^\mu + a_2 (\tilde{\phi} \partial_t \partial_t \tilde{\phi} - \partial_t \tilde{\phi} \partial_t \tilde{\phi}) \tilde{B}^{00} + a_3 \tilde{\phi}^2 \tilde{\beta} + a_4 \tilde{\phi} \partial_t \tilde{\phi} \tilde{B}^0 \end{aligned} \quad (3.7)$$

where $\tilde{f} = e^{\frac{1}{2} V_{00}^{11} (\partial_t^2 - 1)} f$. Other interaction terms involving level 2 fields, for example β^3 or $B_\mu B^\mu \phi$, would contribute to the effective action at higher powers of ϕ . The coefficients a_1, \dots, a_4 are real numbers and can be expressed via the appropriate matter and ghost Neumann coefficients,

$$\begin{aligned} a_1 &= -\frac{V_{11}^{11}}{\sqrt{2}} \approx 0.130946, & a_2 &= \sqrt{2} (V_{01}^{12})^2 \approx 0.419026, \\ a_3 &= X_{11}^{11} \approx 0.407407, & a_4 &= -6V_{02}^{12} \approx 0.628539. \end{aligned} \quad (3.8)$$

Following the procedure described above we write down the equations of motion for the massive fields, and plug them into (3.7). We then obtain a quartic term in the tachyonic effective action,

$$\begin{aligned} & g^2 e^{-3V_{00}^{11}} \int \prod_{i=1}^4 (2\pi dw_i) \phi(w_i) \delta(\sum w_i) \frac{\exp(-V_{00}^{11} [w_1^2 + w_2^2 + w_3^2 + w_4^2 + w_1 w_2 + w_3 w_4])}{1 - (w_1 + w_2)^2} \\ & \left(b_1 + b_2 w_2 (w_2 - w_1) + b_3 w_1 w_4 (w_2 - w_1) (w_4 - w_3) + b_4 w_2 w_4 \right), \end{aligned} \quad (3.9)$$

where we have denoted

$$\begin{aligned} b_1 &= \frac{1}{2}(13(V_{11}^{11})^2 - (X_{11}^{11})^2), & b_2 &= -V_{11}^{11}(V_{01}^{12})^2, \\ b_3 &= (V_{01}^{12})^4, & b_4 &= 18(V_{02}^{12})^2. \end{aligned} \quad (3.10)$$

We have explicitly computed the terms to order ϕ^7 in the effective tachyon action in the $L = 2$ truncation. One can in principle continue the procedure further, increasing both the level of truncation and the powers of ϕ in the effective action. Explicit calculation, however, becomes laborious as we take into account more and more string field components; the oscillator method [20], described in the appendix C is more efficient for high-level computations. In the next section we proceed to find the solution of the equations of motion from the effective tachyon action.

3.2.2 Solving the equations of motion in the effective theory

We now outline the process for solving the tachyon equation of motion for the effective theory, and we compute the first perturbative corrections to the free solution. The variation of the effective action (3.4) gives equations of the form

$$(\partial_t^2 - 1)\phi = \sum_{n=2}^{\infty} g^{n-1} K_n(\phi, \dots, \phi) \quad (3.11)$$

where the nonlinear terms of order ϕ^n are denoted by K_n . The specific form of the K_n follow by differentiating (3.4) with respect to $\phi(t)$. The functions Ξ_n appearing in (3.4), and thus the corresponding K_{n-1} 's can in principle be explicitly computed for arbitrary n at any finite level of truncation using the method described in the previous subsection. An alternate approach which is more efficient for computing K_n at small n but large truncation level is reviewed in Appendix A. In general, independently of the method used to compute it, K_n will be a complicated momentum-dependent function of its arguments.

The solution of the linearized equations of motion which satisfies the boundary condition $\phi \rightarrow 0$ as $t \rightarrow -\infty$ is $\phi(t) = c_1 e^t$. As discussed above, we wish to use

perturbation theory to find a rolling solution which is defined by this asymptotic condition as $t \rightarrow -\infty$. Note that this asymptotic form places a condition on all derivatives of ϕ in the limit $t \rightarrow -\infty$, as appropriate for a solution of an equation with an unbounded number of time derivatives. If we now assume that the full solution can be computed by solving (3.11) using perturbation theory, at least in some region $t < t_{\max}$, it can be easily seen that the successive corrections to the asymptotic solution $\phi_1(t) = c_1 e^t$ are of the form $\phi_n(t) = c_n e^{nt}$. In other words, to solve the equations of motion using perturbation theory we expand $\phi(g, t)$ in powers of g

$$\phi(g, t) = \phi_1(t) + g\phi_2(t) + g^2\phi_3(t) + \dots \quad (3.12)$$

where

$$\phi_n(t) = c_n e^{nt}. \quad (3.13)$$

As we will see, our assumption leads to a power series which seems to be convergent for all t and all g . Note that since $g^n e^{nt} = e^{n(t+\log(g))}$, the coupling constant can be set to 1 by translating the time variable and rescaling ϕ , so convergence for fixed g and all t implies convergence for all t and for all g . Plugging (3.12) into (3.11) we find

$$(\partial_t^2 - 1)\phi_n = (n^2 - 1)c_n e^{nt} = \sum_p \sum_{m_1+m_2+\dots+m_p=n} K_p(\phi_{m_1}, \phi_{m_2}, \dots, \phi_{m_p}). \quad (3.14)$$

These equations allow us to solve for $c_{n>1}$ iteratively in n . Having solved the equations for c_2, \dots, c_{n-1} we can plug them in via (3.13) on the right hand side of (3.14) to determine c_n .

As an example, let us consider the first correction $\phi_2(t) = c_2 e^{2t}$ to the linearized solution $\phi_1(t) = c_1 e^t$. The equation of motion at quadratic order arising from K_2 is

$$(\partial_t^2 - 1)\phi = -e^{\frac{1}{2}V_{00}^{11}(\partial_t^2-1)}(e^{\frac{1}{2}V_{00}^{11}(\partial_t^2-1)}\phi)^2. \quad (3.15)$$

Plugging in $\phi_1 = c_1 e^t$, $\phi_2 = c_2 e^{2t}$ we find

$$c_2(\partial_t^2 - 1)e^{2t} = -c_1^2 e^{\frac{1}{2}V_{00}^{11}(\partial_t^2 - 1)}(e^{\frac{1}{2}V_{00}^{11}(\partial_t^2 - 1)}e^t)^2 \quad (3.16)$$

and therefore

$$c_2 = -\frac{1}{3}e^{\frac{3}{2}V_{00}^{11}}c_1^2. \quad (3.17)$$

If we normalize $c_1 = 1$ then the solution to order e^{2t} is

$$\phi(t) = e^t - \frac{64}{243\sqrt{3}}e^{2t} + \dots \quad (3.18)$$

The quartic interaction term in the effective action would contribute to coefficients $c_n e^{nt}$ with $n \geq 3$ with the leading order contribution being $c_3 e^{3t}$. From equation (3.14) we have

$$c_3 = \frac{e^{-3t}}{8} (2c_2 K_2(e^t, e^{2t}) + K_3(e^t, e^t, e^t)). \quad (3.19)$$

where K_3 is obtained by differentiating (3.9) with respect to $\phi(t)$. The two summands in (3.19) represent contributions from the cubic and quartic terms in the effective action. The numerical values of these contributions are

$$(\delta c_3)_{\text{cubic}} \cong 0.0021385, \quad (\delta c_3)_{\text{quartic}} \cong 0.0000492826. \quad (3.20)$$

It is perhaps surprising that the contribution to c_3 from the quartic term in the effective action is merely 0.2% of the contribution from the cubic term. Adding the contributions we get the rolling solution to second order in perturbation theory in level 2 truncation

$$\phi(t) \cong e^t - 0.152059e^{2t} + 0.002187e^{3t} + \dots \quad (3.21)$$

In this section we have explicitly demonstrated our procedure for the calculation of the rolling tachyon solution. We considered the CSFT action truncated to fields with level less or equal than two and computed the first two corrections to the solution of

the linearized equations of motion. The next section is dedicated to the more detailed numerical analysis of the rolling tachyon solution.

3.3 Numerical results

In this section we describe the results of using the level-truncated effective action $S[\phi]$ to compute approximate perturbative solutions to the equation of motion through (3.14). We are testing the convergence of the solution in two respects. In subsection 3.3.1 we check that the solution converges nicely at fixed t when we take into account successively higher powers of ϕ in a perturbative expansion of the effective action while keeping the truncation level fixed at $L = 2$. In subsection 3.3.2 we check that the solution converges well for fixed t when we keep the order of perturbation theory fixed while increasing the truncation level.

3.3.1 Convergence of perturbation theory at $L = 2$

The equation (3.14) allows us to find the successive perturbative contributions to the solution of the equations of motion, given an explicit expression for the terms in the effective action. The solution takes the form

$$\phi(t) = \sum_n c_n e^{nt}. \quad (3.22)$$

Since all the derivatives of e^{nt} are straightforward to compute, as in (3.17), we can replace these derivatives in any operator through $f(\partial_t)e^{nt} \rightarrow f(n)e^{nt}$. This manipulation is justified as long as f is regular at n .

We have computed the functions Ξ_n^{CSFT} and the resulting K_{n-1} 's by solving the equations of motion up to $n = 7$ and integrating out all fields at truncation level $L = 2$ as described in subsection 2.1. We have used these K_n 's to compute the resulting approximate coefficients c_n , with $n \leq 6$. To compute the coefficient c_n one needs the effective tachyon action computed to order $n + 1$; higher terms in the action contribute only to higher order coefficients. The $L = 2$ approximation to the solution

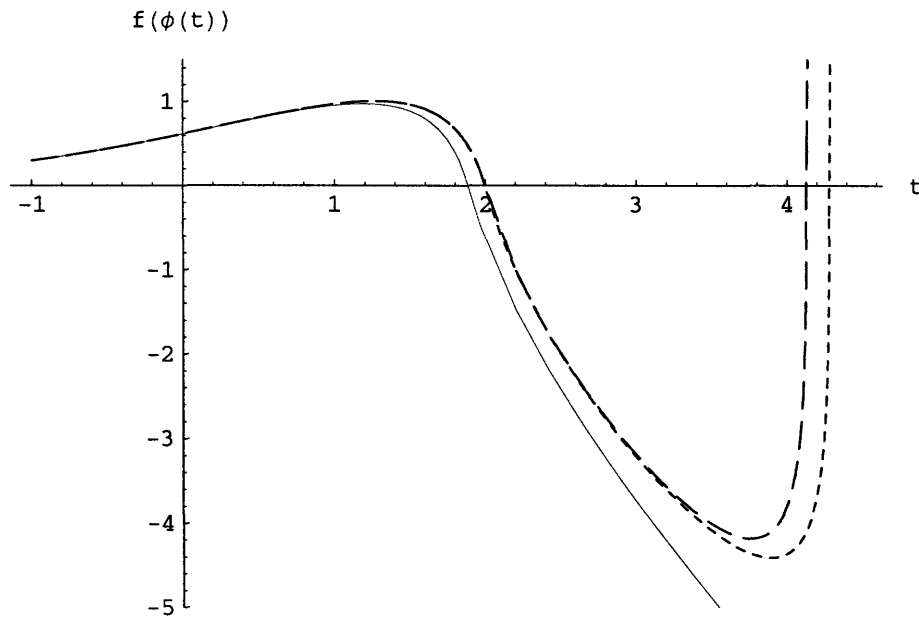


Figure 3-1: The solution $\phi(t)$ including the first two turnaround points, including fields up to level $L = 2$. The solid line graphs the approximation $\phi(t) = e^t + c_2 e^{2t}$. The long dashed line graphs $\phi(t) = e^t + c_2 e^{2t} + c_3 e^{3t}$. The approximate solutions computed up to e^{4t} , e^{5t} and e^{6t} are very close in this range of t and are all represented in the short dashed line. One can see that after going through the first turnaround point with coordinates $(t, \phi(t)) \sim (1.27, 1.8)$ the solution decreases, reaching the second turnaround at around $(t, \phi(t)) \sim (3.9, -81)$. The function $f(\phi(t)) = \text{sign}(\phi(t)) \log(1 + |\phi(t)|)$ is graphed to show both turnaround points clearly on the same scale.

for the tachyon field is

$$\phi(t) \cong e^t - \frac{64 e^{2t}}{243 \sqrt{3}} + 0.002187 e^{3t} - 3.9258 \cdot 10^{-6} e^{4t} + 4.9407 \cdot 10^{-10} e^{5t} + 6.3227 \cdot 10^{-12} e^{6t}. \quad (3.23)$$

Plotting the result we observe that for small enough t the term e^t dominates and the solution decays as e^t at $-\infty$. Then, as t grows, the second term in (3.23) becomes important. The solution turns around and $\phi(t)$ becomes negative, with the major contribution coming from e^{2t} . Then the next mode, e^{3t} becomes dominating and so on. The solution $\phi(t)$ around the first two turnaround points is shown on the figure 3-1. Note that the trajectory passes through the minimum of the static potential, which is at $\phi \sim 0.545$ [127, 5], well before the first turnaround point. The positions of

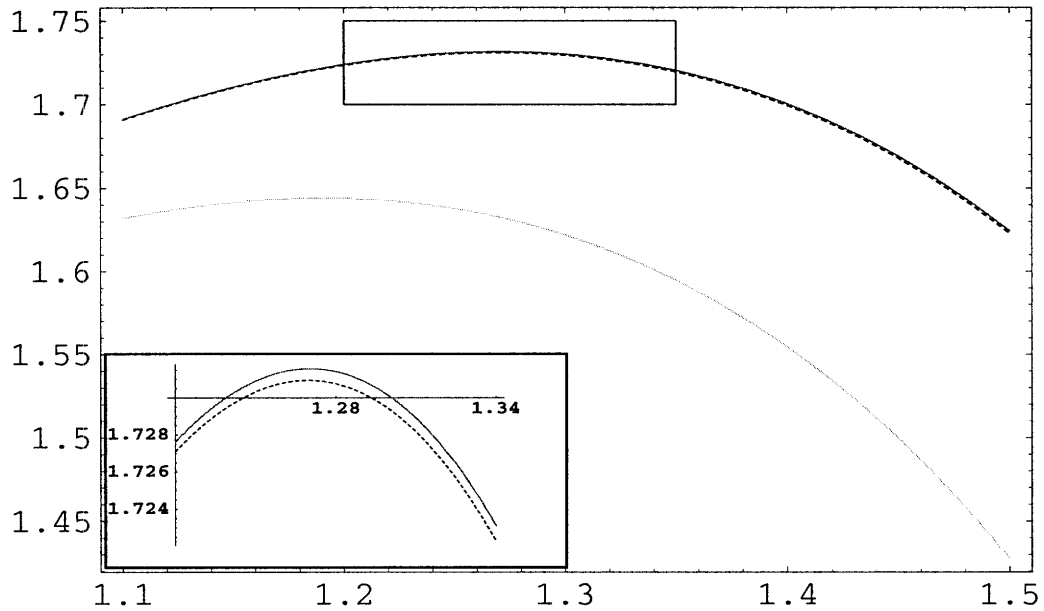


Figure 3-2: First turnaround point for the solution in $L = 2$ truncation scheme. The large plot shows the approximations with ϕ^3 (the gray line), ϕ^4 (black solid), and ϕ^5 (dashed lines) terms in the action taken into account. The smaller plot zooms in on the approximations with ϕ^4 and ϕ^5 terms taken into account. The corrections from higher powers of ϕ are very small and the corresponding plots are indistinguishable from the one of the ϕ^5 approximation.

the first 2 turnaround points are quite accurately determined by taking into account

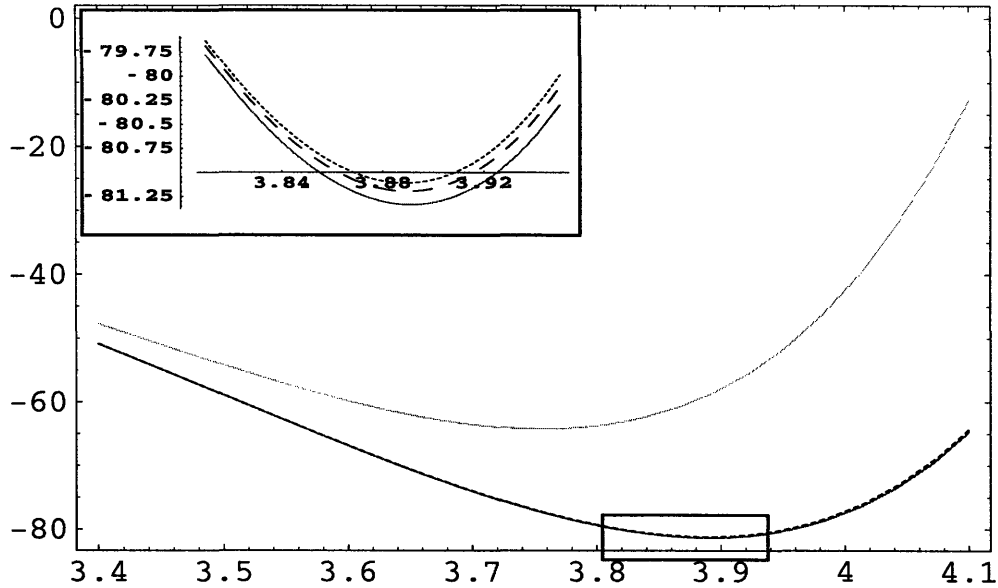


Figure 3-3: Second turnaround point for the solution in $L = 2$ truncation scheme. The gray line on the large plot shows the solution computed with the effective action including terms up to ϕ^4 . The black solid and dashed lines represent higher order corrections. On the small plot the solid line includes ϕ^5 corrections, the dotted line includes corrections from the ϕ^6 term and the dashed line takes into account the ϕ^7 term.

the effective action terms up to ϕ^5 . The inclusion of the higher order terms in the action changes the position of the first 2 turnaround points only slightly. Figures 3-2 and 3-3 illustrate the dependence of the position of the first two turnaround points on the powers of ϕ included in effective action. We interpret these results as strong evidence that, at least for the effective action at truncation level $L = 2$, the solution (3.22) is given by a perturbative series in e^t which converges at least as far as the second turnaround point, and plausibly for all t .

3.3.2 Convergence of level truncation

From the results of the previous subsection, we have confidence that the first two points where the tachyon trajectory turns around are well determined by the ϕ^4 and ϕ^5 terms in the effective action. To check whether these oscillations are truly part of a well-defined trajectory in the full CSFT, we must check to make sure that the turnaround points are stable as our level of truncation is increased and the terms

in the effective action are computed more precisely. From previous experience with level truncated calculations of the static effective tachyon potential and the vector field effective action [5, 20, 43], where coefficients in the effective action generically converge well, with errors of order $1/L$ at truncation level L , we expect that the full tachyon effective action will also converge well and will lead to convergent values of c_n within a factor of order 1 of the $L = 2$ results computed explicitly.

We have computed the ϕ^4 term in the effective action at levels of truncation up to $L = 16$. The results of this computation for the approximate trajectory $\phi(t)$ are shown in Figure 3-4, which demonstrates the behavior of the first turnaround point as we include higher level fields. This computation shows that the first turnaround point is already determined to within less than 1% by the level $L = 2$ truncation. This turnaround point is also in close agreement with the computations of [25].¹ We take this computation as giving strong evidence that this turnaround point is real. We expect from analogy with other level truncation computations of effective actions and potentials that the other terms in the effective action considered here will also generally converge well. Combining the explicit result for the ϕ^4 term at high levels of truncation with the computation of the previous subsection, we have (to us) convincing evidence that the perturbative expansion e^{nt} for the rolling tachyon solution is valid well past the first turnaround point, and that the level truncation procedure converges to a trajectory containing this turnaround point. Extrapolating the results of this computation, we believe that the qualitative phenomenon of wild oscillations revealed by the level $L = 2$ computation is a correct feature of the time-dependent tachyon trajectory in CSFT, and that more precise calculations at higher level will only shift the positions of the turnaround points mildly, leaving the qualitative behavior intact. It is interesting to compare the behavior of the perturbative expansion of this time-dependent tachyon solution with a perturbative expansion of the effective tachyon potential $V(\phi)$. As found in [5], the power series expansion for $V(\phi)$ fails to converge beyond $|\phi| \sim 0.1$ due to a branch point singularity at negative ϕ

¹Barton Zwiebach has pointed out that the position of the first turnaround point for the $\cosh(nt)$ solution of [25] is very close to the first turnaround point of the e^{nt} solution which we have computed here, and that comparing results with two terms in the expansion gives agreement to within 1%.

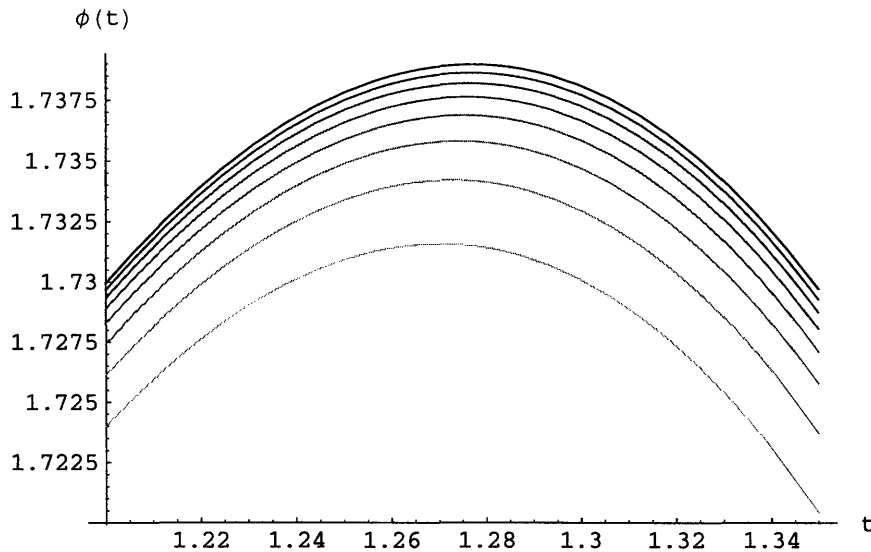


Figure 3-4: The figure shows the convergence of the solution around the first turnaround point as we increase the truncation level. Bottom to top the graphs represent the approximate solutions computed with the effective action computed up to ϕ^4 and truncation level increasing in steps of 2 from $L = 2$ to $L = 16$. We observe that the turnaround point is determined to a very high precision already at the level 2. Similar behavior is observed for the second turnaround point.

where the Feynman-Siegel gauge choice breaks down [128]. Although the potential can be continued for positive ϕ past the radius of convergence using the method of Padé approximants [129], another branch point associated with the breakdown of Feynman-Siegel gauge is encountered at a positive ϕ just past the minimum near $\phi \sim 0.545$. Because of these branch points, the expansion for the effective potential is badly behaved past these points; unlike the time-dependent solution we have studied here, there is no sense in which the potential $V(\phi)$ converges for a general fixed value of ϕ . While we initially thought that the wild oscillations of the low-level computation of the tachyon trajectory $\phi(t)$ might indicate a similar breakdown of the perturbative expansion, our results at higher levels seem to give conclusive evidence that this is not the case. This suggests that the Feynman-Siegel gauge choice is valid in the region of field space containing the trajectory $\phi(t)$ for all t even though the corresponding static ϕ lies outside the region of gauge validity.

3.4 Taming the tachyon with a field redefinition

Now that we have confirmed that CSFT gives a well-defined but highly oscillatory time-dependent solution, we want to understand the physics of this solution. Although the oscillations seem quite unnatural from the point of view of familiar theories with only quadratic kinetic terms and a potential, the story is much more subtle in CSFT due to the higher-derivative terms in the action. For example, while the tachyon field apparently² rolls into a region with $V(\phi) \gg V(0) = 0$, the energy of the perturbative rolling tachyon solution we have found is conserved, as we have verified by a perturbative calculation of the energy including arbitrary derivative terms, along the lines of similar calculations in [25].

To understand the apparently odd behavior of the rolling tachyon in CSFT, it is useful to consider a related story. In [43] we computed the effective action for the massless vector field on a D-brane in CSFT by integrating out the massive fields. The resulting action did not take the expected form of a Born-Infeld action, but included various extra terms with higher derivatives which appeared because the degrees of freedom natural to CSFT are not the natural degrees of freedom expected for the CFT on a D-brane, but are related to those degrees of freedom by a complicated field redefinition with arbitrary derivative terms. In principle, we expect such a field redefinition to be necessary any time one wishes to compare string field theory computations (or any other background-independent formulation) with CFT computations in any particular background. The necessity for considering such field redefinitions was previously discussed in [31, 130].

Thus, to compare the complicated time-dependent trajectory we have found for CSFT with the marginal e^t perturbation of the boundary CFT found in [22, 23], we must relate the degrees of freedom of BSFT and CSFT through a field redefinition which can include arbitrary derivative terms. Given an explicit form $S[T]$ for the BSFT effective tachyon action which admits a solution $T(t) = e^t$, we can construct a

²This is suggested by the effective tachyon potential at low levels of truncation, which is well-defined into the region where $V(\phi) > 0$; due to a breakdown of Feynman-Siegel gauge at large *constant* positive ϕ at higher levels of truncation, as mentioned at the end of Section 3, the static potential is not well-defined in the region of ϕ encountered by the rolling solution in this gauge.

perturbative field redefinition $\phi(t) = \Phi(T(t))$ which maps the BSFT effective action $S[T]$ to the CSFT effective tachyon action $S[\phi]$. Since such a field redefinition must map a solution of the field equations in one picture to a solution in the other picture, it follows that this map takes the BSFT solution $T(t) = e^t$ to the perturbative solution $\phi(t)$ of the CSFT effective action. In this section we use an explicit formulation of the BSFT effective action to compute the leading terms of the field redefinition relating the effective field theories for $T(\phi)$, the tachyon field in boundary string field theory and ϕ , the tachyon in cubic string field theory. This computation shows in a concrete example how the complicated dynamics we have found for the tachyon in CSFT maps to the simple dynamics of BSFT associated with the marginal deformation e^t .

In our explicit computations, we use the effective tachyonic action of BSFT computed up to cubic order in [131]; another approach to computing the BSFT action which may apply more generally was developed in [132]. As we have just discussed, we expect that a similar field redefinition can be constructed for any BSFT effective tachyon action. The BSFT action is determined via the partition function for the boundary SFT and the tachyon's beta function. Thus the particular form of the action depends on the renormalization scheme for the boundary CFT. The BSFT tachyon T we use here is, therefore, the renormalized tachyon with the renormalization scheme of [131]. We now proceed to construct a perturbative field redefinition relating the CSFT and BSFT effective actions. We then will check explicitly that the field redefinition maps the rolling tachyon solution $T(t) = e^t$ to the leading terms in the perturbative solution $\phi(t) = e^t - \frac{64}{243\sqrt{3}}e^{2t} + \dots$ which we have computed in the previous section. The fact that the field redefinition is nonsingular at $T = e^t$ is consistent with the Ansatz $\sum_n c_n e^{nt}$ for the rolling tachyon solution in CSFT.

In parallel with (3.4) we write the action for the boundary tachyon T as

$$S[T] = \sum_n \frac{g^{n-2}}{n!} \int \prod_{i=1}^n (2\pi dw_i) \delta\left(\sum_i w_i\right) \Xi_n^{BSFT}(w_1, \dots, w_n) T(w_1) \dots T(w_n) \quad (3.24)$$

where the functions $\Xi_n^{BSFT}(w_1, \dots, w_n)$ define the derivative structure of the term of

n 'th power in T . The kernel for the quadratic terms is

$$\Xi_2^{\text{BSFT}}(w_1, w_2) = \frac{\Gamma(2 - 2w_1w_2)}{\Gamma^2(1 - w_1w_2)}. \quad (3.25)$$

where Γ is the Euler gamma function. Denoting $a_1 = -w_2w_3$, $a_2 = -w_1w_3$, $a_3 = -w_2w_3$ the kernel for the cubic term can be written as

$$\Xi_3^{\text{BSFT}}(w_1, w_2, w_3) = 2(1 + a_1 + a_2 + a_3)I(w_1, w_2, w_3) + J(w_1, w_2, w_3) \quad (3.26)$$

where functions $I(a_1, a_2, a_3)$ and $J(a_1, a_2, a_3)$ are defined by

$$\begin{aligned} I(a_1, a_2, a_3) &= \frac{\Gamma(1 + a_1 + a_2 + a_3)\Gamma(1 + 2a_1)\Gamma(1 + 2a_2)\Gamma(1 + 2a_3)}{\Gamma(1 + a_1)\Gamma(1 + a_2)\Gamma(1 + a_3)\Gamma(1 + a_1 + a_2)\Gamma(1 + a_1 + a_3)\Gamma(1 + a_2 + a_3)}, \\ J(a_1, a_2, a_3) &= -\frac{\Gamma(1 + 2a_1)\Gamma(2 + 2a_2 + 2a_3)}{\Gamma^2(1 + a_1)\Gamma^2(1 + a_2 + a_3)} + \text{cyclic}. \end{aligned} \quad (3.27)$$

We are interested in the field redefinition that relates $S[T]$ with the CSFT action $S[\phi]$ given in (3.4), (3.5), (3.6). A generic time-dependent field redefinition can be written in momentum space as

$$\begin{aligned} \phi(w_1) &= \int dw_2 \delta(w_1 - w_2) f_1(w_1, w_2) T(w_2) + \\ &\quad \int dw_2 dw_3 f_2(w_1, w_2, w_3) T(w_2) T(w_3) \delta(w_1 - w_2 - w_3) + \dots \end{aligned} \quad (3.28)$$

Note that adding to f_2 a term antisymmetric under exchange of w_2 and w_3 does not change the field redefinition. Thus, we can choose f_2 to be symmetric under $w_2 \leftrightarrow w_3$.

The requirement that this field redefinition maps the CSFT action to the BSFT action,

$$S[\phi(T)] = S[T], \quad (3.29)$$

imposes conditions on the functions $f_i(w_1, \dots, w_{i+1})$. In order to match the quadratic terms, f_1 must satisfy the equation

$$\Xi_2^{\text{BSFT}}(w_1, w_2) - f_1(w_1, w_1) f_1(w_2, w_2) \Xi_2^{\text{CSFT}}(w_1, w_2) \approx 0. \quad (3.30)$$

In this equation the approximate sign means that the left hand side becomes equal to the right hand side when inserted into $\int dw_1 dw_2 \delta(w_1 + w_2) \phi(w_1) \phi(w_2)$ for arbitrary $\phi(w)$.³ Solving equation (3.30) we find

$$f_1(w, w) \equiv f_1(w) = \sqrt{\frac{1}{1+w^2} \frac{\Gamma(2+2w^2)}{\Gamma^2(1+w^2)}}. \quad (3.31)$$

The analogous equation for f_2 is

$$\begin{aligned} \frac{1}{3} \Xi_3^{\text{BSFT}}(w_1, w_2, w_3) \approx \frac{1}{3} f_1(w_1) f_1(w_2) f_1(w_3) \Xi_3^{\text{CSFT}}(w_1, w_2, w_3) + \\ f_1(w_1) f_2(-w_1, w_2, w_3) \Xi_2^{\text{CSFT}}(-w_1, w_1). \end{aligned} \quad (3.32)$$

In constructing a consistent perturbative field redefinition, we further require that the field redefinition must map the mass-shell states correctly, by keeping the mass-shell component of any $\phi(w)$ intact. In other words the mass-shell component of the Fourier expansion of $\phi(t)$ should not be affected by the higher-order terms f_2 , *etc.* This translates to a restriction on f_2

$$f_2(-w_1, w_2, w_3)|_{w_1^2=-1} = 0. \quad (3.33)$$

This constraint is crucial for the field redefinition to correctly relate the on-shell scattering amplitudes for T with those for ϕ . It also ensures that the solution of the classical equations of motion for T maps to the solution of the equations of motion for ϕ .

Equation (3.32) can be simplified by making a substitution

$$f_2(-w_1, w_2, w_3) = \frac{\Xi_3^{\text{BSFT}}(w_1, w_2, w_3)/f_1(w_1) - \Xi_3^{\text{CSFT}}(w_1, w_2, w_3) f_1(w_2) f_1(w_3)}{\Xi_2^{\text{CSFT}}(-w_1, w_1)} A_2(w_1, w_2, w_3) \quad (3.34)$$

³When matching the quadratic terms this condition implies strict equality since both Ξ_2 's are symmetric, but in general the condition is less restrictive. Considering a discrete analogue, it is easy to see that the equation $M_{kl} c_k c_l = 0$ is equivalent to $M_{kl} + M_{lk} = 0$. Similarly, the equation $M_{n_1, \dots, n_k} c_{n_1} \dots c_{n_k} = 0$ is equivalent to the sum over permutations σ on n elements $\sum_{\sigma} M_{\sigma(n_1, \dots, n_k)} = 0$.

giving a simple equation for $A_2(w_1, w_2, w_3)$

$$A_2(w_1, w_2, w_3) \approx \frac{1}{3}. \quad (3.35)$$

Thus, we would now like to find a function $A(w_1, w_2, w_3)$ on the momentum conservation hyperplane $-w_1 + w_2 + w_3 = 0$, symmetric (by choice) under the exchange of w_2 and w_3 and satisfying

$$A_2(w_1, w_2, w_3) + A_2(w_2, w_3, w_1) + A_2(w_3, w_1, w_2) = 1, \quad (3.36)$$

with the constraint⁴

$$A_2(w_1, w_2, w_3)|_{w_1^2=-1} = 0. \quad (3.37)$$

It is sufficient for our needs here to consider a discrete case, where w_1, w_2, w_3 are (imaginary) integers. Indeed, as we are expanding in powers of e^t , we are restricting attention to fields expressed in modes with $w = in$. It is easy to check that the discretized form of A given by

$$A(w_1, w_2, w_3) = \begin{cases} \frac{1}{3}, & w_{1,2,3} \neq \pm i \\ 0, & w_1 = \pm i \\ \frac{1}{2}, & w_2 = \pm i, w_{1,3} \neq \pm i \quad \text{or} \quad w_3 = \pm i, w_{1,2} \neq \pm i \\ \frac{1}{3}, & w_1 = -2i, w_{2,3} = i \quad \text{or} \quad w_1 = 2i, w_{2,3} = -i \\ 1 & w_1 = 0, w_2 = -w_3 = \pm i \end{cases} \quad (3.38)$$

is a solution to (3.36), (3.37). Of course, to define a consistent field redefinition for the complete field theory for all functions ϕ on $t \in (-\infty, \infty)$ we would need to construct

⁴One can check that the prescription used here is correct on a simple example. The simplest example is a polynomial system with a finite number of degrees of freedom and no time dependence. For a system with time-dependence, consider mapping the action of the harmonic oscillator to the action of an anharmonic oscillator with a cubic potential term $-\frac{1}{3}\phi^3$. With the choice of A that preserves the mass-shell modes one gets a field redefinition that correctly maps the solution of the harmonic oscillator e^{it} to the perturbative solution of the anharmonic oscillator $e^{it} - \frac{1}{3}e^{2it} + \dots$. Attempting to choose, for example, a completely symmetric A gives rise to an unwanted additional factor of $1/3$.

a continuous function A , satisfying the above conditions. Since this is not crucial for the development of this chapter we relegate a brief discussion of the construction of such a function to Appendix C.

Let us make a few comments on the field redefinition.

- While $f_1(w)$ is smooth at the mass-shell point due to a cancelation of poles at $w^2 = -1$, there is a pole at $w^2 = -3/2$ below which the expression under the square root becomes negative. This means that the field redefinition (3.28) is only well defined on the subspace $T(w)$ with $w^2 > -3/2$. Within this region $f_1(w)$ is smooth without any zeroes or poles. The mass-shell point, $w^2 = -1$ lies within this region. Related observations were made in [132].
- The function A_2 represents a universal part of f_2 and is independent of the particular properties of the CSFT and BSFT actions. For example to map the action of a harmonic oscillator to the action of an anharmonic oscillator we could use the same A .
- The term multiplying $A_2(w_1, w_2, w_3)$ in (3.34) has a number of poles. However it is non-singular in two important cases. The first case is for spatially dependent fields with $k_{1,2,3}^2 = 1$, when the tachyon fields in both frames $T(k)$ and $\phi(k)$ are on mass-shell. At this point the two summands in the numerator of (3.34) cancel, and there is no pole at this point. The requirement of this cancelation was used in [131] to fix the normalization of BSFT action.

The second case is the one of the rolling tachyon. In this case $T(w_2)$ and $T(w_3)$ are on mass-shell: $w_2^2 = w_3^2 = -1$, while $\phi(w_1)$ is not: $w_1^2 = -4$. There is a potential singularity in the term $\Xi_3^{\text{BSFT}}(w_1, w_2, w_3)/f_1(w_1)$ in the numerator. $f_1(w_1)$ has a zero at $w_1^2 = -4$, but the functions I and J in the Ξ_3^{BSFT} have a stronger zero resulting in a zero at that point.

Finally, we want to check that the field redefinition maps the rolling tachyon solution of BSFT into the perturbative solution that we have found in section 3.2.2. Plugging the rolling solution $T_{\text{rolling}}(t) = e^t$ into the field redefinition and computing

the numerical values we obtain

$$\phi(t) = e^t - \frac{64 e^{2t}}{243 \sqrt{3}} + \dots \quad (3.39)$$

which exactly reproduces the leading order terms in the perturbative CSFT solution found in section 3.2. As we include higher powers of ϕ in the field redefinition we should continue to generate the higher power terms e^{nt} in the perturbative solution.

3.5 Discussion

In this chapter we have confirmed and expanded on the earlier results of [25] and [109], which suggested that in CSFT the rolling tachyon oscillates wildly rather than converging to the stable vacuum. We have shown that the oscillatory trajectory is stable when higher-level fields are included and thus correctly represents the dynamics of CSFT. We have found that the energy of this oscillatory solution is conserved. We have further shown that this dynamics is not in conflict with the more physically intuitive e^t dynamics of BSFT by explicitly demonstrating a field redefinition, including arbitrary derivative terms, which (perturbatively) maps the CSFT action to the BSFT action and the oscillatory CSFT solution to the e^t BSFT solution.

This resolves the outstanding puzzle of the apparently different behavior of the rolling tachyon in these two descriptions of the theory. On the one hand, this serves as further validation of the CSFT framework, which has the added virtue of background-independence, and which has been shown to include disparate vacua at finite points in field space. On the other hand, the results of this chapter serve as further confirmation of the complexity of using the degrees of freedom of CSFT to describe even simple physics. Further insight into the physical properties of the solution we have computed here, such as an understanding of the pressure of the rolling tachyon field, would require new insight or substantial computation. As noted in previous work, many phenomena which are very easy to describe with the degrees of freedom natural to CFT, such as marginal deformations [133], and the low-energy Yang-Mills/Born-Infeld

dynamics of D-branes [43] are extremely obscure in the variables natural to CSFT. This is in some sense possibly an unavoidable consequence of attempting to work with a background-independent theory: the degrees of freedom natural to any particular background arise in complicated ways from the underlying degrees of freedom of the background-independent theory. This problem becomes even more acute in the known formulations of string field theory, which require a canonical choice of background to expand around, when attempting to describe the physics of a background far from the original canonical background choice, such as when describing the physics of the true vacuum using the CSFT defined around the perturbative vacuum [9, 128]. The complexity of the field redefinitions needed to relate even simple backgrounds such as the rolling tachyon discussed in this chapter to the natural CFT variables make it clear that powerful new tools are needed to take string field theory from its current form to a framework in which relevant physics in a variety of backgrounds can be clearly computed and interpreted.

Chapter 4

Conclusions and future directions

String Field Theory has proven to be a useful tool in studying non-perturbative string phenomena, such as transitions between vacua with different geometrical properties. In this thesis we have studied the physics of unstable bosonic D25-brane from the standpoint of string field theory. In particular, we have computed the low energy effective action for the vector field A_μ that describes the oscillations of the D25-brane up to terms of the order of F^4 in powers of $F_{\mu\nu}$ both in abelian and nonabelian cases. We found that a complicated field redefinition is required to relate the SFT variables to the worldsheet string variables. We have computed to the leading order the field redefinition that relates the two pictures. After the field redefinition is performed, the resulting action agrees with the one computed by worldsheet methods. We have also computed the rolling tachyon solution in string field theory and related it to the worldsheet rolling tachyon solution of Sen via the appropriate field redefinition. We found that although in SFT variables the solution develops growing oscillations, in worldsheet variables we obtain well behaved rolling tachyon solution of Sen. The recent paper [135] agrees with our calculations in Chapter 3 and extends the comparison of CSFT and worldsheet approaches to quartic order in ϕ .

There are several directions to continue and extend the results described in this thesis. One direction in which the progress would be much desired is to get some insight on the analytical properties of the field redefinition which relates SFT to the worldsheet actions. The freedom of making a field redefinition is present in almost

any problem we can pose in string field theory and analytical grasp on it's properties would be very helpful for calculating physically useful results from string field theory. A recent paper [136] contains analysis of restrictions imposed on the field redefinitions by T-duality in cubic string field theory. It then discusses the classical solutions in the closed bosonic string field theory.

One of the interesting things to do would be to extend the calculation of effective action to the supersymmetric case. There are several proposals for the supersymmetric SFT's which include Berkovits quartic SFT [73], the modified Witten's approach [74, 75, 76], and Zwiebach and Okawa's heterotic string field theory [137]. The Yang-Mills part of the 10-dimensional D-brane vector field effective action was derived in the context of Berkovits SFT by Berkovits and Schnabl [65]. It would be very interesting to get more insight on the non-abelian analogue of supersymmetric Born-Infeld action from string field theory. The difficulty in doing this is in the field redefinition that relates SFT variables to worldsheet variables. Getting more analytical insight on the structure of the field redefinition would be very helpful for such analysis.

Another interesting direction of research is to compute the effective action for the tachyon field and the gauge field combined. It has been argued from the worldsheet point of view [78] that the combined abelian vector-tachyon effective action is

$$S = V(\phi)\sqrt{-\text{Det}(\eta_{\mu\nu} + F_{\mu\nu} + \partial_\mu\partial_\nu\phi)}. \quad (4.1)$$

To verify this result from SFT point of view and get some information on it's non-abelian analogue is an interesting topic for research.

It would also be interesting to generalize the calculations in this thesis to the closed string field theory [138], [139]. Just as the electromagnetic action is the leading order term in the expansion of the Born-Infeld action, the Einstein action is the leading term in the full non-linear stringy effective action for the gravitational field. It would be interesting to calculate first few terms in the gravitational effective action from string field theory. Some progress in this direction was made in [140]. See [136] for analysis of classical solutions in closed bosonic string field theory.

Appendix A

Neumann Coefficients

In this Appendix we give explicit expressions for and properties of the Neumann coefficients that we use throughout this thesis. First we define coefficients A_n and B_n by the series expansions

$$\left(\frac{1+iz}{1-iz}\right)^{1/3} = \sum_{n \text{ even}} A_n z^n + i \sum_{n \text{ odd}} A_n z^n, \quad (\text{A.1})$$

$$\left(\frac{1+iz}{1-iz}\right)^{2/3} = \sum_{n \text{ even}} B_n z^n + i \sum_{n \text{ odd}} B_n z^n. \quad (\text{A.2})$$

In terms of A_n and B_n we define the coefficients $N_{mn}^{r,\pm s}$ as follows:

$$\begin{aligned} N_{nm}^{r,\pm r} &= \frac{1}{3(n \pm m)} \begin{cases} (-1)^n (A_n B_m \pm B_n A_m) & m+n \in 2\mathbb{Z}, \quad m \neq n \\ 0 & m+n \in 2\mathbb{Z} + 1 \end{cases}, \\ N_{nm}^{r,\pm(r+1)} &= \frac{1}{6(n \pm m)} \begin{cases} (-1)^{n+1} (A_n B_m \pm B_n A_m) & m+n \in 2\mathbb{Z}, \quad m \neq n \\ \sqrt{3} (A_n B_m \mp B_n A_m) & m+n \in 2\mathbb{Z} + 1 \end{cases}, \\ N_{nm}^{r,\pm(r-1)} &= \frac{1}{6(n \mp m)} \begin{cases} (-1)^{n+1} (A_n B_m \mp B_n A_m) & m+n \in 2\mathbb{Z}, \quad m \neq n \\ -\sqrt{3} (A_n B_m \pm B_n A_m) & m+n \in 2\mathbb{Z} + 1 \end{cases}. \end{aligned} \quad (\text{A.3})$$

The coefficients V_{mn}^{rs} are then given by

$$V_{nm}^{rs} = \sqrt{mn} (N_{nm}^{r,s} + N_{nm}^{r,-s}) \quad m \neq n, m, n > 0, \quad (\text{A.4a})$$

$$V_{nn}^{rr} = \frac{1}{3} \left(2 \sum_{k=0}^n (-1)^{n-k} A_k^2 - (-1)^n - A_n^2 \right), \quad n \neq 0, \quad (\text{A.4b})$$

$$V_{nn}^{r(r+1)} = V_{nn}^{r(r+2)} = -\frac{1}{2} ((-1)^n + V_{nn}^{rr}) \quad n \neq 0, \quad (\text{A.4c})$$

$$V_{0n}^{rs} = \sqrt{2n} (N_{0n}^{r,s} + N_{0n}^{r,-s}) \quad n \neq 0, \quad (\text{A.4d})$$

$$V_{00}^{rr} = -\ln(27/16). \quad (\text{A.4e})$$

The analogous expressions for the ghost Neumann coefficients are

$$\begin{aligned} \mathcal{N}_{nm}^{r,\pm r} &= \frac{1}{3(n \pm m)} \begin{cases} (-1)^{n+1} (B_n A_m \pm A_n B_m) & m+n \in 2\mathbb{Z}, m \neq n \\ 0 & m+n \in 2\mathbb{Z}+1 \end{cases}, \\ \mathcal{N}_{nm}^{r,\pm(r+1)} &= \frac{1}{6(n \pm m)} \begin{cases} (-1)^n (B_n A_m \pm A_n B_m) & m+n \in 2\mathbb{Z}, m \neq n \\ -\sqrt{3} (B_n A_m \mp A_n B_m) & m+n \in 2\mathbb{Z}+1 \end{cases}, \\ \mathcal{N}_{nm}^{r,\pm(r-1)} &= \frac{1}{6(n \mp m)} \begin{cases} (-1)^n (B_n A_m \mp A_n B_m) & m+n \in 2\mathbb{Z}, m \neq n \\ \sqrt{3} (B_n A_m \pm A_n B_m) & m+n \in 2\mathbb{Z}+1 \end{cases}. \end{aligned} \quad (\text{A.5})$$

Observe that the ghost formulae (A.5) are related to matter ones (A.4a) by $A_m \rightarrow -B_m$, $B_m \rightarrow A_m$. The ghost Neumann coefficients are expressed via \mathcal{N}_{nm}^{rs} as

$$X_{nm}^{rs} = m (\mathcal{N}_{nm}^{r,s} + \mathcal{N}_{nm}^{r,-s}) \quad m \neq n, m > 0, \quad (\text{A.6a})$$

$$X_{nn}^{rr} = -\frac{2}{3} (-1)^n A_n B_n + \frac{1}{3} \left(2 \sum_{k=0}^n (-1)^{n-k} A_k^2 - (-1)^n - A_n^2 \right) \quad n \neq 0, \quad (\text{A.6b})$$

$$X_{nn}^{rs} = X_{nn}^{rs} = -\frac{1}{2} ((-1)^n + X_{nn}^{rr}), \quad r \neq s, n \neq 0, \quad (\text{A.6c})$$

The exponential in the vertex $\langle V_3 |$ does not contain X_{n0} , so we have not included an expression for this coefficient; alternatively, we can simply define this coefficient to

vanish and include c_0 in the exponential in $\langle V_3 \rangle$.

Now we describe some algebraic properties satisfied by V^{rs} and X^{rs} . Define $M_{mn}^{rs} = CV^{rs}$, $\mathcal{M}_{mn}^{rs} = \sqrt{\frac{n}{m}}CX_{mn}^{rs}$. The matrices M and \mathcal{M} satisfy symmetry and cyclicity properties

$$M^{r+1s+1} = M^{rs}, \quad \mathcal{M}^{r+1s+1} = \mathcal{M}^{rs}, \quad (\text{A.7a})$$

$$(M^{rs})^T = M^{rs}, \quad (\mathcal{M}^{rs})^T = \mathcal{M}^{rs}, \quad (\text{A.7b})$$

$$CM^{rs}C = M^{sr}, \quad C\mathcal{M}^{rs}C = \mathcal{M}^{sr}. \quad (\text{A.7c})$$

This reduces the set of independent matter Neumann matrices to M^{11} , M^{12} , M^{21} and similarly for ghosts. These matrices commute and in addition satisfy

$$M^{11} + M^{12} + M^{21} = -1, \quad \mathcal{M}^{11} + \mathcal{M}^{12} + \mathcal{M}^{21} = -1, \quad (\text{A.8a})$$

$$M^{12}M^{21} = M^{11}(M^{11} + 1), \quad \mathcal{M}^{12}\mathcal{M}^{21} = \mathcal{M}^{11}(\mathcal{M}^{11} - 1). \quad (\text{A.8b})$$

These relations imply that there is only one independent Neumann matrix.

Appendix B

Perturbative computation of effective tachyon action

We have used two methods to compute the coefficients in the effective action $S[\phi(t)]$. The first method, as described in the main text, consists of solving the equations of motion for each field perturbatively in ϕ . The second method consists of computing the effective action by summing diagrams which can be computed using the method of level truncation on oscillators. This approach is summarized briefly here, and applied to the computation of the term of order ϕ^4 in the effective action.

The classical effective action for the tachyon can be perturbatively computed as a sum over all tree-level connected Feynman diagrams. A method for computing such diagrams to high levels of truncation in string field theory was presented in [20], and used in [43] to compute the effective action for the massless vector field. A review of this approach is given in [134]. Using this method, the contribution of a given Feynman diagram with n vertices, $n - 1$ propagators and $n + 2$ external fields is given by an integral of the form

$$\delta S = \int \prod_{i=1}^n dk_i (2\pi)^d \delta(\sum k_i) \int \prod_{j=1}^{n-3} \frac{d\sigma_j}{\sigma_j^2} \text{Det} \left(\frac{1 - \mathcal{X}\mathcal{P}}{(1 - \mathcal{V}\mathcal{P})^{13}} \right) \exp(k_i \mathcal{Q}^{ij} k_j) \phi(k_1) \dots \phi(k_j). \quad (\text{B.1})$$

In this formula \mathcal{V} and \mathcal{X} are $n \times n$ block matrices whose blocks are matter and

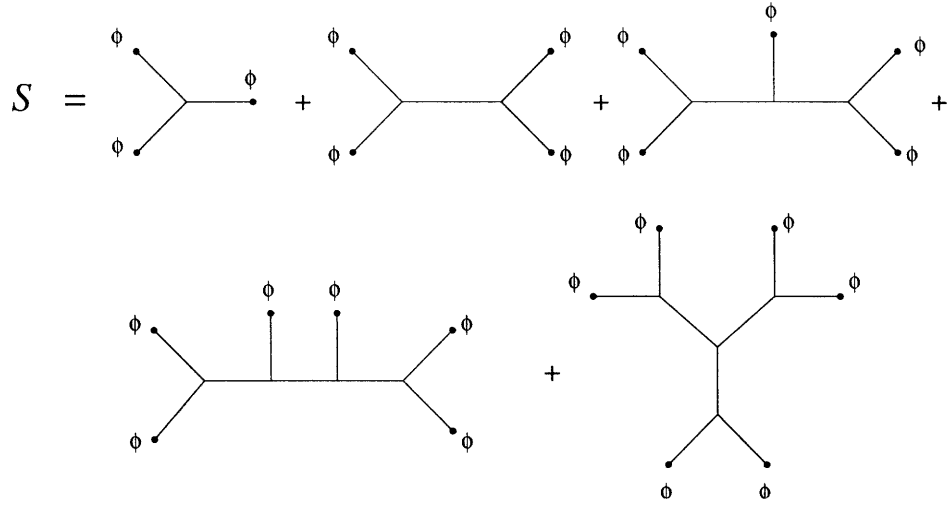


Figure B-1: The first few diagrams contributing to the effective action

ghost Neumann coefficients V^{rs} and X^{rs} of the cubic string field theory vertex. More precisely

$$\mathcal{V} = \begin{pmatrix} V^{r_1 s_1} & 0 & \dots & 0 \\ 0 & V^{r_2 s_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & V^{r_n s_n} \end{pmatrix}, \quad \mathcal{X} = \begin{pmatrix} X^{r_1 s_1} & 0 & \dots & 0 \\ 0 & X^{r_2 s_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & X^{r_n s_n} \end{pmatrix}. \quad (\text{B.2})$$

When using level truncation V^{rs} and X^{rs} become $3L \times 3L$ matrices of real numbers. The matrix \mathcal{P} encodes information about propagators, external states and the graph structure of the diagram. We define it as

$$\mathcal{P} = K^T \hat{\mathcal{P}} K. \quad (\text{B.3})$$

Here $\hat{\mathcal{P}}$ is a block-diagonal matrix of the form

$$\hat{\mathcal{P}} = \begin{pmatrix} P(\sigma_1) & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & P(\sigma_2) & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & P(\sigma_{n-1}) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}. \quad (\text{B.4})$$

The diagonal blocks $P(\sigma_i)$ correspond to propagators. In the level truncation scheme the block $P(\sigma)$ of $\hat{\mathcal{P}}$ is the $2L \times 2L$ matrix

$$P(\sigma) = \begin{pmatrix} 0 & P_{12}(\sigma) \\ P_{21}(\sigma) & 0 \end{pmatrix} \quad (\text{B.5})$$

where

$$P_{12}(\sigma) = P_{21}(\sigma) = \begin{pmatrix} \sigma & 0 & \dots & 0 \\ 0 & \sigma^2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sigma^L \end{pmatrix}. \quad (\text{B.6})$$

The last $n + 2$ rows and columns of $\hat{\mathcal{P}}$ are filled with zeroes which correspond to external tachyon states. The matrix K is the block permutation matrix that encodes

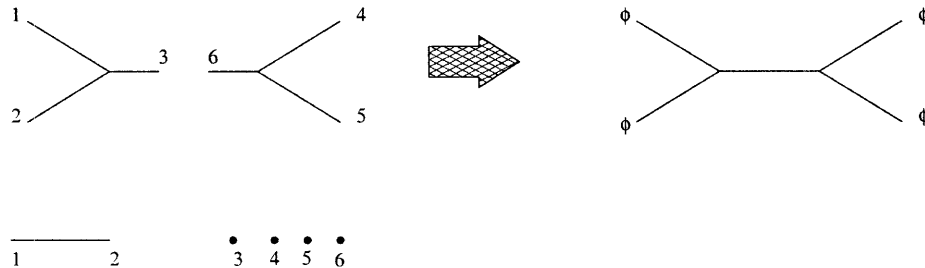


Figure B-2: To construct the 4 point diagram we label consecutively the edges of vertices on one hand and propagators and external states on the other. The matrix K corresponds to a permutation which glues them into one diagram.

information on the graph structure of the diagram. The corresponding permutation κ connects the external states and propagators to vertices as illustrated for the 4-point diagram in Figure B-2. The vertices' edges which are labeled 1 through 6 are connected by permutation to the propagator edges labeled 1 and 2 and the external points labeled 3, 4, 5 and 6. As we can see a suitable choice of a permutation is

$$\kappa : (1 \ 2 \ 3 \ 4 \ 5 \ 6) \rightarrow (3 \ 6 \ 1 \ 2 \ 4 \ 5), \quad (\text{B.7})$$

which corresponds to

$$K = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \quad (\text{B.8})$$

For example, multiplying matrices for the 4-point amplitude we find

$$\mathcal{VP} = (\tilde{V}^{11})^2, \quad \mathcal{XP} = (\tilde{X}^{11})^2, \quad (\text{B.9})$$

where

$$\tilde{V}_{mn}^{11} = \sigma^{\frac{m+n}{2}} V_{mn}^{11}, \quad \tilde{X}_{mn}^{11} = \sigma^{\frac{m+n}{2}} X_{mn}^{11}. \quad (\text{B.10})$$

The contribution from the Feynman diagram with 4 external tachyons is given by [43]

$$\frac{e^{-3V_{00}^{11}} g^2}{2} \int \prod_{i=1}^4 (2\pi dw_i) \phi(w_i) \delta(\sum w_i) \int \frac{d\sigma}{\sigma^2} \text{Det} \left(\frac{1 - (\tilde{X}^{11})^2}{[1 - (\tilde{V}^{11})^2]^{13}} \right) \sigma^{-\frac{1}{2}[(w_1+w_2)^2 + (w_3+w_4)^2]} \exp(-w_i \mathcal{Q}^{ij} w_j), \quad (\text{B.11})$$

with \mathcal{Q}^{ij} defined as

$$\mathcal{Q}^{ij} = \mathcal{U}_0^{i3} \frac{1}{1 - (\tilde{V}^{11})^2} \tilde{V}^{11} \mathcal{U}_0^{3j} + \mathcal{U}_{00}^{ij} \quad i, j = 1, 2 \text{ or } i, j = 3, 4, \quad (\text{B.12})$$

$$\mathcal{Q}^{ij} = -\mathcal{U}_0^{i3} \frac{1}{1 - (\tilde{V}^{11})^2} C \mathcal{U}_0^{3j} \quad (i = 1, 2 \text{ and } j = 3, 4) \text{ or } (i = 3, 4 \text{ and } j = 1, 2) \quad (\text{B.13})$$

where \mathcal{U}^{ij} is given by

$$\mathcal{U}^{ij} = \begin{pmatrix} V_{00}^{ij} - V_{00}^{i3} - V_{00}^{3j} + V_{00}^{33} & \tilde{V}_{0n}^{ij} - \tilde{V}_{0n}^{3j} \\ \tilde{V}_{m0}^{ij} - \tilde{V}_{m0}^{i3} & \tilde{V}_{mn}^{ij} \end{pmatrix}, \quad (\text{B.14})$$

and $C_{mn} = \delta_{mn}(-1)^n$. Considering only the contribution coming from level 2 fields, we have to consider only these Neumann coefficients whose powers and products sum up to a total oscillator level of 2, i.e. V_{01}, V_{11}, V_{02} and X_{11} [20]¹. Doing so equation (B.11) simplifies a lot and the integral over the modular parameter reduces to

$$\int d\sigma \sigma^{-\frac{1}{2}} [(w_1 + w_2)^2 + (w_3 + w_4)^2]. \quad (\text{B.15})$$

Performing this integral it is easy to get the same result as in formula (3.9).

¹If we want to calculate the quartic term in the effective action we have to subtract the contribution from the tachyon in the propagator.

Appendix C

Construction of $A(w_1, w_2, w_3)$ in the continuous case

As we have discussed in section 3.4, in order to construct the field redefinition from BSFT to CSFT that preserves general solutions to the equations of motion we need a continuous function $A(w_1, w_2, w_3)$, defined on the plane $-w_1 + w_2 + w_3 = 0$ and satisfying

$$A_2(w_1, w_2, w_3) + A_2(w_2, w_3, w_1) + A_2(w_3, w_1, w_2) = 1, \quad (\text{C.1})$$

and

$$A_2(w_1, w_2, w_3)|_{w_1^2=-1} = 0. \quad (\text{C.2})$$

Figure C-1 illustrates the construction of the desired function.

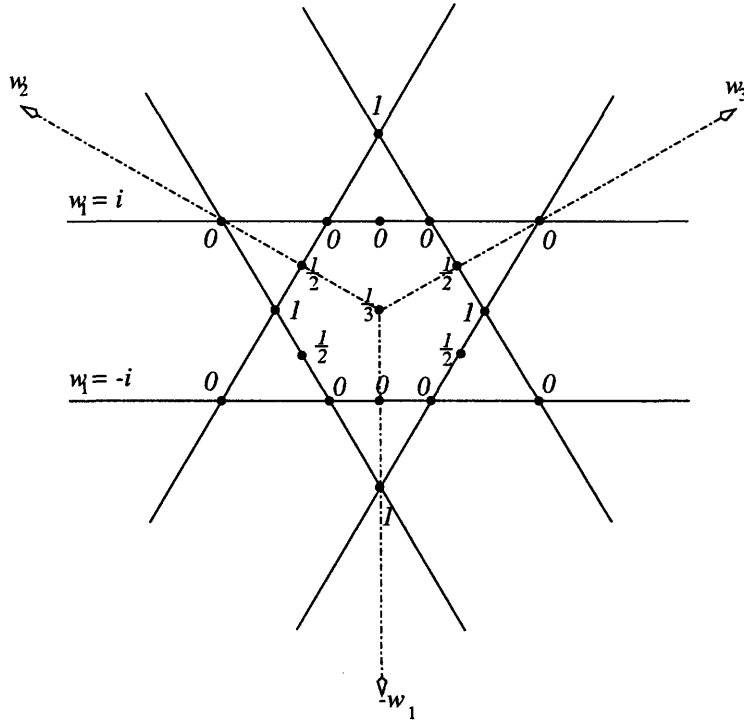


Figure C-1: Construction of a continuous $A(w_1, w_2, w_3)$: The figure shows the plane $-w_1 + w_2 + w_3 = 0$ coincident with the plane of the paper. Dashed lines are the coordinate axes $-w_1, w_2, w_3$, going at equal angles out of the plane of the paper. The two solid horizontal lines are intersections of the plane $-w_1 + w_2 + w_3 = 0$ with the planes $w_1 = \pm i$. According to (C.2) the function A is zero along these lines. Clearly, in this projection the cyclic shift of momenta w_i corresponds to a 60 degree rotation. Thus, the condition (C.1) implies that the sum of the values of A over the vertices of any equilateral triangle centered at the origin is one. Together with the reflection symmetry $w_2 \leftrightarrow w_3$ this allows us to fix the value of A at several discrete points. The values are shown on the figure. The slanted solid lines show the locus of the vertices of equilateral triangles with one vertex fixed on the lines $w_1^2 = \pm i$. The assignment of a value for A on one of the slanted lines defines the values on the other line, related by 60° rotation. These assignments can be made continuously along the lines while taking the values 0, 1, and $\frac{1}{2}$ at the symmetrically positioned points. One can then continuously extend A into the rest of the plane, while maintaining (C.1) by interpolating between the values of A at the boundaries.

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