Kazhdan–Lusztig Polynomials and Cells for Affine Weyl Groups and Unequal Parameters

by

Kirsten Bremke

Diplom, RWTH Aachen (1992)

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Mathematics

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 1996

Author ....
Department of Mathematics
February 6, 1996

Certified by ....
George Lusztig
Professor of Mathematics
Thesis Supervisor

Accepted by ....
David A. Vogan
Chairman, Departmental Committee on Graduate Students

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Abstract

Let W_a be an affine Weyl group with a set S_a of simple reflections and parameter set $\{c_s \mid s \in S_a\}$. We study Kazhdan–Lusztig polynomials and cells for the cases that not all parameters are equal.

Denote by $\mathcal{H}$ the generic Hecke algebra corresponding to (W_a, S_a) and $\{c_s \mid s \in S_a\}$. We show the existence of a canonical basis for a certain $\mathcal{H}$-module $\mathcal{M}^0$. The coefficients of the basis elements are generically inverses of the Kazhdan–Lusztig polynomials. We establish a formula for Kazhdan–Lusztig polynomials in terms of certain alcove polynomials. We also obtain a formula involving an analogue of Kostant's partition function.

We explicitly describe the lowest generalized two–sided cell. We find reduced expressions for its elements, provide a geometric interpretation of this cell, and we give a description in terms of a numerical function $\boldsymbol{a}$ on W_a. We also prove that the lowest generalized two–sided cell consists of at most $|W_0|$ generalized left cells where W_0 denotes the finite Weyl group corresponding to W_a. For parameters coming from graph automorphisms, we show that this bound is exact. For these parameters we also characterize all generalized left cells.

Thesis Supervisor: George Lusztig
Title: Professor of Mathematics

To my parents

Acknowledgements

First of all, I wish to thank Professor George Lusztig who guided me through my Ph.D. studies. His patience, generosity, and invaluable insights made this thesis possible. I feel very fortunate to have been his student.

I am very grateful to Professor David Vogan for his comments and suggestions, which led to several improvements of this thesis. I also want to thank Professor Richard Stanley for his kind interest and support.

I thank C. Kenneth Fan for many stimulating discussions and all his advice. I also thank Meinolf Geck without whom I would not be here and on whose continuing support I could always count.

Many more persons contributed in various ways to the success of this thesis. A few of them are my colleagues Ana Canas da Silva, Sang Chin, Mathieu Gagné, Mark Joshi, Jozsef Losonczy, Diko Mihov, and Richard Stone.

Finally, I want to thank my parents for their love, their understanding, for always being there for me, and for taking care of so many things.

Financial support was provided by a scholarship from the Studienstiftung des deutschen Volkes, a teaching assistantship from the Mathematics Department, and a research assistantship from Professor George Lusztig, which I gratefully acknowledge.

Contents

INTRODUCTION

The concept of cells for an arbitrary Coxeter system (W, S) was introduced by Kazhdan and Lusztig in [10]. They defined left, right, and two-sided cells. The definition involves a canonical basis of the generic Hecke algebra $\mathcal{H}$ of W. The coefficients of the canonical basis elements with respect to a standard basis of $\mathcal{H}$ are the Kazhdan–Lusztig polynomials.

Cells in Coxeter groups have been intensively studied. They are important because they give rise to representations of $\mathcal{H}$. For example, if W is the finite Weyl group of type A_r, $r \in \mathbb{N}$, the representations attached to the left cells of W are all the irreducible representations of $\mathcal{H}$, and similarly for right cells.

In [16], Lusztig extends the concept of cells to Coxeter systems (W, S) equipped with a parameter set $\{c_s \in \mathbb{N} \mid s \in S\}$ such that $c_s = c_{s'}$ if s and s' are conjugate. We refer to these cells as generalized left, right and two-sided cells. If all parameters c_s are equal, we get the same cells as before.

Generalized cells give rise to representations of the generic Hecke algebra corresponding to (W, S) and parameter set $\{c_s \mid s \in S\}$. If W is a finite resp. affine Weyl group the representation theory of corresponding Hecke algebras is very relevant for the representation theory of reductive groups over finite resp. p-adic fields.

In this thesis, we are mostly concerned with Hecke algebras corresponding to affine Weyl groups. Let W_a be the affine Weyl group with a set S_a of simple reflections and parameter set $\{c_s \mid s \in S_a\}$. Denote by $\mathcal{H}$ the corresponding Hecke algebra over the ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ where v is an indeterminate. Let W_0 be the finite Weyl group corresponding to W_a.

For equal parameters, the cells in W_a have been explicitly described for type $\tilde{A}_r$, $r \in \mathbb{N}$ (see [27], [17]), rank 2, 3 (see [18], [2], [6]), and types $\tilde{B}_4$, $\tilde{C}_4$, $\tilde{D}_4$ (see [29], [30], [5]), and a lot of important results are obtained in [18] – [21]. In particular, Lusztig shows that the two-sided cells of W_a are in bijective correspondence with the unipotent conjugacy classes in a simple algebraic group over $\mathbb{C}$ of type dual to that of W_0.

Several problems regarding Kazhdan–Lusztig polynomials and cells arise for arbitrary parameters but have only been solved for equal parameters. In this thesis, we solve some of these problems for unequal parameters.

The following is an outline of the contents of this thesis.

Chapter 1 contains background material on affine Weyl groups and affine Hecke algebras. In particular, the affine Weyl group W_a is realized as a group of affine motions of a Euclidean space V as well as in terms of certain alcoves in V. Let X be this set of alcoves.

In Chapter 2, we consider a certain $\mathcal{H}$-module $\mathcal{M}^0$. Generalizing [14], we prove the existence of an $\mathcal{A}$-basis $\{D_C \mid C \in X\}$ whose characterization is similar to the characterization of the canonical basis for $\mathcal{H}$. We show that the coefficients of the elements D_C, $C \in X$, expressed with respect to X are generically inverses of the Kazhdan–Lusztig polynomials and that they can be calculated by a finite induction. We obtain patterns which generalize Jantzen's generic decomposition patterns for Weyl modules of a simply connected almost simple algebraic group over an algebraically closed field of prime characteristic p (for p large).

In Chapter 3, we prove formulas involving Kazhdan–Lusztig polynomials, their inverses, and alcove polynomials. (Alcove polynomials generically equal Kazhdan–Lusztig polynomials.) This generalizes work by Kato and Andersen ([9] and [1]). The main statement is a formula which expresses Kazhdan–Lusztig polynomials for dominant elements in terms of alcove polynomials. We also find a way to express certain Kazhdan–Lusztig polynomials in terms of an analogue of Kostant's partition function.

Chapter 4 deals with generalized cells. We first consider the lowest generalized two-sided cell. This cell contains nearly all elements of W_a. We give different descriptions, and show that this cell consists of at most $|W_0|$ generalized left cells. (References for the equal parameter case are given in Chapter 4.) We then study the case that the parameters c_s, $s \in S_a$, come from a graph automorphism. In this situation, the coefficients of the Kazhdan–Lusztig polynomials and of the structure constants of the canonical basis of $\mathcal{H}$ can be interpreted in terms of intersection cohomology sheaves. We derive a characterization of all generalized left cells. This characterization implies that there are only finitely many generalized left cells. We also show that for parameters coming from a graph automorphism, the lowest generalized two-sided cell consists of exactly $|W_0|$ generalized left cells. The results in this chapter partially answer questions raised in [31].

1. Preliminaries

In this chapter, we collect some basic material about affine Weyl groups and affine Hecke algebras which will be needed later on. The exposition follows [9], [14] and [31]. For more details and proofs, we refer to these publications.

1.1. **The affine Weyl group.** Let V be a Euclidean space of finite dimension $r \geq 1$. Let $\Phi \subset V$ be an irreducible root system of rank r and $\check{\Phi} \subset V^*$ the dual root system. We denote the coroot corresponding to $\alpha \in \Phi$ by $\check{\alpha}$, and we write $\langle x, y \rangle$ for the value of $y \in V^*$ at $x \in V$.

Let Q be the root lattice and P the weight lattice. The Weyl group W_0 of Φ acts on Q and P (on the left), so we can form the semidirect products

$$W_a = W_0 \ltimes Q$$

and

$$\widetilde{W}_a = W_0 \ltimes P.$$

The group W_a is the affine Weyl group of type $\check{\Phi}$, and $\widetilde{W}_a$ is the extended affine Weyl group associated to a simply connected, simple algebraic group G (over $\mathbb{C}$) of root system Φ. When $\lambda \in P$ is regarded as an element of $\widetilde{W}_a$, we will also write p_λ instead of λ. The reflection in W_0 along the hyperplane orthogonal to $\alpha \in \Phi$ will be denoted by s_α.

Geometrically, W_a can be described as follows. (We will not distinguish between V and the underlying affine space.) Fix a set of positive roots $\Phi^+ \subset \Phi$, and let $\Pi \subseteq \Phi^+$ be the set of simple roots. For $\alpha \in \Phi^+$ and $n \in \mathbb{Z}$, we define a hyperplane

$$H_{\alpha,n} = \{x \in V \mid \langle x, \check{\alpha} \rangle = n\}$$

and write $\sigma_{\alpha,n} = \sigma_{H_{\alpha,n}}$ for the reflection along $H_{\alpha,n}$. Let $\alpha_0 \in \Phi^+$ be such that $\check{\alpha}_0$ is the highest coroot in $\check{\Phi}$. Mapping s_α, $\alpha \in \Pi$, to $\sigma_{\alpha,0}$ and $s_0 \stackrel{\text{def}}{=} p_{\alpha_0} s_{\alpha_0}$ to $\sigma_{\alpha_0,1}$ establishes an isomorphism from W_a to the group Ω generated by $\sigma_{\alpha,0}$, $\alpha \in \Pi$, and $\sigma_{\alpha_0,1}$.

Denote the set of simple reflections in W_0 by S_0. The group W_a is a Coxeter group with generating set $S_a = S_0 \cup \{s_0\}$.

The extended affine Weyl group $\widetilde{W}_a$ can be written as the semidirect product $N \ltimes W_a$ where N is the normalizer of S_a in $\widetilde{W}_a$.

We also need the following realization of W_a (cf. [14]). Let

$$\mathcal{F} = \{H_{\alpha,n} \mid \alpha \in \Phi^+,\, n \in \mathbb{Z}\},$$

and let X be the set of connected components of $V - \bigcup_{H \in \mathcal{F}} H$. The elements of X are called *alcoves*. The group Ω acts on the set of faces of alcoves, and we denote the set of Ω-orbits by S'_a. If f is a face contained in the orbit $t \in S'_a$, we say f *is of type* t. For $A \in X$ and $t \in S'_a$, there is a unique alcove $tA \in X$, $tA \neq A$, such that tA shares with A its face of type t. The involutions $\sigma_t : A \mapsto tA$ on X for $t \in S'_a$ generate a group W'_a. There is an isomorphism from W_a to W'_a, which can be described as follows. Let

$$A^+ = \{x \in V \mid \langle x, \check{\alpha} \rangle > 0 \text{ for all } \alpha \in \Pi,\, \langle x, \check{\alpha}_0 \rangle < 1\}.$$

For $s = s_\alpha$, $\alpha \in \Pi$, resp. $s = s_0$, the hyperplane $H_{\alpha,0}$ resp. $H_{\alpha_0,1}$ contains a unique face of A^+, whose orbit in S'_a we denote by t_s. The isomorphism sends $s \in S_a$ to the involution σ_{t_s}.

Identifying W_a with Ω yields an action of W_a on V and thereby on X, which we consider as a right action. We also identify W_a with W'_a from now on and write the action of W_a on X resulting from this identification on the left. The two actions of W_a on X can be seen to commute and to be simply transitive.

We fix parameters $c_s \in \mathbb{N}$ for $s \in S_a$ such that $c_s = c_t$ whenever s and t are conjugate in W_a.

Lemma 1.1.1. *Let H be a hyperplane in $\mathcal{F}$, and suppose H supports faces of types $s, t \in S_a$. Then s and t are conjugate in W_a.*

Proof. The assumptions imply that there are alcoves A, $A' \in X$ such that $sA = A\sigma_H$ and $tA' = A'\sigma_H$. Because of the transitivity of the left action of W_a on X, we can find an element $w \in W_a$ such that $A' = wA$. We have

$$twA = tA' = A'\sigma_H = wA\sigma_H = wsA,$$

and hence $tw = ws$, i.e. s and t are conjugate via w.

As a consequence of this lemma, we can associate a parameter $c_H \in \mathbb{N}$ to $H \in \mathcal{F}$ where $c_H = c_s$ if H supports a face of type s.

For a 0-dimensional facet λ of an alcove, we define

$$m(\lambda) = \sum_{H \in \mathcal{F}, H \ni \lambda} c_H,$$

and we call λ a *special point* if $m(\lambda)$ is maximal. Note that, in general, the set of 0-dimensional facets of alcoves contains P as a proper subset.

Let $T \subset V$ be the set of all special points. If all parameters are equal, the notion of special points coincides with the notion in [14], so $T = P$ and $m(\lambda) = |\Phi^+|$ for $\lambda \in T$. The next lemma will enable us to determine T in all cases.

Let Γ be the Coxeter graph of (W_a, S_a), and identify the set of vertices of Γ with S_a. If Γ is of type $\tilde{A}_1$ or $\tilde{C}_r$, $r \geq 2$, there is a unique nontrivial automorphism $\tilde{}$ on Γ.

Lemma 1.1.2. *Let H, H' be parallel hyperplanes in $\mathcal{F}$ and let $s, s' \in S_a$. If H supports a face of type s and H' supports a face of type s', we either have*

(i) *Γ is of type $\tilde{A}_1$ or $\tilde{C}_r$, $r \geq 2$, and $\{s, s'\} = \{s_0, \tilde{s}_0\}$ or*

(ii) *s and s' are conjugate in W_a.*

Proof. Suppose (i) does not hold (and $s \neq s'$). W.l.o.g. we can assume that there exists an element $t \in S_a$ such that $(st)^3 = \mathrm{id}$.

Let A be an alcove having its face of type s on H, and let H'' be the hyperplane containing the face of A of type t. Then H'' intersects H and hence H' at an angle $\pm\frac{\pi}{3}$, which implies $H''\sigma_{H'\sigma_{H''}} = H'$. Therefore, H' supports a face of type t, and $s' \sim t$ by Lemma 1.1.1. Since $(sts)s(sts) = t$, we have $s \sim t$. Thus $s' \sim s$.

Throughout this paper, we refer to the situation in which the Coxeter graph Γ is of type $\tilde{A}_1$ or $\tilde{C}_r$, $r \geq 2$, and $c_{s_0} \neq c_{\tilde{s}_0}$, as **Case 1** and all other situations as **Case 2**.

In Case 1, let λ_r be the fundamental weight such that P is generated by Q and λ_r.

Claim 1.1.3. *We have $T = Q$ or $T = \lambda_r + Q$ in Case 1 and $T = P$ in Case 2.*

We first notice that if λ is a special point and $\mu \in Q$, the point $\lambda + \mu = \lambda p_\mu$ is a special point as well.

Next, since according to the definition of the weight lattice, P consists of all points $\lambda \in V$ that lie in the intersection of $|\Phi^+|$ hyperplanes in $\mathcal{F}$, we have $T \subseteq P$.

Now suppose we are in Case 1 and Γ is of type $\tilde{C}_r$. (For the following data about roots and weights see e.g. [4], Ch. VI.) Take an orthonormal basis $\{e_1, \dots, e_r\}$ of V and write $\alpha_i = e_i - e_{i+1}$, $1 \leq i \leq r-1$, and $\alpha_r = e_r$ for the simple roots in Π. Then $\lambda_r = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + r\alpha_r) = \frac{1}{2}(e_1 + \cdots + e_r)$ and $\alpha_0 = \alpha_1 + \cdots + \alpha_r = e_1$. Hence $\langle \lambda_r, \check{\alpha}_0 \rangle = 1$, i.e. $\lambda_r \in H_{\alpha_0,1}$. More generally, we have $\lambda_r \in H_{\alpha,1}$ for all short roots $\alpha \in \Phi^+$. Since each $H_{\alpha,1}$, $\alpha \in \Phi^+$, α short, supports a face of type s_0, we conclude $T = Q$ if $c_{s_0} < c_{\tilde{s}_0}$ and $T = \lambda_r + Q$ if $c_{s_0} > c_{\tilde{s}_0}$. (The case Γ of type $\tilde{A}_1$ follows by a simpler computation.)

In Case 2, parallel hyperplanes have the same parameter, so the special points are the same as in [14].

For the remainder of this paper, we assume that in Case 1 we have $c_{s_0} < c_{\tilde{s}_0}$, so $T = Q$. (We can always make this true by labeling the simple reflections accordingly.)

We define

$$\widehat{W}_a = W_0 \ltimes T.$$

The right action of W_a on X naturally extends to a right action of $\widehat{W}_a$ on X.

We need some more notation. If $w \in W_a$ has a reduced expression $w = s_1 s_2 \cdots s_n$, $s_i \in S_a$ for $1 \leq i \leq n$, we set

$$m(w) = \sum_{i=1}^{n} c_{s_i},$$

which is independent of the chosen reduced expression. (This follows from the fact that any two reduced expressions for w can be transformed into each other by a sequence of braid relations. See e.g. [4], Ch. IV.) We remark that the length function l on W_a and the function m on W_a extend to functions on $\widetilde{W}_a$ via $l(nw) = l(w)$ and $m(nw) = m(w)$ for $n \in N$ and $w \in W_a$.

Let W_λ, $\lambda \in T$, be the stabilizer of the set of alcoves containing λ in their closure with respect to the left action of W_a. (The definition of W_0 is consistent.) It can be shown that this group is a maximal parabolic subgroup of W_a and that $m(w_\lambda) = m(\lambda)$ for the longest element $w_\lambda \in W_\lambda$.

A hyperplane $H = H_{\alpha,n} \in \mathcal{F}$ divides $V - H$ into the two parts

$$V_H^+ = \{x \in V \mid \langle x, \check{\alpha}\rangle > n\}$$

and

$$V_H^- = \{x \in V \mid \langle x, \check{\alpha}\rangle < n\}.$$

For $\lambda \in T$, a *quarter with vertex* λ is a connected component of

$$V - \bigcup_{H \in \mathcal{F}, H \ni \lambda} H.$$

Hyperplanes which are adjacent to a quarter $\mathcal{C}$ are called *walls of* $\mathcal{C}$. The quarter

$$\bigcap_{H \in \mathcal{F}, H \ni \lambda} V_H^+$$

will be denoted by $\mathcal{C}_\lambda^+$, and A_λ^+ is the unique alcove in $\mathcal{C}_\lambda^+$ such that λ lies in the closure $\overline{A_\lambda^+}$. Set $A_\lambda^- = w_\lambda A_\lambda^+$, and let $\mathcal{C}_\lambda^-$ be the quarter with vertex λ containing A_λ^-. For $\lambda = 0$, we also write $\mathcal{C}^+$, A^+ etc. Let $v_\lambda \in W_a$ be such that $A_\lambda^+ v_\lambda = A_\lambda^-$.

Let $\mathcal{F}^*$ be the set of hyperplanes $H \in \mathcal{F}$ such that H is a wall of $\mathcal{C}_\lambda^+$ for some $\lambda \in T$. The connected components of $V - \bigcup_{H \in \mathcal{F}^*} H$ will be called *boxes*. We denote by Π_λ the box containing A_λ^+ for $\lambda \in T$.

To an alcove $A \in X$, we associate the subset $\mathcal{L}(A)$ of S_a containing all $s \in S_a$ such that, if $H \in \mathcal{F}$ is the hyperplane supporting the face of type s of A, then $A \subset V_H^+$, $sA \subset V_H^-$.

We define two integers $d(A, B)$ and $c(A, B)$ for A, $B \in X$. Consider the set of hyperplanes $H \in \mathcal{F}$ separating A from B. For each such hyperplane, we set $\varepsilon_H = 1$ if $A \subset V_H^-$, $B \subset V_H^+$ and $\varepsilon_H = -1$ if $A \subset V_H^+$, $B \subset V_H^-$. The integer $d(A, B)$ is the sum of all ε_H, and $c(A, B)$ is the sum of all $\varepsilon_H c_H$.

A *length function on* X is a function $\delta : X \to \mathbb{Z}$ such that

$$d(A, B) = \delta(B) - \delta(A)$$

for all A, $B \in X$. Similarly, we call a function $\gamma : X \to \mathbb{Z}$ a *weighted length function on* X if it satisfies

$$c(A,B) = \gamma(B) - \gamma(A)$$

for all A, $B \in X$.

We have the following partial order on X (cf. [14]). For A, $B \in X$, we say $A \leq B$ if there exists a sequence $A = A_0, A_1, \ldots, A_n = B$ of alcoves such that $d(A_{i-1}, A_i) = 1$ and $A_i = A_{i-1}\sigma_{H_i}$ for some $H_i \in \mathcal{F}$ for all $1 \leq i \leq n$.

1.2. **The affine Hecke algebra.** Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials in an indeterminate v, and set $\mathcal{A}^+ = \mathbb{Z}[v]$. The generic Hecke algebra $\widehat{\mathcal{H}}$ of $\widehat{W}_a$ with parameters c_s, $s \in S_a$, can be defined as a free $\mathcal{A}$-module with basis $\{T_w \mid w \in \widehat{W}_a\}$ and relations

$$(T_s - v^{2c_s})(T_s + 1) = 0$$

for $s \in S_a$ and

$$T_w T_{w'} = T_{ww'}$$

for w, $w' \in \widehat{W}_a$ with $l(w) + l(w') = l(ww')$. We denote by $\mathcal{H}$ the subalgebra of $\widehat{\mathcal{H}}$ generated by T_s for $s \in S_a$. Note that in Case 1, the algebras $\mathcal{H}$ and $\widehat{\mathcal{H}}$ coincide.

We have

$$T_s^{-1} = v^{-2c_s} T_s + v^{-2c_s} - 1$$

for $s \in S_a$ and

$$T_n^{-1} = T_{n^{-1}}$$

for $n \in N$, which implies the invertibility of all elements T_w, $w \in W_a$ resp. $w \in \widehat{W}_a$, in $\mathcal{H}$ resp. $\widehat{\mathcal{H}}$.

Let $^{-} \colon \widehat{\mathcal{H}} \to \widehat{\mathcal{H}}$ be the unique involutive automorphism sending v to v^{-1} and T_w to $T_{w^{-1}}^{-1}$, $w \in \widehat{W}_a$.

For $w \in W_a$, there exists a unique element $C_w^* \in \mathcal{H}$ such that $\overline{C_w^*} = v^{-2m(w)} C_w^*$ and

$$C_w^* = \sum_{y \leq w} P_{y,w} T_y$$

where the degree of $P_{y,w} \in \mathcal{A}^+$ is smaller than $m(w) - m(y)$ for $y < w$ and $P_{w,w} = 1$. (This is proven in [16] for $w \in W_a$. The extension to $\widehat{W}_a$ is straightforward.)

The polynomials $P_{y,w}$ for y, $w \in \widehat{W}_a$ are the Kazhdan–Lusztig polynomials. It is easily seen that the elements C_w^* for $w \in \widehat{W}_a$ form an $\mathcal{A}$-basis for $\widehat{\mathcal{H}}$.

Let $\mathcal{M}$ be the free $\mathcal{A}$-module on X. There is a unique $\mathcal{H}$-module structure on $\mathcal{M}$ such that

$$T_s A = \begin{cases} sA & \text{if } s \notin \mathcal{L}(A), \\ v^{2c_s} sA + (v^{2c_s} - 1)A & \text{if } s \in \mathcal{L}(A) \end{cases}$$

for $A \in X$ and $s \in S_a$ (comp. [31]).

For $\lambda \in T$, we set

$$e_\lambda = \sum_{A \in X,\, \bar{A} \ni \lambda} A \in \mathcal{M}$$

and denote by $\mathcal{M}_\lambda$ the $\mathcal{H}$-submodule of $\mathcal{M}$ generated by e_λ. (See Remark 2.3.2 for a motivation of this definition.) The $\mathcal{H}$-submodule of $\mathcal{M}$ generated by all elements e_λ will be called $\mathcal{M}^0$.

Let $T^+ = T \cap \overline{\mathcal{C}^+}$ be the set of dominant weights in T. For $\lambda \in T$, we choose an element $\mu \in T^+$ such that $\lambda + \mu \in T^+$ and set

$$\check{T}_\lambda = T_{\lambda+\mu} T_\mu^{-1}.$$

This is a well–defined element in $\widehat{\mathcal{H}}$ (cf. [22]). For $w \in \widehat{W_a}$ of the form $w = xp_\lambda$, $x \in W_0$, $\lambda \in T$, we set

$$\check{T}_w = T_x \check{T}_\lambda.$$

Proposition 1.2.1. *(comp.* [9], *Proposition 1.10)*

(i) *The set* $\{\check{T}_w \mid w \in W_a\}$ *is an* $\mathcal{A}$*-basis for* $\mathcal{H}$*.*

(ii) *The map* $\phi : \mathcal{H} \to \mathcal{M}$ *sending* $f \in \mathcal{H}$ *to* $f(A^-) \in \mathcal{M}$ *is an* $\mathcal{H}$*-module isomorphism. (Consider* $\mathcal{H}$ *as a left* $\mathcal{H}$*-module.)*

For the proof, we refer the reader to the proof of Proposition 1.10 in *loc. cit.* The generalization of the arguments used there is straightforward.

We remark that ϕ^{-1} sends A^-w to $\check{T}_w$ for $w \in W_a$.

In *loc. cit.*, Kato introduces the *generic length function* $g : \widetilde{W_a} \to \mathbb{Z}$ as follows. Let $\mathrm{ht} : Q \to \mathbb{Z}$ be the linear function satisfying $\mathrm{ht}(\alpha) = 1$ for all $\alpha \in \Pi$. This function uniquely extends to a linear function $\mathrm{ht} : P \to \frac{1}{2}\mathbb{Z}$. The generic length function is given by

$$g(xp_\lambda) = l(x) + 2\mathrm{ht}(\lambda)$$

for $x \in W_0$, $\lambda \in P$. Kato shows that this function satisfies

$$g(w) = d(A^-, A^-w)$$

for $w \in \widetilde{W_a}$.

Similarly, let $\widetilde{\mathrm{ht}} : T \to \frac{1}{2}\mathbb{Z}$ be the linear function satisfying $\widetilde{\mathrm{ht}}(\alpha) = c_{s_\alpha}$ for $\alpha \in \Pi$. We set

$$h(xp_\lambda) = m(x) + 2\widetilde{\mathrm{ht}}(\lambda)$$

for $x \in W_0$, $\lambda \in T$. Arguments analogous to the ones in *loc. cit.* then show that

$$h(w) = c(A^-, A^-w)$$

for $w \in \widehat{W}_a$. We call the function $h : \widehat{W}_a \to \mathbb{Z}$ the *generic weighted length function.* The restriction of h to T will again be denoted by h. Note that $h : T \to \mathbb{Z}$ is linear.

For $\lambda \in T$, we set

$$\hat{T}_\lambda = v^{-h(\lambda)}\check{T}_\lambda.$$

(As shown in [22], the sets $\{T_w\hat{T}_\lambda \mid w \in W_0,\ \lambda \in T\}$ and $\{\hat{T}_\lambda T_w \mid w \in W_0,\ \lambda \in T\}$ are $\mathcal{A}$-bases of $\widehat{\mathcal{H}}$; for equal parameters, the elements $\hat{T}_\lambda$ were introduced by Bernstein.)

2. A canonical basis for $\mathcal{M}^0$

The main result of this chapter is Theorem 2.3.1, which generalizes the main theorem of [14] to unequal parameters c_s, $s \in S_a$. We follow the proof in *loc. cit.* Sections 2.1 and 2.2 deal with those parts of the proof whose generalization is not straightforward. Section 2.3 also contains the generalization of several other statements in *loc. cit.*

2.1. **Intertwining operators.** Our main goal in this section is to define an element $\theta_w \in \mathrm{End}_{\mathcal{H}}(\mathcal{M}^0)$ for $w \in W_a$. We do this in Proposition 2.1.3.

For $\lambda \in T$, we set

$$d_\lambda = T_{n(p_\lambda)^{-1}}\,\Big(\sum_{x \in W_0} T_x\Big)\,\check{T}_\lambda,$$

where $n(w)$ for $w \in \widehat{W}_a$ denotes the element in N such that $n(w)^{-1}w \in W_a$. The elements d_λ lie in $\mathcal{H}$ (cf. [9] for equal parameters; the general case is completely analogous). Also, the arguments in *loc. cit.* show that $\phi(d_\lambda) = e_\lambda$.

The $\mathcal{H}$-submodule of $\mathcal{H}$ generated by all elements d_λ will be called $\mathcal{H}^0$. We first define an element $\Theta_w \in \mathrm{End}_{\mathcal{H}}(\mathcal{H}^0)$ for $w \in W_a$. For equal parameters, this has been done in *loc. cit.*

We extend the definition of the automorphism $\tilde{}$ by saying that $\tilde{}$ is the identity on the Coxeter graph Γ (and on S_0) whenever Γ is not of type $\widetilde{A}_1$ or $\widetilde{C}_r$. For $\alpha \in \Pi$ and $s = s_\alpha$ the corresponding simple reflection, we define

$$\tilde{I}_s = T_s(1 - \hat{T}_{-2\alpha}) - \hat{T}_{-\alpha}(v^{c_s + c_{\tilde{s}}} - v^{c_s - c_{\tilde{s}}}) + 1 - v^{2c_s} \in \mathcal{H}.$$

This relates to the element I_s in *loc. cit.* as follows. Suppose $s \in S_0$ with $c_s = c_{\tilde{s}}$. Then

$$\tilde{I}_s = I_s(1 + \hat{T}_{-\alpha})$$

where

$$I_s = T_s(1 - \hat{T}_{-\alpha}) + 1 - v^{2c_s}.$$

Proposition 2.1.1. *For $\lambda \in T$ and $s \in S_0$, we have*

$$\hat{T}_\lambda \tilde{I}_s = \tilde{I}_s \hat{T}_{\lambda s}$$

Proof. Let U be the set of coroots in Case 1 and the set of coweights in Case 2. Since $(T, U, \Phi, \check{\Phi}, \Pi)$ is a root system, as defined in [22], Proposition 3.6 in *loc. cit.* implies the stated identity.

Let $\alpha \in \Pi$, $s = s_\alpha$. In *loc. cit.*, Lusztig defines an element

$$\mathcal{G}(\alpha) = \frac{(\hat{T}_\alpha v^{c_s + c_{\tilde{s}}} - 1)(\hat{T}_\alpha v^{c_s - c_{\tilde{s}}} + 1)}{\hat{T}_{2\alpha} - 1},$$

which lies in the quotient field of the $\mathcal{A}$-submodule of $\mathcal{H}$ generated by the elements $\hat{T}_\lambda$ for $\lambda \in Q$. (Note the similarity between $\mathcal{G}(\alpha)$ and the function c_α in [26], p. 98, which is used in order to construct intertwining operators between the principal series representations of $\mathcal{H}$; also, see [25], p. 51.)

In terms of $\mathcal{G}(\alpha)$, we can express $\tilde{I}_s$ as

$$\tilde{I}_s = (T_s + 1 - \mathcal{G}(\alpha))(1 - \hat{T}_{-2\alpha}).$$

We define

$$\tilde{\mathcal{G}}(\alpha) = (\hat{T}_{-\alpha} v^{c_s + c_{\tilde{s}}} - 1)(\hat{T}_{-\alpha} v^{c_s - c_{\tilde{s}}} + 1)$$

and denote by $\tilde{G}(\alpha) \in \mathrm{End}_{\mathcal{H}}(\mathcal{H})$ right multiplication with $\tilde{\mathcal{G}}(\alpha)$.

To $s \in S_0$, we associate an $\mathcal{H}$-endomorphism $\tilde{\Theta}_s$ of $\mathcal{H}$, which maps $f \in \mathcal{H}$ to

$$\tilde{\Theta}_s(f) = f \tilde{I}_s.$$

Lemma 2.1.2. *Let $\alpha \in \Pi$, $s = s_\alpha$, and $\lambda \in T$. Then*

$$\tilde{\Theta}_s((T_s + 1)\check{T}_\lambda) = -v^{h(\lambda - \lambda s)} \tilde{G}(\alpha)((T_s + 1)\check{T}_{\lambda s}).$$

Proof. Using Proposition 2.1.1, we calculate

$$\begin{aligned}
&\tilde{\Theta}_s((T_s + 1)\check{T}_\lambda) \\
&= (T_s + 1)\check{T}_\lambda \tilde{I}_s \\
&= v^{h(\lambda - \lambda s)}(T_s + 1)\tilde{I}_s \check{T}_{\lambda s} \\
&= v^{h(\lambda - \lambda s)}(T_s + 1)(v^{2c_s}(1 - \hat{T}_{-2\alpha}) - \hat{T}_{-\alpha}(v^{c_s + c_{\tilde{s}}} - v^{c_s - c_{\tilde{s}}}) + 1 - v^{2c_s})\check{T}_{\lambda s} \\
&= -v^{h(\lambda - \lambda s)}(T_s + 1)(\hat{T}_{-2\alpha} v^{2c_s} + \hat{T}_{-\alpha}(v^{c_s + c_{\tilde{s}}} - v^{c_s - c_{\tilde{s}}}) - 1)\check{T}_{\lambda s} \\
&= -v^{h(\lambda - \lambda s)}(T_s + 1)\, \tilde{\mathcal{G}}(\alpha)\check{T}_{\lambda s} \\
&= -v^{h(\lambda - \lambda s)} \tilde{G}(\alpha)((T_s + 1)\check{T}_{\lambda s}),
\end{aligned}$$

which is what we claimed.

We deduce from this lemma that

$$\tilde{\Theta}_s(d_\lambda) = -v^{h(\lambda-\lambda s)}\tilde{G}(\alpha)d_{\lambda s}.$$

Thus $\tilde{\Theta}_s$ induces an $\mathcal{H}$-endomorphism on $\mathcal{H}^0$, for which we write $\tilde{\Theta}_s$ as well. Since $\tilde{G}(\alpha)$ is an injective endomorphism on $\mathcal{H}$ (and on $\mathcal{H}^0$), we can define

$$\Theta_s = -\tilde{G}(\alpha)^{-1}\tilde{\Theta}_s \in \mathrm{End}_{\mathcal{H}}(\mathcal{H}^0).$$

If $w = s_1 \cdots s_n \in W_0$ with $s_i \in S_0$, $1 \le i \le n$, we set $\Theta_w = \Theta_{s_n} \cdots \Theta_{s_1} \in \mathrm{End}_{\mathcal{H}}(\mathcal{H}^0)$.

Now take an arbitrary element $w = xp_\lambda$, $x \in W_0$, $\lambda \in Q$, in W_a. Right multiplication with $\hat{T}_\lambda$ yields an element Θ_λ in $\mathrm{End}_{\mathcal{H}}(\mathcal{H}^0)$ (and in $\mathrm{End}_{\mathcal{H}}(\mathcal{H})$). We let $\Theta_w = \Theta_\lambda \cdot \Theta_x$ and thereby obtain an element in $\mathrm{End}_{\mathcal{H}}(\mathcal{H}^0)$, which satisfies

$$\Theta_w(d_\lambda) = v^{h(\lambda-\lambda w)}d_{\lambda w}$$

for $\lambda \in T$.

Proposition 2.1.3. *(comp.* [14]*, Proposition 2.8) For any $w \in W_a$, there is a unique $\mathcal{H}$-linear isomorphism θ_w of $\mathcal{M}^0$ such that*

$$\theta_w(e_\lambda) = v^{c(A^+_{\lambda w}, A^+_\lambda)}e_{\lambda w}$$

for any $\lambda \in T$.

Proof. We have

$$\begin{aligned}
\phi\,\Theta_w\phi^{-1}(e_\lambda) &= \phi\,\Theta_w(d_\lambda)\\
&= v^{h(\lambda-\lambda w)}\phi(d_{\lambda w})\\
&= v^{c(A^-_{\lambda w}, A^-_\lambda)}e_{\lambda w}\\
&= v^{c(A^+_{\lambda w}, A^+_\lambda)}e_{\lambda w},
\end{aligned}$$

and the map $\theta_w \overset{\mathrm{def}}{=} \phi\,\Theta_w\phi^{-1}$ has the desired properties. The uniqueness is clear.

Note the equality $\theta_w\theta_{w'} = \theta_{ww'}$ for w, $w' \in W_a$.

There are further maps introduced in [14], which we will need later on. Let $q = v^2$. Replacing q by the appropriate q^{c_s}, $s \in S_a$, and δ, a length function, by γ, a weighted length function, will prove the following statements in this section.

Lemma 2.1.4. *(comp.* [14]*, Lemma 2.10) For $\lambda \in T$, the map $\varphi_\lambda : \mathcal{M} \to \mathcal{M}$ defined by*

$$\varphi_\lambda\Big(\sum_{A\in X} c_A A\Big) = \sum_{A\in X} \bar{c}_A A v_\lambda,$$

$c_A \in \mathcal{A}$ for $A \in X$, is $\mathcal{H}$-antilinear.

Theorem 2.1.5. *(comp. [14], Theorem 2.12) Let γ be a weighted length function on X.*

(i) *There is a unique $\mathcal{H}$-antilinear map $\Phi_\gamma : \mathcal{M}^0 \to \mathcal{M}^0$ such that*

$$\Phi_\gamma(e_\lambda) = v^{-2\gamma(A_\lambda^+)} e_\lambda$$

for all $\lambda \in T$.

(ii) *The map Φ_γ is an involution.*

(iii) *If $\gamma' = \gamma + n$, $n \in \mathbb{Z}$, is another weighted length function on X we have $\Phi_{\gamma'} = v^{-2n}\Phi_\gamma$.*

Corollary 2.1.6. *(comp. [14], Corollary 2.13) Let $\lambda \in T$ and γ_λ be the weighted length function satisfying $\gamma_\lambda(A_\lambda^+) = 0$. Then $\Phi_{\gamma_\lambda}(m) = \varphi_\lambda(m)$ for all $m \in \mathcal{M}_\lambda$.*

2.2. **Some estimates.** We now generalize the degree estimates in [14], Section 4. We achieve this by means of a more detailed analysis of the arguments in *loc. cit.*

Fix a special point λ. If A is an alcove such that $\lambda \in \bar{A}$ and if $y \in W_a$ is such that $y(A_\lambda^+) \subset \mathcal{C}_\lambda^+$, we write

$$T_y(A) = \sum_{B \in X} \pi_{B,y}^A B \, (\in \mathcal{M}).$$

Proposition 2.2.1. *(comp. [14], Proposition 4.2) Let A and y be as above.*

(i) *The coefficient $\pi_{B,y}^A$ of $B \in X$ is zero unless $B \leq y(A)$, in which case $\pi_{B,y}^A$ is a polynomial in q of degree at most $\frac{1}{2}c(B, y(A_\lambda^+))$.*

(ii) *Let $c^* = \min_{s \in S_a} c_s$, and suppose $y(A_\lambda^+) \subseteq \Pi_\lambda$. If $B < y(A)$, the polynomial $\pi_{B,y}^A$ has degree at most $\frac{1}{2}(c(B, y(A_\lambda^+)) - c^*)$. If $B = y(A)$, we have $\pi_{B,y}^A = 1$.*

(iii) *If $c(B, y(A_\lambda^+))$ is even, the coefficient of $q^{\frac{1}{2}c(B,y(A_\lambda^+))}$ in $\pi_{B,y}^A$ is non-negative.*

(iv) *In part (ii), if $c(B, y(A_\lambda^+)) - c^*$ is even, the coefficient of $q^{\frac{1}{2}(c(B,y(A_\lambda^+))-c^*)}$ in $\pi_{B,y}^A$ is non-negative.*

In order to prove this proposition, we need the following result, which is Corollary 3.4 in *loc. cit.*

Lemma 2.2.2. *Let $A \in X$ and $s_1, s_2, \ldots, s_k \in S_a$ be such that $d(A, s_k \cdots s_2 s_1(A)) = k$. For any sequence $1 \leq i_1 < \cdots < i_p \leq k$, we have $s_{i_p} \cdots s_{i_1}(A) \leq s_k \cdots s_2 s_1(A)$.*

Proof of Proposition 2.2.1. Let $s_1, \ldots, s_k \in S_a$. Let $\mathcal{I}$ be the collection of all $I = \{i_1, \ldots, i_{p_I}\}$ such that $1 \leq i_1 < \cdots < i_{p_I} \leq k$ and

$$s_{i_t} \cdots \hat{s}_{i_{t-1}} \cdots \hat{s}_{i_2} \cdots \hat{s}_{i_1} \cdots s_1(A) < \hat{s}_{i_t} \cdots \hat{s}_{i_{t-1}} \cdots \hat{s}_{i_2} \cdots \hat{s}_{i_1} \cdots s_1(A)$$

for all $t = 1, \ldots, p_I$. For $I \in \mathcal{I}$, define

$$\tilde{p}_I = \sum_{j=1}^{p_I} c_{s_{i_j}},$$

$$\widetilde{\Pi}_I = \prod_{j=1}^{p_I} (q^{c_{s_{i_j}}} - 1) \text{ and}$$

$$\tilde{m}_I = \sum_{j} c_{s_j}$$

where the last sum runs over all j, $1 \le j \le k$, such that $j \notin I$ and

$$s_j s_{j-1} \cdots \hat{s}_{i_2} \cdots \hat{s}_{i_1} \cdots s_1(A) < s_{j-1} \cdots \hat{s}_{i_2} \cdots \hat{s}_{i_1} \cdots s_1(A).$$

(In the last expression we omit all s_i such that $i \in I$ and $i < j$.)

One verifies by induction on k that

$$(*) \qquad T_{s_k} \cdots T_{s_1}(A) = \sum_{I \in \mathcal{I}} q^{\tilde{m}_I} \widetilde{\Pi}_I \, s_k \cdots \hat{s}_{i_{p_I}} \cdots \hat{s}_{i_2} \cdots \hat{s}_{i_1} \cdots s_1(A).$$

We now take a reduced expression $s_k \cdots s_1 = y$. If $s'_1, \ldots, s'_n \in S_a \cap W_\lambda$ are such that $s'_n \cdots s'_1(A) = A^+_\lambda$ and $d(A, A^+_\lambda) = n$, we have

$$d(A, ys'_n \cdots s'_1(A)) = d(A, y(A^+_\lambda)) = n + k = l(ys'_n \cdots s'_1).$$

We conclude with Lemma 2.2.2 that

$$s_k \cdots \hat{s}_{i_{p_I}} \cdots \hat{s}_{i_2} \cdots \hat{s}_{i_1} \cdots s_1(A) \le y(A^+_\lambda)$$

for all $I = \{i_1, \ldots, i_{p_I}\}$, $1 \le i_1 < \cdots < i_{p_I} \le k$. Suppose $I \in \mathcal{I}$. We set

$$B = s_k \cdots \hat{s}_{i_{p_I}} \cdots \hat{s}_{i_2} \cdots \hat{s}_{i_1} \cdots s_1(A).$$

Then

$$d(A, B) = k - p_I - 2m_I$$

and

$$c(A, B) = \sum_{i=1}^{k} c_{s_i} - \tilde{p}_I - 2\tilde{m}_I.$$

We deduce

$$\begin{aligned} c(B, y(A^+_\lambda)) &= c(A, y(A^+_\lambda)) - c(A, B) \\ &= \sum_{i=1}^{k} c_{s_i} + \sum_{i=1}^{n} c_{s'_i} - \sum_{i=1}^{k} c_{s_i} + \tilde{p}_I + 2\tilde{m}_I \\ &= \sum_{i=1}^{n} c_{s'_i} + \tilde{p}_I + 2\tilde{m}_I. \end{aligned}$$

In view of $(*)$, it remains to prove

$$\sum_{i=1}^{n} c_{s'_i} \geq \tilde{p}_I$$

for part (i), and

$$\sum_{i=1}^{n} c_{s'_i} - c^* \geq \tilde{p}_I$$

(under the additional assumption that $y(A_\lambda^+) \subseteq \Pi_\lambda$) for part (ii). Parts (iii) and (iv) then follow.

Let $\bar{\mathcal{F}}$ be the set of directions of hyperplanes in $\mathcal{F}$. We denote the direction of a hyperplane $H \in \mathcal{F}$ by $i(H)$. If $\mathcal{C}$ is a quarter, the set $\mathcal{I}(\mathcal{C})$ is said to contain all directions $i \in \bar{\mathcal{F}}$ such that $\mathcal{C} \subseteq V_H^-$ for some $H \in \mathcal{F}$ with $i(H) = i$.

The proof in *loc. cit.* makes use of the following two facts.

(2.2.3) Let $\mathcal{C}$ be the quarter with vertex λ containing A. Then

$$n = |\mathcal{I}(\mathcal{C})|.$$

(2.2.4) If $\mathcal{C}$ is any quarter and $H \in \mathcal{F}$ with direction $i(H) \in \mathcal{I}(\mathcal{C})$, then

$$|\mathcal{I}(\mathcal{C}\sigma_H)| < |\mathcal{I}(\mathcal{C})|.$$

We will need stronger versions. For $i \in \bar{\mathcal{F}}$, set

$$c_i = \max_{H \in \mathcal{F}, i(H) = i} c_H,$$

and if $\mathcal{J} \subseteq \bar{\mathcal{F}}$, we write

$$m(\mathcal{J}) = \sum_{i \in \mathcal{J}} c_i.$$

Claim 2.2.5. *Let $\mathcal{C}$ be the quarter with vertex λ containing A. We have*

$$\sum_{j=1}^{n} c_{s'_j} = m(\mathcal{I}(\mathcal{C}))$$

Indeed, let H_j, $1 \leq j \leq n$, be the hyperplane in $\mathcal{F}$ separating $s'_{j-1} \cdots s'_1(A)$ from $s'_j \cdots s'_1(A)$. Since $s'_n \cdots s'_1(A) = A_\lambda^+ \subset V_{H_j}^+$ for all $H_j \in \mathcal{F}$, $1 \leq j \leq n$, we have $\mathcal{C} \subseteq V_{H_j}^-$ for all $1 \leq j \leq n$, i.e. $\mathcal{I}(\mathcal{C}) = \{i(H_j) \mid 1 \leq j \leq n\}$. Besides, for each j, $1 \leq j \leq n$, the hyperplane H_j passes through $\lambda \in T$, so $c_{i(H_j)} = c_{H_j}$, and since H_j supports a face of type $s'_j \in W_0$, the assertion follows.

Claim 2.2.6. *If $\mathcal{C}$ is a quarter with vertex in T and if $H \in \mathcal{F}$ has direction $i = i(H) \in \mathcal{I}(\mathcal{C})$, there is an injective map $\mathcal{I}(\mathcal{C}\sigma_H) \to \mathcal{I}(\mathcal{C}) - \{i\}$ which preserves the parameter.*

W.l.o.g., let $\mathcal{C}$ be the quarter with vertex λ containing A. We distinguish two cases.

First, suppose H passes through λ, hence H equals one of the above hyperplanes H_j for some $1 \leq j \leq n$. We have $A\sigma_H \subset \mathcal{C}\sigma_H$ and $\lambda \in \overline{A\sigma_H}$. Since $s'_j \cdots s'_1(A) = s'_{j-1} \cdots s'_1(A\sigma_H)$, we get

$$A_\lambda^+ = s'_n \cdots s'_1(A) = s'_n \cdots s'_{j+1} s'_{j-1} \cdots s'_1(A\sigma_H).$$

Therefore, any hyperplane H' separating A_λ^+ and $A\sigma_H$ supports a face of type s'_k for some $1 \leq k \leq n$, $k \neq j$, and we can map $i' = i(H')$ to $i(H_k)$. (Note that the expression $s'_n \cdots s'_{j+1} s'_{j-1} \cdots s'_1$ does not have to be reduced.)

Next, suppose H does not pass through λ, i.e. $H = H_j\, p_\mu$ for some $1 \leq j \leq n$ and some $\mu \in P$. Then

$$\mathcal{C}\sigma_H = \mathcal{C}\sigma_{H_j\, p_\mu} = (\mathcal{C}\sigma_{H_j})\, p_{2\mu}.$$

Since $\lambda\, p_{2\mu}$ is again a special point and $\mathcal{I}(\mathcal{C}) = \mathcal{I}(\mathcal{C}\, p_{2\mu})$, we can replace $\mathcal{C}$ by $\mathcal{C}\, p_{2\mu}$ and argue as before.

By Claim 2.2.5, the proposition becomes a consequence of the following statement.

Lemma 2.2.7. *(comp.* [14], *Lemma 4.3) Let λ be a special point, A an alcove containing λ in its closure and $\mathcal{C}$ the quarter with vertex λ containing A. Let $s_1, \ldots, s_k \in S_a$ be such that $d(A_\lambda^+, s_k \cdots s_1(A_\lambda^+)) = k$, and let $1 \leq i_1 < \cdots < i_p \leq k$ be such that*

$$s_{i_t} \cdots \hat{s}_{i_{t-1}} \cdots \hat{s}_{i_1} \cdots s_1(A) < \hat{s}_{i_t} \cdots \hat{s}_{i_{t-1}} \cdots \hat{s}_{i_1} \cdots s_1(A)$$

for all $t = 1, \ldots, p$. Write $\tilde{p} = \sum_{j=1}^{p} c_{s_{i_j}}$.

(i) *We have $\tilde{p} \leq m(\mathcal{I}(\mathcal{C}))$.*

(ii) *If $s_k \cdots s_1(A_\lambda^+) \subseteq \Pi_\lambda$ and $A \neq A_\lambda^+$, then $\tilde{p} \leq m(\mathcal{I}(\mathcal{C})) - c^*$.*

Proof. The following facts can be found in *loc. cit.*

Let $w \in W_a$ be such that $\lambda w = \lambda$ and $A_\lambda^+ w = A$. We denote by H_j, $1 \leq j \leq k$, the hyperplane in $\mathcal{F}$ separating $s_{j-1} \cdots s_1(A_\lambda^+)$ from $s_j \cdots s_1(A_\lambda^+)$. Let $1 \leq j \leq k$.

- If $s_j s_{j-1} \cdots s_1(A) < s_{j-1} \cdots s_1(A)$ we have $i(H_j w) \in \mathcal{I}(\mathcal{C})$.
- If $s_k \cdots s_1(A_\lambda^+) \subseteq \Pi_\lambda$ the hyperplane $H_j w$ is not parallel to any of the walls of $\mathcal{C}$.

The proof proceeds by induction on $|\mathcal{I}(\mathcal{C})|$. If $\mathcal{I}(\mathcal{C}) = \emptyset$ we have $A = A_\lambda^+$ and $p = \tilde{p} = 0$. There is nothing to prove for part (ii).

Now let $|\mathcal{I}(\mathcal{C})| \geq 1$. The case $p = 0$ is clear, so we assume $p \geq 1$. Let H be the unique hyperplane in $\mathcal{F}$ separating $s_{i_1} \cdots s_2 s_1(A)$ from $\hat{s}_{i_1} \cdots s_2 s_1(A)$. We set

$\lambda' = \lambda\sigma_H$, $A' = A\sigma_H$, $\mathcal{C}' = \mathcal{C}\sigma_H$. Then $d(A^+_{\lambda'}, s_k \cdots s_1(A^+_{\lambda'})) = k$, and $\hat{s}_{i_1} \cdots s_2 s_1(A) = s_{i_1} \cdots s_2 s_1(A')$ implies

$$s_{i_t} \cdots \hat{s}_{i_{t-1}} \cdots \hat{s}_{i_2} \cdots s_{i_1} \cdots s_1(A') < \hat{s}_{i_t} \cdots \hat{s}_{i_{t-1}} \cdots \hat{s}_{i_2} \cdots s_{i_1} \cdots s_1(A')$$

for $t = 2, 3, \ldots, p$.

We have $|\mathcal{I}(\mathcal{C}')| < |\mathcal{I}(\mathcal{C})|$. Applying the inductive assumption, part (i), to λ', A', $s_1, \ldots, s_k$ and $i_2, \ldots, i_p$ yields

$$\sum_{j=2}^{p} c_{s_{i_j}} \leq m(\mathcal{I}(\mathcal{C}\sigma_H)),$$

and therefore

$$\begin{aligned} \sum_{j=1}^{p} c_{s_{i_j}} &\leq m(\mathcal{I}(\mathcal{C}\sigma_H)) + c_{s_{i_1}} \\ &= m(\mathcal{I}(\mathcal{C}\sigma_H)) + c_H \\ &\leq m(\mathcal{I}(\mathcal{C})). \end{aligned}$$

We used Claim 2.2.6 for the last inequality. (Note that it is feasible that

$$m(\mathcal{I}(\mathcal{C}\sigma_H)) + c_H < m(\mathcal{I}(\mathcal{C}\sigma_H)) + c_{i(H)} < m(\mathcal{I}(\mathcal{C})).)$$

Under the assumptions of part (ii), if $\mathcal{C}' \neq \mathcal{C}^+_{\lambda'}$ it follows by induction that

$$\sum_{j=2}^{p} c_{s_{i_j}} \leq m(\mathcal{I}(\mathcal{C}\sigma_H)) - c^*,$$

and therefore

$$\sum_{j=1}^{p} c_{s_{i_j}} \leq m(\mathcal{I}(\mathcal{C})) - c^*.$$

Otherwise, we have

$$\sum_{j=2}^{p} c_{s_{i_j}} = 0.$$

But then, according to the two facts mentioned at the beginning of the proof, $\mathcal{I}(\mathcal{C})$ contains the direction of Hw, which is not parallel to any wall of $\mathcal{C}$, and also the direction of some wall of $\mathcal{C}$. Consequently,

$$\sum_{j=1}^{p} c_{s_{i_j}} = c_{s_{i_1}} = c_H \leq m(\mathcal{I}(\mathcal{C})) - c^*,$$

which completes the proof of the lemma.

Note that $\tilde{p} = m(\mathcal{I}(\mathcal{C}))$ implies $p = |\mathcal{I}(\mathcal{C})|$. We therefore obtain the following corollary of the proof of the above lemma in the same way as for equal parameters.

Corollary 2.2.8. *(comp.* [14], *Corollary 4.4) Let A and y be as in Proposition 2.2.1 (i), and let B be an alcove such that $B \leq y(A_\lambda^+)$ and such that $\pi_{B,y}^A$ has degree equal to $\frac{1}{2}c(B, y(A_\lambda^+))$. Then $B = y(A_\lambda^+)\tau$ for some translation $\tau \in W_a$.*

2.3. **The basis elements.** The following results all have their counterpart in [14]. In view of the two previous sections, most proofs in *loc. cit.* can easily be generalized, and we refer the reader to *loc. cit.* whenever a statement in this section is presented without proof.

Theorem 2.3.1. *(comp.* [14], *Theorem 2.15) Let λ be a special point and C an alcove in Π_λ. There exists a unique element $D_C \in \mathcal{M}_\lambda$ such that*

(i) $D_C = \sum_{A \leq C} Q_{A,C} A$ *where $Q_{A,C}$ is a polynomial in v with integer coefficients of degree less than $c(A, C)$ if $A < C$ and $Q_{C,C} = 1$ and*

(ii) $v^{2c(A_\lambda^+, C)} \overline{Q_{A,C}} = Q_{Av_\lambda, C}$.

(We set $Q_{A,C} = 0$ for A, $C \in X$, $A \not\leq C$.)

Remark 2.3.2. With λ and C as above, let $B = Cv_\lambda$ and $D^B = \varphi_\lambda(D_C)$. The element D^B is characterized by the conditions $D^B \in \mathcal{M}_\lambda$ and

(i) $D^B = \sum_{A \geq B} \overline{Q^{B,A}} A$ where $Q^{B,A}$ is a polynomial in v with integer coefficients of degree less than $c(B, A)$ if $B < A$ and $Q^{B,B} = 1$ and

(ii) $v^{2c(B, A_\lambda^-)} \overline{Q^{B,A}} = Q^{B, Av_\lambda}$.

Let G be a simply connected almost simple algebraic group of type Φ over an algebraically closed field of characteristic $p > 1$. Assume that p is sufficiently large.

In *loc. cit.*, Lusztig conjectured that the integer $Q^{B,A}(1)$, $A \geq B$, equals the multiplicity of a certain irreducible G-module in a Jordan–Hölder series of a certain Weyl module of G associated to A. He pointed out that for $v = 1$ the condition $D^B \in \mathcal{M}_\lambda$ becomes the condition that

$$\sum_{A \geq B} Q^{B,A}(1) A$$

is invariant under the stabilizer Ω_λ of λ in W_a acting on X on the right. Assuming Lusztig's conjecture, this Ω_λ-invariance is due to Jantzen (cf. [7]).

The Ω_λ-invariance of $\sum_{A \geq B} Q^{B,A}(1) A$ also implies the symmetry condition (ii) for $v = 1$.

(The above conjecture by Lusztig is equivalent to his conjecture on the irreducible characters of rational G-modules in [13]. It is now known to be true – due to work involving affine Kac–Moody Lie algebras as well as quantum groups.)

For $\lambda \in Q$, define

$$Y_\lambda = \{yp_\mu \mid y \in W_0,\, \mu \in Q,\, \mu \preceq \lambda\}$$

where $\preceq$ denotes the usual partial order on Q. A subset $K \subset W_a$ is called *x-bounded*, $x \in W_0$, if

$$K \subseteq Y_\lambda\, x$$

for some $\mu \in Q$. Instead of e-bounded, e being the identity in W_0, we also say *bounded.* For $x \in W_0$, let $\hat{\mathcal{H}}_x$ be the set of formal sums $f = \sum_{w \in W_a} a_w \check{T}_w$, $a_w \in \mathcal{A}$, such that $\operatorname{Supp} f \stackrel{\text{def}}{=} \{w \in W_a \mid a_w \neq 0\}$ is x-bounded. The sets $\hat{\mathcal{H}}_x$ can naturally be regarded as $\mathcal{H}$-modules. Similarly, let $\hat{\mathcal{M}}_x$ be the $\mathcal{H}$-module which contains all formal sums $f = \sum_{A \in X} b_A A$, $b_A \in \mathcal{A}$, such that $\{w \in W_a \mid b_{A-w} \neq 0\}$ is x-bounded. We set $\hat{\mathcal{H}} = \hat{\mathcal{H}}_e$ and $\hat{\mathcal{M}} = \hat{\mathcal{M}}_e$. In this section we will only need $\hat{\mathcal{M}}$.

Choose a weighted length function γ on X.

As in *loc. cit.*, we can extend $\Phi_\gamma : \mathcal{M}^0 \to \mathcal{M}^0$ to a map $\hat{\Phi}_\gamma : \hat{\mathcal{M}} \to \hat{\mathcal{M}}$, and we define elements $\mathcal{R}_{B,A} \in \mathcal{A}$ for $A,\, B \in X$ by

$$\hat{\Phi}_\gamma(A) = v^{-2\gamma(A)} \sum_{B \in X} (-1)^{d(A,B)} \mathcal{R}_{B,A} B.$$

We remark that $\mathcal{R}_{B,A} = 0$ for all $B \not\leq A$ and that $\mathcal{R}_{A,A} = 1$.

The following statement strengthens the uniqueness part of Theorem 2.3.1.

Theorem 2.3.3. *(comp.* [14], *Theorem 7.3) For any $C \in X$, there is a unique element $D_C \in \hat{\mathcal{M}}$ such that $\hat{\Phi}_\gamma(D_C) = v^{-2\gamma(C)} D_C$ and*

$$D_C = \sum_{A \leq C} Q_{A,C} A,$$

where $Q_{A,C} \in \mathcal{A}^+$ has degree less than $c(A,C)$ if $A < C$ and $Q_{C,C} = 1$.

(It follows from Theorem 2.1.5 and Corollary 2.1.6 that the conditions $\hat{\Phi}_\gamma(D_C) = v^{-2\gamma(C)} D_C$ and $v^{2c(A_\lambda^+, C)} \overline{Q_{A,C}} = Q_{Av_\lambda, C}$ are equivalent.)

For $C \in X$, let $\lambda \in T$ and $w \in W_a$ be such that $C \subseteq \Pi_\lambda$ and $w(A_\lambda^+) = C$. The element D_C can be shown to satisfy

$$\begin{aligned} D_C &= \sum_{y \leq ww_\lambda,\, l(yw_\lambda) = l(y) + l(w_\lambda)} P_{yw_\lambda, ww_\lambda} T_y\Big(\sum_{z \in W_\lambda} T_z(A_\lambda^-)\Big) \\ &= \sum_{y' \leq ww_\lambda} P_{y', ww_\lambda} T_{y'}(A_\lambda^-) \\ &= C^*_{ww_\lambda}(A_\lambda^-). \end{aligned}$$

This representation of D_C is used to prove the next two corollaries. (We preserve the notation.)

Corollary 2.3.4. *(comp. [14], Corollary 5.3) With $y \in W_a$ such that $A = y(A_\lambda^-)$, we have*

$$Q_{A,C}(1) = P_{y,ww_\lambda}(1)$$

Corollary 2.3.5. *(comp. [14], Corollary 5.4) Let $\mathcal{O} \subset X$ be an orbit under the translation subgroup Q of W_a. For $C \in X$, the sum*

$$\sum_{A \in \mathcal{O}} Q_{A,C}$$

is independent of the choice of $\mathcal{O}$, and it is equal to

$$\sum_{y \leq ww_\lambda,\, l(yw_\lambda)=l(y)+l(w_\lambda)} v^{2m(y)} P_{yw_\lambda, ww_\lambda}.$$

Theorem 2.3.3 is the main ingredient of the following statement.

Corollary 2.3.6. *(comp. [14], Corollary 7.4) Let $\tau \in T$ be a translation on X. For any $A \leq C$ in X, we have $Q_{A,C} = Q_{A\tau,C\tau}$.*

Finally note that for A, $C \in X$ and $\lambda \in T$ such that $C \subseteq \Pi_\lambda$, the polynomial $Q_{A,C}$ is non–zero only if $Cv_\lambda \leq A$.

In analogy with *loc. cit.*, we now want to further examine the elements D_C. The aim is to find an inductive formula.

For A, $C \in X$ and $s \in S_a$ such that $sA < A < C < sC$, we define elements $M^s_{A,C} \in \mathcal{A}$ by the conditions

(i) $\displaystyle\sum_{A \leq B < C,\, sB < B} v^{c(B,C)} Q_{A,B} M^s_{B,C} - v^{c_s} Q_{A,C}$ has degree less than $c(A,C)$ and

(ii) $M^s_{A,C} = \overline{M^s_{A,C}}$.

(We set $M^s_{A,C} = 0$ in all other cases.) This definition is similar to the definition of $M^s_{y,w} \in \mathcal{A}$ for y, $w \in W_a$ and $s \in S_a$ with $sy < y < w < sw$ in [16], Section 3.

Suppose C is an alcove in Π_λ, $\lambda \in T$. We claim that $M^s_{A,C} = 0$ unless $Cv_\lambda \leq A \leq C$. In particular, for any $C \in X$, there are only finitely many $M^s_{A,C}$, $A \in X$, $s \in S_a$, which are non–zero.

Indeed, assume $A \not\leq C$. Then $Q_{A,C} = 0$, and of course there is no $B \in X$ satisfying $A < B < C$. Thus $M^s_{A,C} = 0$ for any $s \in S_a$.

Next assume $Cv_\lambda \not\leq A$ or equivalently $Av_\lambda \not\leq C$. Again, we know that $Q_{A,C} = 0$, and if $B \in X$, $A < B < C$, is such that $Q_{A,B} \neq 0$, then $Av_\lambda \leq B$, hence $Av_\lambda \leq C$, a contradiction. The claim follows.

Theorem 2.3.7. *(comp. [14], Theorem 8.2) Let $C \in X$ and $s \in S_a$. We have*

$$T_s D_C = \begin{cases} -D_C + D_{sC} + \displaystyle\sum_{A<C,\, s\in\mathcal{L}(A)} v^{c(A,C)+c_s} M^s_{A,C} D_A & \text{if } s \notin \mathcal{L}(C), \\ v^{2c_s} D_C & \text{if } s \in \mathcal{L}(C). \end{cases}$$

Note that the sum is finite.

Proof. The case $s \in \mathcal{L}(C)$ is dealt with as in *loc. cit.* Consider the case $s \notin \mathcal{L}(C)$. We set

$$D'_{sC} = (T_s + 1)D_C - \sum_{A<C,\, s\in\mathcal{L}(A)} v^{c(A,C)+c_s} M^s_{A,C} D_A.$$

As in *loc. cit.*, we deduce that

$$\Phi_\gamma(D'_{sC}) = v^{-2\gamma(sC)} D'_{sC}.$$

The coefficient of $B \in X$ in D'_{sC} equals

$$\tilde{Q}_B = \left\{ \begin{array}{ll} v^{2c_s} Q_{sB,C} + Q_{B,C}, & \text{if } sB > B \\ Q_{sB,C} + v^{2c_s} Q_{B,C}, & \text{if } sB < B \end{array} \right\} - \sum_{A<C,\, s\in\mathcal{L}(A)} v^{c(A,C)+c_s} Q_{B,A} M^s_{A,C}.$$

We have $\tilde{Q}_{sC} = Q_{C,C} = 1$, and since the condition $B \le sC$ is equivalent to $sB \le sC$, the coefficient $\tilde{Q}_B$ is zero if $B \not\le sC$.

Suppose $B < sC$. If $sB > B$, the definition of $M^s_{sB,C}$ implies that

$$\sum_{A<C,\, s\in\mathcal{L}(A)} v^{c(A,C)} Q_{sB,A} M^s_{A,C} - v^{c_s} Q_{sB,C}$$

has degree less than $c(sB, C)$. From $T_s D_A = v^{2c_s} D_A$ for $A \in X$ with $sA < A$, we derive $Q_{B,A} = Q_{sB,A}$ for any $B \le A$. Consequently, the degree of

$$\sum_{A<C,\, s\in\mathcal{L}(A)} v^{c(A,C)} Q_{B,A} M^s_{A,C} - v^{c_s} Q_{B,C}$$

is less than $c(B,C) - c_s$, and therefore $\tilde{Q}_B$ has degree less than $c(B,C) < c(B, sC)$.

If $sB < B$ the degree of

$$\sum_{A<C,\, s\in\mathcal{L}(A)} v^{c(A,C)} Q_{B,A} M^s_{A,C} - v^{c_s} Q_{B,C}$$

is less than $c(B,C)$. Thus $\tilde{Q}_B$ has again degree less than $c(B, sC)(= c(B,C) + c_s)$.

Now we apply the uniqueness statement in Theorem 2.3.3 to conclude that $D'_{sC} = D_{sC}$ (and $\tilde{Q}_B = Q_{B,sC}$), hence the assertion.

Notice one equality which occurred in the above proof.

Corollary 2.3.8. *(comp. [14], Corollary 8.4 (a)) For $A, C \in X$ with $A \le C$ and $s \in \mathcal{L}(C)$, we have $Q_{A,C} = Q_{sA,C}$.*

The following statement can be deduced from Theorem 2.3.7.

Corollary 2.3.9. *(comp.* [14], *Corollary 8.3) The elements D_C, $C \in X$, form an $\mathcal{A}$-basis of $\mathcal{M}^0$.*

For A, $C \in X$, we write $A \lhd C$ if for some (or equivalently any) box Π the alcoves A', C' in Π obtained from A, C by translation under T satisfy $A' < C'$.

Proposition 2.3.10. *(comp.* [14], *Corollaries 10.5 and 10.6) Let $C \in X$ and $s \in S_a - \mathcal{L}(C)$ such that sC and C lie in the same box. Then $(T_s + 1)D_C - D_{sC}$ is an $\mathcal{A}^+$-linear combination of elements D_A such that $A < C$, $A \lhd C$. In particular, if $A \in X$ is such that $sA < A < C$ and $M^s_{A,C} \neq 0$, then $A \lhd C$.*

This result enables us to give an algorithm to compute the elements D_C by a finite induction.

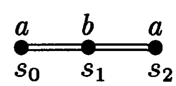
Let $C \in X$. There exists a unique $\lambda \in T$ such that $C \subseteq \Pi_\lambda$, and we can find elements $s_1, s_2, \ldots, s_{n_C}$ in S_a such that $C = s_1 s_2 \cdots s_{n_C} A^+_\lambda$ and $d(A^+_\lambda, C) = n_C$. We proceed by induction on n_C.

If $n_C = 0$ we have $C = A^+_\lambda$, hence $D_C = e_\lambda$ and $M^s_{A,C} = 0$ for all $A \in X$, $s \in S_a$. Now assume $n_C \geq 1$. Let $C' = s_2 \cdots s_{n_C}(A^+_\lambda)$, which lies in Π_λ. By Propositions 2.3.7 and 2.3.10, we can write

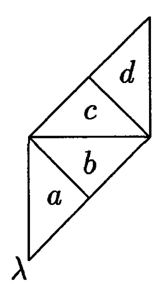
$$D_C = (T_{s_1} + 1)D_{C'} - \sum_{A \lhd C',\, s_1 \in \mathcal{L}(A)} v^{c(A,C') + c_{s_1}} M^{s_1}_{A,C'} D_A.$$

Since $n_{C'} < n_C$ and $A \lhd C'$ implies $n_A < n_{C'}$, the elements $D_{C'}$ and D_A for $A \lhd C'$, as well as the polynomials $M^s_{A,C'}$ are known by induction. Thus D_C and by induction on $d(A, C)$ the polynomials $M^s_{A,C}$ for $A \in X$, $s \in S_a$, can be calculated.

Example 2.3.11. Suppose W_a is of type $\tilde{C}_2$ with parameters



and



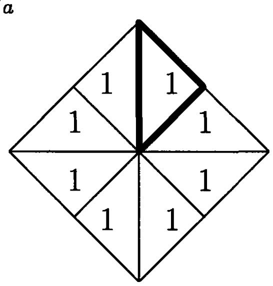
represents a box Π_λ, $\lambda \in T$. (So if λ lies in the same Ω-orbit as 0 then $a = A^+_\lambda$, $b = s_0 A^+_\lambda$, $c = s_1 s_0 A^+_\lambda$, and $d = s_2 s_1 s_0 A^+_\lambda$.) Since $c_{s_0} = c_{s_2}$, we can assume that λ and 0 lie in the same Ω-orbit.

Set $x = v^{2a}$ and $y = v^{2b}$. We obtain

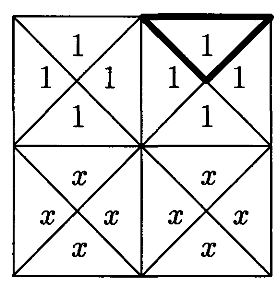
$$
\begin{aligned}
D_b &= (T_{s_0}+1)D_a,\\
D_c &= (T_{s_1}+1)D_b - \left\{\begin{array}{ll} x+y & \text{if } a<b \\ x & \text{if } a=b \\ 0 & \text{if } a>b \end{array}\right\} D_a,\\
D_d &= (T_{s_2}+1)D_c - \left\{\begin{array}{ll} 0 & \text{if } a<b \\ x & \text{if } a=b \\ x+y & \text{if } a>b \end{array}\right\} D_b - \left\{\begin{array}{ll} 0 & \text{if } 2a<b \\ -x^2 & \text{if } 2a=b \\ -x^2-y & \text{if } a<b<2a \\ 0 & \text{if } a=b \\ xy+y & \text{if } a>b \end{array}\right\} D_a.
\end{aligned}
$$

The following patterns describe the elements D_C for $C \in \{a, b, c, d\}$. Each pattern has center λ. The alcove C is singled out, and the entry of an alcove A is the polynomial $Q_{A,C}$.

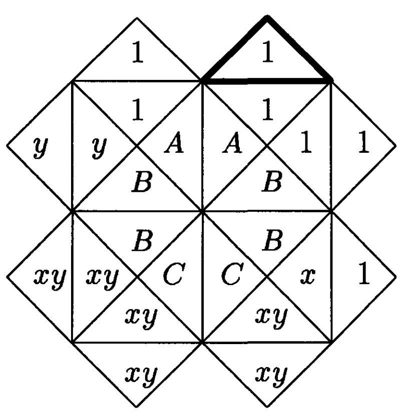
D_a



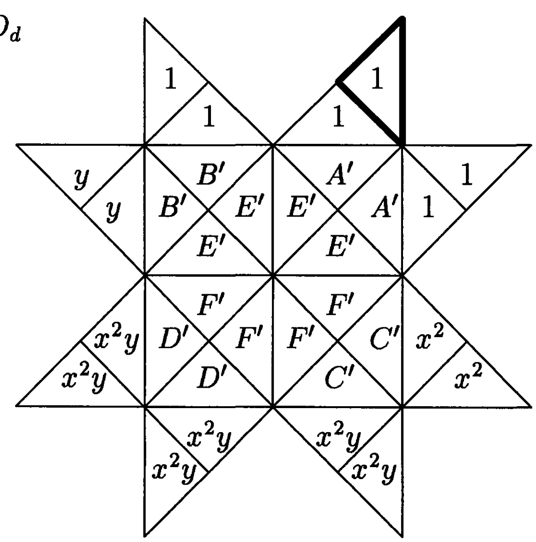
D_b



D_c



D_d



where

$$A = \begin{cases} 1-x \\ 1 \\ 1+y \end{cases}, \quad B = \begin{cases} 0 \\ x \\ x+y \end{cases}, \quad C = \begin{cases} xy-y \\ x^2 \\ xy+y \end{cases},$$

$$A' = \begin{cases} 1+x \\ 1 \\ 1-y \end{cases}, \quad B' = \begin{cases} x+y \\ x \\ 0 \end{cases}, \quad C' = \begin{cases} x^2+xy \\ x^2 \\ 0 \end{cases}, \quad D' = \begin{cases} xy+x^2y \\ x^3 \\ x^2y-x^2 \end{cases}$$

for

$$\begin{cases} a<b \\ a=b \\ a>b \end{cases}$$

and

$$E' = \begin{cases} x-x^2 \\ x \\ x+y \\ x \\ x-y \end{cases}, \quad F' = \begin{cases} xy-y \\ xy-y+x^2 \\ xy+x^2 \\ x^2 \\ 0 \end{cases} \quad \text{for} \begin{cases} 2a<b \\ 2a=b \\ a<b<2a \\ a=b \\ a>b \end{cases}.$$

(For W_a of type $\tilde{G}_2$, we get 21 cases for the relationship between a and b.)

We now introduce another right action of W_a on X. Let $A \in X$, $A = y(A_\lambda^+) \subseteq \Pi_\lambda$ for some $y \in W_a$ and some $\lambda \in T$. For $w \in W_0$, we define

$$A * w = y(A_{\lambda w}^+) \subseteq \Pi_{\lambda w}.$$

Proposition 2.3.12. *(comp. [14], Proposition 8.7) For $C \in X$ and $w \in W_a$, we have*

$$\theta_w(D_C) = v^{c(C*w,C)} D_{C*w}.$$

For $C \in X$, let $\tilde{D}_C = v^{-\gamma(C)} D_C$, and define

$$\tilde{M}_{A,C}^s = \begin{cases} M_{A,C}^s & \text{if } A<C, \\ 1 & \text{if } A = sC > C, \\ 0 & \text{otherwise} \end{cases}$$

for A, $C \in X$ and $s \in S_a$. We rewrite some of the previous statements in terms of the $\mathcal{A}$-basis $\{\tilde{D}_C \mid C \in X\}$ of $\mathcal{M}^0$.

Corollary 2.3.13. *(comp. [14], Corollary 8.9) Let $C \in X$, $w \in W_a$ and $s \in S_a$. We have*

$$\Phi_\gamma(\tilde{D}_C) = \tilde{D}_C,$$

$$\tilde{D}_C \theta_w = \tilde{D}_{C*w},$$

and

$$T_s\tilde{D}_C = \begin{cases} -\tilde{D}_C + v^{c_s} \sum\limits_{A<C,\, s\in\mathcal{L}(A)} \tilde{M}^s_{A,C}\tilde{D}_A & \text{if } s \notin \mathcal{L}(C), \\ v^{2c_s}\tilde{D}_C & \text{if } s \in \mathcal{L}(C). \end{cases}$$

Corollary 2.3.14. *(comp. [14], Corollary 8.10) For A, $C \in X$, $s \in S_a$, and $w \in W_a$, we have $\tilde{M}^s_{A,C} = \tilde{M}^s_{A*w,C*w}$.*

Remark 2.3.15. Suppose we specialize q to a prime power p^s. Let λ be a homomorphism of the group of translations Q in W_a to $\mathbb{C}^*$. The $\mathbb{C}$-vector space $\mathcal{M}_\lambda$ spanned by all infinite formal linear combinations

$$\tilde{D}^*_C = \sum_{\tau\in Q} \tilde{D}_{C\tau}\lambda(\tau^{-1}).$$

has dimension equal to the number of orbits of Q on X, and it has a natural $\mathcal{H}$-action. For generic λ, the $\mathcal{H}$-module $\mathcal{M}_\lambda$ is isomorphic to the principal series representation defined by Matsumoto in [26].

We conclude this section with a few results concerning the polynomials $\mathcal{R}_{B,A}$ and $Q_{A,B}$, A, $B \in X$. For more details, we refer to [14].

Define polynomials $R_{y,w} \in \mathcal{A}^+$, y, $w \in W_a$, by

$$T^{-1}_{w^{-1}} = v^{-2m(w)} \sum_{y\in W_a} (-1)^{l(w)-l(y)} R_{y,w} T_y$$

and polynomials $\tilde{Q}_{y,w} \in \mathcal{A}^+$ for y, $w \in W_a$, $y(A^+) \subset \mathcal{C}^-$, $w(A^+) \subset \mathcal{C}^-$ by

$$\sum_{y\le z\le w,\, z(A^+)\subset\mathcal{C}^-} (-1)^{l(z)-l(y)} P_{y,z}\tilde{Q}_{z,w} = \delta_{y,w}.$$

Theorem 2.3.16. *(comp. [14], Theorem 11.6 and Corollary 11.9) Suppose A, $B \in X$ and $\lambda \in T$ are such that A, $B \subset \mathcal{C}^-_\lambda$ and A, B are sufficiently far from the walls of $\mathcal{C}^-_\lambda$. Let y, $w \in W_a$ be such that $y(A^+_\lambda) = B$, $w(A^+_\lambda) = A$. Then*

$$\mathcal{R}_{B,A} = \sum_{b\in W_\lambda} R_{w,yb}$$

and

$$Q_{A,B} = \tilde{Q}_{y,w}.$$

We remark that as in *loc. cit.*, the function defined by

$$(A, B) \mapsto (-1)^{d(A,B)} Q_{A,B}(0)$$

for A, $B \in X$ is the Möbius function of the partially ordered set $(X, \le)$. In particular, $Q_{A,B}(0)$ does not depend on the choice of parameters.

Proposition 2.3.17. *(comp.* [14], *Proposition 11.15) Let $\lambda \in T$ and y, $w \in W_\lambda$. We have*

$$\mathcal{R}_{y(A_\lambda^-),w(A_\lambda^-)} = R_{y,w}$$

and

$$Q_{y(A_\lambda^-),w(A_\lambda^-)} = P_{y,w}.$$

3. A formula for the Kazhdan–Lusztig polynomials

In Corollary 3.3.3, we express the Kazhdan–Lusztig polynomials in terms of alcove polynomials (see Section 3.2 for the definition). For equal parameters this has been achieved by Kato in [9], and we follow his approach. Statements which are not proven here are obtained via a straightforward generalization of the proof of the corresponding statement in *loc. cit.*

3.1. **Definitions.** We need some more notation from *loc. cit.*

Let $\psi : \mathcal{H} \to \mathcal{H}$ be the map such that

$$\psi(f) = \bar{f} T_{w_0}$$

for $f \in \mathcal{H}$. This map is $\mathcal{H}$-antilinear, and it satisfies $\psi(\mathcal{H}^0) = \mathcal{H}^0$.

Proposition 3.1.1. *(comp.* [9], *Proposition 2.8) There exists a unique involutive $\mathcal{H}$-antilinear map $\Psi^0 : \mathcal{H}^0 \to \mathcal{H}^0$ such that*

$$\Psi^0(d_\lambda) = v^{-2h(w_0 p_\lambda)} d_\lambda$$

for any $\lambda \in T$. We have

$$\Psi^0 = v^{-2m(w_0)} \psi\, \Theta_{w_0}.$$

Remark 3.1.2. Let γ be the weighted length function on X defined by

$$\gamma(A^- w) = h(w)$$

for $w \in W_a$. It turns out that under the isomorphism $\phi : \mathcal{H} \to \mathcal{M}$, the map ψ corresponds to the map $\varphi \stackrel{\text{def}}{=} \varphi_0$ and Ψ^0 corresponds to Φ_γ. From now on, we fix γ and also the length function δ on X defined by

$$\delta(A^- w) = g(w)$$

for $w \in W_a$.

We define $\tilde{\mathcal{G}}(\alpha)$ for arbitrary $\alpha \in \Phi$. Let $\alpha \in \Phi$. If α lies in the W_0-orbit of a simple root β, we set $c_\alpha = c_{s_\beta}$, $\tilde{c}_\alpha = c_{\tilde{s_\beta}}$, and

$$\tilde{\mathcal{G}}(\alpha) = (\hat{T}_{-\alpha} v^{c_\alpha + \tilde{c}_\alpha} - 1)(\hat{T}_{-\alpha} v^{c_\alpha - \tilde{c}_\alpha} + 1).$$

Besides, we let

$$\mathbf{e}(\alpha) = -\tilde{G}(\alpha)$$

where $\tilde{G}(\alpha)$ is again right multiplication with $\tilde{\mathcal{G}}(\alpha)$.

Let $x \in W_0$. For $\lambda \in Q$, we can regard Θ_λ as an element of $\mathrm{End}_{\mathcal{H}}(\hat{\mathcal{H}}_x)$ (and θ_λ as an element of $\mathrm{End}_{\mathcal{H}}(\hat{\mathcal{M}}_x)$). The operators of the form $1 - p\,\Theta_\alpha$, $p \in \mathcal{A}$, $\alpha \in \Phi$, are invertible on $\hat{\mathcal{H}}_x$ with inverses as follows. Consider the formal sums

$$(1 - p\,\Theta_\alpha)_+^{-1} = \begin{cases} \sum_{n \geq 0} p^n \Theta_{n\alpha} & \text{if } \alpha > 0, \\ -p\,\Theta_\alpha \sum_{n \geq 0} p^{-n} \Theta_{-n\alpha} & \text{if } \alpha < 0 \end{cases}$$

and

$$(1 - p\,\Theta_\alpha)_-^{-1} = \begin{cases} -p\,\Theta_\alpha \sum_{n \geq 0} p^{-n} \Theta_{-n\alpha} & \text{if } \alpha > 0, \\ \sum_{n \geq 0} p^n \Theta_{n\alpha} & \text{if } \alpha < 0. \end{cases}$$

Then

$$(1 - p\,\Theta_\alpha)^{-1} = \begin{cases} (1 - p\Theta_\alpha)_+^{-1} & \text{if } \alpha > 0,\ \alpha x^{-1} < 0 \quad \text{or if } \alpha < 0,\ \alpha x^{-1} > 0, \\ (1 - p\Theta_\alpha)_-^{-1} & \text{otherwise.} \end{cases}$$

The map $\tilde{\Theta}_s$, $s \in S_0$, can be regarded as an element of $\mathrm{Hom}_{\mathcal{H}}(\hat{\mathcal{H}}_x, \hat{\mathcal{H}}_{xs})$. For $x \in W_0$, $s = s_\alpha$, $\alpha \in \Pi$, we can therefore define an operator

$$\Theta_s^+ = \mathbf{e}(\alpha)^{-1} \tilde{\Theta}_s$$

belonging to $\mathrm{Hom}_{\mathcal{H}}(\hat{\mathcal{H}}_x, \hat{\mathcal{H}}_{xs})$. By definition, we have $\Theta_s^+ |_{\mathcal{H}^0} = \Theta_s$. The corresponding operator in $\mathrm{Hom}_{\mathcal{H}}(\hat{\mathcal{M}}_x, \hat{\mathcal{M}}_{xs})$ will be denoted by θ_s^+.

Let $w_0 = s_1 \cdots s_n$, $s_i \in S_0$, $1 \leq i \leq n$, be a reduced expression, and define

$$\Theta_{w_0}^+ = \Theta_{s_n}^+ \cdots \Theta_{s_1}^+ \in \mathrm{Hom}_{\mathcal{H}}(\hat{\mathcal{H}}, \hat{\mathcal{H}}_{w_0})$$

(which can be shown to be independent of the chosen reduced expression). Each Θ_{s_i}, $1 \leq i \leq n$, is a map in $\mathrm{Hom}_{\mathcal{H}}(\hat{\mathcal{H}}_x, \hat{\mathcal{H}}_{xs_i})$ for some $x \in W_0$ such that $xs_i > x$. Hence $(-\alpha_i)(xs_i)^{-1} > 0$ (where $\alpha_i \in \Pi$ is such that $s_i = s_{\alpha_i}$), and

$$\Theta_{s_i}^+ = (1 - v^{c_{\alpha_i} + \tilde{c}_{\alpha_i}} \Theta_{-\alpha_i})_+^{-1} (1 + v^{c_{\alpha_i} - \tilde{c}_{\alpha_i}} \Theta_{-\alpha_i})_+^{-1} \tilde{\Theta}_{s_i}.$$

Since we also have $(1-p\,\Theta_{-\alpha})^{-1}=(1-p\,\Theta_{-\alpha})_+^{-1}$ on $\hat{\mathcal{H}}_{w_0}$ for any $\alpha>0$, we deduce, using Proposition 2.1.1, that

$$\Theta_{w_0}^+=\prod_{\alpha>0}\mathbf{e}(\alpha)^{-1}\,\tilde{\Theta}_{w_0}$$

where $\tilde{\Theta}_{w_0}=\tilde{\Theta}_{s_n}\cdots\tilde{\Theta}_{s_1}$. We now set

$$\Psi=v^{-2m(w_0)}\psi\,\Theta_{w_0}^+,$$

an $\mathcal{H}$-antilinear map on $\hat{\mathcal{H}}$.

Let $\preceq$ be the partial order on W_a given by $y\preceq w$, y, $w\in W_a$, if and only if $y(A^-)\le w(A^-)$. We point out that, since $x(A^-)=A^-x$ for $x\in W_a$, this definition agrees with the definition of $\preceq$ on $Q\subset W_a$.

Lemma 3.1.3. *For $w\in W_a$, we have*

$$\Psi(\check{T}_w)=v^{-2h(w)}\sum_{y\preceq w}(-1)^{g(w)-g(y)}\mathcal{R}_{y(A^-),w(A^-)}\check{T}_y.$$

(This follows from the correspondence between Ψ and the $\mathcal{H}$-antilinear map $\hat{\Phi}_\gamma$ on $\hat{\mathcal{M}}$.)

3.2. **Alcove polynomials.** We introduce elements $E_C\in\hat{\mathcal{M}}$, $C\in X$, whose coefficients generically equal the Kazhdan–Lusztig polynomials.

Definition 3.2.1. The *alcove polynomials* $\hat{P}_{y,w}\in\mathcal{A}^+$, y, $w\in W_a$, are defined by the conditions

(i) $v^{-2h(w)}\hat{P}_{y,w}=\sum_{y\preceq z\preceq w}v^{-2h(y)}\overline{\mathcal{R}_{z(A^+),y(A^+)}}\,\overline{\hat{P}_{z,w}}$ and

(ii) the degree of $\hat{P}_{y,w}$ is less than $h(w)-h(y)$ if $y\prec w$ and $\hat{P}_{w,w}=1$.

For $A=y(A^-)$ and $C=w(A^-)$, we set $\hat{P}_{A,C}=\hat{P}_{y,w}$.

Let

$$D=\{w\in W_a\mid w(A^-)\subset\mathcal{C}^+\}.$$

Elements in ND are called *dominant*. If y, $w\in D$ and $y(A^-)$, $w(A^-)$ are sufficiently far from the walls of $\mathcal{C}^+$, it turns out that $\hat{P}_{y,w}=P_{y,w}$.

For $\alpha\in\Phi$, put

$$\mathbf{f}(\alpha)=\begin{cases}1-v^{-2c_\alpha}\theta_{-\alpha} & \text{if } c_\alpha=\tilde{c}_\alpha,\\ (1-v^{-(c_\alpha+\tilde{c}_\alpha)}\theta_{-\alpha})(1+v^{-(c_\alpha-\tilde{c}_\alpha)}\theta_{-\alpha}) & \text{otherwise.}\end{cases}$$

(The condition $c_\alpha=\tilde{c}_\alpha$ is equivalent to the condition $\check{\alpha}\notin 2U$ where U is as in the proof of Proposition 2.1.1.)

Since all $\mathbf{f}(\alpha)$, $\alpha \in \Phi^+$, are invertible on $\hat{\mathcal{M}}$, we can define elements $E_C \in \hat{\mathcal{M}}$, $C \in X$, by

$$E_C = \prod_{\alpha>0} \mathbf{f}(\alpha)^{-1} D_C.$$

In the remainder of this section, we prove the following theorem.

Theorem 3.2.2. *(comp.* [9], *Theorem 3.5) For any $C \in X$, we have*

$$E_C = \sum_{A \leq C} \hat{P}_{A,C} A.$$

Let $C \in X$. We set

$$E'_C = \sum_{A \leq C} \hat{P}_{A,C} A \in \hat{\mathcal{M}},$$

and we define an $\mathcal{H}$-antilinear map $\tilde{\Phi}$ on $\hat{\mathcal{M}}$ given by

$$\tilde{\Phi}(A) = \sum_{B \in X} v^{-2\gamma(B)} \overline{\mathcal{R}_{Aw_0,Bw_0}} B$$

for $A \in X$.

The element E'_C can then be characterized by the conditions

(i) $\tilde{\Phi}(E'_C) = v^{-2\gamma(C)} E'_C$ and

(ii) $E'_C = \sum_{A \leq C} P'_{A,C} A$ where $P'_{A,C} \in \mathcal{A}^+$ has degree less than $c(A, C)$ if $A < C$ and $P'_{C,C} = 1$.

We first show that E_C satisfies (ii).

Let $\alpha > 0$. Remember that the inverse of $1 - p\,\theta_{-\alpha}$, $p \in \mathcal{A}$, on $\hat{\mathcal{M}}$ is given by

$$(1 - p\,\theta_{-\alpha})^{-1} = \sum_{n \geq 0} p^n \theta_{-n\alpha}.$$

First, suppose $c_\alpha = \tilde{c}_\alpha$. The degree of the coefficient of $Ap_{-n\alpha}$, $A \in X$, $n \geq 0$, in $v^{-2nc_\alpha} \theta_{-n\alpha} A$ is $c(Ap_{-n\alpha}, A) - 2nc_\alpha$.

Next, suppose $c_\alpha \neq \tilde{c}_\alpha$. The degree of the coefficient of $Ap_{-(m+n)\alpha}$, $A \in X$, $m, n \geq 0$, in $v^{-m(c_\alpha + \tilde{c}_\alpha)} \theta_{-m\alpha} (-1)^n v^{-n(c_\alpha - \tilde{c}_\alpha)} \theta_{-n\alpha} A$ is $c(Ap_{-(m+n)\alpha}, A) - m(c_\alpha + \tilde{c}_\alpha) - n(c_\alpha - \tilde{c}_\alpha)$.

So if $\beta = \sum_{\alpha>0} n_\alpha \alpha \neq 0$, $n_\alpha \geq 0$, then the degree of the coefficient of Ap_β in $\prod_{\alpha>0} \mathbf{f}(\alpha)^{-1} A$, $A \in X$, is less than $c(Ap_\beta, A)$. (Also, notice that in both cases the degree is non–negative.) This, together with the facts that $Q_{A,C}$ has degree less than $c(A, C)$ if $A < C$ and $Q_{C,C} = 1$ shows that E_C satisfies condition (ii).

Condition (i) is implied by the following theorem.

Theorem 3.2.3. *(comp. [9], Theorem 3.7) We can write $\tilde{\Phi}$ as*
$$\tilde{\Phi} = \prod_{\alpha>0} \mathbf{f}(\alpha)^{-1}\hat{\Phi}_\gamma \prod_{\alpha>0} \mathbf{f}(\alpha).$$

The rest of this section will be taken by the proof of this theorem.

We have an $\mathcal{A}$-valued symmetric bilinear form $(\cdot\,,\cdot)$ on $\mathcal{H}$ satisfying
$$(\check{T}_w, \check{T}_y) = v^{2h(w)}\delta_{y,w}$$
where $y,\, w \in W_a$. We also write $(\cdot\,,\cdot)$ for the corresponding bilinear form on $\mathcal{M}$.

Let $\mathcal{H}^*$ be the $\mathcal{A}$-module consisting of all formal sums $\sum_{w\in W_a} a_w T_w$, $a_w \in \mathcal{A}$ for $w \in W_a$. The domain of $(\cdot,\cdot)$ can be extended to $\mathcal{H}^* \times \mathcal{H}$ (and $\mathcal{H} \times \mathcal{H}^*$), and we can consider $\mathcal{H}^*$ as the dual space of $\mathcal{H}$ via $f(g) = (f,g)$ for $f \in \mathcal{H}^*$, $g \in \mathcal{H}$. For an arbitrary $\mathcal{A}$-linear map $\pi : \mathcal{H} \to \mathcal{H}^*$, the formal adjoint operator $\pi^* : \mathcal{H} \to \mathcal{H}^*$ is then defined by
$$(\pi(\check{T}_w), \check{T}_y) = (\check{T}_w, \pi^*(\check{T}_y))$$
for $y,\, w \in W_a$. We will also need to extend the domain of π^* to $\hat{\mathcal{H}}_x$ for some $x \in W_0$. If, for example, π is of the form
$$\pi(\check{T}_w) = \sum_{y\succeq w} a_{y,w}\check{T}_y$$
with $a_{y,w} \in \mathcal{A}$, $y,\, w \in W_a$, we have
$$\pi^*(\check{T}_y) = v^{2h(y)} \sum_{w\preceq y} a_{y,w} v^{-2h(w)}\check{T}_w$$
for $y \in W_a$. Hence π^* can be regarded as an element of $\mathrm{End}_{\mathcal{A}}(\hat{\mathcal{H}})$.

Formal adjoint operators on $\mathcal{M}$ (or $\hat{\mathcal{M}}_x$, $x \in W_0$) are defined accordingly.

We define an involutive ring automorphism j on $\mathcal{H}$ by
$$j(\sum_{w\in W_a} a_w T_w) = \sum_{w\in W_a} \overline{a_w}(-1)^{l(w)} v^{2m(w)},$$
$a_w \in \mathcal{A}$ for $w \in W_a$. The corresponding map in $\mathrm{Aut}_{\mathbb{Z}}(\mathcal{M})$ will also be denoted by j.

Let $w = xp_\lambda \in W_a$, $x \in W_0$, $\lambda \in Q$, and let $\mu \in Q \cap T^+$ be such that $\lambda + \mu \in T^+$. We compute
$$\begin{aligned} j(\check{T}_w) &= j(T_x\check{T}_\lambda) = j(T_x T_{\lambda+\mu} {T_\mu}^{-1}) \\ &= (-1)^{l(x)+l(p_{\lambda+\mu})-l(p_\mu)}\, v^{-2m(x)-2m(p_{\lambda+\mu})+2m(p_\mu)} T_x T_{\lambda+\mu} T_\mu^{-1} \\ &= (-1)^{g(w)} v^{-2h(w)}\check{T}_w. \end{aligned}$$

It follows that
$$j(A) = (-1)^{\delta(A)} v^{-2\gamma(A)} A$$
for $A \in X$.

We will also consider j as a map on $\hat{\mathcal{H}}_x$ and $\hat{\mathcal{M}}_x$, $x \in W_0$. Note that $j^* = j$.

The following three lemmas contain several relations needed for the proof of Theorem 3.2.3. The calculations are straightforward, and we only show the first lemma.

Lemma 3.2.4. *(comp. [9], Lemma 3.10) On $\mathcal{H}$ resp. $\mathcal{M}$, we have*

(i) $j\,\Theta_\lambda\, j = \Theta_\lambda$ *for* $\lambda \in Q$,
(ii) $j\,\tilde{\Theta}_s\, j = -v^{-2c_s}\tilde{\Theta}_s$ *for* $s \in S_0$,
(iii) $j\,\psi\, j = (-1)^{l(w_0)}v^{-2m(w_0)}\,\psi$ *and*
(iv) $j\,\varphi\, j = (-1)^{l(w_0)}v^{-2m(w_0)}\,\varphi$.

Moreover, (i) holds on $\hat{\mathcal{H}}_x$, $x \in W_0$, (ii) is an equality of operators in $\mathrm{Hom}_{\mathcal{H}}(\hat{\mathcal{H}}_x, \hat{\mathcal{H}}_{xs})$, *$x \in W_0$, and (iii) resp. (iv) is an equality of $\mathcal{H}$-antilinear operators in* $\mathrm{Hom}_{\mathbb{Z}}(\hat{\mathcal{H}}, \hat{\mathcal{H}}_{w_0})$. *resp.* $\mathrm{Hom}_{\mathbb{Z}}(\hat{\mathcal{M}}, \hat{\mathcal{M}}_{w_0})$.

Proof. (i) Since

$$j(\hat{T}_\lambda) = v^{h(\lambda)}(-1)^{2\mathrm{ht}(\lambda)}v^{-2h(\lambda)}\check{T}_\lambda = \hat{T}_\lambda,$$

we obtain

$$j\,\Theta_\lambda\, j(f) = j(j(f)\hat{T}_\lambda) = f\, j(\hat{T}_\lambda) = \Theta_\lambda(f)$$

for $f \in \hat{\mathcal{H}}_x$, $x \in W_0$.

(ii) If $s = s_\alpha$, $\alpha \in \Pi$, we have

$$\begin{aligned} j(\tilde{I}_s) &= -v^{-2c_s}T_s(1-\hat{T}_{-2\alpha}) - \hat{T}_{-\alpha}(v^{-(c_s+\tilde{c}_s)} - v^{-(c_s-\tilde{c}_s)}) + 1 - v^{2c_s} \\ &= -v^{-2c_s}(T_s(1-\hat{T}_{-2\alpha}) - \hat{T}_{-\alpha}(v^{c_s+\tilde{c}_s} - v^{c_s-\tilde{c}_s}) + 1 - v^{-2c_s}) \\ &= -v^{-2c_s}\tilde{I}_s. \end{aligned}$$

For $f \in \hat{\mathcal{H}}_x$, $x \in W_0$, we conclude

$$j\,\tilde{\Theta}_s\, j(f) = j(j(f)\tilde{I}_s) = -v^{-2c_s}\tilde{\Theta}_s(f).$$

(iii) We calculate

$$\begin{aligned} j\,\psi\, j(\check{T}_w) &= j\,\psi((-1)^{g(w)}v^{-2h(w)}\check{T}_w) \\ &= j((-1)^{g(w)}v^{2h(w)}\overline{\check{T}}_w T_{w_0}) \\ &= (-1)^{l(w_0)}v^{-2m(w_0)}\overline{\check{T}}_w T_{w_0} \\ &= (-1)^{l(w_0)}v^{-2m(w_0)}\psi(\check{T}_w) \end{aligned}$$

for $w \in W_a$.

(iv) This is equivalent to (iii).

Lemma 3.2.5. *(comp. [9], Lemma 3.11) The map $\tilde{\Phi}$ can be expressed as*

$$\tilde{\Phi} = v^{-2m(w_0)}j\,(j\,\varphi\,\hat{\Phi}_\gamma\,\varphi)^*.$$

Notice that $(\varphi\, \hat{\Phi}_\gamma\, \varphi)^*$ and therefore $j\,(j\,\varphi\hat{\Phi}_\gamma\,\varphi)^*$ can be considered as $\mathcal{H}$-antilinear elements of $\mathrm{End}_{\mathbb{Z}}(\hat{\mathcal{M}})$.

Lemma 3.2.6. *(comp. [9], Lemma 3.13) On $\mathcal{H}$ resp. $\mathcal{M}$ and moreover as elements of* $\mathrm{Hom}_{\mathbb{Z}}(\hat{\mathcal{H}}, \hat{\mathcal{H}}_{w_0})$ *resp.* $\mathrm{Hom}_{\mathbb{Z}}(\hat{\mathcal{M}}, \hat{\mathcal{M}}_{w_0})$*, we have*

(i) $(j\,\psi)^* = (-1)^{l(w_0)} j\,\psi$ *and*

(ii) $(j\,\varphi)^* = (-1)^{l(w_0)} j\,\varphi$.

Let

$$\tilde{\Psi} = v^{-2m(w_0)} j\,(j\,\psi\,\Psi\,\psi)^*$$

be the operator on $\hat{\mathcal{H}}$ corresponding to $\tilde{\Phi}$. By the previous lemma, we get

$$\begin{aligned}\tilde{\Psi} &= j\,(j\,\psi\,\psi\,\Theta^+_{w_0}\,\psi)^* \\ &= j\,(j\,\psi)^*\,(j\,\Theta^+_{w_0}\,j)^* \\ &= (-1)^{l(w_0)}\psi\,(j\,\Theta^+_{w_0}\,j)^*.\end{aligned}$$

For $s \in S_0$, we define $\mathcal{A}$-linear maps β_s and ρ_s on $\mathcal{H}$ by

$$\beta_s(T_x\check{T}_\lambda) = T_x T_s \check{T}_\lambda$$

and

$$\rho_s(T_x\check{T}_\lambda) = v^{h(\lambda-\lambda s)} T_x \check{T}_{\lambda s}$$

where $x \in W_0$ and $\lambda \in Q$. The map β_s can be considered as an element of $\mathrm{End}_{\mathcal{A}}(\hat{\mathcal{H}}_x)$ and ρ_s as an element of $\mathrm{Hom}_{\mathcal{A}}(\hat{\mathcal{H}}_x, \hat{\mathcal{H}}_{xs})$, $x \in W_0$. Let $s = s_\alpha$, $\alpha \in \Pi$. For $x \in W_0$, $\lambda \in Q$, we have

$$\begin{aligned}&\{(\beta_s(1-\Theta_{-2\alpha}) - \Theta_{-\alpha}(v^{c_s+c_{\tilde{s}}} - v^{c_s-c_{\tilde{s}}}) + 1 - v^{2c_s})\rho_s\}(T_x\check{T}_\lambda) \\ &= v^{h(\lambda-\lambda s)}(\beta_s(1-\Theta_{-2\alpha}) - \Theta_{-\alpha}(v^{c_s+c_{\tilde{s}}} - v^{c_s-c_{\tilde{s}}}) + 1 - v^{2c_s})(T_x\check{T}_{\lambda s}) \\ &= v^{h(\lambda-\lambda s)} T_x(T_s\check{T}_{\lambda s}(1-\hat{T}_{-2\alpha}) - \check{T}_{\lambda s}\hat{T}_{-\alpha}(v^{c_s+c_{\tilde{s}}} - v^{c_s-c_{\tilde{s}}}) + \check{T}_{\lambda s}(1-v^{2c_s})) \\ &= v^{h(\lambda-\lambda s)} T_x \tilde{I}_s \check{T}_{\lambda s} \\ &= T_x\check{T}_\lambda \tilde{I}_s\end{aligned}$$

where the last equality follows from Proposition 2.1.1. So

$$\tilde{\Theta}_s = (\beta_s(1-\Theta_{-2\alpha}) - \Theta_{-\alpha}(v^{c_s+c_{\tilde{s}}} - v^{c_s-c_{\tilde{s}}}) + 1 - v^{2c_s})\rho_s.$$

Arguments analogous to the ones in *loc. cit.* then demonstrate the following lemma.

Lemma 3.2.7. *(comp. [9], Lemma 3.15) On $\mathcal{H}$ and as elements of* $\mathrm{Hom}_{\mathcal{A}}(\hat{\mathcal{H}}_x, \hat{\mathcal{H}}_{xs})$, *$x \in W_0$, $s \in S_0$, we have*

$$(\tilde{\Theta}_s)^* = \tilde{\Theta}_s.$$

The next lemma contains a few more equalities which are all easy consequences of the definitions.

Lemma 3.2.8. *On $\mathcal{H}$, we have*

(i) $\Theta_\lambda^* = \Theta_{-\lambda}$ *for* $\lambda \in Q$,
(ii) $\tilde{\Theta}_s \Theta_\alpha = \Theta_{\alpha s} \tilde{\Theta}_s$ *for* $s \in S_0$, $\alpha \in \Phi$, *and*
(iii) $\psi\, \Theta_\alpha = \Theta_{-\alpha} \psi$ *for* $\alpha \in \Phi$.

Furthermore, (i) and (ii) hold on $\hat{\mathcal{H}}_x$, $x \in W_0$, *and (iii) on* $\hat{\mathcal{H}}$.

We remark that (ii) implies

$$(*) \qquad \Theta_{w_0}^+ \Theta_\alpha = \Theta_{-\alpha} \Theta_{w_0}^+$$

for $\alpha \in \Phi$.

We now prove Theorem 3.2.3 or rather the equivalent on $\hat{\mathcal{H}}$.

Recall that $\Theta_{w_0}^+ = \prod_{\alpha>0} \mathbf{e}(\alpha)^{-1} \tilde{\Theta}_{w_0}$ and that for $p \in \mathcal{A}$ and $\alpha > 0$, we have

$$(1 - p\,\Theta_{-\alpha})^{-1} = -p\,\Theta_{-\alpha} \sum_{n \geq 0} p^{-n} \Theta_{n\alpha}$$

on $\hat{\mathcal{H}}_{w_0}$ and

$$(1 - p\,\Theta_{\alpha})^{-1} = -p\,\Theta_{\alpha} \sum_{n \geq 0} p^{-n} \Theta_{-n\alpha}$$

on $\hat{\mathcal{H}}$. Therefore, using Lemma 3.2.4 (ii) and (i), Lemma 3.2.7 and Lemma 3.2.8 (i), we can write

$$(j\,\Theta_{w_0}^+ j)^* \;=\; (-1)^{l(w_0)} v^{-2m(w_0)} \tilde{\Theta}_{w_0} \prod_{\alpha>0} (1 - v^{-(c_\alpha + \tilde{c}_\alpha)} \Theta_\alpha)^{-1} (1 + v^{-(c_\alpha - \tilde{c}_\alpha)} \Theta_\alpha)^{-1}.$$

Thus

$$\begin{aligned}
\tilde{\Psi} &= (-1)^{l(w_0)} \psi\,(j\,\Theta_{w_0}^+ j)^* \\
&= v^{2m(w_0)} \psi\, \tilde{\Theta}_{w_0} \prod_{\alpha>0} (1 - v^{-(c_\alpha + \tilde{c}_\alpha)} \Theta_\alpha)^{-1} (1 + v^{-(c_\alpha - \tilde{c}_\alpha)} \Theta_\alpha)^{-1} \\
&= v^{2m(w_0)} \psi \prod_{\alpha>0} (1 - v^{c_\alpha + \tilde{c}_\alpha} \Theta_{-\alpha})(1 + v^{c_\alpha - \tilde{c}_\alpha} \Theta_{-\alpha})\, \Theta_{w_0}^+ \\
&\qquad \cdot \prod_{\alpha>0} (1 - v^{-(c_\alpha + \tilde{c}_\alpha)} \Theta_\alpha)^{-1} (1 + v^{-(c_\alpha - \tilde{c}_\alpha)} \Theta_\alpha)^{-1}.
\end{aligned}$$

Set $\Phi_1^+ = \{\alpha \in \Phi^+ \mid c_\alpha = \tilde{c}_\alpha\}$ and $\Phi_2^+ = \{\alpha \in \Phi^+ \mid c_\alpha \neq \tilde{c}_\alpha\}$. Then

$$
\begin{aligned}
\tilde{\Psi} &= v^{2m(w_0)}\psi \prod_{\alpha\in\Phi_1^+}(1-v^{2c_\alpha}\Theta_{-\alpha}) \prod_{\alpha\in\Phi_2^+}(1-v^{c_\alpha+\tilde{c}_\alpha}\Theta_{-\alpha})(1+v^{c_\alpha-\tilde{c}_\alpha}\Theta_{-\alpha}) \\
&\quad \cdot \Theta_{w_0}^+ \prod_{\alpha\in\Phi_1^+}(1+\Theta_\alpha) \prod_{\alpha\in\Phi_1^+}(1+\Theta_\alpha)^{-1} \\
&\quad \cdot \prod_{\alpha\in\Phi_1^+}(1-v^{-2c_\alpha}\Theta_\alpha)^{-1} \prod_{\alpha\in\Phi_2^+}(1-v^{-(c_\alpha+\tilde{c}_\alpha)}\Theta_\alpha)^{-1}(1+v^{-(c_\alpha-\tilde{c}_\alpha)}\Theta_\alpha)^{-1} \\
&= v^{2m(w_0)}\psi \prod_{\alpha\in\Phi_1^+}(1-v^{-2c_\alpha}\Theta_{-\alpha})^{-1} \prod_{\alpha\in\Phi_2^+}(1-v^{-(c_\alpha+\tilde{c}_\alpha)}\Theta_{-\alpha})^{-1}(1+v^{-(c_\alpha-\tilde{c}_\alpha)}\Theta_{-\alpha})^{-1} \\
&\quad \cdot \Theta_{w_0}^+ \prod_{\alpha\in\Phi_1^+}(1-v^{2c_\alpha}\Theta_\alpha) \prod_{\alpha\in\Phi_2^+}(1-v^{c_\alpha+\tilde{c}_\alpha}\Theta_\alpha)(1+v^{c_\alpha-\tilde{c}_\alpha}\Theta_\alpha)
\end{aligned}
$$

For the above calculations, we used consequence $(*)$ of Lemma 3.2.8 (ii). Lemma 3.2.8 (iii) now yields

$$
\begin{aligned}
\tilde{\Psi} &= v^{2m(w_0)} \prod_{\alpha\in\Phi_1^+}(1-v^{2c_\alpha}\Theta_\alpha)^{-1} \prod_{\alpha\in\Phi_2^+}(1-v^{c_\alpha+\tilde{c}_\alpha}\Theta_\alpha)^{-1}(1+v^{c_\alpha-\tilde{c}_\alpha}\Theta_\alpha)^{-1} \\
&\quad \cdot \psi\, \Theta_{w_0}^+ \prod_{\alpha\in\Phi_1^+}(1-v^{2c_\alpha}\Theta_\alpha) \prod_{\alpha\in\Phi_2^+}(1-v^{c_\alpha+\tilde{c}_\alpha}\Theta_\alpha)(1+v^{c_\alpha-\tilde{c}_\alpha}\Theta_\alpha).
\end{aligned}
$$

Using

$$
\sum_{\alpha\in\Phi_1^+} 2c_\alpha + \sum_{\alpha\in\Phi_2^+}(c_\alpha+\tilde{c}_\alpha+c_\alpha-\tilde{c}_\alpha) = \sum_{\alpha>0} 2c_\alpha = 2m(w_0),
$$

we can rewrite the last expression for $\tilde{\Psi}$ as

$$
\begin{aligned}
\tilde{\Psi} &= v^{2m(w_0)}v^{-2m(w_0)}(-1)^{l(w_0)} \prod_{\alpha\in\Phi_1^+}\Theta_\alpha^{-1} \prod_{\alpha\in\Phi_2^+}\Theta_{2\alpha}^{-1} \\
&\quad \cdot \prod_{\alpha\in\Phi_1^+}(1-v^{-2c_\alpha}\Theta_{-\alpha})^{-1} \prod_{\alpha\in\Phi_2^+}(1-v^{-(c_\alpha+\tilde{c}_\alpha)}\Theta_{-\alpha})^{-1}(1+v^{-(c_\alpha-\tilde{c}_\alpha)}\Theta_{-\alpha})^{-1} \\
&\quad \cdot \psi\, \Theta_{w_0}^+ v^{2m(w_0)}(-1)^{l(w_0)} \prod_{\alpha\in\Phi_1^+}\Theta_\alpha \prod_{\alpha\in\Phi_2^+}\Theta_{2\alpha} \\
&\quad \cdot \prod_{\alpha\in\Phi_1^+}(1-v^{-2c_\alpha}\Theta_{-\alpha}) \prod_{\alpha\in\Phi_2^+}(1-v^{-(c_\alpha+\tilde{c}_\alpha)}\Theta_{-\alpha})(1+v^{-(c_\alpha-\tilde{c}_\alpha)}\Theta_{-\alpha}) \\
&= v^{-2m(w_0)} \prod_{\alpha\in\Phi_1^+}(1-v^{-2c_\alpha}\Theta_{-\alpha})^{-1} \prod_{\alpha\in\Phi_2^+}(1-v^{-(c_\alpha+\tilde{c}_\alpha)}\Theta_{-\alpha})^{-1}(1+v^{-(c_\alpha-\tilde{c}_\alpha)}\Theta_{-\alpha})^{-1} \\
&\quad \cdot \psi\, \Theta_{w_0}^+ \prod_{\alpha\in\Phi_1^+}(1-v^{-2c_\alpha}\Theta_{-\alpha}) \prod_{\alpha\in\Phi_2^+}(1-v^{-(c_\alpha+\tilde{c}_\alpha)}\Theta_{-\alpha})(1+v^{-(c_\alpha-\tilde{c}_\alpha)}\Theta_{-\alpha}).
\end{aligned}
$$

We conclude

$$\begin{aligned}\tilde{\Psi} &= \prod_{\alpha\in\Phi_1^+}(1-v^{-2c_\alpha}\Theta_{-\alpha})^{-1}\prod_{\alpha\in\Phi_2^+}(1-v^{-(c_\alpha+\tilde{c}_\alpha)}\Theta_{-\alpha})^{-1}(1+v^{-(c_\alpha-\tilde{c}_\alpha)}\Theta_{-\alpha})^{-1}\Psi\\ &\quad\cdot\prod_{\alpha\in\Phi_1^+}(1-v^{-2c_\alpha}\Theta_{-\alpha})\prod_{\alpha\in\Phi_2^+}(1-v^{-(c_\alpha+\tilde{c}_\alpha)}\Theta_{-\alpha})(1+v^{-(c_\alpha-\tilde{c}_\alpha)}\Theta_{-\alpha}),\end{aligned}$$

which is equivalent to Theorem 3.2.3.

3.3. **The Kazhdan–Lusztig polynomials.** We now generalize the formula in [9] for Kazhdan–Lusztig polynomials. They appear as coefficients of elements $F_C \in \hat{\mathcal{M}}$, $C \in X$. At the end of the section, we describe an analogue of Kostant's partition function, generalizing part of [8].

We start with introducing a third right action of W_0 on X. Let $A = y(A_\lambda^+) \subseteq \Pi_\lambda$, $y \in W_a$, $\lambda \in T$. For $s \in S_0$ with $c_s = c_{\tilde{s}}$, we set

$$A \circledast s = y(A^+_{(\lambda+\rho)s-\rho}) \subseteq \Pi_{(\lambda+\rho)s-\rho}.$$

For $s \in S_0$ with $c_s \neq c_{\tilde{s}}$, we set

$$A \circledast s = y(A^+_{(\lambda+2\rho)s-2\rho}) \subseteq \Pi_{(\lambda+2\rho)s-2\rho}.$$

Hence for $w \in W_0$, we have

$$A \circledast w = y(A^+_{(\lambda+\tilde{\rho})w-\tilde{\rho}}) \subseteq \Pi_{(\lambda+\tilde{\rho})w-\tilde{\rho}}$$

where

$$\tilde{\rho} = \frac{1}{2}\sum_{\alpha\in\Phi_1^+}\alpha + \sum_{\alpha\in\Phi_2^+}\alpha.$$

If $A = A^- y$, $C = A^- w$, y, $w \in W_a$, we let $P_{A,C} = P_{y,w}$. The next theorem and corollary contain the main statements of this section.

Theorem 3.3.1. *(comp. [9], Theorem 4.2) Let C be an alcove in $\mathcal{C}^+$. The coefficient of any alcove $A \subset \mathcal{C}^+$ in*

$$F_C = \sum_{x\in W_0}(-1)^{l(x)}v^{c(C\circledast x, C)}E_{C\circledast x}$$

equals $P_{A,C}$.

Remark 3.3.2. As in *loc. cit.*, the above theorem implies that $P_{A,C}$, A, $C \subset \mathcal{C}^+$, can be calculated from $P_{A',C'}$ for A', $C' \subset \Pi$. Furthermore, for equal parameters, the above formula generalizes the q-analogue of Kostant's weight multiplicity formula found in [8].

Corollary 3.3.3. *(comp. [9], Corollary 4.3) For any alcoves A, $C \subset \mathcal{C}^+$, we have*

$$P_{A,C} = \sum_{x \in W_0} (-1)^{l(w)} v^{c(C \circledast x, C)} \hat{P}_{A, C \circledast x}.$$

The corollary is an immediate consequence of the preceding theorem. In order to prove the theorem, we first have to generalize a result of Andersen in [1]. Let $\mathcal{N}$ be the $\mathcal{H}$-submodule of $\mathcal{M}$ generated by $A^+ + A^+ s$ where s runs over S_0, and let

$$\hat{\mathcal{N}}^\perp = \{f \in \hat{\mathcal{M}}_{w_0} \mid (n, f) = 0 \text{ for all } n \in \mathcal{N}\}.$$

Instead of considering $\mathcal{M}$, Andersen introduces a module $\mathcal{M}_-$ which is a free $\mathcal{A}$-module on $\mathcal{C}^-$ and has an $\mathcal{H}$-module structure given by

$$T_s A = \begin{cases} -A & \text{if } sA \not\subset \mathcal{C}^-, \\ sA & \text{if } sA \in \mathcal{C}^- \text{ and } sA > A, \\ v^{2c_s} sA + (v^{2c_s} - 1)A & \text{if } sA < A \end{cases}$$

for $s \in S_a$ and $A \subset \mathcal{C}^-$. It turns out that $\mathcal{M}_-$ is isomorphic to $\mathcal{M}/\mathcal{N}$. By modifying [14], Andersen obtains elements D'_C for all alcoves $C \subset \mathcal{C}^-$ whose coefficients $Q'_{A,C}$ are exactly inverse Kazhdan–Lusztig polynomials. The polynomials $Q'_{A,C}$ satisfy an induction formula which coincides with a formula for the composition factors of Weyl modules. Andersen obtains the latter formula by "inverting" the Lusztig conjecture in [13].

The result we need is the following one, which for equal parameters reformulates [1], Theorem 7.2.

Theorem 3.3.4. *For alcoves A, $B \subset \mathcal{C}^-$, set*

$$\mathcal{R}'_{B,A} = \sum_{x \in W_0} R_{y,wx}$$

where y, $w \in D$ are such that $y(A^+) = A$ and $w(A^+) = B$. Then

$$(\hat{\Phi}_\gamma(A), f) = v^{-2\gamma(A)} \sum_{B \subset \mathcal{C}^-, B \leq A} (-1)^{d(A,B)} \mathcal{R}'_{B,A}\,(B, f)$$

for any $f \in \hat{\mathcal{N}}^\perp$.

Note that $\mathcal{R}'_{B,A} = 0$ unless $B \leq A$.

We comment on those parts of the proof of the above theorem which require real changes.

Let $\alpha \in \Pi$. We denote by $\mathcal{N}_\alpha$ the $\mathcal{H}$-submodule of $\mathcal{M}$ generated by $A^+ + A^+ s_\alpha$, and for $n \in \mathbb{Z}$, we set

$$U_n = \{v \in V \mid n < \langle v, \check{\alpha} \rangle < n + 1\}.$$

(Note *loc. cit.*, Remark 7.4.) For any alcove A in U_0, we define alcoves A_n, $n \in \mathbb{Z}$, inductively by setting $A = A_0$ and $A_n = A_{n-1}\sigma_{\alpha,n}$. We write

$$f_{A_n} = A_n + A_{n+1}$$

where $n \in \mathbb{Z}$ and

$$\begin{aligned} g_n(A) &= v^{2c(A,A_{n-1})}A_{-n} + \sum_{i=1}^{n-1}(v^{2c(A,A_{n-i})} - v^{2c(A,A_{n-i-1})})A_{2i-n-1} + A_{n-1}. \\ &= v^{2c(A,A_{n-1})}f_{A_{-n}} - \sum_{i=1}^{n-1} v^{2c(A,A_{n-i-1})}(f_{A_{2i-n-1}} - f_{A_{2i-n}}) \end{aligned}$$

where $n \in \mathbb{N}$.

Lemma 3.3.5. *(comp. [1], Lemma 4.3)*

(i) *The elements $g_n(A)$, A an alcove in U_0 and $n \in \mathbb{N}$, constitute a basis for $\mathcal{N}_\alpha$ as an $\mathcal{A}$-module.*

(ii) *Let A be an alcove in U_0. We have*

$$v^{2c(A,A_{n-1})}f_{A_{-n}} + f_{A_{n-2}} \in \mathcal{N}_\alpha$$

if either $c_\alpha = \tilde{c}_\alpha$ and $n \in \mathbb{N}$ arbitrary, or if $c_\alpha \neq \tilde{c}_\alpha$ and $n \in \mathbb{N}$ odd or equal to 2. If $c_\alpha \neq \tilde{c}_\alpha$ and $n \in \mathbb{N}$ even, we have

$$v^{2c(A,A_{n-1})}f_{A_{-n}} + v^{2c(A,A_1)}(v^{2c(A_1,A_{n-2})}f_{A_{-n+2}} + f_{A_{n-4}}) + f_{A_{n-2}} \in \mathcal{N}_\alpha.$$

Proof. (i) for any $\alpha \in \Pi$ and (ii) for $\alpha \in \Pi \cap \Phi_1^+$ follow as in *loc. cit.*

Suppose $\alpha \in \Pi \cap \Phi_2^+$. For $n \in \mathbb{N}$, we denote the respective expression in (ii) by $F_n(A)$. (We take the first expression for $n = 2$. The second expression then equals $2F_2(A)$.) It is easily checked that

$$\begin{aligned} F_1(A) &= v^{2c(A,A_0)}f_{A_{-1}} + f_{A_{-1}} = 2g_1(A) \in \mathcal{N}_\alpha, \\ F_2(A) &= v^{2c(A,A_1)}f_{A_{-2}} + f_A = g_2(A) + g_1(A) \in \mathcal{N}_\alpha, \\ F_3(A) &= v^{2c(A,A_2)}f_{A_{-3}} + f_{A_1} = g_3(A) + F_2(A) - v^{2c(A,A_1)}g_1(A) \in \mathcal{N}_\alpha, \text{ and} \\ F_4(A) &= v^{2c(A,A_3)}f_{A_{-4}} + v^{2c(A,A_1)}(v^{2c(A_1,A_2)}f_{A_{-2}} + f_A) + f_{A_2} \\ &= g_4(A) + F_3(A) + v^{2c(A,A_1)}g_1(A) \in \mathcal{N}_\alpha. \end{aligned}$$

For $n \geq 5$ we claim

$$F_n(A) = \begin{cases} g_n(A) + (F_{n-1}(A) - v^{2c(A,A_1)}F_{n-2}(A)) - v^{2c(A,A_2)}g_{n-4}(A) & \text{if } n \text{ odd,} \\ g_n(A) + (F_{n-1}(A) + v^{2c(A,A_1)}F_{n-3}(A)) - v^{2c(A,A_2)}g_{n-4}(A) & \text{if } n \text{ even.} \end{cases}$$

Indeed, if n is odd we obtain

$$\begin{aligned}
\text{RHS} &= v^{2c(A,A_{n-1})} f_{A_{-n}} - \sum_{i=1}^{n-1} v^{2c(A,A_{n-i-1})}(f_{A_{2i-n-1}} - f_{A_{2i-n}}) \\
&\quad + v^{2c(A,A_{n-2})} f_{A_{-n+1}} + v^{2c(A,A_{n-3})} f_{A_{-n+3}} + v^{2c(A,A_1)} f_{A_{n-5}} + f_{A_{n-3}} \\
&\quad - v^{2c(A,A_1)} v^{2c(A,A_{n-3})} f_{A_{-n+2}} - v^{2c(A,A_1)} f_{A_{n-4}} \\
&\quad - v^{2c(A,A_{n-3})} f_{A_{-n+4}} + \sum_{i=3}^{n-3} v^{2c(A,A_{n-i-1})}(f_{A_{2i-n-1}} - f_{A_{2i-n}}) \\
&= v^{2c(A,A_{n-1})} f_{A_{-n}} - v^{2c(A,A_{n-2})}(f_{A_{-n+1}} - f_{A_{-n+2}}) - v^{2c(A,A_{n-3})}(f_{A_{-n+3}} - f_{A_{-n+4}}) \\
&\quad - v^{2c(A,A_1)}(f_{A_{n-5}} - f_{A_{n-4}}) - (f_{A_{n-3}} - f_{A_{n-2}}) \\
&\quad + v^{2c(A,A_{n-2})} f_{A_{-n+1}} + v^{2c(A,A_{n-3})} f_{A_{-n+3}} + v^{2c(A,A_1)} f_{A_{n-5}} + f_{A_{n-3}} \\
&\quad - v^{2c(A,A_{n-2})} f_{A_{-n+2}} - v^{2c(A,A_1)} f_{A_{n-4}} - v^{2c(A,A_{n-3})} f_{A_{-n+4}} \\
&= v^{2c(A,A_{n-1})} f_{A_{-n}} + f_{A_{n-2}} \\
&= F_n(A).
\end{aligned}$$

For n even, we obtain

$$\begin{aligned}
\text{RHS} &= v^{2c(A,A_{n-1})} f_{A_{-n}} - \sum_{i=1}^{n-1} v^{2c(A,A_{n-i-1})}(f_{A_{2i-n-1}} - f_{A_{2i-n}}) \\
&\quad + v^{2c(A,A_{n-2})} f_{A_{-n+1}} + f_{A_{n-3}} + v^{2c(A,A_{n-3})} f_{A_{-n+3}} + v^{2c(A,A_1)} f_{A_{n-5}} \\
&\quad - v^{2c(A,A_{n-3})} f_{A_{-n+4}} + \sum_{i=3}^{n-3} v^{2c(A,A_{n-i-1})}(f_{A_{2i-n-1}} - f_{A_{2i-n}}) \\
&= v^{2c(A,A_{n-1})} f_{A_{-n}} - v^{2c(A,A_{n-2})}(f_{A_{-n+1}} - f_{A_{-n+2}}) - v^{2c(A,A_{n-3})}(f_{A_{-n+3}} - f_{A_{-n+4}}) \\
&\quad - v^{2c(A,A_1)}(f_{A_{n-5}} - f_{A_{n-4}}) - (f_{A_{n-3}} - f_{A_{n-2}}) \\
&\quad + v^{2c(A,A_{n-2})} f_{A_{-n+1}} + f_{A_{n-3}} + v^{2c(A,A_{n-3})} f_{A_{-n+3}} + v^{2c(A,A_1)} f_{A_{n-5}} \\
&\quad - v^{2c(A,A_{n-3})} f_{A_{-n+4}} \\
&= v^{2c(A,A_{n-1})} f_{A_{-n}} + v^{2c(A,A_{n-2})} f_{A_{-n+2}} + v^{2c(A,A_1)} f_{A_{n-4}} + f_{A_{n-2}} \\
&= F_n(A).
\end{aligned}$$

So $F_n(A) \in \mathcal{N}_\alpha$ for any $n \in \mathbb{N}$.

We need this lemma for the following proposition.

Proposition 3.3.6. *(comp. [1], Proposition 5.1) Let $\lambda \in T$ and $w \in W_0$. We have*

$$e_\lambda - (-1)^{l(w)} v^{c(A^+_{\lambda w}, A^+_\lambda)} e_{\lambda w} \in \mathcal{N}.$$

Proof. As in *loc. cit.*, it is enough to prove that

$$e_\lambda + v^{c(A^+_{\lambda\sigma_\alpha}, A^+_\lambda)} e_{\lambda\sigma_\alpha} \in \mathcal{N}_\alpha \subseteq \mathcal{N}$$

for all $\alpha \in \Pi$. Let $\alpha \in \Pi$ and $n \in \mathbb{Z}$ be such that $\lambda \in H_{\alpha,n}$. We write

$$e_\lambda = \sum_{B \in X, \bar{B} \ni \lambda} B = \sum_{B \in X, \bar{B} \ni \lambda, B < B\sigma_{\alpha,n}} (B + B\sigma_{\alpha,n}) = \sum_A f_{A_{n-1}}$$

where the last sum runs over all alcoves A in U_0 such that $\lambda \in \overline{A_{n-1}}$. Similarly, we have

$$e_{\lambda\sigma_\alpha} = \sum_A f_{A_{-n-1}}$$

where the last sum runs over the same set of alcoves A as before.

According to the definition of T, if $c_\alpha \neq \tilde{c}_\alpha$ the integer n is necessarily even. So, since $c(A^+_{\lambda\sigma_\alpha}, A^+_\lambda) = 2c(A, A^+_\lambda)$ where A is the alcove in U_0 such that $A_n = A^+_\lambda$ (hence $A^+_{\lambda\sigma_\alpha} = A_{-n}$), the claim follows from the previous lemma.

Notice that this proposition shows that the operator $\theta_w \in \mathrm{Aut}_{\mathcal{H}}(\mathcal{M}^0)$, $w \in W_0$, maps $\mathcal{M}^0 \cap \mathcal{N}$ into itself and therefore induces an $\mathcal{H}$-linear automorphism of $\mathcal{M}^0/\mathcal{M}^0 \cap \mathcal{N}$. Theorem 3.3.4 can now be proven as in *loc. cit.*

Back to the proof of Theorem 3.3.1, it remains to check that

$$j\,\varphi(F_C) \in \hat{\mathcal{N}}^\perp.$$

(The other arguments in [9] have a straightforward generalization.)

The left ideal $\mathcal{I}$ of $\mathcal{H}$ generated by $T_{w_0} + T_{w_0 s}$ for any $s \in S_0$ is the submodule of $\mathcal{H}$ corresponding to $\mathcal{N}$. Hence the condition

$$j\,\varphi(f) \in \hat{\mathcal{N}}^\perp$$

for $f \in \hat{\mathcal{M}}$ is equivalent to the condition

$$(\mathcal{I}, j\,\psi(g)) = 0$$

for $g = \phi^{-1}(f) \in \hat{\mathcal{H}}$. And since

$$j\,\psi(T_{w_0} + T_{w_0 s}) = j(1 + T_s) = 1 - v^{-2c_s} T_s,$$

the latter condition is equivalent to

$$(\mathcal{H}(T_s - v^{2c_s}), g) = 0.$$

If $s = s_\alpha$, $\alpha \in \Pi$, the operator $\tilde{\Theta}_s - \mathbf{e}(\alpha)$ on $\mathcal{H}$ is right multiplication with

$$\begin{aligned} &T_s(1 - \hat{T}_{-2\alpha}) - \hat{T}_{-\alpha}(v^{c_s + c_{\tilde{s}}} - v^{c_s - c_{\tilde{s}}}) + 1 - v^{2c_s} - (1 - \hat{T}_{-\alpha} v^{c_s + c_{\tilde{s}}})(1 + \hat{T}_{-\alpha} v^{c_s - c_{\tilde{s}}}) \\ &= \; T_s(1 - \hat{T}_{-2\alpha}) - v^{2c_s} + \hat{T}_{-2\alpha} v^{2c_s}, \end{aligned}$$

and therefore $(1-\Theta_{-2\alpha})^{-1}(\tilde{\Theta}_s - \mathbf{e}(\alpha))$ is equal to the map $h \mapsto h(T_s - v^{2c_s})$, $h \in \mathcal{H}$. Hence the formal adjoint operator of this map is

$$((1-\Theta_{-2\alpha})^{-1}(\tilde{\Theta}_s - \mathbf{e}(\alpha))^* = (\tilde{\Theta}_s - \mathbf{e}(-\alpha))(1-\Theta_{2\alpha})^{-1},$$

which we consider as a map on $\hat{\mathcal{H}}$. The condition on g becomes

$$((\tilde{\Theta}_s - \mathbf{e}(\alpha))(1-\Theta_{2\alpha})^{-1}g, \mathcal{H}) = 0,$$

i.e.

$$\tilde{\Theta}_s(1-\Theta_{2\alpha})^{-1}g = \mathbf{e}(-\alpha)(1-\Theta_{2\alpha})^{-1}g.$$

We can apply $1-\Theta_{-2\alpha} = -\Theta_{-2\alpha}(1-\Theta_{2\alpha})$ on both sides and obtain the equivalent condition

$$\tilde{\Theta}_s g = -\mathbf{e}(-\alpha)\Theta_{-2\alpha}g.$$

Now, $\tilde{\Theta}_s = \mathbf{e}(\alpha)\,\Theta_s^+$ and consequently

$$\begin{aligned} \tilde{\Theta}_s g &= (1-\Theta_{-\alpha}v^{c_s+c_{\tilde{s}}})(1+\Theta_{-\alpha}v^{c_s-c_{\tilde{s}}})\Theta_s^+ g \\ &= -\Theta_{-2\alpha}v^{2c_s}(1-v^{-(c_s+c_{\tilde{s}})}\Theta_\alpha)(1+v^{-(c_s-c_{\tilde{s}})}\Theta_\alpha)\Theta_s^+ g \\ &= -\Theta_{-2\alpha}v^{2c_s}\Theta_s^+(1-v^{-(c_s+c_{\tilde{s}})}\Theta_{-\alpha})(1+v^{-(c_s-c_{\tilde{s}})}\Theta_{-\alpha})g. \end{aligned}$$

On the other hand,

$$-\mathbf{e}(-\alpha)\Theta_{-2\alpha}g = v^{2c_s}(1-v^{-(c_s+c_{\tilde{s}})}\Theta_{-\alpha})(1+v^{-(c_s-c_{\tilde{s}})}\Theta_{-\alpha})g.$$

So we can write the condition on $g \in \hat{\mathcal{H}}$ in the form

$$\begin{aligned} &\Theta_{-2\alpha}\Theta_s^+(1-v^{-(c_s+c_{\tilde{s}})}\Theta_{-\alpha})(1+v^{-(c_s-c_{\tilde{s}})}\Theta_{-\alpha})g \\ &\quad = -(1-v^{-(c_s+c_{\tilde{s}})}\Theta_{-\alpha})(1+v^{-(c_s-c_{\tilde{s}})}\Theta_{-\alpha})g, \end{aligned}$$

and the initial condition $j\,\varphi(f) \in \hat{\mathcal{N}}^\perp$ is equivalent to

$$\begin{aligned} &\theta_{-2\alpha}\theta_s^+(1-v^{-(c_s+c_{\tilde{s}})}\theta_{-\alpha})(1+v^{-(c_s-c_{\tilde{s}})}\theta_{-\alpha})f \\ &\quad = -(1-v^{-(c_s+c_{\tilde{s}})}\theta_{-\alpha})(1+v^{-(c_s-c_{\tilde{s}})}\theta_{-\alpha})f. \end{aligned}$$

We defined $E_C = \prod_{\beta>0} \mathbf{f}(\beta)^{-1}D_C$ for $C \in X$, and according to Proposition 2.3.12 we have $\theta_s D_C = v^{c(C*s,C)}D_{C*s}$.

Suppose $C = y(A_\lambda^+) \subseteq \Pi_\lambda$, $y \in W_a$, $\lambda \in T$. We first consider the case that $c_s = c_{\tilde{s}}$. We have

$$C \circledast s = y(A^+_{(\lambda+\tilde{\rho})s-\tilde{\rho}}) = y(A^+_{\lambda s-\alpha}) = (C*s)p_{-\alpha}$$

and

$$c(C \circledast s, C) = \gamma(A^+p_\alpha) + c(C*s,C) = h(p_\alpha) + c(C*s,C).$$

Hence

$$
\begin{aligned}
&\theta_{-2\alpha}\theta_s^+(1-v^{-(c_s+c_{\bar s})}\theta_{-\alpha})(1+v^{-(c_s-c_{\bar s})}\theta_{-\alpha})E_C\\
&= \theta_{-\alpha}(1+\theta_\alpha)\theta_{-\alpha}\theta_s^+\prod_{\beta>0,\,\beta\neq\alpha}\mathbf{f}(\beta)^{-1}D_C\\
&= (1+\theta_{-\alpha})\prod_{\beta>0,\,\beta\neq\alpha}\mathbf{f}(\beta)^{-1}v^{c(C*s,C)-h(p_{-\alpha})}D_{(C*s)p_{-\alpha}}\\
&= (1+\theta_{-\alpha})\prod_{\beta>0,\,\beta\neq\alpha}\mathbf{f}(\beta)^{-1}v^{c(C\circledast s,C)}D_{C\circledast s}\\
&= (1-v^{-(c_s+c_{\bar s})}\theta_{-\alpha})(1+v^{-(c_s-c_{\bar s})}\theta_{-\alpha})v^{c(C\circledast s,C)}E_{C\circledast s}.
\end{aligned}
$$

Secondly, if $c_s\neq c_{\bar s}$ we have

$$C\circledast s = y(A^+_{\lambda s-2\alpha}) = (C*s)p_{-2\alpha}$$

and

$$c(C\circledast s,C) = h(p_{2\alpha}) + c(C*s,C).$$

Hence

$$
\begin{aligned}
&\theta_{-2\alpha}\theta_s^+(1-v^{-(c_s+c_{\bar s})}\theta_{-\alpha})(1+v^{-(c_s-c_{\bar s})}\theta_{-\alpha})E_C\\
&= \prod_{\beta>0,\,\beta\neq\alpha}\mathbf{f}(\beta)^{-1}v^{c(C*s,C)-h(p_{-2\alpha})}D_{(C*s)p_{-2\alpha}}\\
&= (1-v^{-(c_s+c_{\bar s})}\theta_{-\alpha})(1+v^{-(c_s-c_{\bar s})}\theta_{-\alpha})v^{c(C\circledast s,C)}E_{C\circledast s}.
\end{aligned}
$$

In both cases, we thereby obtain

$$
\begin{aligned}
&\theta_{-2\alpha}\theta_s^+(1-v^{-(c_s+c_{\bar s})}\theta_{-\alpha})(1+v^{-(c_s-c_{\bar s})}\theta_{-\alpha})F_C\\
&= \theta_{-2\alpha}\theta_s^+(1-v^{-(c_s+c_{\bar s})}\theta_{-\alpha})(1+v^{-(c_s-c_{\bar s})}\theta_{-\alpha})\sum_{x\in W_0}(-1)^{l(x)}v^{c(C\circledast x,C)}E_{C\circledast x}\\
&= (1-v^{-(c_s+c_{\bar s})}\theta_{-\alpha})(1+v^{-(c_s-c_{\bar s})}\theta_{-\alpha})\sum_{x\in W_0}(-1)^{l(x)}v^{c(C\circledast x,C)+c(C\circledast x\circledast s,C\circledast x)}E_{C\circledast x\circledast s}\\
&= -(1-v^{-(c_s+c_{\bar s})}\theta_{-\alpha})(1+v^{-(c_s-c_{\bar s})}\theta_{-\alpha})\sum_{x\in W_0}(-1)^{l(xs)}v^{c(C\circledast xs,C)}E_{C\circledast xs}\\
&= -(1-v^{-(c_s+c_{\bar s})}\theta_{-\alpha})(1+v^{-(c_s-c_{\bar s})}\theta_{-\alpha})F_C.
\end{aligned}
$$

Thus $j\,\varphi(F_C)\in\hat{\mathcal{N}}^\perp$, which finishes the proof of Theorem 3.3.1.

We conclude this section with the generalization of Theorem 1.8 in [8]. If $K=(k_\alpha)_\alpha\in\mathbb{Z}_+^{|\Phi_1^+|}$, $M=(m_\alpha)_\alpha\in\mathbb{Z}_+^{|\Phi_2^+|}$ and $N=(n_\alpha)_\alpha\in\mathbb{Z}_+^{|\Phi_2^+|}$, we set

$$\sigma(N)=\sum_{\alpha\in\Phi_2^+}n_\alpha$$

and

$$\sigma'(K,M,N) = 2\sum_{\alpha\in\Phi_1^+} k_\alpha c_\alpha + \sum_{\alpha\in\Phi_2^+}(m_\alpha(c_\alpha+\tilde{c}_\alpha)+n_\alpha(c_\alpha-\tilde{c}_\alpha)).$$

For $\kappa \in T$, $\kappa \geq 0$, we then define

$$\mathcal{P}^*(\kappa; v) = \sum_{K,M,N}(-1)^{\sigma(N)}v^{\sigma'(K,M,N)}$$

where the sum runs over all $K = (k_\alpha) \in \mathbb{Z}_+^{|\Phi_1^+|}$, $M = (m_\alpha) \in \mathbb{Z}_+^{|\Phi_2^+|}$, $N = (n_\alpha) \in \mathbb{Z}_+^{|\Phi_2^+|}$ such that

$$\sum_{\alpha\in\Phi_1^+} k_\alpha\alpha + \sum_{\alpha\in\Phi_2^+}(m_\alpha+n_\alpha)\alpha = \kappa.$$

Let $n_\lambda \in \widehat{W}_a$, $\lambda \in T^+$, be the element of maximal length in $W_0 p_\lambda W_0$.

Theorem 3.3.7. *Let λ, $\mu \in T^+$ with $\lambda \geq \mu$. We have*

$$v^{h(\lambda-\mu)}P_{n_\mu,n_\lambda}(v^{-1}) = \sum_{x\in W_0}(-1)^{l(x)}\mathcal{P}^*((\lambda+\tilde{\rho})x-(\mu+\tilde{\rho}); v).$$

Proof. Since

$$P_{n_\mu,n_\lambda} = P_{A^-n_\mu,A^-n_\lambda} = P_{A_\mu^+,A_\lambda^+},$$

we calculate the coefficient of A_μ^+ in $F_{A_\lambda^+}$.

Remember that

$$F_{A_\lambda^+} = \sum_{x\in W_0}(-1)^{l(x)}v^{c(A_{(\lambda+\tilde{\rho})x-\tilde{\rho}},A_\lambda^+)}E_{A^+_{(\lambda+\tilde{\rho})x-\tilde{\rho}}}$$

and

$$\begin{aligned} E_{A^+_{(\lambda+\tilde{\rho})x-\tilde{\rho}}} &= \prod_{\alpha>0}\mathrm{f}(\alpha)^{-1}e_{(\lambda+\tilde{\rho})x-\tilde{\rho}} \\ &= \prod_{\alpha\in\Phi_1^+}(\sum_{k\geq 0}v^{-2kc_\alpha}\theta_{-k\alpha})\prod_{\alpha\in\Phi_2^+}(\sum_{m\geq 0}v^{-m(c_\alpha+\tilde{c}_\alpha)}\theta_{-m\alpha}) \\ &\quad\cdot\prod_{\alpha\in\Phi_2^+}(\sum_{n\geq 0}(-1)^n v^{-n(c_\alpha-\tilde{c}_\alpha)}\theta_{-n\alpha})e_{(\lambda+\tilde{\rho})x-\tilde{\rho}}. \end{aligned}$$

We have

$$\prod_{\alpha\in\Phi_1^+}\theta_{-k_\alpha\alpha}\prod_{\alpha\in\Phi_2^+}\theta_{-m_\alpha\alpha}\prod_{\alpha\in\Phi_2^+}\theta_{-n_\alpha\alpha}(A^+_{(\lambda+\tilde{\rho})x-\tilde{\rho}}) = A_\mu^+$$

for some $K = (k_\alpha)_\alpha \in \mathbb{Z}_+^{|\Phi_1^+|}$, $M = (m_\alpha)_\alpha \in \mathbb{Z}_+^{|\Phi_2^+|}$, $N = (n_\alpha)_\alpha \in \mathbb{Z}_+^{|\Phi_2^+|}$ if and only if

$$\sum_{\alpha\in\Phi_1^+} k_\alpha\alpha + \sum_{\alpha\in\Phi_2^+}(m_\alpha+n_\alpha)\alpha = (\lambda+\tilde{\rho})x-(\mu+\tilde{\rho}),$$

and the contribution to the coefficient of A_μ^+ in $E_{A^+_{(\lambda+\tilde\rho)x-\tilde\rho}}$ is

$$\prod_{\alpha\in\Phi_2^+}(-1)^{n_\alpha}\prod_{\alpha\in\Phi_1^+}v^{k_\alpha h(\alpha)-2k_\alpha c_\alpha}\prod_{\alpha\in\Phi_2^+}v^{m_\alpha h(\alpha)-m_\alpha(c_\alpha+\tilde c_\alpha)}\prod_{\alpha\in\Phi_2^+}v^{n_\alpha h(\alpha)-n_\alpha(c_\alpha-\tilde c_\alpha)}$$
$$=(-1)^{\sigma(N)}\,v^{h((\lambda+\tilde\rho)x-(\mu+\tilde\rho))-\sigma'(K,M,N)}.$$

Since

$$c(A^+_{(\lambda+\tilde\rho)x-\tilde\rho},A^+_\lambda)+h((\lambda+\tilde\rho)x-(\mu+\tilde\rho))=c(A^+_\mu,A^+_\lambda),$$

the claim follows.

4. Some results on generalized cells

In the first section, we briefly introduce generalized cells and the $\boldsymbol{a}$-function. In the second section, we examine a particular generalized two–sided cell of W_a. Here, Lemma 2.2.7 will play an important role. The third section contains a description of the generalized left cells in a Coxeter group for one class of parameters. We conclude this chapter with an example.

4.1. **Definitions and basic properties.** For equal parameters, the following definitions and facts can be found in [18]. In [16], Lusztig generalizes the concept of cells to unequal parameters.

Let W be a Coxeter group with generating set S and parameters c_s, $s\in S$, subject to the condition that $c_s=c_t$ whenever s and t are conjugate in S. The definition of the corresponding Hecke algebra $\mathcal{H}$ is analogous to the definition in Section 1.2. Here, it will be more convenient to work with the $\mathcal{A}$-basis $\{\tilde T_w\mid w\in W\}$ (instead of $\{T_w\mid w\in W\}$) where

$$\tilde T_w=v^{-m(w)}T_w.$$

Multiplication in terms of these elements is given by

$$(\tilde T_s-v^{c_s})(\tilde T_s+v^{-c_s})=0$$

for $s\in S$ and

$$\tilde T_w\tilde T_{w'}=\tilde T_{ww'}$$

for w, $w'\in W$, $l(w)+l(w')=l(ww')$.

The $\mathcal{A}$-basis of $\mathcal{H}$ consisting of the elements

$$C_w=\sum_{y\le w}(-1)^{l(w)-l(y)}v^{m(w)-m(y)}P_{y,w}(v^{-1})\tilde T_y$$

for $w\in W$ will be denoted by $\mathcal{C}$ (see *loc. cit.*).

For y, $w \in W$ and $s \in S$ such that $sy < y < w < sw$, we inductively define polynomials $M^s_{y,w} \in \mathcal{A}$ by

(i) $\displaystyle\sum_{y \leq z < w,\, sz < z} v^{m(z)-m(y)} P_{y,z} M^s_{z,w} - v^{c_s+m(w)-m(y)} P_{y,w} \in \mathbb{Z}[v^{-1}]$ and

(ii) $\overline{M^s_{y,w}} = M^s_{y,w}$.

It turns out that

$$(*) \qquad C_s C_w = \begin{cases} -(v^{c_s} + v^{-c_s}) C_w & \text{if } sw < w, \\ C_{sw} - \displaystyle\sum_{z<w, sz<z} (-1)^{l(w)-l(z)} M^s_{z,w} C_z & \text{if } sw > w. \end{cases}$$

We remark that

$$C_w \in \tilde{T}_w + v \sum_{y<w} \mathcal{A}^+ \tilde{T}_y$$

and

$$\tilde{T}_w \in C_w + v \sum_{y<w} \mathcal{A}^+ C_y.$$

For y, $w \in W$, $y \leq w$, define polynomials $Q_{y,w} \in \mathcal{A}^+$ by

$$\sum_{y \leq z \leq w} (-1)^{l(z)-l(y)} P_{y,z} Q_{z,w} = \delta_{y,w},$$

and set

$$D_y = \sum_{y \leq w} v^{m(w)-m(y)} Q_{y,w}(v^{-1}) \tilde{T}_w,$$

which is an element in the set $\mathcal{H}^*$ of formal $\mathcal{A}$-linear combinations of the elements $\tilde{T}_w$, $w \in W$.

We have an $\mathcal{A}$-linear map $\tau : \mathcal{H}^* \to \mathcal{A}$, given by

$$\tau(\sum_{w \in W} a_w \tilde{T}_w) = a_e,$$

$a_w \in \mathcal{A}$ for $w \in W$. It is easy to check that

$$\tau(\tilde{T}_x \tilde{T}_y) = \delta_{x,y^{-1}}$$

and

$$\tau(C_x D_y) = \tau(D_y C_x) = \delta_{x,y^{-1}}$$

for x, $y \in W$.

Let $\leq_L$ be the preorder on W which is generated by $x \leq_L y$ for x, $y \in W_a$, if there exists some $s \in S$ such that $\tau(C_s C_y D_{x^{-1}}) \neq 0$. The associated equivalence relation is denoted by $\sim_L$, and the equivalence classes with respect to $\sim_L$ are called generalized left cells. Similarly, we define $\leq_R$, $\sim_R$, and generalized right cells. We say $x \leq_{LR} y$ for x, $y \in W$, if and only if there exists a sequence

$$x = x_0,\, x_1,\, \ldots,\, x_n = y$$

such that for all $0 \leq i \leq n-1$ we have $x_i \leq_L x_{i+1}$ or $x_i \leq_R x_{i+1}$. We write $\sim_{LR}$ for the associated equivalence relation, and the equivalence classes are called generalized two–sided cells. (We use the attribute *generalized* whenever unequal parameters are involved.)

For $y \in W$, define

$$\mathcal{L}(y) = \{s \in S \mid sy < y\}$$

and

$$\mathcal{R}(y) = \{s \in S \mid ys < y\}.$$

Remark 4.1.1. Let x, $y \in W$.

(i) ([31], Corollary 1.20) If $x \leq_L y$, then $\mathcal{R}(x) \supseteq \mathcal{R}(y)$. Therefore, $x \sim_L y$ implies $\mathcal{R}(x) = \mathcal{R}(y)$.

(ii) If $C_x D_y \neq 0$, then $y^{-1} \leq_L x$. (Use [31], Corollary 1.15 (a).)

A function

$$\boldsymbol{a} : W \to \mathbb{N}_0 \cup \{\infty\}$$

is defined as follows (cf. [18] for equal parameters and [23] for unequal parameters). Let $w \in W$. For x, $y \in W$, express $\tilde{T}_x \tilde{T}_y$ with respect to the basis $\mathcal{C}$, and consider the coefficient of $C_{w^{-1}}$. If the order of the pole at 0 of these coefficients is bounded as x and y vary, we set $\boldsymbol{a}(w)$ equal to the largest such order. Otherwise, $\boldsymbol{a}(w) = \infty$. For equal parameters, this function is an important tool in the study of representations of Hecke algebras in [18].

Remark 4.1.2. Let M be an abelian group acting on W in a way such that $m(S) = S$ for all $m \in M$. Consider the semidirect product $W' = M \ltimes W$. The definitions in this section can be extended to W' (comp. [21]). Generalized cells in W' are of the form $\{(m, w) \mid m \in M,\, w \in \Gamma\}$ where Γ is a generalized cell in W.

4.2. **The lowest generalized two–sided cell in W_a.** Suppose $W = W_a$. (We use the notation of Chapter 1.)

Let $\nu = l(w_0)$, $\tilde{\nu} = m(w_0)$, and set $\xi_s = v^{c_s} - v^{-c_s}$ for $s \in S$. For x, $y \in W_a$, we write

$$\tilde{T}_x \tilde{T}_y = \sum_{z \in W_a} m_{x,y,z} \tilde{T}_{z^{-1}}.$$

Note that $\tilde{T}_s^2 = \xi_s \tilde{T}_s + 1$ for $s \in S$, so any $m_{x,y,z}$ is a polynomial in ξ_s, $s \in S$.

Theorem 4.2.1. *Let $x,\ y,\ z \in W_a$.*

(i) *As a polynomial in ξ_s, $s \in S_a$, the degree of $m_{x,y,z}$ is at most ν, and the coefficients are non–negative integers.*

(ii) *The degree of $m_{x,y,z}$ in v is at most $\tilde{\nu}$.*

The proof of part (i) is analogous to the proof of Theorem 7.2 in [18]. Part (ii) is approached in a similar way, using Lemma 2.2.7 instead of [14], Lemma 4.3.

We will need the following two corollaries, whose proofs are again analogous to the ones in [18].

Corollary 4.2.2. *(comp.* [18], *Corollary 7.3) For $w \in W_a$, we have $\mathbf{a}(w) \leq \tilde{\nu}$.*

Corollary 4.2.3. *(comp.* [18], *Corollary 7.10) For any $x,\ y,\ z \in W_a$, the elements $v^{\tilde{\nu}}\tau(\tilde{T}_x\tilde{T}_y\tilde{T}_z)$ and $v^{\tilde{\nu}}\tau(\tilde{T}_x\tilde{T}_yD_z)$ are in $\mathcal{A}^+$ and have the same constant term.*

Now, for any $w, x, y \in W_a$, the notation $w = x \cdot y$ means that $w = xy$ and $l(w) = l(x) + l(y)$ (and similarly for $w = x \cdot y \cdot z$, $w, x, y, z \in W_a$.) For $\lambda \in T$, let

$$M_\lambda = \{z \in W_a \mid w_\lambda z = w_\lambda \cdot z\},$$

and define

$$N_{\lambda,z} = \{w \in W_a \mid w = z' \cdot w_\lambda \cdot z,\ z' \in W_a\}$$

where $z \in M_\lambda$. Note that according to [28], Lemma 3.2, the condition $w_\lambda z = w_\lambda \cdot z$ implies $z^{-1}w_\lambda z = z^{-1} \cdot w_\lambda \cdot z$.

Theorem 4.2.4. *Let $\lambda \in T$ and $z \in M_\lambda$. The set $N_{\lambda,z}$ is contained in a generalized left cell.*

Proof. Let $x \in W_a$, and suppose $x = s_k \cdots s_1$, $s_i \in S_a$ for $1 \leq i \leq k$, is a reduced expression. Let $y \in W_a$. We denote by $\mathcal{I}_y$ the collection of all $I = \{i_1, \ldots, i_{p_I}\}$ such that $1 \leq i_1 < \cdots < i_{p_I} \leq k$ and

$$s_{i_t} \cdots \hat{s}_{i_{t-1}} \cdots \hat{s}_{i_1} \cdots s_1 y < \hat{s}_{i_t} \cdots \hat{s}_{i_{t-1}} \cdots \hat{s}_{i_1} \cdots s_1 y$$

for all $1 \leq t \leq p_I$. For $I \in \mathcal{I}_y$, we write

$$\tilde{T}_I = \tilde{T}_{s_k \cdots \hat{s}_{i_{p_I}} \cdots \hat{s}_{i_1} \cdots s_1 y}.$$

Induction on k shows that

$$\tilde{T}_x\tilde{T}_y = \sum_{I \in \mathcal{I}_y} (\prod_{j=1}^{p_I} \xi_{s_{i_j}})\tilde{T}_I.$$

Now take $y \in N_{\lambda,z}$, $y = z' \cdot w_\lambda \cdot z$, for some $z' \in W_a$, and $x = z^{-1}w_\lambda z$. (So $x \in N_{\lambda,z}$.) Say $x = s_k \cdots s_1$ is a reduced expression such that $w_\lambda = s_m \cdots s_n$ and $s_{n-1} \cdots s_1 = z$

for some $k \geq m \geq n \geq 1$. If we set $I = \{n, \dots, m\}$, we have $I \in \mathcal{I}_{y^{-1}}$ and $\tilde{p}_I = \tilde{\nu}$ (see Lemma 2.2.7 for the definition of $\tilde{p}_I$). Hence

$$v^{\tilde{\nu}}\tau(\tilde{T}_x\tilde{T}_{y^{-1}}\tilde{T}_{(z^{-1}zy^{-1})^{-1}}) = v^{\tilde{\nu}}\tau(\tilde{T}_x\tilde{T}_{y^{-1}}\tilde{T}_y)$$

has non–zero constant term. Using

$$\tau(\tilde{T}_x\tilde{T}_{y^{-1}}\tilde{T}_y) = \tau(\tilde{T}_{y^{-1}}\tilde{T}_y\tilde{T}_x) = \tau(\tilde{T}_y\tilde{T}_x\tilde{T}_{y^{-1}})$$

and Corollary 4.2.3, we see that the polynomials $v^{\tilde{\nu}}\tau(C_{y^{-1}}C_yD_x)$ and $v^{\tilde{\nu}}\tau(C_yC_xD_{y^{-1}})$ have non–zero constant terms, in particular, C_yD_x and $C_xD_{y^{-1}}$ are non–zero. Thus, by Remark 4.1.1 (ii), we have $x = x^{-1} \leq_L y$ and $y \leq_L x$, i.e. $y \sim_L x$ for all $y \in N_{\lambda,z}$.

The next corollary is an immediate consequence of the above theorem for $z = e$ and Remark 4.1.1 (i). For equal parameters, this is Corollary 8.5 in [18].

For $\lambda \in T$, let $S_\lambda = W_\lambda \cap S_a$ and write $\mathcal{R}(S_\lambda) = \{w \in W_a \mid ws < w \text{ for all } s \in S_\lambda\}$.

Corollary 4.2.5. *The set $\mathcal{R}(S_\lambda)$, $\lambda \in T$, is a generalized left cell in W_a.*

We also get the following result, which for Case 2 is Theorem 3.22 in [31]. Let

$$W_T = \{w \in W_a \mid w = z' \cdot w_\lambda \cdot z,\ z,\ z' \in W_a,\ \lambda \in T\}.$$

Corollary 4.2.6. *The set W_T is a generalized two–sided cell.*

Proof. We only need to look at Case 1, in which case

$$W_T = \bigcup_{z \in M_\lambda} N_{\lambda,z}$$

for some fixed $\lambda \in T$. Let $z'w_\lambda z$ and $y'w_\lambda y$ be elements in W_T, $z'w_\lambda z = z' \cdot w_\lambda \cdot z$ and $y'w_\lambda y = y' \cdot w_\lambda \cdot y$. Using Theorem 4.2.4, together with its version for generalized right cells, we obtain

$$z'w_\lambda z \sim_L w_\lambda z \sim_R w_\lambda y \sim_L y'w_\lambda y,$$

Thus W_T is contained in a generalized two–sided cell. The other inclusion is proven in the same way as in *loc. cit.*

We remark that W_T is the lowest generalized two–sided cell in W_a with respect to $\leq_{LR}$.

Theorem 4.2.7. *The set W_T contains at most $|W_0|$ generalized left cells.*

Proof. For $\lambda \in T$, let

$$M'_\lambda = \{z \in W_a \mid w_\lambda z = w_\lambda \cdot z,\ sw_\lambda z \notin W_T \text{ for all } s \in S_\lambda\}.$$

Following [28], we choose a set of representatives for the Ω-orbits on T and denote it by R. Then

$$W_T = \bigcup_{\lambda \in R, z \in M'_\lambda} N_{\lambda,z},$$

so the number of generalized left cells in W_T is at most the number of pairs (λ, z), $\lambda \in R$, $z \in M'_\lambda$.

As in *loc. cit.*, we see that $z \in M'_\lambda$ for some $\lambda \in T$ implies $z^{-1}(A^+_\lambda) \subseteq \Pi_\lambda$. Since all $z^{-1}(A^+_\lambda)$, $\lambda \in R$, $z \in M'_\lambda$, are different, the number of pairs (λ, z) is thereby bounded by the cardinality of $\{A \in X \mid A \subseteq \Pi_\lambda \text{ for some } \lambda \in R\}$. The latter set is easily seen to be a fundamental domain for the action of the translation subgroup of Ω on X, so its cardinality is $|W_0|$. The assertion follows.

There is another way to describe the set W_T, which is similar to the description in [3]. Let $i \in \bar{\mathcal{F}}$, and denote by $\mathcal{F}_i$ the set of all hyperplanes $H \in \mathcal{F}$ of direction i such that $c_H = c_i$. The connected components of

$$V - \bigcup_{H \in \mathcal{F}_i} H$$

are called strips. We write

$$\mathcal{U}(A) = \bigcup_{U \text{ strip}, U \supseteq A} U$$

for $A \in X$.

Lemma 4.2.8. *We have*

$$W_T = \{w \in W_a \mid w(A^+) \not\subseteq \mathcal{U}(A^+)\}.$$

Notice that instead of A^+, we could have chosen any other alcove $A \in X$ since $\mathcal{U}(A^+v) = \mathcal{U}(A^+)v$ for any $v \in W_a$.

Proof. First, let $w \in W_a$ be such that $w(A^+) \not\subseteq \mathcal{U}(A^+)$. The alcove $w(A^+)$ lies in some connected component $\mathcal{C}$ of $V - \cup_{\alpha \in \Phi^+} H_{\alpha,0}$. This quarter $\mathcal{C}$ with vertex 0 can be described as

$$\mathcal{C} = \{x \in V \mid \langle x, \check{\alpha} \rangle > 0 \text{ for } \alpha \in y.\Pi\}$$

where $y \in W_0$ maps $\mathcal{C}^+$ to $\mathcal{C}$. So there are r linearly independent positive roots $\alpha_1, \dots, \alpha_r$ and some $k \in \{1, \dots, r\}$ such that

$$\mathcal{C} = \{x \in V \mid \langle x, \check{\alpha}_i \rangle < 0 \text{ for } 1 \leq i \leq k,\ \langle x, \check{\alpha}_i \rangle > 0 \text{ for } k+1 \leq i \leq r\}.$$

We remove from $\mathcal{C}$ all alcoves which lie in $\mathcal{U}(A^+)$ and obtain the quarter

$$\mathcal{C}' = \{x \in V \mid \langle x, \check{\alpha}_i \rangle < 0 \text{ for } 1 \leq i \leq k,\ \langle x, \check{\alpha}_i \rangle > b_i \text{ for } k+1 \leq i \leq r\}$$

where

$$b_i = \begin{cases} 1 & \text{if } c_{\alpha_i} = \tilde{c}_{\alpha_i}, \\ 2 & \text{otherwise} \end{cases}$$

for $k+1 \leq i \leq r$. (The set $\mathcal{C}'$ is a translate of $\mathcal{C}$.) So $w(A^+) \subset \mathcal{C}'$.

Let λ be the vertex of $\mathcal{C}'$, and let $z \in W_a$ be such that $z(A^+)$ is the unique alcove in $\mathcal{C}'$ containing λ in its closure. Since $\mathcal{C}' \subset \mathcal{C}$, any other element $v \in W_a$ with $v(A^+) \subset \mathcal{C}'$ satisfies $v = v' \cdot z$ for some $v' \in W_a$, in particular $w = w' \cdot z$ for some $w' \in W_a$.

Let $s \in S_\lambda$, so there is a wall of $\mathcal{C}$ containing a face of type s. For each wall of $\mathcal{C}'$, the alcoves A^+ and $z(A^+)$ lie on different sides. Since $l(v)$ for $v \in W_a$ counts the number of hyperplanes such that A^+ and $v(A^+)$ lie on different sides, we obtain $sz < z$. We conclude $z = w_\lambda \cdot z'$ for some $z' \in W_a$ and $w = w' \cdot w_\lambda \cdot z'$.

Conversely, let $w \in W_T$, $w = z' \cdot w_\lambda \cdot z$ for some z', $z \in W_a$ and $\lambda \in T$. If we are in Case 1, we set $A = z^{-1}(A^+)$. Using [28], Lemma 4.2, we get

$$A = z^{-1}(A^+) \subset \mathcal{C}^+$$

and

$$w(A) = z' w_\lambda (A^+) = z'(A^-) \subset \mathcal{C}^-.$$

Hence $\langle x, \check{\alpha} \rangle > 0$ and $\langle y, \check{\alpha} \rangle < 0$ for all $x \in A$, $y \in w(A)$, $\alpha \in \Phi^+$, which implies $w(A) \not\subseteq \mathcal{U}(A)$.

If we are in Case 2, we set $A = z^{-1}(A_\lambda^+)$. We obtain $\langle x, \check{\alpha} \rangle \neq \langle x', \check{\alpha} \rangle$ for all $x \in A$, $x' \in w(A)$, $\alpha \in \Phi^+$, and this again implies $w(A) \not\subseteq \mathcal{U}(A)$.

Note that the connected components of $V - \mathcal{U}(A^+)$ turn out to be precisely the quarters of the form $\mathcal{C}'$.

Remark 4.2.9. Define

$$W_{(\tilde{\nu})} = \{ w \in W_a \mid \boldsymbol{a}(w) = \tilde{\nu} \}.$$

For $w \in W_T$, we obtain from the proof of Theorem 4.2.4 that $\boldsymbol{a}(w) \geq \tilde{\nu}$ and from Corollary 4.2.2 that $\boldsymbol{a}(w) \leq \tilde{\nu}$. Thus $W_T \subseteq W_{(\tilde{\nu})}$. At the end of the next section, we will see that this is actually an equality.

We also point out that we can use the procedure in [19] to attach a based ring J_{W_T} to the lowest generalized two–sided cell W_T. For fixed T, the ring J_{W_T} does not depend on the parameters.

4.3. **Parameters coming from graph automorphisms.** We now place ourselves in the setting of [15], Section 8. Let $\widetilde{W}$ be a finite or affine Weyl group with a set $\widetilde{S}$ of simple reflections and $\alpha : \widetilde{W} \to \widetilde{W}$ a non–trivial automorphism such that $\alpha(\widetilde{S}) = \widetilde{S}$. Let W be the fixed point set of $\widetilde{W}$ under α. This is again a finite or affine Weyl group and has a set S of simple reflections corresponding to the orbits of α on $\widetilde{S}$. For $s \in S$, let c_s be the length of the longest element in the subgroup of $\widetilde{W}$ generated by the orbit corresponding to s. We thereby get parameters $c_s \in \mathbb{N}$ such that $c_s = c_t$ whenever s and t are conjugate in W.

We will write $P_{y,w}$, C_w, $\boldsymbol{a}$ etc. for W and $\widetilde{P}_{y,w}$, $\widetilde{C}_w$, $\widetilde{\boldsymbol{a}}$ etc. for $\widetilde{W}$.

For x, y, $z \in W$, set

$$h_{x,y,z} = \tau(C_x C_y D_{z^{-1}}),$$

and correspondingly

$$\widetilde{h}_{x,y,z} = \widetilde{\tau}(\widetilde{C}_x \widetilde{C}_y \widetilde{D}_{z^{-1}})$$

for x, y, $z \in \widetilde{W}$.

Let x, y, $z \in W$ be such that $x \leq y$. According to [11], [16], and [24], the coefficients of $\widetilde{P}_{x,y}$ and $\widetilde{h}_{x,y,z}$ can (up to a sign) be interpreted as dimensions of certain vector spaces on which α acts, and the corresponding coefficients of $P_{x,y}$ and $h_{x,y,z}$ are the traces of α on these vector spaces. We need the following facts which are consequences of these interpretations.

4.3.1. We have $\deg P_{x,y} \leq \deg \widetilde{P}_{x,y}$.

4.3.2. If the coefficient of v^i, $i \in \mathbb{Z}$, in $h_{x,y,z}$ is non–zero, then the coefficient of v^i in $\widetilde{h}_{x,y,z}$ is non–zero as well. In particular, $\boldsymbol{a}(z) \leq \widetilde{\boldsymbol{a}}(z)$.

4.3.3. If the coefficient of v^i, $i \in \mathbb{Z}$, in $\widetilde{h}_{x,y,z}$ is ± 1, then the coefficient of v^i in $h_{x,y,z}$ is non–zero.

For x, y, $z \in W$, let $c_{x,y,z}$, be the integer given by

$$v^{\boldsymbol{a}(z)} h_{x,y,z^{-1}} - c_{x,y,z} \in v\mathcal{A}^+,$$

and write $\delta(z)$ for the degree of the polynomial $P_{e,z}$ in v. Similarly, we define $\widetilde{c}_{x,y,z}$ and $\widetilde{\delta}(z)$ for x, y, $z \in \widetilde{W}$, and we set $\widetilde{\gamma}_{x,y,z} = (-1)^{\widetilde{\boldsymbol{a}}(z)} \widetilde{c}_{x,y,z}$.

Lemma 4.3.4. *We have $\boldsymbol{a}(z) \leq m(z) - \delta(z)$ for any $z \in W$.*

Proof. According to [19], the corresponding inequality holds for $\widetilde{W}$. In view of 4.3.2 and 4.3.1, we can therefore conclude

$$(*) \qquad \boldsymbol{a}(z) \leq \widetilde{\boldsymbol{a}}(z) \leq \widetilde{l}(z) - \widetilde{\delta}(z) \leq m(z) - \delta(z).$$

Remark 4.3.5. Let

$$\widetilde{\mathcal{D}} = \{z \in \widetilde{W} \mid \widetilde{\boldsymbol{a}}(z) = \widetilde{l}(z) - \widetilde{\delta}(z)\}$$

be the set of distinguished involutions in $\widetilde{W}$. We set

$$\mathcal{D} = \{z \in W \mid \boldsymbol{a}(z) = m(z) - \delta(z)\}.$$

From (*), we derive that $\boldsymbol{a}(d) = \widetilde{\boldsymbol{a}}(d)$ for all $d \in \mathcal{D}$ and that $\mathcal{D} \subset \widetilde{\mathcal{D}}$.

The following two results can be found in [19].

4.3.6. Any left cell $\widetilde{\Gamma}$ of $\widetilde{W}$ contains a unique $d \in \widetilde{\mathcal{D}}$. For $z \in \widetilde{\Gamma}$, we have $\widetilde{\gamma}_{z^{-1},z,d} = 1$, and d is the unique element in $\widetilde{W}$ such that $\widetilde{\gamma}_{z^{-1},z,d} \neq 0$.

4.3.7. We have $\widetilde{\gamma}_{x,y,z} = \widetilde{\gamma}_{y,z,x}$ for all x, y, $z \in \widetilde{W}$.

Theorem 4.3.8. *The generalized left cells of W are the fixed point sets of the left cells of $\widetilde{W}$ under α.*

Proof. Let $z \in W$. It follows from the definition of C_w, $w \in W$, that

$$\tau(C_{z^{-1}}C_z) \in 1 + v\mathcal{A}^+.$$

On the other hand,

$$\begin{aligned}\tau(C_{z^{-1}}C_z) &= \sum_{w\in W} h_{z^{-1},z,w}\tau(C_w)\\ &= \sum_{w\in W} h_{z^{-1},z,w}(-1)^{l(w)}v^{m(w)}P_{e,w}(v^{-1}).\end{aligned}$$

Lemma 4.3.4 shows that there exists some $d \in \mathcal{D}$ such that $c_{z^{-1},z,d} \neq 0$. Since $\boldsymbol{a}(d) = \widetilde{\boldsymbol{a}}(d)$, we conclude $\widetilde{c}_{z^{-1},z,d} \neq 0$ and hence $\widetilde{\gamma}_{z^{-1},z,d} \neq 0$. We have $d \in \widetilde{\mathcal{D}}$, and using 4.3.6 and 4.3.7, we arrive at

$$\widetilde{\gamma}_{z^{-1},z,d} = \widetilde{\gamma}_{z,d,z^{-1}} = 1.$$

Thus, according to 4.3.3, both elements C_zD_d and $C_dD_{z^{-1}}$ are non–zero, which in turn implies $d \leq_L z \leq_L d$ (cf. Remark 4.1.1 (ii)), i.e. $d \sim_L z$. This means that any generalized left cell in W contains a distinguished involution.

As a consequence of 4.3.2, any generalized left cell in W is contained in the fixed point set of some left cell in $\widetilde{W}$ (comp. [15]). Since there is only one distinguished involution in each left cell in $\widetilde{W}$, the assertion follows.

In [19], Lusztig proves that if $\widetilde{W}$ is an affine Weyl group, $\widetilde{W}$ consists of only finitely many left cells. So we obtain the following statement.

Corollary 4.3.9. *If W is an affine Weyl group (with parameters as above) the number of generalized left cells in W is finite.*

Note that the proof of Theorem 4.3.8 shows that $\mathcal{D}$ is the fixed point set of $\widetilde{\mathcal{D}}$ and that $\boldsymbol{a}(z) = \widetilde{\boldsymbol{a}}(z)$ for any $z \in W$. Several statements in [19] then carry over to W. We will need the following one for the proof of Theorem 4.3.13.

Corollary 4.3.10. *(comp. [19], Corollary 1.9 (b)) For any z', $z \in W$, if $z' \leq_L z$ and $\boldsymbol{a}(z') = \boldsymbol{a}(z)$ then $z' \sim_L z$.*

Remark 4.3.11. As we will see in the next section, a generalized two–sided cell in W does not have to be the fixed point set of some two–sided cell in $\widetilde{W}$.

We now again use the notations of Chapter 1 (taking $W = W_a$).

Theorem 4.3.12. *If we have parameters coming from a graph automorphism, W_T consists of exactly $|W_0|$ generalized left cells.*

Proof. Let

$$N = \{z^{-1}w_\lambda z \mid \lambda \in R,\ z \in M'_\lambda\}.$$

As in [28], we see that $N \subseteq \mathcal{D}$. In view of Theorem 4.2.7, it therefore suffices to show that $|N| = |W_0|$.

It follows from the proof of Lemma 4.2.8 that $V - \mathcal{U}(A^+)$ has $|W_0|$ connected components. According to the same lemma, we have

$$V - \mathcal{U}(A^+) = W_T.A^+,$$

which in turn equals

$$\bigcup_{\lambda \in R,\, z \in M'_\lambda} N_{\lambda,z}.A^+.$$

In the proof of Theorem 4.2.7, we saw that $|\{(\lambda, z) \mid \lambda \in R,\ z \in M'_\lambda\}| \leq |W_0|$, and since $N_{\lambda,z}.A^+$ is connected, we obtain $|\{(\lambda, z) \mid \lambda \in R,\ z \in M'_\lambda\}| = |W_0|$. We have $z^{-1}w_\lambda z.A^+ \subseteq N_{\lambda,z}.A^+$ for $\lambda \in R$, $z \in M'_\lambda$, so all $z^{-1}w_\lambda z.A^+$ and thus all $z^{-1}w_\lambda z$ are different. We conclude $|N| = |W_0|$.

Finally, we prove the equality indicated in Remark 4.2.9. (This holds for any parameters.)

Theorem 4.3.13. *We have*

$$W_{(\tilde{\nu})} = W_T.$$

Proof. It remains to show that $W_{(\tilde{\nu})} \subseteq W_T$.

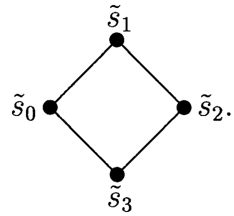
We first assume that we have parameters coming from a graph automorphism. Let $z \in W_a$ be such that $\boldsymbol{a}(z) = \tilde{\nu}$. Choose some $\lambda \in T$, and let $y \in W_a$ be the element of minimal length in $W_\lambda z$. Then $z = x \cdot y$ for some $x \in W_\lambda$, and by induction on

$l(w_\lambda) - l(x)$ we see that we can find $s_1, \dots, s_n \in S_\lambda$ such that $s_n \cdots s_1 z \in W_T$ and $s_i \cdots s_1 z > s_{i-1} \cdots s_1 z$ for all $1 \leq i \leq n$, i.e. there exists an element $z' \in W_T$ such that $z' \leq_L z$. By Corollary 4.3.10, we get $z' \sim_L z$, thus $z \in W_T$.

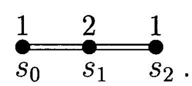
For arbitrary parameters c_s, $s \in S_a$, one can check that there always are parameters c'_s, $s \in S_a$, such that the corresponding set $W_{(\tilde{\nu}')}$ resp. $W_{T'}$ equals $W_{(\tilde{\nu})}$ resp. W_T. (Use Corollary 4.2.3 for $W_{(\tilde{\nu}')}$.) The claim follows.

4.4. **The case $\tilde{C}_2$.** In this section, we explicitly determine the generalized left and two–sided cells for the case W of type $\tilde{C}_2$ $(= \tilde{B}_2)$ and $\widetilde{W}$ of type $\tilde{A}_3$. More cases can be found in the appendix.

Let $\widetilde{W}$ be a Coxeter group of type $\tilde{A}_3$ with Dynkin diagram



Let α be the automorphism on $\widetilde{W}$ which fixes $\tilde{s}_0$ and $\tilde{s}_2$ and interchanges $\tilde{s}_1$ and $\tilde{s}_3$. Then $W = \widetilde{W}^\alpha$ is of type $\tilde{C}_2$ with parameters



The cells for Weyl groups $\mathcal{A}_r$, $r \geq 2$, of type $\tilde{A}_{r-1}$ are given in [27]. In particular, Lusztig and Shi established a bijective correspondence between the two–sided cells of $\mathcal{A}_r$ and the partitions of r, which we describe now.

We realize $\mathcal{A}_r$ as the group of all permutations σ on $\mathbb{Z}$ such that $\sigma(i+r) = \sigma(i)+r$ for all $i \in \mathbb{Z}$ and $\sum_{i=1}^r \sigma(i) = \sum_{i=1}^r i$ by letting the simple reflection s_i, $0 \leq i \leq r-1$ act as

$$s_i(j) = \begin{cases} j+1 & \text{if } j \equiv i \pmod{r} \\ j-1 & \text{if } j \equiv i+1 \pmod{r} \\ j & \text{if } j \not\equiv i,\, i+1 \pmod{r}. \end{cases}$$

A map π from $\mathcal{A}_r$ to the set of partitions of r is defined as follows. Let $w \in \mathcal{A}_r$. We denote by d_k, $k \in \mathbb{N}$, the maximal cardinality of a subset of $\mathbb{Z}$ whose elements are incongruent to each other mod r and which is a disjoint union of k subsets each of which has its natural order reversed by w. (We set $d_0 = 0$.) Let $n \in \mathbb{N}$ be minimal such that $d_n = r$. For $1 \leq k \leq n$, we define $\lambda_k = d_k - d_{k-1}$. Then $\sum_{k=1}^n \lambda_k = r$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ (cf. [27]). We set $\pi(w) = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$. The two–sided cells in $\mathcal{A}_r$ then coincide with the fibers of π.

Now take $\widetilde{W}$ and W as at the beginning of this section. We use the above description of the two–sided cells in order to determine the generalized left cells in W. The fixed point sets of the two–sided cells in $\widetilde{W}$ are unions of generalized left and two–sided cells in W.

The lowest generalized two–sided cell $\underline{c}_0$ consists of the eight generalized left cells

$$N_{0,1},\ N_{0,s_0},\ N_{0,s_0s_1},\ N_{0,s_0s_1s_2},\ N_{\frac{1}{2}\alpha_0,1},\ N_{\frac{1}{2}\alpha_0,s_2},\ N_{\frac{1}{2}\alpha_0,s_2s_1},\ N_{\frac{1}{2}\alpha_0,s_2s_1s_0}.$$

The two–sided cell corresponding to the partition $(1 \geq 1 \geq 1 \geq 1)$ contains only the identity element. The fixed point set $\underline{c}_1 = \{1\}$ is a generalized left and two–sided cell.

As noted in [12], the two–sided cell corresponding to $(2 \geq 1 \geq 1)$ consists of all elements in $\widetilde{W}$ with a unique reduced expression. Its set of fixed points is therefore $\{s_0,\, s_2\}$. Remark 4.1.1 (i) implies that $\underline{c}_2 = \{s_0\}$ and $\underline{c}_3 = \{s_2\}$ are generalized left and right cells and hence generalized two–sided cells. (This is an example where the generalized two–sided cells in W are not the fixed point sets of the two–sided cells in $\widetilde{W}$.)

We are left with the elements in $\underline{c}_4 = \underline{c}_4^1 \cup \underline{c}_4^2 \cup \underline{c}_4^3 \cup \underline{c}_4^4$ where

$$\begin{aligned}
\underline{c}_4^1 &= \{s_1(s_0s_2s_1)^n, s_2s_1(s_0s_2s_1)^n, s_0s_1(s_0s_2s_1)^n, (s_0s_2s_1)^{n+1}; n \geq 0\},\\
\underline{c}_4^2 &= \{s_2s_0(s_1s_2s_0)^n, (s_1s_2s_0)^{n+1}, s_2(s_1s_2s_0)^{n+1}, s_0(s_1s_2s_0)^{n+1}; n \geq 0\},\\
\underline{c}_4^3 &= \{s_1s_2(s_0s_1s_2)^n, s_2s_1s_2(s_0s_1s_2)^n, (s_0s_1s_2)^{n+1}, s_2(s_0s_1s_2)^{n+1}; n \geq 0\},\\
\underline{c}_4^4 &= \{s_1s_0(s_2s_1s_0)^n, s_0s_1s_0(s_2s_1s_0)^n, (s_2s_1s_0)^{n+1}, s_0(s_2s_1s_0)^{n+1}; n \geq 0\}
\end{aligned}$$

and $\underline{c}_5 = \underline{c}_5^1 \cup \underline{c}_5^2 \cup \underline{c}_5^3 \cup \underline{c}_5^4$ where

$$\begin{aligned}
\underline{c}_5^1 &= \{s_1s_2s_1(s_0s_1s_2s_1)^n, (s_0s_1s_2s_1)^{n+1}, s_1(s_0s_1s_2s_1)^{n+1}, s_2s_1(s_0s_1s_2s_1)^{n+1}; n \geq 0\},\\
\underline{c}_5^2 &= \{s_1s_0s_1(s_2s_1s_0s_1)^n, (s_2s_1s_0s_1)^{n+1}, s_1(s_2s_1s_0s_1)^{n+1}, s_0s_1(s_2s_1s_0s_1)^{n+1}; n \geq 0\},\\
\underline{c}_5^3 &= \{(s_1s_0s_1s_2)^n, s_2(s_1s_0s_1s_2)^n, s_1s_2(s_1s_0s_1s_2)^n, s_0s_1s_2(s_1s_0s_1s_2)^n; n \geq 1\},\\
\underline{c}_5^4 &= \{(s_1s_2s_1s_0)^n, s_0(s_1s_2s_1s_0)^n, s_1s_0(s_1s_2s_1s_0)^n, s_2s_1s_0(s_1s_2s_1s_0)^n; n \geq 1\}.
\end{aligned}$$

It can be shown by induction on n that π sends the elements in $\underline{c}_4$ to $(2 \geq 2)$ and the elements in $\underline{c}_5$ to $(3 \geq 1)$. (For example, the element $(s_0s_2s_1)^n$, $n \in \mathbb{N}$, as permutation on $\mathbb{Z}$ maps

$$\begin{array}{lll}
1 & \text{to} & 1+2n,\\
2 & \text{to} & 2-2n,\\
3 & \text{to} & 3+2n,\\
4 & \text{to} & 4-2n.
\end{array}$$

So sets of integers which are incongruent mod 4 and whose order is reversed by $(s_0s_2s_1)^n$ are of the form

$$\{1+4k, 2+4l\},\ \{1+4k, 4l\},\ \{2+4k, 3+4l\},\ \text{ or } \{3+4k, 4l\}$$

for certain k, $l \in \mathbb{Z}$. Thus $\pi((s_0s_2s_1)^n) = (2 \geq 2)$.)

We conclude that $\widetilde{\boldsymbol{a}}$ is constant on $\underline{c}_4$ and $\underline{c}_5$, respectively, and that $\underline{c}_4$ and $\underline{c}_5$ are unions of generalized left and two–sided cells. (It is shown in [18] that $\widetilde{\boldsymbol{a}}$ is constant on two–sided cells.)

It follows from the multiplication formula $(*)$ in Section 4.1 that

$$s_1(s_0s_2s_1)^n \geq_L \left\{ \begin{array}{c} s_2s_1(s_0s_2s_1)^n \\ s_0s_1(s_0s_2s_1)^n \end{array} \right\} \geq_L (s_0s_2s_1)^{n+1} \geq_L s_1(s_0s_2s_1)^{n+1}.$$

for $n \geq 0$. Similar relationships hold for the elements in $\underline{c}_4^i$, $i \geq 2$, and $\underline{c}_5^j$, $j \geq 1$. We derive from Corollary 4.3.10 that elements x, $y \in W$ with $x \leq_L y$ and $\pi(x) = \pi(y)$ lie in the same generalized left cell. Therefore, any $\underline{c}_4^i$ and any $\underline{c}_5^j$ is contained in a generalized left cell. Remark 4.1.1 (i) then implies that $\underline{c}_4^i$ for $i \geq 1$ and $\underline{c}_5^j$ for $j \geq 3$ are generalized left cells.

According to [27], Theorem 16.1.2, elements in $\mathcal{A}_r$, $r \geq 2$, lie in the same left cells if and only if they have the same generalized right τ-invariant (for the definition see [27], p.18). Since the elements in $\mathcal{A}_4$ corresponding to $s_1s_2s_1 \in \underline{c}_5^1$ and $s_1s_0s_1 \in \underline{c}_5^2$ do not have the same generalized right τ-invariant (take the right star operation with respect to $\{\tilde{s}_2, \tilde{s}_3\}$), they do not lie in the same left cell. It follows that $\underline{c}_5^1$ and $\underline{c}_5^2$ are generalized left cells.

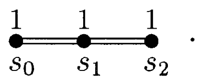
Since generalized two–sided cells are unions of generalized left and right cells and since $(\underline{c}_n^1)^{-1} \cap \underline{c}_n^i \neq \emptyset$ for $n \in \{4, 5\}$ and all $i \geq 1$, the sets $\underline{c}_4$ and $\underline{c}_5$ are generalized two–sided cells.

We remark that in this case, as well as in the cases listed in the appendix, each generalized two–sided cell has a non–trivial intersection with a maximal parabolic subgroup W_λ of W for some special point λ.

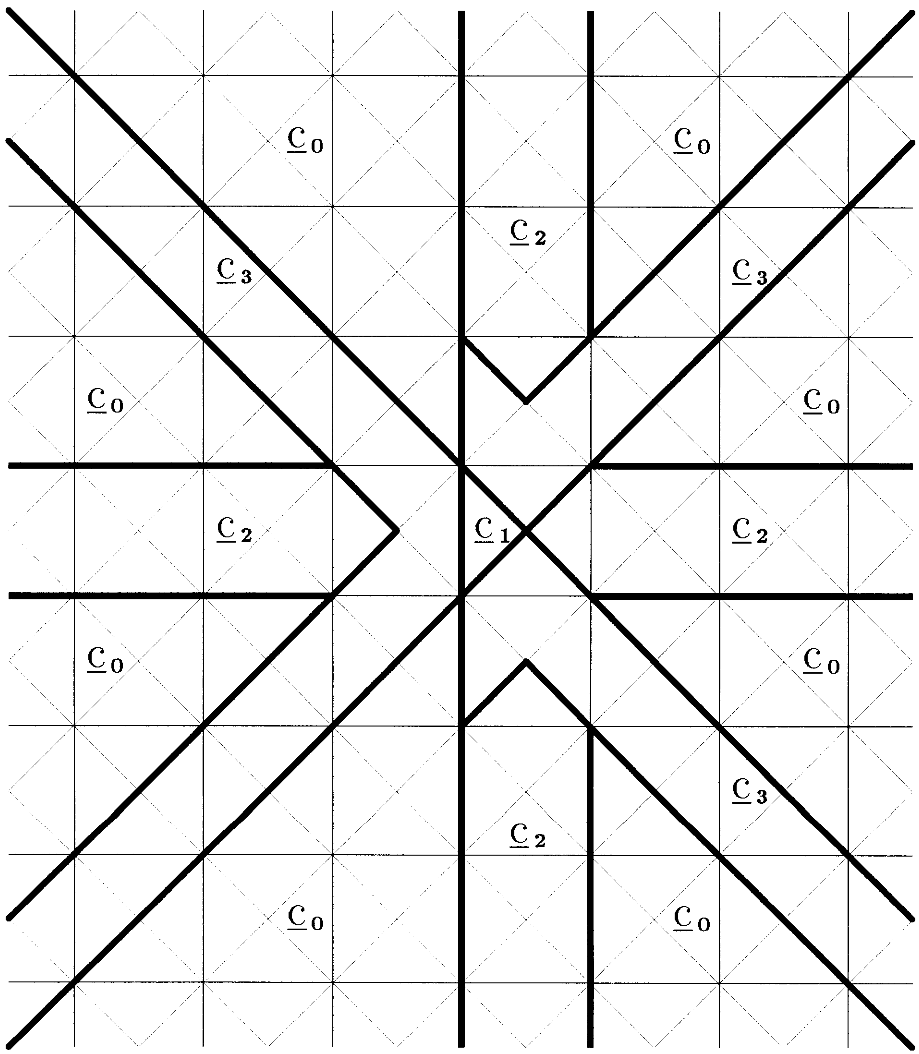
Appendix

For W of type $\tilde{C}_2$, we describe generalized left and two–sided cells using the realization of W in terms of alcoves as in Section 2.3. The generalized left cells are obtained as the connected components after removing the thick lines. The generalized two–sided cells are the unions of all generalized left cells with the same label $\underline{c}_i$, $i \geq 0$. In all cases, the elements in the lowest generalized two–sided cell are labeled $\underline{c}_0$. The alcove corresponding to the identity element is labeled $\underline{c}_1$.

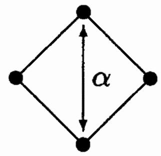
We start with the equal parameter case, which is taken from [18]. In the unequal parameter cases, we indicate $\widetilde{W}$ as well as the non–trivial actions of α. (The notation is that of Section 4.3.) The calculations are similar to those in Section 4.4. (For $\widetilde{W}$ of type $\tilde{B}_3$ resp. $\tilde{D}_4$, we use [6] resp. [5], [30].)

W: 

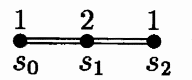
The 4 two–sided cells consist of $8 + 1 + 3 + 4 = 16$ left cells (in this order).



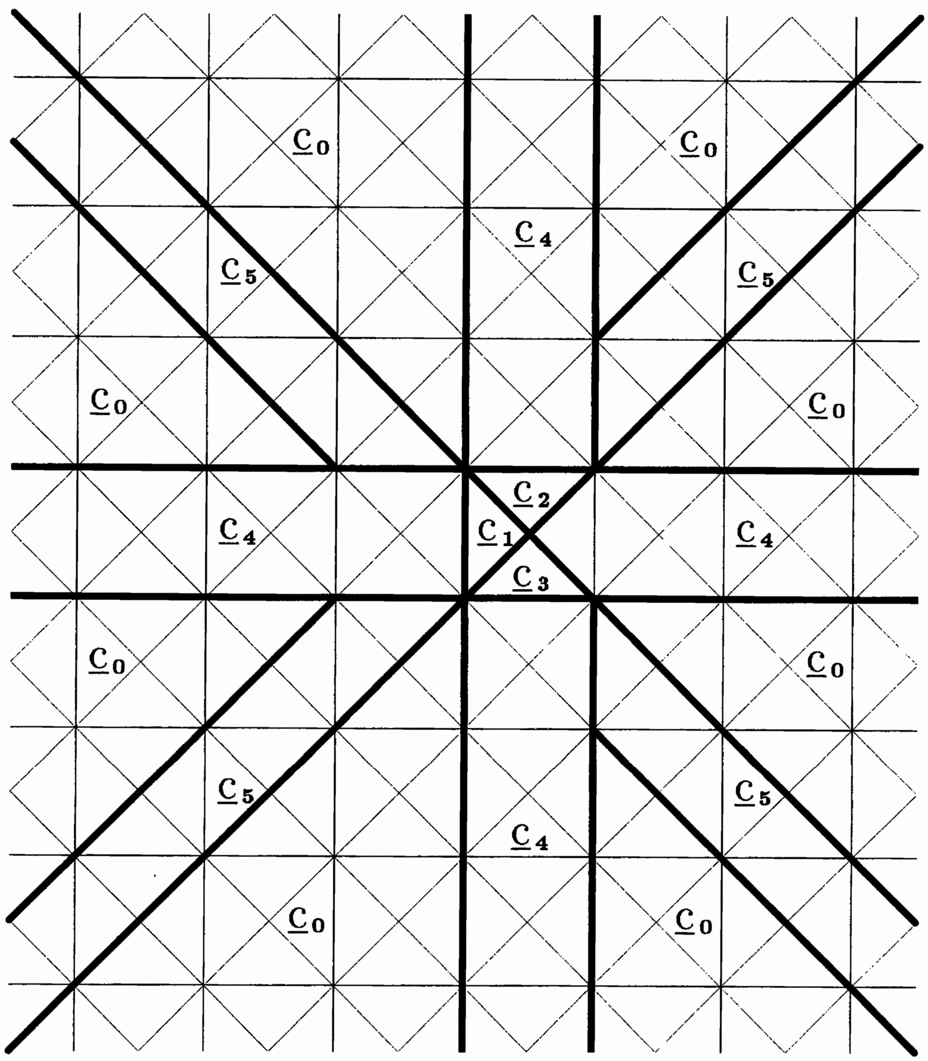
$\widetilde{W}$:



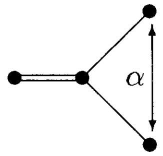
W:



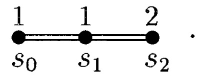
The 6 generalized two–sided cells consist of $8+1+1+1+4+4=19$ generalized left cells.



$\widetilde{W}$:



W:



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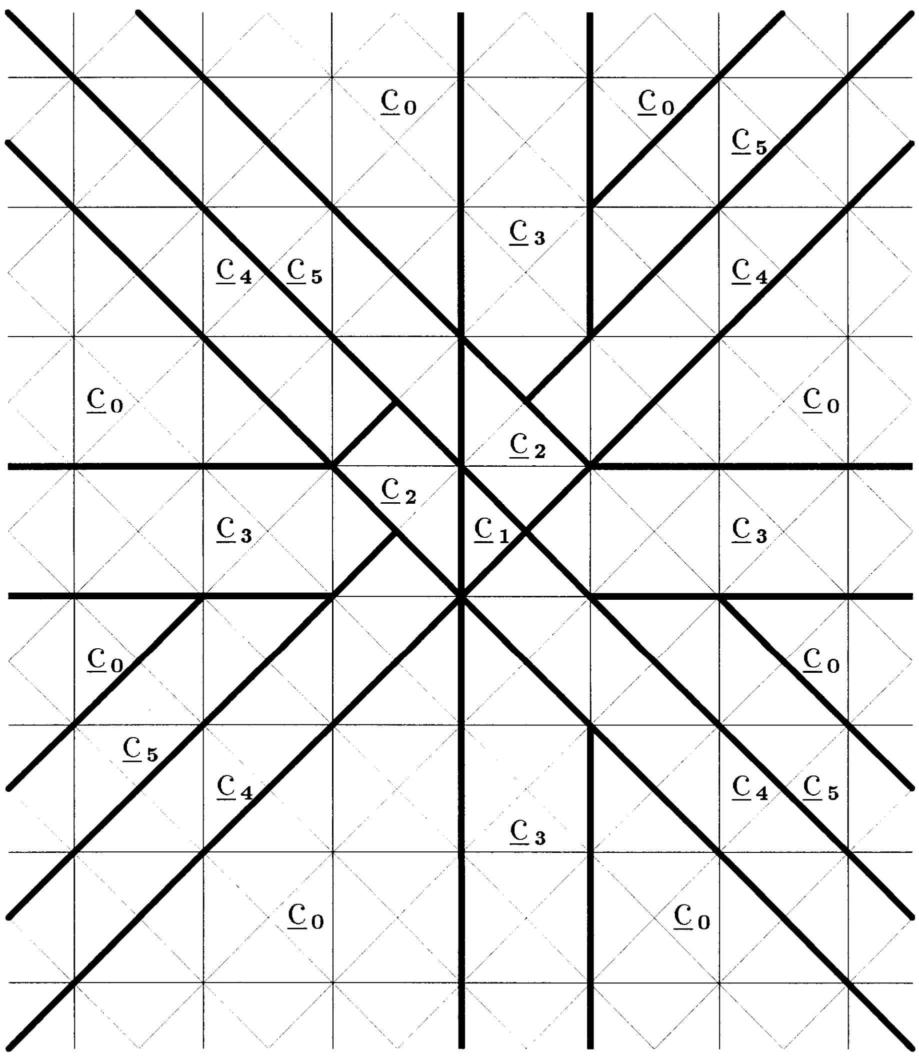
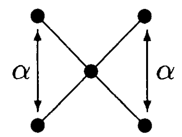
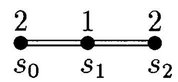
The 6 generalized two–sided cells consist of $8 + 1 + 2 + 4 + 4 + 4 = 23$ generalized left cells.



$\widetilde{W}$:

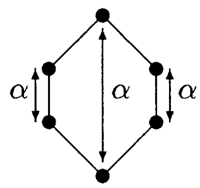


W:

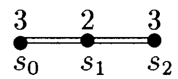


and

$\widetilde{W}$:



W:



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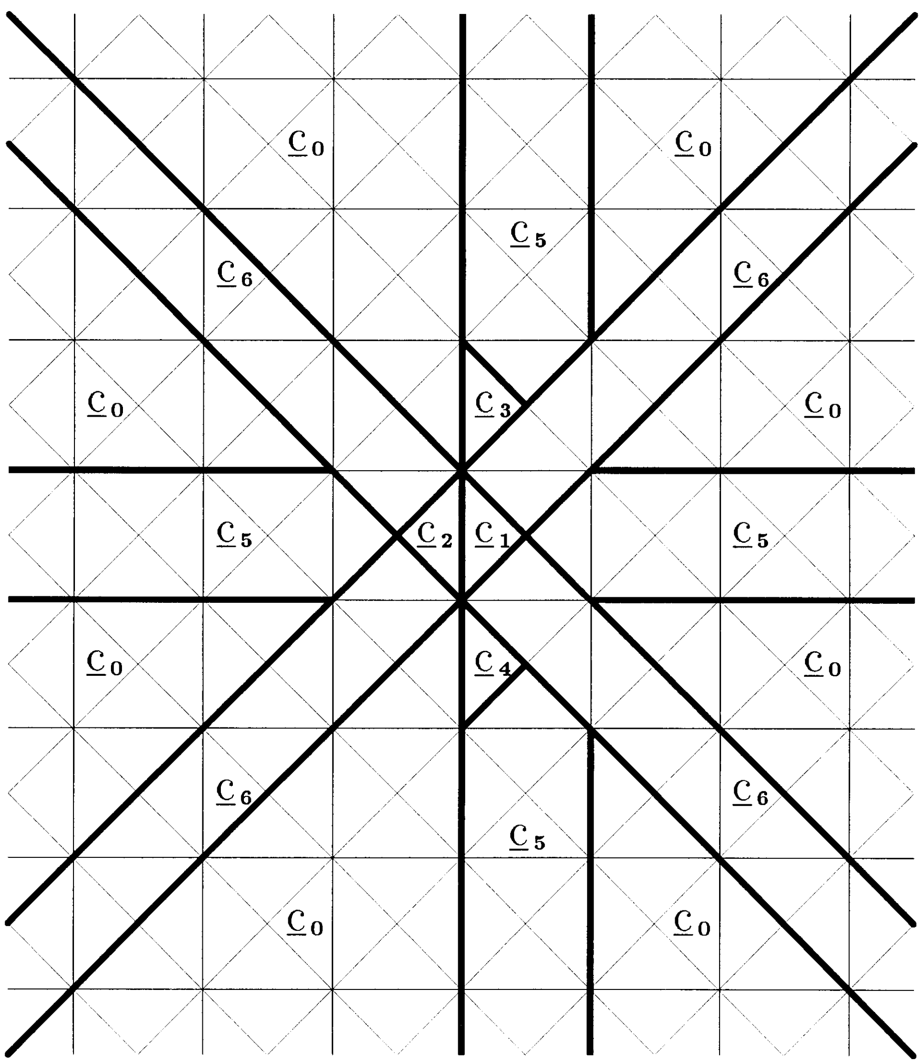
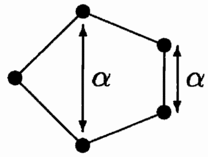
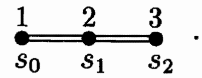
The 7 generalized two-sided cells consist of $8+1+1+1+1+4+4=20$ generalized left cells.



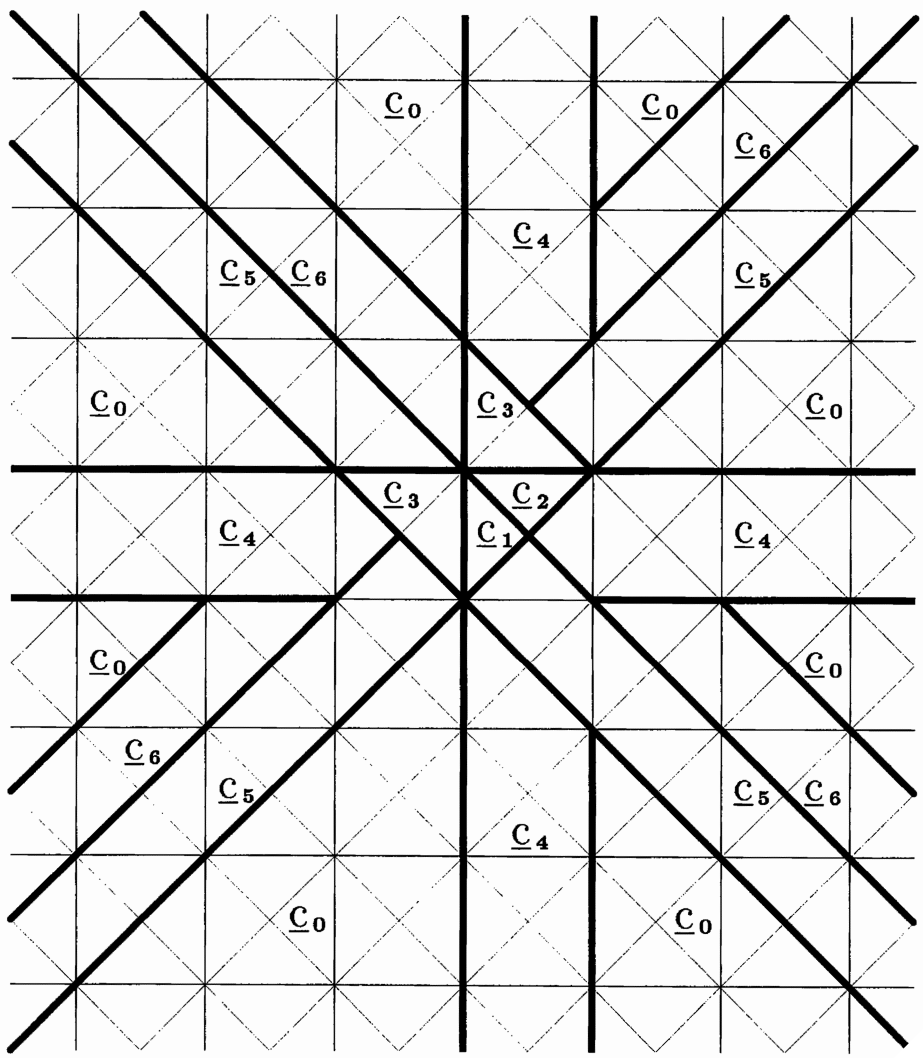
$\widetilde{W}$:



W:



The 7 generalized two–sided cells consist of $8+1+1+2+4+4+4 = 24$ generalized left cells.



References

[1] H.H. Andersen. *An inversion formula for the Kazhdan-Lusztig polynomials for affine Weyl groups.* Adv. in Math. **60** (1986), 125–153.

[2] R. Bédard. *Cells for two Coxeter groups.* Comm. Alg. **14** (1988), 1253–1286.

[3] R. Bédard. *The lowest two-sided cell for an affine Weyl group.* Comm. Alg. **16(6)** (1988), 1113–1132.

[4] N. Bourbaki. *Groupes et Algèbres de Lie, IV, V, VI.* Hermann, Paris, 1968.

[5] C. Chen. *The decomposition into left cells of the affine Weyl group of type $\tilde{D}_4$.* J. Alg. **163** (1994), 692–728.

[6] J. Du. *The decomposition into cells of the affine Weyl group of type $\tilde{B}_3$.* Comm. Alg. **16** (1988), 1383–1409.

[7] J.C. Jantzen. *Über das Dekompositionsverhalten gewisser modularer Darstellungen halbeinfacher Gruppen und ihrer Lie–Algebren.* J. Alg. **49** (1977), 441–469.

[8] S.I. Kato. *Spherical functions and a q-analogue of Kostant's weight multiplicity formula.* Invent. Math. **66** (1982), 461–468.

[9] S.I. Kato. *On the Kazhdan-Lusztig polynomials for affine Weyl groups.* Adv. in Math. **55** (1985), 103–130.

[10] D. Kazhdan and G. Lusztig. *Representations of Coxeter groups and Hecke algebras.* Invent. Math. **53** (1979), 165–184.

[11] D. Kazhdan and G. Lusztig. *Schubert varieties and Poincaré duality.* Proc. Symp. Pure Math. **36**, Amer. Math. Soc. (1980), 185–203.

[12] G. Lawton. *Two-sided cells in affine Weyl groups of type $\tilde{A}_{n-1}$.* J. Algebra **120** (1989), 74–89.

[13] G. Lusztig. *Some problems in the representation theory of finite Chevalley groups.* Proc. Symp. Pure Math. **37** (1980), 313–317.

[14] G. Lusztig. *Hecke algebras and Jantzen's generic decomposition patterns.* Adv. in Math. **37** (1980), 121–164.

[15] G. Lusztig. *Singularities, character formulas, and a q-analog of weight multiplicities.* Astérisque **101-102** (1983), 208–227.

[16] G. Lusztig. *Left cells in Weyl groups.* Lecture Notes in Mathematics **1024**, Springer-Verlag (1983), 99–111.

[17] G. Lusztig. *The two-sided cells of the affine Weyl group of type A_n.* In "Infinite dimensional groups with applications", Springer–Verlag, New York (1985), 275–287.

[18] G. Lusztig. *Cells in affine Weyl groups.* In "Algebraic Groups and Related Topics", Advanced Studies in Pure Math. **6**, Kinokuniya and North Holland, Amsterdam, 1985.

[19] G. Lusztig. *Cells in affine Weyl groups, II.* J. Algebra **109** (1987), 536–548.

[20] G. Lusztig. *Cells in affine Weyl groups, III.* J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. **34** (1987), 223–243.

[21] G. Lusztig. *Cells in affine Weyl groups, IV.* J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. **36** (1989), 297–328.

[22] G. Lusztig. *Affine Hecke algebras and their graded version.* J. Amer. Math. Soc. **2** (1989), 599–635.

[23] G. Lusztig. Lecture at MIT, Spring 1993.

[24] G. Lusztig. Personal communication.

[25] I.G. Macdonald. *Spherical Functions on a Group of p-adic Type.* Publ. Ramanujan Inst., no. 2, Madras, 1971.

[26] H. Matsumoto. *Analyse Harmonique dans les Systémes de Tits Bornologiques de Type Affine.* Lecture Notes in Mathematics **590**, Springer-Verlag, 1977.

[27] J.-Y. Shi. *The Kazhdan-Lusztig Cells in Certain Affine Weyl Groups.* Lecture Notes in Mathematics **1179**, Springer-Verlag, 1986.

[28] J.-Y. Shi. *A two-sided cell in an affine Weyl group, II.* J. London Math. Soc.(2) **37** (1988), 253–264.

[29] J.-Y. Shi. *Left cells in affine Weyl groups.* Tohoku Math. Jour. **46** (1994), no. 1, 105–124.

[30] J.-Y. Shi. *Left cells in the affine Weyl group $W_a(\tilde{D}_4)$.* Osaka J. Math. **31** (1994), no. 1, 27–50.

[31] N. Xi. *Representations of Affine Hecke Algebras.* Lecture Notes in Mathematics **1587**, Springer-Verlag, 1994.

Index of Notation