Quantization of Nilpotent Coadjoint Orbits

by

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SCIENCE

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Abstract

Let G be a complex reductive group. We study the problem of associating Dixmier algebras to nilpotent (co)adjoint orbits of G, or, more generally, to orbit data for G.

If $\mathfrak{g} = \mathfrak{h} + \mathfrak{n} + \bar{\mathfrak{n}}$ is a triangular decomposition of $\mathfrak{g}$ and $\mathcal{O}$ is a nilpotent orbit, we consider the irreducible components of $\mathcal{O} \cap \mathfrak{n}$, which are Lagrangian subvarieties of $\mathcal{O}$. The main idea is to construct, starting with certain "good" components of $\mathcal{O} \cap \mathfrak{n}$, a Dixmier algebra which should be associated to some cover of $\mathcal{O}$. We carry out the construction if the orbit $\mathcal{O}$ is *small*. Then we apply this result to certain simple groups and obtain the Dixmier algebras associated to a variety of nilpotent orbits. A particularly interesting example is a non-commutative orbit datum which we call the Clifford orbit datum. By modifying our main construction a bit we obtain a Dixmier algebra which should be associated to that datum.

Thesis Supervisor: David A. Vogan
Title: Professor of Mathematics

To my parents

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Quantization of Nilpotent Coadjoint Orbits

Contents

1 Introduction

The goal of the method of coadjoint orbits of Kirillov and Kostant is to establish a close relation between the unitary dual (i.e. the set of irreducible unitary representations) of a Lie group $G_{\mathbb{R}}$ and the set of orbits of $G_{\mathbb{R}}$ in $\mathfrak{g}_{\mathbb{R}}^*$ (the coadjoint orbits). If $G_{\mathbb{R}}$ is nilpotent, Kirillov has shown in [12] that the unitary dual of $G_{\mathbb{R}}$ is in a natural bijection with the set of coadjoint orbits of $G_{\mathbb{R}}$. In [1], Kostant and Auslander have generalized Kirillov's results to give a complete description of the unitary dual when $G_{\mathbb{R}}$ is a type I solvable group. However, extending these results to the case when $G_{\mathbb{R}}$ is reductive has proven to be very difficult. After initiating a program that eventually established, for complex solvable groups $G_{\mathbb{C}}$, a bijection between coadjoint orbits and primitive ideals of $U(\mathfrak{g}_{\mathbb{C}})$, Dixmier suggested that, in the reductive case, one could postpone the construction of a unitary representation associated with the real orbit $\mathcal{O}_{\mathbb{R}}$ and instead construct a primitive ideal I in the enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$. The ideal $I = I(\mathcal{O}_{\mathbb{C}})$ should depend only on the complexification $\mathcal{O}_{\mathbb{C}}$ of $\mathcal{O}_{\mathbb{R}}$, and the unitary representation $\pi(\mathcal{O}_{\mathbb{R}})$ associated to $\mathcal{O}_{\mathbb{R}}$ should be an $U(\mathfrak{g}_{\mathbb{C}})/I(\mathcal{O}_{\mathbb{C}})$-module. Associating an ideal in $U(\mathfrak{g}_{\mathbb{C}})$ to $\mathcal{O}_{\mathbb{C}}$ is an easier but nevertheless very interesting problem that should provide useful ideas for the construction of the representations $\pi(\mathcal{O}_{\mathbb{R}})$.

The relation between the associated unitary representation $\pi(\mathcal{O}_{\mathbb{R}})$ and the primitive ideal $I(\mathcal{O}_{\mathbb{C}})$ is illustrated by the case of a nilpotent group G. Then $I(\mathcal{O}_{\mathbb{C}}) = \operatorname{Ann} \pi(\mathcal{O}_{\mathbb{R}})$ and the quotient algebra $U(\mathfrak{g}_{\mathbb{C}})/I(\mathcal{O}_{\mathbb{C}})$ is an algebra of twisted differential operators on a polarizing manifold for $\mathcal{O}_{\mathbb{C}}$. It is a filtered algebra and the associated graded algebra is the same as the associated algebra to the filtered algebra of functions on the orbit.

The main tool for associating primitive ideals to coadjoint orbits of a complex reductive group $G_{\mathbb{C}}$ is parabolic induction. Using it, one can reduce the problem to associating primitive ideals to nilpotent coadjoint orbits. However, there are many non-induced nilpotent orbits, and we need a different method for dealing with them. Furthermore, Borho has given an example of a nilpotent orbit that can be obtained in two different ways by induction such that the two corresponding induced ideals are different. Thus both of these ideals ought to be attached to the same orbit so the orbit method map seems to become multi-valued. The problem in this case is caused by the topology of the orbit — it admits a double cover, so one of the ideals should be associated to the orbit and the other one to the orbit cover. These considerations have lead Vogan to formulate in [17] the following conjecture: instead of "coadjoint orbits" we should consider the slightly more general notion of "orbit data" — algebras that are extensions of the ring of functions on a coadjoint orbit (rings of functions on orbit covers give examples of orbit data). Similarly, "primitive ideals" should be replaced by "Dixmier algebras", which are extensions of primitive quotients of $U(\mathfrak{g}_{\mathbb{C}})$. The orbit method should construct a bijective correspondence (Dixmier map) between a certain set of orbit data (which should include

all rings of functions on orbit covers) and a corresponding set of Dixmier algebras. If an orbit datum R corresponds to a Dixmier algebra A, they should admit G-stable filtrations such that the associated graded algebras are isomorphic (in particular R and A should have the same G-module structure). The precise definitions will be given in Section 2.

The main goal of this thesis is to illustrate a new method for constructing Dixmier algebras. It has been recognized for a long time that Lagrangian subvarieties of the orbit $\mathcal{O}$ play an important role in understanding the geometry of the orbit. In case G is reductive it is Vogan's idea that a Lagrangian subvariety stable under a Borel subgroup B of G should give rise to a Dixmier algebra. Our main result is in Section 6 where we show how, starting with a nilpotent orbit $\mathcal{O}$ of certain type (we call such orbits *small*) and a B-stable Lagrangian in $\mathcal{O}$ satisfying some conditions, we can construct a Dixmier algebra which should be associated to a certain cover of $\mathcal{O}$. In Section 7 we give examples of nilpotent orbits to which we can apply this construction: all spherical orbits of $Sp(2n, \mathbb{C})$, certain spherical orbits of $SO(2n, \mathbb{C})$, the double cover of the maximal spherical orbit of $Sp(4n, \mathbb{C})$ (recall that an orbit $\mathcal{O}$ is called *spherical* if some Borel subgroup of G has a dense open orbit in $\mathcal{O}$).

Another application that we give is obtaining the Dixmier algebra associated to a certain non-commutative datum (the Clifford orbit datum) which appears naturally as follows. Recall that the Dixmier correspondence conjectured by Vogan should be a bijection between a certain set of orbit data (including all orbit covers) and a corresponding set of Dixmier algebras. One condition that should be imposed, if one hopes to have a nice correspondence, is that both the orbit data and the Dixmier algebras must be completely prime (i.e. with no zero divisors). However, there are certain completely prime primitive ideals that cannot be associated to any orbit cover, so one expects that they should have an associated non-commutative orbit datum. One such example appears when $G = Sp(4n, \mathbb{C})$. In [15] McGovern has constructed a completely prime Dixmier algebra A_G which is a model representation for G, i.e. each finite-dimensional representation of G appears exactly once in A_G. A_G is a 2-fold extension of the quotient of $U(\mathfrak{g})$ sitting in it. It is easy to see (by comparing the G-module structures) that neither A_G nor the quotient of $U(\mathfrak{g})$ that it contains can be associated to any orbit cover. We will construct a non-commutative completely prime orbit datum — the Clifford orbit datum R_G — which is also a model representation for G. It is natural to expect that A_G should be associated to R_G under the Dixmier map. As an application of our main construction, we obtain the model Dixmier algebra A_G in a way which strongly suggests that its associated orbit datum is R_G.

Here is an outline of this thesis. In Section 2 we give definitions of orbit data, Dixmier algebras, and discuss the conjectural Dixmier map between them. In Section 3 we recall some well-known methods for quantizing coadjoint orbits (polarizations, change of polar-

ization maps, induction, etc.) and discuss an example which illustrates all steps of our main construction. In the next two sections we develop the machinery that will be needed for the construction. In Section 4 we define induction of Dixmier algebras and prove its basic properties. Section 5 deals with the construction of a "change of polarization" map between Dixmier algebras obtained from a coadjoint orbit by using two different polarizations. In Section 6 we carry out (under favorable circumstances) the construction of a Dixmier algebra which should be associated to the ring of functions on some orbit cover. Finally, in Section 7 we discuss how restrictive the conditions we put on the orbit are, and consider various examples of orbit data for which we are able to construct associated Dixmier algebras: the spherical orbits of $Sp(2n, \mathbb{C})$, some spherical orbits of $SO(2n, \mathbb{C})$, the double cover of the maximal spherical orbit of $Sp(4n, \mathbb{C})$, and the Clifford orbit datum for $Sp(4n, \mathbb{C})$.

2 Orbit data and Dixmier algebras.

Suppose G is a complex algebraic group. As discussed in the Introduction, the attempts to define a nice correspondence between coadjoint orbits for G and primitive ideals in $U(\mathfrak{g})$ have shown that, if such a correspondence is to exist, one has to consider the more general notions of orbit data and Dixmier algebras. In more detail, each orbit $\mathcal{O}$ is replaced by its ring of functions $R[\mathcal{O}]$. This ring is equipped with a G-action, and restricting regular functions from $\mathfrak{g}^*$ to $\mathcal{O}$ gives a G-equivariant map

$$\psi : S(\mathfrak{g}) \longrightarrow R[\mathcal{O}]. \tag{2.1}$$

$\ker \psi$ is a G-invariant ideal of $S(\mathfrak{g})$ and $\mathcal{O}$ can be recovered as the unique dense G-orbit in the closed subvariety of $\mathfrak{g}^*$ corresponding to this ideal. Furthermore, recall from the Introduction that a main reason for introducing orbit data was the observation that non-ramified covers of coadjoint orbits had to be associated to certain primitive ideals of $U(\mathfrak{g})$. If $\widetilde{\mathcal{O}} \to \mathcal{O}$ is one such cover, then pulling back functions from $\mathcal{O}$ to $\widetilde{\mathcal{O}}$ gives an inclusion $R[\mathcal{O}] \hookrightarrow R[\widetilde{\mathcal{O}}]$ and, after composing with (2.1), we get a map $S(\mathfrak{g}) \to R[\widetilde{\mathcal{O}}]$. These examples lead to the general definition of orbit data [17].

Definiton 2.1 *An* ORBIT DATUM *for G is an algebra R equipped with the following structure:*

- *An algebra homomorphism $S(\mathfrak{g}) \to R$ which sends $S(\mathfrak{g})$ to the center of R.*

- *An algebraic G-action on R by algebra automorphisms (denoted* Ad*) such that the map $S(\mathfrak{g}) \to R$ is G-equivariant.*

As we saw above, any orbit cover gives rise to an orbit datum, namely the ring of regular functions on the cover. Such orbit data are called GEOMETRIC. The SUPPORT of an orbit datum R is the G-variety in $\mathfrak{g}^*$ corresponding to the ideal $\ker \psi$ of $S(\mathfrak{g})$. The orbit data that we will consider will be spaces of global sections of bundles of algebras on a coadjoint orbit $\mathcal{O}$, i.e. they will be supported on the closure of $\mathcal{O}$.

Definiton 2.2 *A* DIXMIER ALGEBRA *for G is an algebra A equipped with the following structure:*

- *An algebra homomorphism $\varphi : U(\mathfrak{g}) \to A$. There is a corresponding $\mathfrak{g}$-action on A, denoted by* ad*, given by $\mathrm{ad}(X)(a) = \varphi(X)a - a\varphi(X)$ for $X \in \mathfrak{g}$ and $a \in A$.*

- *An algebraic G-action on A by algebra automorphisms (denoted* Ad*) such that the map $U(\mathfrak{g}) \to A$ is G-equivariant and the differential of the G-action* Ad *is the $\mathfrak{g}$-action* ad.

This definition of a Dixmier algebra differs slightly from the one in [17]: we do not require that the Dixmier algebra is a finitely generated module over $U(\mathfrak{g})$. The reason is that, if G is not reductive, the algebras that we would like to call Dixmier algebras (e.g. algebras of twisted differential operators on a polarizing manifold for some orbit) are not necessarily finite as $U(\mathfrak{g})$-modules. The same remark applies to the definition of an orbit datum.

Rather than a correspondence between coadjoint orbits and primitive ideals, one expects that there should be a correspondence between a certain set of orbit data and a corresponding set of primitive Dixmier algebras (i.e. algebras such that $\ker \psi$ is a primitive ideal of $U(\mathfrak{g})$). As mentioned in the Introduction, many examples suggest that, if such a correspondence is to have reasonable properties, both the Dixmier algebra and the corresponding orbit datum should be completely prime, i.e. with no zero divisors. The following conjecture was formulated by Vogan in [17]:

Conjecture 2.3 *There is a natural injection from the set of geometric orbit data into the set of completely prime Dixmier algebras. The Dixmier algebra A corresponding to an orbit datum R should satisfy*

- *A and R admit G-invariant filtrations indexed by $\frac{1}{2}\mathbb{N}$ such that the maps $U(\mathfrak{g}) \to A$ and $S(\mathfrak{g}) \to R$ respect the filtrations.*

- *There is a G-equivariant isomorphism $\operatorname{gr} A \to \operatorname{gr} R$ carrying $\operatorname{gr} \varphi$ to $\operatorname{gr} \psi$, i.e.*

$$
\begin{array}{ccc}
\operatorname{gr} A & \xrightarrow{\ \cong\ } & \operatorname{gr} R \\
\uparrow{\scriptstyle \operatorname{gr} \varphi} & & \uparrow{\scriptstyle \operatorname{gr} \psi} \\
\operatorname{gr} U(\mathfrak{g}) & \xrightarrow{\ \cong\ } & S(\mathfrak{g})
\end{array}
$$

- *The algebra $\operatorname{gr} A \cong \operatorname{gr} R$ is completely prime.*

Traditionally, the unitary representation associated to the real coadjoint orbit $\mathcal{O}_{\mathbb{R}}$ is called the quantization of $\mathcal{O}_{\mathbb{R}}$. Analogously to that, if the Dixmier algebra A is associated to the orbit datum R we will often call it the QUANTIZATION of R.

3 An example of the construction

In this section we start by recalling a few fundamental ideas related to quantization of symplectic manifolds. Next we consider an example showing how these ideas are used in the main construction of Dixmier algebras in Section 4. Finally, the example will point out the technical tools needed for the construction.

Even though in recalling the main ideas we do consider real symplectic manifolds and real Lie groups, our main concern are complex algebraic groups and their coadjoint orbits. From Section 3.4 on, all groups considered are complex algebraic.

3.1 Polarizations

The prototypical example of quantization of a symplectic manifold is the following. Let M be a smooth real manifold and let $X = T^*M$ be its cotangent bundle. Then X carries a natural symplectic structure. If we denote the real bundle of half-densities on M by $\mathcal{D}^{1/2}$ then the quantization of X is given by the Hilbert space $L^2(M, \mathcal{D}^{1/2})$. (The reason we should consider $L^2(M, \mathcal{D}^{1/2})$, rather than $L^2(M)$, is that defining $L^2(M)$ involves the choice of a measure on M, while the space $L^2(M, \mathcal{D}^{1/2})$ is defined naturally and carries a natural Hilbert space structure.) To define the Dixmier algebra associated to X, notice that the algebra of differential operators $\mathbb{D}(M, \mathcal{D}^{1/2})$ acts on the smooth sections of the bundle $\mathcal{D}^{1/2}$. Moreover, if a Lie group G acts on M, there is an induced G-action on $\mathcal{D}^{1/2}$ and an induced map $U(\mathfrak{g}) \to \mathbb{D}(M, \mathcal{D}^{1/2})$. That will make the $\mathfrak{g}$-finite part A of $\mathbb{D}(M, \mathcal{D}^{1/2})$ a Dixmier algebra for G (which we associate to X).

This example can be generalized as follows. Let M be a smooth manifold and let $\mathcal{L}$ be a Hermitian line bundle on M. Then, as shown in [13], one can define a twisted cotangent bundle $X = T^*(M, \mathcal{L})$ on M. X will be an affine bundle on M and the fiber at $m \in M$ will be an affine space for T_m^*M. X carries a natural symplectic structure and its quantization is the Hilbert space $L^2(M, \mathcal{L} \otimes \mathcal{D}^{1/2})$. As above, the Dixmier algebra associated to X is the $\mathfrak{g}$-finite part of $\mathbb{D}(M, \mathcal{L} \otimes \mathcal{D}^{1/2})$.

These examples lead to the following method for quantizing symplectic manifolds. If X is a symplectic manifold we would like to find a manifold M and a line bundle $\mathcal{L}$ on M such that $X \cong T^*(M, \mathcal{L})$. Observe that the fibers of $T^*(M, \mathcal{L})$ will give a foliation of X such that the leaves are Lagrangian submanifolds. Therefore, given X, we would like to find a Lagrangian foliation on X and *define* M as the space of leaves. A Lagrangian foliation is called a (real) *polarization* of X. Notice that, if a Lie group G acts on X, we want G to act on the space of leaves of the foliation as well (so that $L^2(M, \mathcal{L} \otimes \mathcal{D}^{1/2})$ would be a representation of G). In other words, we want the polarization to be G-invariant. In the special case when $X = \mathcal{O}$ is a coadjoint orbit, $\lambda \in \mathcal{O}$, and G_λ is the stabilizer of λ in G, this forces the set of leaves to be a homogeneous space G/H for G and the leaf through

λ to be precisely $H \cdot \lambda$, for some subgroup H of G. The other leaves of the foliation will be the G-translates of the leaf through λ, hence the polarization is completely determined by specifying λ and H. The conditions that H must satisfy in order to give a polarization of $\mathcal{O}$ at λ are:

a) $G_\lambda \subseteq H$.

b) λ is a character of $\mathfrak{h}$, i.e. $\lambda|_{[\mathfrak{h},\mathfrak{h}]} = 0$.

c) $\dim G/H = \dim H \cdot \lambda = \frac{1}{2} \dim \mathcal{O}$.

If such an H is given, and if $2\pi i\lambda$ exponentiates to a character of G_λ, we obtain the representation $L^2(G/H, \mathcal{L}_\lambda \otimes \mathcal{D}^{1/2})$ associated to $\mathcal{O}$, where $\mathcal{L}_\lambda$ is the sheaf of sections of the bundle $G \times_H \mathbb{C}_{2\pi i\lambda}$ associated to $2\pi i\lambda$. Notice that if we only want to associate a Dixmier algebra to $\mathcal{O}$, we do not need the integrality condition on λ — the algebra of twisted differential operators $\mathbb{D}(G/H, \mathcal{L}_\lambda \otimes \mathcal{D}^{1/2})$ exists even if λ is not integral.

3.2 Change of polarization

A manifold X may admit more than one polarization. In that case, a natural and important question is whether the associated unitary space (respectively, the associated Dixmier algebra) depends on the choice of a polarization. A typical example is $X = T^*V \cong V \oplus V^*$ where V is a real vector space. There are two obvious polarizations of X: one with leaves parallel to V and another one, with leaves parallel to V^*. The corresponding spaces of leaves are V^* and V, respectively. The associated Hilbert spaces are $L^2(V^*, \mathcal{D}^{1/2})$, respectively $L^2(V, \mathcal{D}^{1/2})$. We know that the Fourier transform gives a natural isomorphism between these two Hilbert spaces. In the setting of Dixmier algebras, the Fourier transform provides an algebra isomorphism $\mathbb{D}(V, \mathcal{D}^{1/2}) \cong \mathbb{D}(V^*, \mathcal{D}^{1/2})$. As this example indicates, we expect that, under favorable conditions, quantization is independent of the choice of polarization.

Actually, $X = V \oplus V^*$ is a coadjoint orbit for the Heisenberg group $V \oplus V^* \oplus \mathbb{R}$ and the Fourier transform actually is an equivariant isomorphism. (This example will be discussed in more detail and used in Section 5.) More generally, the independence of polarization can be proven for polarizations of coadjoint orbits of unipotent groups. However, there is no "independence of polarization" result for arbitrary symplectic manifolds (or even for coadjoint orbits of general Lie groups). One of the reasons such a result is hard to even formulate is that, in general, the representations (and Dixmier algebras) obtained from different polarizations will *not* be isomorphic. To see what can be expected in general, consider the example of the group $G = \mathbb{R}^\times \ltimes \mathbb{R}$ where $\mathbb{R}^\times$ acts on $\mathbb{R}$ by multiplication. There is a dense coadjoint orbit $G \cdot \lambda$ in $\mathfrak{g}^*$, isomorphic to G, and the subgroups $\mathbb{R}^\times$ and

$\mathbb{R}$ of G give two polarizations of $G \cdot \lambda$. We cannot expect that there is an isomorphism of Dixmier algebras

$$\mathbb{D}(G/\mathbb{R}^\times, \mathcal{L}_\lambda \otimes \mathcal{D}^{1/2}) \cong \mathbb{D}(G/\mathbb{R}, \mathcal{L}_\lambda \otimes \mathcal{D}^{1/2}).$$

(The algebras of symbols of these two algebras of differential operators are not isomorphic.) However, from results of Dixmier ([8], Theorem 6.1.4) and Borho and Brylinski ([4],§1), it follows that for any solvable group G, $\lambda \in \mathfrak{g}^*$, and a polarization H of $G \cdot \lambda$, the kernel of the map $U(\mathfrak{g}) \longrightarrow \mathbb{D}(G/H, \mathcal{L}_\lambda \otimes \mathcal{D}^{1/2})$ is independent of the choice of polarization. In the example above, the map

$$U(\mathfrak{g}) \longrightarrow \mathbb{D}(G/\mathbb{R}^\times, \mathcal{L}_\lambda \otimes \mathcal{D}^{1/2})$$

is surjective, so, rather than an isomorphism, we have an *injection* of Dixmier algebras

$$\mathbb{D}(G/\mathbb{R}^\times, \mathcal{L}_\lambda \otimes \mathcal{D}^{1/2}) \hookrightarrow \mathbb{D}(G/\mathbb{R}, \mathcal{L}_\lambda \otimes \mathcal{D}^{1/2})$$

(and the image of $U(\mathfrak{g})$ is the same in both algebras, which is precisely Dixmier's result).

In Section 5 we will construct a similar change of polarization map for coadjoint orbits of not necessarily solvable groups (but the orbits considered will be of very special kind). That map will be an essential tool in our construction of Dixmier algebras.

3.3 Induction

Induction is a generalization of the method of using polarizations. The basic idea of induction is the following. Let H be a subgroup of G, let X_H be a symplectic manifold with a Hamiltonian H-action, and let $\mu : X_H \to \mathfrak{h}^*$ be the moment map. Suppose the unitary representation π_H is the quantization of X_H. We can consider the induced representation $\pi_G = \operatorname{Ind}_H^G(\pi_H \otimes \mathcal{D}^{1/2})$ and a natural question is to determine the manifold X_G whose quantization is π_G. Roughly, the answer is

$$X_G = G \times_H p^{-1}(X_H) \tag{3.1}$$

where $p^{-1}(X_H)$ denotes the pullback of X_H under the projection $p : \mathfrak{g}^* \to \mathfrak{h}^*$.

To see how this extends the method of polarizations, observe that if H is a polarization of $G \cdot \lambda$ at λ then $\lambda|_{\mathfrak{h}}$ is fixed by H, i.e. its orbit $\mathcal{O}_H$ consists of a single point. The unitary representation of H associated to $\mathcal{O}_H$ will be the one-dimensional Hermitian vector space $\pi_H = \mathbb{C}_{2\pi i \lambda}$ (where H acts by the unitary character $\exp(2\pi i \lambda)$) and the induced representation from H to G will be $L^2(G/H, \mathcal{L}_\lambda \otimes \mathcal{D}^{1/2})$, which is precisely the quantization of $G \cdot \lambda$. Now it is easy to see that the induced space X_G is equipped with a map $\mu_G : X_G \to \mathfrak{g}^*$ which, at least if G is unipotent, gives an isomorphism $X_G \cong G \cdot \lambda$. In other words, using the polarization H to quantize $G \cdot \lambda$ is the same as simultaneously

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inducing the unitary representation $\pi_H = \mathbb{C}_{2\pi i\lambda}$ and its corresponding coadjoint orbit $H \cdot \lambda|_{\mathfrak{h}}$ from H to G.

We will be interested in induction mostly in the context of Dixmier algebras and orbit data. One can define induction of orbit data in a way that is consistent with and motivated by (3.1). Similarly, one can define induction of Dixmier algebras. This is done in [17] for the case of parabolic induction, i.e. induction from a parabolic subgroup to a reductive group. We will give the corresponding definitions in Section 4. We expect that, at least under some favorable conditions, quantization should commute with induction.

3.4 Dense orbits

From now on all groups considered will be complex and algebraic. Let H be a closed subgroup of G. Suppose $\mathcal{O}_H = H \cdot \lambda$ is dense in the coadjoint orbit $\mathcal{O}_G = G \cdot \lambda$. Let R_H be the ring of functions on $\mathcal{O}_H$ and let R_G be the ring of functions on $\mathcal{O}_G$ (so that $R_G \subseteq R_H$). The action of $\mathfrak{g}$ on R_G extends to an action on R_H (the elements of $\mathfrak{g}$ act as vector fields on $\mathcal{O}_H$) and R_G is precisely the G-finite part of R_H (i.e. the space of functions in R_H which transform finitely under $\mathfrak{g}$ and on which the $\mathfrak{g}$-action exponentiates to a G-action). To see this, first recall that regular functions on algebraic varieties equipped with an algebraic G-action transform finitely under G, hence R_G consists of G-finite functions. To prove the converse implication, suppose f is a regular function on $\mathcal{O}_H$ (hence a rational function on $\mathcal{O}_G$) which transforms finitely under G. Suppose f is not regular on $\mathcal{O}_G$ and let P_f denote the divisor of poles of f (i.e. points on $\mathcal{O}_G$ where f is not defined). Let V denote the (finite-dimensional) vector space of rational functions on $\mathcal{O}_G$ spanned by G-translates of f. Then V has a finite basis over $\mathbb{C}$, hence the set of polar divisors of functions in V is finite. On the other hand, any G-translate of P_f will be the polar divisor of the corresponding G-translate of f, and the set of such translates is clearly infinite. This contradiction shows that f cannot have poles, i.e. it must be a regular function on $\mathcal{O}_G$.

Suppose now that A_H is the Dixmier algebra for H which quantizes R_H. Then we expect to have $A_G \subseteq A_H$ and, furthermore, A_G should be precisely the G-finite part of A_H for the adjoint $\mathfrak{g}$-action on A_H coming from the map $\mathfrak{g} \to A_G \to A_H$. In particular, in order to construct A_G, it will be sufficient to construct A_H and the map $\mathfrak{g} \to A_H$, and then define A_G as the G-finite part of A_H.

3.5 Using Lagrangian coverings instead of polarizations

If G is reductive, only a small fraction of the nilpotent coadjoint orbits admit a polarization. One would like to find a "generalization" of the notion of polarization which would give a procedure for quantizing non-polarizable orbits. A possible approach is developed

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by Graham and Vogan in [10]. The idea is to replace the Lagrangian foliation of $\mathcal{O}$ by a family of Lagrangian submanifolds of $\mathcal{O}$ which are allowed to overlap. Recall that if $\mathcal{O} = G \cdot \lambda$ is polarizable and if $P \cdot \lambda$ is the leaf of the foliation through λ then, by a result of Ozeki and Wakimoto [16], P is a parabolic subgroup of G. Then $P \cdot \lambda$ is an irreducible Lagrangian subvariety of $\mathcal{O}$ stable under some Borel subgroup B of G and the space of leaves of the foliation is precisely the space of G-translates of $P \cdot \lambda$. While few nilpotent orbits admit polarizations, all have B-stable Lagrangian subvarieties ([9]). Furthermore, it is known that the set of irreducible B-stable Lagrangian subvarieties of $\mathcal{O}$ is precisely the set of irreducible components of $\mathcal{O} \cap \mathfrak{n}$ (where $\mathfrak{n}$ is the nilradical of $\mathfrak{b} = \mathrm{Lie}(B)$). We can try to obtain the quantization of $\mathcal{O}$ (or of orbit covers of $\mathcal{O}$, or, even more generally, of orbit data supported on $\mathcal{O}$) from any irreducible component L of $\mathcal{O} \cap \mathfrak{n}$. To be more specific, let the parabolic subgroup Q be the stabilizer of L in $\mathcal{O}$. Then the space of G-translates of L is G/Q. The translates of L will overlap unless Q is a polarization of $\mathcal{O}$ but we may still hope to obtain the quantization of $\mathcal{O}$ from the space of L^2-sections of some well chosen line bundle $\mathcal{L}$ on G/Q. Unless Q is a polarization of $\mathcal{O}$, the whole space of L^2-sections of $\mathcal{L}$ on G/Q is "too big" for this purpose. One expects to find a G-invariant subspace of $\Gamma(G/Q, \mathcal{L})$ which will be the quantization of $\mathcal{O}$. (How one should define this subspace is explained in [10].)

This invariant subspace will be preserved by $U(\mathfrak{g})$ and, one expects, by $\mathbb{D}(G/Q, \mathcal{L})$. Therefore we expect to obtain the Dixmier algebra associated to $\mathcal{O}$ as a certain quotient of $\mathbb{D}(G/Q, \mathcal{L})$. Of course, it is not clear how one should choose the bundle $\mathcal{L}$ and how to construct the quotient map. We will show how to do this in the example in the next section. In Section 6 we will generalize the construction to a certain class of nilpotent orbits.

3.6   The example

The following example will demonstrate the ideas involved in the construction of the Dixmier algebras.

Let $G = Sp(4n, \mathbb{C})$, $\mathfrak{g} = \mathrm{Lie}(G)$. Denote by $e_1, \ldots, e_{2n}, f_1, \ldots, f_{2n}$ the standard basis for $\mathbb{C}^{4n}$ and let W and W' denote the span of the vectors $e_1, \ldots, e_{2n}$ and $f_1, \ldots, f_{2n}$. Then W and W' will be complementary maximal isotropic subspaces and the symplectic form identifies W' with the dual of W.

Each element of $\mathfrak{g}$ can be written in the form $\begin{pmatrix} A & S \\ T & -{}^t A \end{pmatrix}$ with $A \in \mathfrak{gl}(W)$, $S \in S^2(W)$, $T \in S^2(W')$. Let $P = \left\{ \begin{pmatrix} A & S \\ 0 & {}^t A^{-1} \end{pmatrix} \mid A^{-1} S \in S^2(W) \right\}$ be the stabilizer of W, a

parabolic subgroup of G with Levi decomposition $P \cong GL(W) \ltimes S^2(W)$, and let

$$\lambda = \begin{pmatrix} 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

be an element of $\mathfrak{u}_P$ (we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using the trace form). Since $\mathfrak{u}_P \cong S^2(W)$, λ gives a non-degenerate symmetric form on W' and an induced non-degenerate symmetric form on W. If V is the span of $e_1, \ldots, e_n$ and V' is the span of $e_{n+1}, \ldots, e_{2n}$ then V and V' will be complementary maximal isotropic (with respect to the symmetric form) subspaces of W. Let Q be the stabilizer of V in G. Q is written in the form

$$Q = \begin{pmatrix} A & * & * \\ 0 & B & * \\ 0 & 0 & {}^tA^{-1} \end{pmatrix} \text{ with } A \in GL(V), \ B \in Sp(2n, \mathbb{C}),$$

in basis $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}, f_{n+1}, \ldots, f_{2n}, f_1, \ldots, f_n\}$. From now on, when writing elements in matrix form, we will be using this basis.

Consider the coadjoint orbit $\mathcal{O}$ of λ. The stabilizer of λ is $O(W) \ltimes U_P \subseteq P$, therefore $\mathcal{O}$ has a double cover $\widetilde{\mathcal{O}} \cong G/[SO(W) \ltimes U_P]$. P is a polarization of $\mathcal{O}$ at λ, hence the Dixmier algebra associated to $\mathcal{O}$ is $\mathbb{D}(G/P, \rho_{G/P})$. A more difficult question is how to obtain the Dixmier algebra associated to $\widetilde{\mathcal{O}}$ and that is what we will show. Let $\widetilde{\lambda}$ be a point in the pre-image of λ in $\widetilde{\mathcal{O}}$.

The special case when the group G is $Sp(4, \mathbb{C})$ suggests how to obtain the quantization of $\widetilde{\mathcal{O}}$. In that case the parabolic Q considered above is a polarization of $\widetilde{\mathcal{O}}$ at $\widetilde{\lambda}$ so the Dixmier algebra associated to $\widetilde{\mathcal{O}}$ is $\mathbb{D}(G/Q, \rho_{G/Q})$. (Notice that Q is not a polarization of $\mathcal{O}$ since $G_\lambda \cong O(W) \ltimes U_P$ is not contained in Q – only $G_{\widetilde{\lambda}} = SO(W) \ltimes U_P$ is). Also, $\mathcal{O} \cap \mathfrak{n} = P \cdot \lambda \cup Q \cdot \lambda$, so, in accord with the general idea of Lagrangian coverings outlined in Section 3.5, we have obtained quantizations (of $\mathcal{O}$ and $\widetilde{\mathcal{O}}$) from each of the two irreducible components of $\mathcal{O} \cap \mathfrak{n}$.

In the $G = Sp(4n, \mathbb{C})$ case Q is not a polarization of $\widetilde{\mathcal{O}}$ since $G_{\widetilde{\lambda}}$ is not contained in Q. However $Q \cdot \lambda$ is a Lagrangian subvariety of $\mathcal{O}$ and we will construct the Dixmier algebra associated to $\widetilde{\mathcal{O}}$ as a quotient of $\mathbb{D}(G/Q, \chi + \rho_{G/Q})$ for some appropriately chosen character χ of Q.

Using the dense $\overline{P}$-orbit on $\mathcal{O}$. An idea on how to proceed is given by the following. Let $\overline{P} = \left\{ \begin{pmatrix} A & 0 \\ T & {}^tA^{-1} \end{pmatrix} \right\}$ be the parabolic opposite to P. Then $\mathcal{O}_{\overline{P}} = \overline{P} \cdot \lambda$ is dense in $\mathcal{O}$.

According to the discussion in Section 3.4 we want to construct a quantization $A_{\overline{P}}$ of the double cover $\widetilde{\mathcal{O}}_{\overline{P}}$ of $\mathcal{O}_{\overline{P}}$, as well as a map

$$\mathbb{D}(G/Q, \chi \otimes \rho_{G/Q}) \to A_{\overline{P}} \tag{3.2}$$

for some character χ of Q.

Since $SO(W) \ltimes U_{\overline{P}}$ is a polarization of $\widetilde{\mathcal{O}}_{\overline{P}}$ at $\widetilde{\lambda}$, the Dixmier algebra associated to $\widetilde{\mathcal{O}}_{\overline{P}}$ will be

$$A_{\overline{P}} = \mathbb{D}(\overline{P}/[SO(W) \ltimes U_{\overline{P}}], \lambda).$$

(The top exterior power of the cotangent bundle to $\overline{P}/[SO(W) \ltimes U_{\overline{P}}]$ is trivial which implies $\rho_{\overline{P}/[SO(W) \ltimes U_{\overline{P}}]} = 0$.) It remains to construct the map (3.2).

Change of polarization. The first step towards obtaining such a map is easy: $\overline{P}/[\overline{P} \cap Q]$ is the open $\overline{P}$-cell on G/Q, so there is a restriction map

$$\mathbb{D}(G/Q, \chi \otimes \rho_{G/Q}) \longrightarrow \mathbb{D}(\overline{P}/[\overline{P} \cap Q], \chi \otimes \rho_{G/Q}). \tag{3.3}$$

Now we need a map

$$\mathbb{D}(\overline{P}/[\overline{P} \cap Q], \chi \otimes \rho_{G/Q}) \longrightarrow \mathbb{D}(\overline{P}/[SO(W) \ltimes U_{\overline{P}}], \lambda).$$

Again, the special case $G = Sp(4, \mathbb{C})$ suggests what to do further. In that case (in basis e_1, e_2, f_2, f_1)

$$\overline{P} \cap Q = \left\{ \begin{pmatrix} a & m & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & n & b^{-1} & * \\ 0 & 0 & 0 & a^{-1} \end{pmatrix} \right\}, \quad SO(W) \ltimes U_{\overline{P}} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ x & y & a & 0 \\ z & x & 0 & a^{-1} \end{pmatrix} \right\}.$$

($*$ denotes an entry which is entirely determined by the rest of the entries in the matrix). $\overline{P} \cap Q$ and $SO(W) \ltimes \overline{U}_P$ are two different polarizations of $\widetilde{\mathcal{O}}_{\overline{P}}$ at λ. As discussed in Section 3.2, we expect that there is a change of polarization map

$$\mathbb{D}(\overline{P}/[\overline{P} \cap Q], \rho_{G/Q}) \longrightarrow \mathbb{D}(\overline{P}/[SO(W) \ltimes U_{\overline{P}}], \lambda).$$

(The weight λ does not have to appear on the left because $\lambda|_{\mathfrak{q}} = 0$.) Such a map will be constructed in Section 5.

So, in the general $G = Sp(4n, \mathbb{C})$ case, we need a similar change of polarization map. We have

$$\overline{P} \cap Q = \left\{ \begin{pmatrix} A & M & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & N & {}^tB^{-1} & * \\ 0 & 0 & 0 & {}^tA^{-1} \end{pmatrix} \right\}, \quad SO(W) \ltimes U_{\overline{P}} = \exp \left\{ \begin{pmatrix} A & S & 0 & 0 \\ T & -{}^tA & 0 & 0 \\ X & Y & A & * \\ Z & {}^tX & * & -{}^tA \end{pmatrix} \right\}$$

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where S and T are skew matrices and Y and Z are symmetric matrices.

Let $\mathsf{Q}_{SO} = \left\{ \begin{pmatrix} A & S \\ 0 & {}^tA^{-1} \end{pmatrix} \right\} \subseteq O(W)$ be the intersection of Q and $O(W)$, i.e. the parabolic in $O(W)$ which stabilizes V. Using the results from Section 5 we obtain a change of polarization map

$$\mathbb{D}(\overline{P}/[\overline{P} \cap Q], \tfrac{1-n}{2}\operatorname{tr}_V \otimes \rho_{G/Q}) \longrightarrow \mathbb{D}(\overline{P}/[\mathsf{Q}_{SO} \ltimes U_{\overline{P}}], \lambda). \tag{3.4}$$

The composition of (3.3) and (3.4) gives an injection

$$\mathbb{D}(G/Q, \tfrac{1-n}{2}\operatorname{tr}_V \otimes \rho_{G/Q}) \longrightarrow \mathbb{D}(\overline{P}/[\mathsf{Q}_{SO} \ltimes U_{\overline{P}}], \lambda). \tag{3.5}$$

Notice that our choice of the character χ of q is determined at this stage — $\chi = \tfrac{1-n}{2}\operatorname{tr}_V$ is forced upon us by the change of polarization map.

Constructing the quotient map. Next, we need a map

$$\mathbb{D}(\overline{P}/[\mathsf{Q}_{SO} \ltimes U_{\overline{P}}, \lambda]) \longrightarrow \mathbb{D}(\overline{P}/[SO(W) \ltimes U_{\overline{P}}], \lambda). \tag{3.6}$$

Notice that we have a fiber bundle

$$\overline{P}/[\mathsf{Q}_{SO} \ltimes U_{\overline{P}}] \longrightarrow \overline{P}/[SO(W) \ltimes U_{\overline{P}}]$$

with fibers isomorphic to the projective space $SO(W)/\mathsf{Q}_{SO}$. Global differential operators on $SO(W)/\mathsf{Q}_{SO}$ act on $\mathbb{C}$ (the space of regular functions on $SO(W)/\mathsf{Q}_{SO}$) by multiplication by scalars, which gives rise to a quotient map

$$\mathbb{D}(SO(W)/\mathsf{Q}_{SO}) \longrightarrow \mathbb{C}. \tag{3.7}$$

Locally on the base $\overline{P}/[SO(W) \cap U_{\overline{P}}]$, an element of $\mathbb{D}(\overline{P}/[\mathsf{Q}_{SO} \ltimes U_{\overline{P}}], \lambda)$ is of the form $\partial = \sum \partial_1 \otimes \partial_2$ with ∂_1 a locally defined differential operator in $\mathcal{D}(\overline{P}/[SO(W) \ltimes U_{\overline{P}}], \lambda)$ and $\partial_2 \in \mathbb{D}(SO(W)/\mathsf{Q}_{SO})$. If c_2 denotes the image of ∂_2 under the map (3.7), then we define the map (3.6) by

$$\partial \mapsto \sum \partial_1 \otimes c_2.$$

This argument is made rigorous in Section 4 by using induction of Dixmier algebras.

Finally, composing (3.5) and (3.6), we obtain a map

$$\mathbb{D}(G/Q, \tfrac{1-n}{2}\operatorname{tr}_V \otimes \rho_{G/Q}) \longrightarrow \mathbb{D}(\overline{P}/[SO(W) \ltimes U_{\overline{P}}], \lambda) = A_{\overline{P}}. \tag{3.8}$$

As discussed in Section 3.4, we now use the composition map
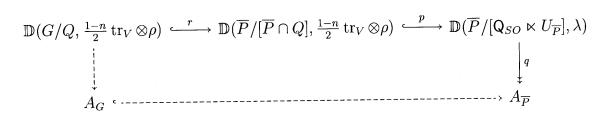
$$U(\mathfrak{g}) \longrightarrow \mathbb{D}(G/Q, \tfrac{1-n}{2}\operatorname{tr}_V \otimes \rho_{G/Q}) \longrightarrow \mathbb{D}(\overline{P}/[SO(W) \ltimes U_{\overline{P}}], \lambda) = A_{\overline{P}}$$

to define a $\mathfrak{g}$-action on $A_{\overline{P}}$, and we define $A_G \subseteq A_{\overline{P}}$ to be the subalgebra of G-finite elements (see Section 3.4). A_G will be the quantization of the double cover $\widetilde{\mathcal{O}}$ of $\mathcal{O}$.

The construction of A_G can be summarized in the following diagram:

$$\mathbb{D}(G/Q, \tfrac{1-n}{2}\operatorname{tr}_V \otimes \rho) \xrightarrow{\;r\;} \mathbb{D}(\overline{P}/[\overline{P} \cap Q], \tfrac{1-n}{2}\operatorname{tr}_V \otimes \rho) \xrightarrow{\;p\;} \mathbb{D}(\overline{P}/[\mathbf{Q}_{SO} \ltimes U_{\overline{P}}], \lambda)$$

with vertical maps to A_G and q to $A_{\overline{P}}$.

The map r is a restriction map, p is a change of polarization map (constructed using results from Section 5), and q is a quotient map.

4 Induction of Dixmier algebras and orbit data

The results from this section will provide the machinery needed for the construction of Dixmier algebras in Section 6. More specifically, we will define induction of Dixmier algebras and prove the transitivity of induction (Proposition 4.16) and the Mackey isomorphism (Proposition 4.15). We start by recalling from [2] the basic definitions and results about D-algebras (possibly equipped with a group action) on algebraic varieties. Dixmier algebras for a group H are the same as D-algebras with H-action on a variety consisting of a single point. An important result from [2] is that, if $X \to Y$ is a principal H-bundle, there is an equivalence of categories between D-algebras on Y and D-algebras with H-action on X. That enables us to obtain, from a Dixmier algebra A_H for H, a D-algebra $\mathcal{A}_Y$ on Y. If G is a complex algebraic group and H is a subgroup of G, by applying this construction to the principal bundle $G \to G/H$ we define induction of Dixmier algebras in Section 4.3. We prove transitivity and Mackey isomorphism for this induction. These properties will be extensively used in the construction of Dixmier algebras in Section 6.

4.1 D-algebras on algebraic varieties

Let R be a commutative algebra over $\mathbb{C}$ and let M be an R-bimodule. For $r \in R$ we define an endomorphism $\mathrm{ad}(r)$ of M by $\mathrm{ad}(r)m = rm - mr$. A *D-filtration* on M is an increasing filtration such that $M_{-1} = 0$ and $\mathrm{ad}(r)M_i \subseteq M_{i-1}$ for $i \geq 0$. One such filtration can be defined by induction:

$$M_{-1}^{\vee} = 0, \quad M_i^{\vee} = \{m \in M \,|\, \mathrm{ad}(r)m \in M_{i-1} \text{ for any } r \in R\}, \ i \geq 0.$$

The submodule $M^{\vee} = \cup_{i \geq 0} M_i^{\vee}$ is called the *differential part* of M. M is called a *differential bimodule* if $M = M^{\vee}$. Equivalently, let $I \subseteq R \otimes R$ be the kernel of the multiplication map $R \otimes R \to R$, i.e. the ideal of functions vanishing on the diagonal of $\mathrm{Spec}\, R \times \mathrm{Spec}\, R$. Then $M_i^{\vee} = \{m \in M \,|\, I^{i+1}m = 0\}$. If $M^{\sim}$ is the sheaf on $\mathrm{Spec}\, R \times \mathrm{Spec}\, R$ associated to the $R \otimes R$-module M, then $M = M^{\vee}$ if and only if $M^{\sim}$ is supported on the diagonal. The definition of a differential bimodule behaves well with respect to localization: if $f \in R$ and $M_f = R_f \otimes_R M \otimes_R R_f$ is the localized bimodule, then $(M_f)^{\vee} = (M^{\vee})_f$.

Let A be an associative algebra equipped with an algebra morphism $i : R \to A$. An increasing filtration on A is called a *D-ring filtration* if $A_i A_j \subseteq A_{i+j}$, $A_{-1} = 0$, $i(R) \subseteq A_0$, and $i(R)$ lies in the center of the associated graded algebra. If we consider A as an R-bimodule, a D-ring filtration will be a D-filtration as a bimodule. The differential part $A^{\vee}$ of A is defined in the same manner as above. A is called an R-*differential* algebra if $A = A^{\vee}$.

The global versions of these definitions are as follows. Let X be a scheme. A *differential $\mathcal{O}_X$-bimodule M* is a quasi-coherent sheaf on $X \times X$ supported on the diagonal. Equivalently, M is a sheaf of $\mathcal{O}_X$-bimodules on X satisfying:

a) For any open $U \subseteq X$, $M(U)$ is a differential $\mathcal{O}(U)$-bimodule.

b) If U is affine and $f \in \mathcal{O}(U)$ then $M(U_f) = M(U)_f$.

An *$\mathcal{O}_X$-differential algebra* (or simply a *D-algebra* on X) is a quasi-coherent sheaf of associative algebras $\mathcal{A}$ on X, equipped with an algebra morphism $i : \mathcal{O}_X \to \mathcal{A}$ such that $\mathcal{A}$ is a differential bimodule.

Example 4.1 Let M, N be quasi-coherent $\mathcal{O}_X$-modules. A $\mathbb{C}$-linear morphism $f : M \to N$ is called a *differential operator* if, for any affine subset U of X, the morphism $f_U : M(U) \to N(U)$ lies in the differential part of the $\mathcal{O}(U)$-bimodule $\mathrm{Hom}_{\mathbb{C}}(M(U), N(U))$. The differential operators form a sheaf of $\mathcal{O}_X$-bimodules $\mathcal{D}(M, N) \subseteq \mathcal{H}om_{\mathbb{C}}(M, N)$. If M is coherent, then $\mathcal{D}(M, N)$ is a differential $\mathcal{O}_X$-bimodule.

Next we describe D-algebras with a group action. Let G be an algebraic group acting on an algebraic variety X. Consider the "complex Harish-Chandra pair" $(\mathfrak{g} \times \mathfrak{g}, G)$. It acts on $X \times X$ (i.e. on the sheaf of functions on $X \times X$) and preserves the diagonal (the G-action comes from the diagonal embedding $G \hookrightarrow G \times G$). Let M be a differential $\mathcal{O}_X$-bimodule, i.e. a quasi-coherent sheaf on $X \times X$ supported on the diagonal. A *G-action* on M is a lifting of the $(\mathfrak{g} \times \mathfrak{g}, G)$ action on the diagonal to an action on M. In other words, a $(\mathfrak{g} \times \mathfrak{g}, G)$-action on M is a pair (μ_M, α_M) where μ_M is a G-action on M considered as a sheaf on $X \times X$ (G acts diagonally on $X \times X$), and α_M is a $\mathfrak{g} \times \mathfrak{g}$-action on the $\mathcal{O}_X \otimes \mathcal{O}_X$-module M, i.e. a Lie algebra map $\mathfrak{g} \times \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(M)$ such that

$$\alpha_M(\gamma_1, \gamma_2)(f_1 m f_2) = [\gamma_1(f_1) m f_2 + f_1 m \gamma_2(f_2)] + f_1 \alpha_M(\gamma_1, \gamma_2)(m) f_2.$$

These actions should be compatible, i.e. the $\mathfrak{g}$-action on M induced from μ_M should coincide with the $\mathfrak{g}$-action on M coming from the diagonal $\mathfrak{g} \subseteq \mathfrak{g} \times \mathfrak{g}$, and $g^*(\alpha_M(\gamma_1, \gamma_2)m) = \alpha_M(g^*\gamma_1, g^*\gamma_2)g^*(m)$ for $g \in G$, γ_1, $\gamma_2 \in \mathfrak{g}$. Here g^* denotes the μ_M-action of an element $g \in G$.

If $\mathcal{A}$ is a D-algebra on X, a *G-action* on $\mathcal{A}$ is a pair $(\mu_{\mathcal{A}}, \alpha_{\mathcal{A}})$ giving a G-action on $\mathcal{A}$ as a differential bimodule, which is compatible with the algebra structure on $\mathcal{A}$ in the following sense:

a) $g^*(a_1 a_2) = (g^* a_1)(g^* a_2)$, $g^*(1) = 1$ for $g \in G$ and $a_1, a_2 \in \mathcal{A}$.

b) We have $\alpha_{\mathcal{A}}(\gamma, 0)(a_1 a_2) = [\alpha_{\mathcal{A}}(\gamma, 0)(a_1)]a_2$, $\alpha_{\mathcal{A}}(0, \gamma)(a_1 a_2) = a_1[\alpha_{\mathcal{A}}(0, \gamma)(a_2)]$ for $\gamma \in \mathfrak{g}$ and $a_1, a_2 \in \mathcal{A}$.

For $\gamma \in \mathfrak{g}$ define $i_{\mathfrak{g}}(\gamma) = \alpha_{\mathcal{A}}(\gamma, 0)(1) \in \mathcal{A}$. Then $\alpha_{\mathcal{A}}$ is completely determined by $i_{\mathfrak{g}}$ since (b) above implies $\alpha_{\mathcal{A}}(\gamma_1, \gamma_2)a = i_{\mathfrak{g}}(\gamma_1)a - ai_{\mathfrak{g}}(\gamma_2)$. The definition of a G-action on $\mathcal{A}$ can be rewritten in terms of $i_{\mathfrak{g}}$. A G-action on the D-algebra $\mathcal{A}$ is a pair $(\mu_{\mathcal{A}}, i_{\mathfrak{g}})$ where $\mu_{\mathcal{A}}$ is a G-action on $\mathcal{A}$ as a left $\mathcal{O}_X$-module and $i_{\mathfrak{g}} : \mathfrak{g} \to \mathcal{A}$ is a Lie algebra map such that

a) the G-action given by $\mu_{\mathcal{A}}$ is compatible with the algebra structure,

b) $i_{\mathfrak{g}} : \mathfrak{g} \to \mathcal{A}$ is a G-equivariant morphism, and

c) the $\mathfrak{g}$-action on $\mathcal{A}$ induced from $\mu_{\mathcal{A}}$ coincides with $\mathrm{ad}_{i_{\mathfrak{g}}}$.

Example 4.2 If G acts on a coherent module P on X then the sheaf $\mathcal{D}(P)$ of differential operators on P is a D-algebra on X with a G-action.

Example 4.3 If X consists of a single point, a D-algebra on X with a G-action is the same as a Dixmier algebra for G.

The result that we need from [2] about D-algebras with group action is the following.

Proposition 4.4 ([2],1.8.9) *Let* $p : X \to Y$ *be a principal H-bundle. Then there is an equivalence of categories between D-algebras with H-action on X and D-algebras on Y.*

The equivalence can be given explicitly. Let $\mathcal{A}$ be a D-algebra on Y. Consider the sheaf $p^*\mathcal{A} = \mathcal{O}_X \otimes_{p^{-1}\mathcal{O}_Y} p^{-1}\mathcal{A}$ of algebras on X. This is a quasi-coherent left $\mathcal{O}_X$-module with a compatible right $p^{-1}\mathcal{A}$-action. Define $p^{\cdot}\mathcal{A}$ to be the sheaf of all differential operators on $p^*\mathcal{A}$ which commute with the right $p^{-1}\mathcal{A}$-action, i.e. $p^{\cdot}\mathcal{A} = \mathcal{D}_{p^{-1}\mathcal{A}}(p^*\mathcal{A})$. Then $p^{\cdot}\mathcal{A}$ is a D-algebra on X with an H-action.

Let $\mathcal{B}$ be a D-algebra with an H-action on X. Let $\widetilde{\mathfrak{h}_Y} = (\mathcal{O}_X \otimes \mathfrak{h})^H$ be the sheaf on Y associated to the H-module $\mathfrak{h}$. (Here we are using the fact that if $\mathcal{S}$ is a sheaf with an H-action on X, then the H-invariants naturally form a sheaf $\mathcal{S}^H$ on Y.) Define $\widetilde{\mathcal{B}_Y} = (p_*\mathcal{B})^H$, a sheaf of D-algebras on Y. The map $i_{\mathfrak{h}} : \mathfrak{h} \to \mathcal{B}$ induces a map $\widetilde{i_{\mathfrak{h}}} : \widetilde{\mathfrak{h}_Y} \to \widetilde{\mathcal{B}_Y}$ and $\widetilde{\mathcal{B}_Y} \cdot \widetilde{i_{\mathfrak{h}}}(\widetilde{\mathfrak{h}_Y})$ is a 2-sided ideal ($\widetilde{\mathcal{B}_Y}$ is fixed by the H-action on $p_*\mathcal{B}$, therefore the adjoint action of $\widetilde{\mathfrak{h}_Y}$ on $\widetilde{\mathcal{B}_Y}$ is trivial). Define $\mathcal{B}_Y = p^{\cdot}\mathcal{B} = \widetilde{\mathcal{B}_Y}/\widetilde{i_{\mathfrak{h}}}(\widetilde{\mathfrak{h}_Y})\widetilde{\mathcal{B}_Y}$. Then $p^{\cdot}$ and $p_{\cdot}$ establish an equivalence of categories between D-algebras on Y and D-algebras with H-action on X.

We have the following important corollary of Proposition 4.4.

Corollary 4.5 *Any Dixmier algebra A_H for H gives rise to a D-algebra on Y.*

Let A_H be a Dixmier algebra for H, $i : \mathfrak{h} \to A_H$. Then $\mathcal{D}(X) \otimes_{\mathbb{C}} A_H$ is a D-algebra with an H-action on X. Here H acts by $h(\partial \otimes a) = h\partial \otimes h^{-1}a$ and the map $\mathfrak{h} \to \mathcal{D}(X) \otimes_{\mathbb{C}} A_H$ is given by $i_X(a) = a^* \otimes 1 - 1 \otimes i(a)$ (a^* is the vector field on X associated to the element

$a \in \mathfrak{h}$ by the H-action). The equivalence of categories shows that there is a corresponding D-algebra $p_{\cdot}(\mathcal{D}(X) \otimes_{\mathbb{C}} A_H)$ on Y.

In the next section we will give a slightly different version of this construction (Proposition 4.7) and will prove that it is equivalent to the one above.

4.2 A construction of D-algebras

Notation and basic facts. For any variety X, $\mathcal{O}_X$ will denote the sheaf of regular functions on X, $\Theta(X)$ will denote the sheaf of vector fields on X, $\mathcal{D}(X)$ will denote the sheaf of differential operators on X, and $\mathbb{D}(X)$ will denote the algebra of global differential operators on X. All varieties considered will be complex smooth algebraic varieties.

Suppose $X \to Y$ is a principal H-bundle with connected base Y and V is an algebraic H-module. Let $\mathcal{V}$ be the sheaf of sections of the associated vector bundle $X \times_H V$ on Y. In other words, there is an isomorphism $\mathcal{V} \cong (\mathcal{O}_X \otimes_{\mathbb{C}} V)^H$ of sheaves on Y, where the H-action on the sheaf $\mathcal{O}_X \otimes_{\mathbb{C}} V$ of functions from X to V is given by $(hf)(x) = h \cdot f(xh)$. $\mathcal{D}(X)$ acts naturally on $\mathcal{O}_X \otimes V$ and if we denote the action of ∂ on f by $\partial(f)$ we have

$$(h\partial)(hf) = h \cdot \partial(f). \tag{4.1}$$

Thus $p_* \mathcal{D}(X)^H$ acts on $(\mathcal{O}_X \otimes V)^H \cong \mathcal{V}$.

The following proposition describes the above action locally on Y. It will be used in the construction of the D-algebra on Y.

Proposition 4.6 *Let $X \to Y$ be a trivial principal H-bundle.*

- *A choice of a trivialization $X \cong Y \times H$ induces a flat connection $\mathsf{h} : \Theta(Y) \to \Theta(X)^H$ which extends to an inclusion of sheaves of algebras $\mathsf{h} : \mathcal{D}(Y) \longrightarrow p_* \mathcal{D}(X)^H$ and an isomorphism of sheaves of algebras on Y*

$$p_* \mathcal{D}(X)^H \cong \mathcal{D}(Y) \otimes_{\mathbb{C}} U(\mathfrak{h}). \tag{4.2}$$

- *A choice of a trivialization also induces an embedding $V \hookrightarrow \Gamma(\mathcal{V})$ with image V_{h} such that $\mathcal{V} \cong \mathcal{O}_Y \otimes_{\mathbb{C}} V_{\mathsf{h}}$. Explicitly, the embedding is given by $v \mapsto f_v$ where $f_v(y, h) = h^{-1}v$.*

- $V_{\mathsf{h}} \subseteq \Gamma(\mathcal{V})$ *is precisely the subspace of sections horizontal with respect to h.*

In light of the above decompositions the $p_* \mathcal{D}(X)^H$-action on the sheaf $\mathcal{V}$ can be described as follows:

- $\mathcal{D}(Y)$ *acts trivially on V_{h}. Its action on $\mathcal{O}_Y$ is the standard one.*

- $U(\mathfrak{h})$ *acts trivially on* $\mathcal{O}_Y$. *Its action on* $V_\mathfrak{h}$ *is transported from the* H-*action on* V.

The construction. Suppose $p : X \to Y$ is a principal H-bundle and A is a Dixmier algebra for H. We will associate to A a sheaf of algebras $\mathcal{D}_{X \to Y}(A)$ on Y which locally (i.e. if $X \to Y$ is trivial) is isomorphic to $\mathcal{D}(Y) \otimes_{\mathbb{C}} A$. This sheaf is the same as the D-algebra constructed in Corollary 4.5, as Proposition 4.8 shows.

Let $\mathcal{U}$ and $\mathcal{A}$ be the sheaves of sections of the bundles of algebras on Y associated to $U(\mathfrak{h})$ and A respectively. There is an induced homomorphism $\mathcal{U} \to \mathcal{A}$. $\mathcal{U}$ can be identified with the subsheaf of vertical differential operators in $p_* \mathcal{D}(X)^H$, i.e. with the subsheaf generated by H-invariant functions $(\mathcal{O}_X)^H$ on X, and by H-invariant vector fields which annihilate $(\mathcal{O}_X)^H$. Now we define

$$\mathcal{D}_{X \to Y}(A) = p_* \mathcal{D}(X)^H \otimes_\mathcal{U} \mathcal{A}.$$

The following proposition shows that this sheaf has a natural algebra structure.

Proposition 4.7 *The sheaf* $\mathcal{D}_{X \to Y}(A)$ *admits a unique algebra structure such that*

a) *Multiplication respects the left* $p_* \mathcal{D}(X)^H$-*module structure and the right* $\mathcal{A}$-*module structure on* $\mathcal{D}_{X \to Y}(A)$. *More specifically, the canonical maps* $p_* \mathcal{D}(X)^H \to \mathcal{D}_{X \to Y}(A)$ *and* $\mathcal{A} \to \mathcal{D}_{X \to Y}(A)$ *are algebra homomorphisms and* $(\partial \otimes 1) \cdot (1 \otimes a) = \partial \otimes a$ *for* $\partial \in p_* \mathcal{D}(X)^H$ *and* $a \in \mathcal{A}$.

b) *For* $\partial \in \Theta(X)^H$ *and for any section* a *of* $\mathcal{A}$,

$$(\partial \otimes 1)(1 \otimes a) - (1 \otimes a)(\partial \otimes 1) = 1 \otimes \partial(a)$$

where $\partial(a)$ *denotes the action of* $\mathcal{D}(X)^H$ *on* $\mathcal{A}$ *defined in* (4.1).

Proof: If such an algebra structure exists it must be unique. To see this, notice that $\mathcal{D}(X)^H$ is generated (as a sheaf of algebras) by its sub-sheaves of H-invariant vector fields and of H-invariant functions on X. However, if ∂ is an H-invariant vector field and $a \in \mathcal{A}$, we have $(\partial \otimes 1)(1 \otimes a) = \partial \otimes a$ by part a), and $(1 \otimes a)(\partial \otimes 1) = \partial \otimes a - 1 \otimes \partial(a)$ by part b). This, along with $(\partial_1 \otimes 1)(\partial_2 \otimes 1) = \partial_1 \partial_2 \otimes 1$ and $(1 \otimes a_1)(1 \otimes a_2) = 1 \otimes a_1 a_2$, determines the algebra structure, if it exists, uniquely.

The uniqueness shows that it is sufficient to prove the existence locally, i.e. for a trivial principal bundle $X \to Y$ (the local products would then be forced to be compatible by the uniqueness). In that case, a choice of a trivialization $X \cong Y \times H$ induces a flat connection

$$\mathsf{h} : \mathcal{D}(Y) \to p_* \mathcal{D}(X)^H$$

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and isomorphisms

$$\mathcal{U} \cong \mathcal{O}_Y \otimes_{\mathbb{C}} U(\mathfrak{h}), \quad \mathcal{A} \cong \mathcal{O}_Y \otimes_{\mathbb{C}} A. \tag{4.3}$$

The space of horizontal sections of $\mathcal{A}$ is isomorphic to A and

$$
\begin{aligned}
p_* \mathcal{D}(X)^H \otimes_{\mathcal{U}} \mathcal{A} &\cong [\mathcal{D}(Y) \otimes_{\mathbb{C}} U(\mathfrak{h})] \otimes_{\mathcal{U}} \mathcal{A} \\
&\cong \mathcal{D}(Y) \otimes_{\mathcal{O}_Y} \mathcal{A} \cong \mathcal{D}(Y) \otimes_{\mathbb{C}} A
\end{aligned}
\tag{4.4}
$$

as sheaves of $\mathcal{O}_Y$-modules.

The usual algebra structure on $\mathcal{D}(Y) \otimes_{\mathbb{C}} A$ (such that $\mathcal{D}(Y)$ and A commute) gives $p_* \mathcal{D}(X)^H \otimes_{\mathcal{U}} \mathcal{A}$ an algebra structure. Using (4.2), (4.3), and (4.4), it is easy to see that this multiplication respects the left $p_* \mathcal{D}(X)^H$-module structure and the right $\mathcal{A}$-module structure on $p_* \mathcal{D}(X)^H \longrightarrow \mathcal{D}_{X \to Y}(A)$. The last two parts of Proposition 4.6 show that

$$(\partial \otimes 1)(1 \otimes a) - (1 \otimes a)(\partial \otimes 1) = 1 \otimes \partial(a).$$

for any $\partial \in \Theta(X)^H$, $a \in \mathcal{A}$. This proves the existence of the algebra structure locally. The uniqueness implies that it is independent of the choice of trivialization, so the local multiplications are compatible and give $\mathcal{D}_{X \to Y}(A)$ the desired algebra structure. $\square$

Remark. The proof above makes use of the local triviality of the principal bundle $X \to Y$. The bundle is locally trivial in the étale topology, but not in the Zariski topology. However, we only used the decompositions

$$p_* \mathcal{D}(X)^H \cong \mathcal{D}(Y) \otimes_{\mathbb{C}} U(\mathfrak{h}), \quad \mathcal{U} \cong \mathcal{O}_Y \otimes_{\mathbb{C}} U(\mathfrak{h}), \quad \mathcal{A} \cong \mathcal{O}_Y \otimes_{\mathbb{C}} A \tag{4.5}$$

obtained from the trivialization. The following argument shows that such decompositions will actually hold (locally) in the Zariski topology.

Let $\widetilde{Y} \to Y$ be an étale cover of Y, let $\widetilde{X}$ be the pull-back of X to $\widetilde{Y}$, and suppose that the bundle $\widetilde{p} : \widetilde{X} \to \widetilde{Y}$ is trivial. Since any étale cover is dominated by a Galois cover, we may assume that $\widetilde{Y}$ is a Galois cover of Y. Let Γ denote the group of covering transformations. It is easy to see that the Γ-action on $\widetilde{X}$ commutes with the H-action. Therefore the isomorphism $\mathcal{D}(\widetilde{X})^\Gamma \cong \mathcal{D}(X)$ gives $(\widetilde{p}_* \mathcal{D}(\widetilde{X})^H)^\Gamma \cong p_* \mathcal{D}(X)^H$. A choice of a trivialization of $\widetilde{X}$ gives an isomorphism $\widetilde{p}_* \mathcal{D}(\widetilde{X})^H \cong \mathcal{D}(\widetilde{Y}) \otimes_{\mathbb{C}} U(\mathfrak{h})$. From this we obtain

$$p_* \mathcal{D}(X)^H \cong (\widetilde{p}_* \mathcal{D}(\widetilde{X})^H)^\Gamma \cong \mathcal{D}(\widetilde{Y})^\Gamma \otimes_{\mathbb{C}} U(\mathfrak{h}) \cong \mathcal{D}(Y) \otimes_{\mathbb{C}} U(\mathfrak{h}).$$

Similarly, we see that the remaining decompositions in (4.5) are true locally in the Zariski topology.

Proposition 4.8 *Let A_H be a Dixmier algebra for H. Then $\mathcal{D}_{X \to Y}(A_H)$ is isomorphic to the D-algebra $p_*(\mathcal{D}(X) \otimes_{\mathbb{C}} A_H)$ constructed in Corollary 4.5.*

Proof: Let

$$\mathcal{A}_X = \mathcal{D}(X) \otimes_{\mathbb{C}} A_H = \mathcal{D}(X) \otimes_{\mathcal{O}_X} (\mathcal{O}_X \otimes_{\mathbb{C}} A_H)$$

be the D-algebra with H-action on X defined in Corollary 4.5. The algebra homomorphisms $\mathcal{D}(X) \to \mathcal{A}_X$ and $\mathcal{O}_X \otimes_{\mathbb{C}} A_H \to \mathcal{A}_X$ give rise to homomorphisms $p_* \mathcal{D}(X)^H \to (p_* \mathcal{A}_X)^H$ and $\mathcal{A}_H = (p_* \mathcal{O}_X \otimes_{\mathbb{C}} A_H)^H \to (p_* \mathcal{A}_X)^H$ and it is easy to check that

$$(p_* \mathcal{A}_X)^H \cong p_* \mathcal{D}(X)^H \otimes_{\mathcal{O}_Y} \mathcal{A}_H.$$

Furthermore (using the notation from Proposition 4.4), the 2-sided ideal of $(p_* \mathcal{A}_X)^H$ generated by $\tilde{i_{\mathfrak{h}}}(\mathfrak{h}^\sim)$ is the same as the ideal in $p_* \mathcal{D}(X)^H \otimes_{\mathcal{O}_Y} \mathcal{A}_H$ generated generated by $h^\sim \otimes 1 - 1 \otimes \tilde{i_{\mathfrak{h}}}(h^\sim)$, where $h^\sim \in \mathfrak{h}_Y^\sim \subseteq \mathcal{D}(X)^H$ is a vertical vector field on X, and $\tilde{i_{\mathfrak{h}}}(h^\sim)$ is its image in $(\mathcal{O}_X \otimes A_H)^H = \mathcal{A}_H$. Since $p_*(\mathcal{D}(X) \otimes_{\mathbb{C}} A_H)$ is the quotient of $(p_* \mathcal{A}_X)^H$ by this ideal we obtain

$$p_*(\mathcal{D}(X) \otimes_{\mathbb{C}} A_H) \cong p_* \mathcal{D}(X)^H \otimes_{\mathcal{U}} \mathcal{A}_H.$$

$\square$

The following easy proposition will be used in the next section to show that induction of Dixmier algebras preserves complete primality.

Proposition 4.9 *If A is completely prime then $\mathcal{D}_{X \to Y}(A)$ is a sheaf of completely prime algebras.*

Proof: Locally (i.e. for a trivial bundle $X \to Y$) the sheaf $\mathcal{D}_{X \to Y}(A)$ is isomorphic to $\mathcal{D}(Y) \otimes_{\mathbb{C}} A$, which is a sheaf of completely algebras on Y since both A and $\mathcal{D}(Y)$ are completely prime. $\square$

In the examples that follow we will show that various familiar algebras of differential operators on Y can be obtained as $\mathcal{D}_{X \to Y}(A)$ for appropriately chosen Dixmier algebras A for H. All the proofs are nearly identical, so we will sketch the idea and skip the details for the individual cases.

In order to prove that there is an isomorphism of sheaves of algebras

$$\mathcal{D}_{X \to Y}(A) \cong \mathcal{D},$$

it is sufficient to construct a map of sheaves $\mathcal{D}_{X \to Y}(A) \to \mathcal{D}$ and see that, locally on Y, the map is an isomorphism. To obtain such a map, it is sufficient to construct maps $\mathcal{D}(X)^H \to \mathcal{D}$ and $\mathcal{A} \to \mathcal{D}$ (notation as in Proposition 4.7) and that can be done in

each individual case. The local decomposition of $\mathcal{D}_{X \to Y}(A)$ was given in the proof of Proposition 4.7, and an analogous local decomposition of the algebra $\mathcal{D}$ can easily be done for each of the examples that we consider.

Example 4.10 Suppose $A = \mathbb{D}(H)$, $U(\mathfrak{h}) \to \mathbb{D}(H)$ is the natural embedding as right H-invariant differential operators and the H-action on $\mathbb{D}(H)$ is induced from the left translation action of H on itself. Then we have a natural isomorphism

$$\mathcal{D}_{X \to Y}[\mathbb{D}(H)] \cong p_* \mathcal{D}(X). \tag{4.6}$$

Following the idea outlined above, it is sufficient to construct maps $p_* \mathcal{D}(X)^H \to p_* \mathcal{D}(X)$ and $(\mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{D}(H))^H \to p_* \mathcal{D}(X)$ $(A = (\mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{D}(H))^H$ since $A = \mathbb{D}(H))$. The first of these is the canonical inclusion. To get the second map, observe that $\mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{D}(H)$ acts naturally on $\mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{C}[H]$ by $(f_1 \partial)(f_2 g) = f_1 f_2 \cdot \partial(g)$. By taking H-invariants we obtain an action of $(\mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{D}(H))^H$ on $(\mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{C}[H])^H \cong \mathcal{O}_X$, which can be easily seen (locally, by trivializing the principal bundle) to be precisely the action of vertical differential operators on $\mathcal{O}_X$. This action gives a map $(\mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{D}(H))^H \to \mathcal{D}(X)$.

To prove that the resulting map $p_* \mathcal{D}(X)^H \otimes_{\mathbb{C}} (\mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{D}(H))^H \longrightarrow p_* \mathcal{D}(X)$ gives the isomorphism (4.6), it is sufficient to consider the case when the bundle $X \to Y$ is trivial. Then a choice of a trivialization induces isomorphisms $p_* \mathcal{D}(X) \cong \mathcal{D}(Y) \otimes_{\mathbb{C}} \mathbb{D}(H)$ and, as in (4.4),

$$\mathcal{D}_{X \to Y}[\mathbb{D}(H)] \cong \mathcal{D}(Y) \otimes_{\mathbb{C}} \mathbb{D}(H).$$

If we take H-invariants in both sides of (4.6) we obtain

$$\mathcal{D}_{X \to Y}[U(\mathfrak{h})] \cong p_* \mathcal{D}(X)^H.$$

Example 4.11 Let V be an H-module and let $\mathcal{V}$ be the associated vector bundle on Y. Then

$$\mathcal{D}_{X \to Y}(\operatorname{End} V) \cong \mathcal{D}(Y, \mathcal{V}).$$

A bit more generally, let F be an H-manifold, let $E \to Y$ be the fiber bundle with fiber F associated to the bundle $X \to Y$, let $\mathcal{V}_F$ be any H-equivariant vector bundle on F and let $\mathcal{V}_E$ be the H-equivariant vector bundle on E induced by $\mathcal{V}_F$. Then $U(\mathfrak{h})$ maps to $\mathbb{D}(F, \mathcal{V}_F)$ H-equivariantly and

$$\mathcal{D}_{X \to Y}[\mathbb{D}(F, \mathcal{V}_F)] = \mathcal{D}(E, \mathcal{V}_E).$$

Example 4.12 A character λ of $\mathfrak{h}$ gives a Dixmier algebra $\operatorname{End} \mathbb{C}_\lambda$ for H. Then the sheaf $\mathcal{D}_{X \to Y}(\operatorname{End} \mathbb{C}_\lambda)$ is a sheaf of twisted differential operators on Y and will often be denoted by $\mathcal{D}(Y, \lambda)$. Also, if A is a Dixmier algebra, the tensor product Dixmier algebra $A \otimes \operatorname{End}(\mathbb{C}_\lambda)$ will often be denoted by $A \otimes \lambda$.

Remark. If A and B are Dixmier algebras for H then $A \otimes_{\mathbb{C}} B$ has a natural structure of a Dixmier algebra for $H \times H$. The diagonal embedding $H \hookrightarrow H \times H$ makes $A \otimes_{\mathbb{C}} B$ into a Dixmier algebra for H. When $B = \mathrm{End}(\mathbb{C}_\lambda)$ for a character λ of $\mathfrak{h}$, $A \otimes B$ is the twist of A by λ.

Example 4.13 As a special case of Example 4.10 consider the case when H is a finite group. Then $p : X \to Y$ is a finite étale covering with transformation group H. Denote by $\mathcal{F}H$ the (commutative) algebra of functions on H under pointwise multiplication. Then

$$p_* \mathcal{D}(X) = \mathcal{D}_{X \to Y}(\mathcal{F}H).$$

If $\mathcal{O}_X$ is the sheaf of functions on X then $p_* \mathcal{O}_X$ is a vector bundle on Y associated to the H-module $\mathcal{F}H$ and

$$\mathcal{D}(Y, p_* \mathcal{O}_X) \cong \mathcal{D}_{X \to Y}(\mathrm{End}\, \mathcal{F}H).$$

The embedding $p_* \mathcal{D}(X) \hookrightarrow \mathcal{D}(Y, p_* \mathcal{O}_X)$ is induced from the embedding of $\mathcal{F}H$ in $\mathrm{End}\, \mathcal{F}H$ as multiplication operators, so $\mathcal{D}(Y, p_* \mathcal{O}_X)$ is a locally free sheaf of $p_* \mathcal{D}(X)$-modules of rank $|H|$.

The next example will be used in the quantization of the Clifford orbit datum in Section 7.4.

Example 4.14 Suppose W is a $2n$-dimensional complex vector space carrying a non-degenerate symmetric bilinear form, V is a maximal isotropic subspace of W. We fix a maximal isotropic complement to V and identify it with V^* using the bilinear form. $\mathsf{Q}_{SO} \subseteq SO(W)$ is the parabolic stabilizing V^* and Q_{Spin} is the corresponding parabolic in $Spin(W)$. Then $-\mathrm{tr}_V$ is an anti-dominant weight of $\mathfrak{q}_{SO}$ which exponentiates to a character of Q_{SO}, and $-\frac{1}{2}\mathrm{tr}_V$ exponentiates to a character of Q_{Spin}. Denote the associated line bundle on $Pin(W)/\mathsf{Q}_{Spin}$ by $\mathcal{L}_{-\frac{1}{2}\mathrm{tr}_V}$. If $C(W)$ denotes the Clifford algebra of W then

$$C(W) \cong \mathrm{End}[\Gamma(Pin(W)/\mathsf{Q}_{Spin}, \mathcal{L}_{-\frac{1}{2}\mathrm{tr}_V})]. \tag{4.7}$$

To see this, recall that the Clifford algebra $C(W)$ has a unique irreducible representation of dimension 2^n, and that its restriction to the group $Pin(W) \subseteq C(W)$ is also irreducible ([7], 5.2.1). The unique irreducible representation of $Pin(W)$ of dimension 2^n is $\Gamma(Pin(W)/\mathsf{Q}_{Spin}, \mathcal{L}_{-\frac{1}{2}\mathrm{tr}_V})$, which proves the isomorphism (4.7).

The action of global differential operators on global sections of $\mathcal{L}_{-\frac{1}{2}\mathrm{tr}_V}$ gives a map

$$\mathbb{D}(Pin(W)/\mathsf{Q}_{Spin}, -\tfrac{1}{2}\mathrm{tr}_V) \longrightarrow C(W)$$

and $C(W)$ is a 2-fold extension of the image of $\mathbb{D}(Pin(W)/\mathbf{Q}_{Spin}, -\frac{1}{2}\operatorname{tr}_V)$. Since

$$\mathbb{D}(Pin(W)/\mathbf{Q}_{Spin}, -\tfrac{1}{2}\operatorname{tr}_V) = \mathbb{D}(O(W)/\mathbf{Q}_{SO}, -\tfrac{1}{2}\operatorname{tr}_V),$$

we get an algebra homomorphism

$$\mathbb{D}(O(W)/\mathbf{Q}_{SO}, -\tfrac{1}{2}\operatorname{tr}_V) \longrightarrow C(W)$$

and $C(W)$ is a 2-fold extension of the image of $\mathbb{D}(O(W)/\mathbf{Q}_{SO}, -\frac{1}{2}\operatorname{tr}_V)$.

4.3 Induction of Dixmier algebras

Suppose H is a subgroup of G and A_H is a Dixmier algebra for H. Then we can apply the construction from Proposition 4.7 to the principal H-bundle $G \to G/H$. We get a sheaf of algebras $\mathcal{D}_{G \to G/H}(A_H)$ on G/H and define $A_G = \Gamma(G/H, \mathcal{D}_{G \to G/H}(A_H))$. The homomorphism

$$\mathcal{D}(G)^H \longrightarrow \mathcal{D}_{G \to G/H}(A_H).$$

combined with $U(\mathfrak{g}) = \mathbb{D}(G)^G \subseteq \mathbb{D}(G)^H$ gives a map $U(\mathfrak{g}) \to A_G$. Also, G acts on A_G algebraically by algebra automorphisms and the map $U(\mathfrak{g}) \to A_G$ is G-equivariant. Therefore A_G is a Dixmier algebra, the Dixmier algebra for G *induced* from A_H. We call this *non-normalized induction* and denote it by

$$A_G = \operatorname{ind}_H^G(A_H).$$

Now we will describe normalized induction. Let $\rho_{G/H} = \frac{1}{2}\operatorname{tr}_{(\mathfrak{g}/\mathfrak{h})^*}$. Then $\rho_{G/H}$ is a character of $U(\mathfrak{h})$ and gives rise to a Dixmier algebra $\operatorname{End}(\rho_{G/H})$ for H as in Example 4.12. The tensor product $A_H \otimes_{\mathbb{C}} \rho_{G/H}$ is also a Dixmier algebra for H. The Dixmier algebra obtained from A_H by *normalized induction* is

$$\operatorname{Ind}_H^G(A_H) = \operatorname{ind}_H^G(A_H \otimes \rho_{G/H}). \tag{4.8}$$

When $A_H = \operatorname{End}(\mathbb{C}_\lambda)$ for a character λ of $\mathfrak{h}$ we have

$$\operatorname{Ind}_H^G(\lambda) = \mathbb{D}(G/H, \lambda \otimes \rho_{G/H})$$

and we will use both notations.

The Mackey isomorphism and transitivity of induction proved in the next two propositions will be used extensively later on.

Proposition 4.15 (Mackey isomorphism) *Suppose G is an algebraic group and $H \subseteq G$ is a subgroup. Let A_G and B_H be Dixmier algebras for G and H, respectively. Then*

$$\operatorname{ind}_H^G(A_G \otimes_{\mathbb{C}} B_H) \cong A_G \otimes_{\mathbb{C}} \operatorname{ind}_H^G(B_H).$$

The analogous result holds for normalized induction.

Proof: It is sufficient to treat only the case of non-normalized induction (that implies the normalized induction case, if we replace B_H by $B'_H = B_H \otimes_{\mathbb{C}} \rho_{G/H}$).

Let $i_A : \mathfrak{g} \to A_G$ denote the Lie algebra homomorphism. Define

$$A'_G = (\mathcal{O}_G \otimes_{\mathbb{C}} A_G)^G \subseteq \mathcal{D}(G) \otimes_{\mathbb{C}} A_G.$$

Then the map $a \mapsto a' = f_a(g) = \mathrm{Ad}(g)^{-1}a$ gives an isomorphism $A_G \cong A'_G$. For $X \in \mathfrak{g}$ denote by X_L (respectively X_R) the corresponding left-invariant (respectively right-invariant) vector field on G. Further, define $\phi' = \phi$ for $\phi \in \mathcal{O}_G$ and $X'_L = X_L \otimes 1 - 1 \otimes i_A(X)$ for $X \in \mathfrak{g}$ (i_A denotes the map $\mathfrak{g} \to A_G$). Using the decomposition $\mathcal{D}(G) \cong \mathcal{O}_G \otimes_{\mathbb{C}} U(\mathfrak{g}_L)$ we obtain an embedding

$$\mathcal{D}(G) \longrightarrow \mathcal{D}(G)' \subseteq \mathcal{D}(G) \otimes_{\mathbb{C}} A_G,$$

which sends the right-invariant vector field X_R to $X'_R = X_R \otimes 1 - 1 \otimes i_A(X)'$. Now A'_G commutes with $\mathcal{D}(G)'$. To prove this it is enough to see that A'_G commutes with $\mathfrak{g}'_R$ (since $\mathcal{D}(G)' \cong \mathcal{O}_G \otimes_{\mathbb{C}} U(\mathfrak{g}'_R)$). If $X \in \mathfrak{g}$ and $a' \in A'_G$ then $[X'_R, a'] = [X_R, a'] - [i_A(X)', a']$. The action of X_R on $a' = f_a(g)$ is given by $[X_R, a'] = X_R \cdot f_a(g) = f_{\mathrm{ad}(X)\cdot a}(g) = (\mathrm{ad}(X) \cdot a)'$ which is the same as the bracket of $i_A(X)'$ and a'. Therefore $[X'_R, a'] = 0$.

We obtain a new decomposition $\mathcal{D}(G) \otimes_{\mathbb{C}} A_G = A'_G \otimes_{\mathbb{C}} \mathcal{D}(G)'$. It has two advantages: the G-action on A'_G is trivial, and the image of $X \in \mathfrak{g}$ under the map $\mathfrak{g} \to \mathcal{D}(G) \otimes_{\mathbb{C}} A_G$ is $X'_L \subseteq \mathcal{D}(G)'$. This allows us to compute $p_{\cdot}(\mathcal{D}(G) \otimes_{\mathbb{C}} A_G \otimes_{\mathbb{C}} B_H)$ easily (where $p : G \to G/H$). First we have

$$(\mathcal{D}(G) \otimes_{\mathbb{C}} A_G \otimes_{\mathbb{C}} B_H)^H = (A'_G \otimes_{\mathbb{C}} \mathcal{D}(G)' \otimes_{\mathbb{C}} B_H)^H = A'_G \otimes_{\mathbb{C}} (\mathcal{D}(G)' \otimes_{\mathbb{C}} B_H)^H. \tag{4.9}$$

Next, we have to divide by the ideal generated by $\tilde{i}(\tilde{\mathfrak{h}})$ where

$$i : X \mapsto X_L \otimes 1 \otimes 1 - 1 \otimes i_A(X) \otimes 1 - 1 \otimes 1 \otimes i_B(X) = X'_L - 1 \otimes 1 \otimes i_B(X).$$

Clearly, the quotient is isomorphic to the D-algebra $A_G \otimes_{\mathbb{C}} \mathcal{D}_{G \to G/H}(B_H)$ on G/H. Since A_G is a free sheaf on G/H, after taking global sections, we obtain the result we want. $\square$

Proposition 4.16 (Transitivity of induction) *Suppose G is an algebraic group and $K \subseteq H$ are subgroups of G. Let A_K be a Dixmier algebra for K. Then there is a canonical isomorphism*

$$\mathrm{ind}_H^G[\mathrm{ind}_K^H(A_K)] \cong \mathrm{ind}_K^G(A_K).$$

The analogous result holds for normalized induction.

Proof: First we will give the proof in the case of non-normalized induction. Let $A_H = \operatorname{Ind}_K^H A_K$, let π denote the projection $G/K \to G/H$, and denote by $\mathcal{U}_K$ (respectively $\mathcal{U}_H$) the sheaf of algebras on G/K (respectively G/H) associated to $U(\mathfrak{k})$ (respectively $U(\mathfrak{h})$). Then

$$\operatorname{ind}_K^G A_K = \Gamma(G/K, \mathcal{D}(G)^K \otimes_{\mathcal{U}_K} \mathcal{A}_K) = \Gamma(G/H, \pi_*(\mathcal{D}(G)^K \otimes_{\mathcal{U}_K} \mathcal{A}_K))$$

and

$$\operatorname{ind}_H^G A_H = \Gamma(G/H, \mathcal{D}(G)^H \otimes_{\mathcal{U}_H} \mathcal{A}_H).$$

Therefore it is sufficient to show that

$$\pi_*(\mathcal{D}(G)^K \otimes_{\mathcal{U}_K} \mathcal{A}_K) \cong \mathcal{D}(G)^H \otimes_{\mathcal{U}_H} \mathcal{A}_H.$$

This question is (étale) local on G/H, so it suffices to notice that if $X \cong Y \times H \to Y$ is a trivial principal H-bundle and π denotes the projection $Y \times H/K \to Y$ then

$$
\begin{aligned}
\Gamma(Y, \pi_*(\mathcal{D}(X)^K \otimes_{\mathcal{U}_K} \mathcal{A}_K)) &= \Gamma(\pi^{-1}Y, \mathcal{D}(X)^K \otimes_{\mathcal{U}_K} \mathcal{A}_K) \\
&\cong \Gamma(Y \times H/K, \mathcal{D}(Y) \otimes_{\mathbb{C}} \mathcal{D}(H)^K \otimes_{\mathcal{U}_K} \mathcal{A}_K) \\
&\cong \Gamma(Y, \mathcal{D}(Y)) \otimes_{\mathbb{C}} \Gamma(H/K, \mathcal{D}(H)^K \otimes_{\mathcal{U}_K} \mathcal{A}_K) \\
&= \mathbb{D}(Y) \otimes_{\mathbb{C}} A_H \qquad\qquad (4.10) \\
&\cong \Gamma(Y, \mathcal{D}(X)^H \otimes_{\mathcal{U}_H} \mathcal{A}_H).
\end{aligned}
$$

To prove transitivity for normalized induction we will use the Mackey isomorphism. We have

$$
\begin{aligned}
\operatorname{Ind}_K^G A_K &= \operatorname{ind}_K^G (A_K \otimes \rho_{G/K}) \cong \operatorname{ind}_K^G (A_K \otimes \rho_{G/H} \otimes \rho_{H/K}) \\
&\cong \operatorname{ind}_H^G \operatorname{ind}_K^H (A_K \otimes \rho_{H/K} \otimes \rho_{G/H}) \\
&\cong \operatorname{ind}_H^G [\operatorname{ind}_K^H (A_K \otimes \rho_{H/K})] \otimes \rho_{G/H} \qquad (4.11) \\
&\cong \operatorname{Ind}_H^G \operatorname{Ind}_K^H (A_K)
\end{aligned}
$$

which finishes the proof. □

Recall from Conjecture 2.3 that the Dixmier map should attach completely prime Dixmier algebras to completely prime orbit data. The following proposition shows that induction of Dixmier algebras preserves complete primality.

Proposition 4.17 *If A_H is completely prime then so is the induced Dixmier algebra $A_G = \operatorname{ind}_H^G A_H$. The analogous result holds for normalized induction.*

Proof: If A_H is completely prime then so is the twisted Dixmier algebra $A_H \otimes_{\mathbb{C}} \rho_{G/H}$. Therefore it suffices to treat the case of non-normalized induction. In that case, Proposition 4.9 shows that $\mathcal{D}_{G \to G/H}(A_H)$ is a sheaf of completely prime algebras on G/H, hence its space of global sections A_G must be completely prime. □

4.4 Induction of orbit data

We will give the definition of induction of orbit data. The one we give here is very similar to that for parabolic induction given by Vogan in [17].

Suppose R_H is an orbit datum for H, i.e. it is an H-algebra equipped with an algebraic H-action and an H-equivariant algebra homomorphism $S(\mathfrak{h}) \to R_H$ such that the image of $S(\mathfrak{h})$ is in the center of R_H. Define $R_{G/H} = S(\mathfrak{g}) \otimes_{S(\mathfrak{h})} R_H$. It has an algebra structure such that $S(\mathfrak{g}) \otimes 1$ commutes with $1 \otimes R_H$. There is an H-equivariant algebra homomorphism $S(\mathfrak{g}) \to R_{G/H}$ sending $S(\mathfrak{g})$ to the center of $R_{G/H}$. Define a G-equivariant sheaf of algebras

$$\mathcal{R}_G = G \times_H R_{G/H} \tag{4.12}$$

on G/H and put $R_G = \Gamma(G/H, \mathcal{R}_G)$. R_G is equipped with an algebraic G-action and a G-equivariant algebra homomorphism $S(\mathfrak{g}) \to R_G$ sending $S(\mathfrak{g})$ to the center of R_G, i.e. R_G is an orbit datum for G.

Definiton 4.18 R_G *is called the orbit datum induced from R_H, and is denoted $R_G = \operatorname{Ind}_H^G R_H$.*

We can give a more geometric description of induction of orbit data. If we have a commutative ring A, an A-algebra B corresponds to a quasi-coherent sheaf of algebras $B^\sim$ on $\operatorname{Spec} A$. Therefore we can think of R_H as an H-equivariant sheaf of algebras $R_H^\sim$ on $\mathfrak{h}^*$. Consider the map $\pi : \mathfrak{g}^* \to \mathfrak{h}^*$ and define the sheaf $R_{G/H}^\sim = \pi^* R_H^\sim$ of algebras on $\mathfrak{g}^*$ (so that $R_{G/H} = \Gamma(\mathfrak{g}^*, R_{G/H}^\sim)$). There is a corresponding G-equivariant sheaf of algebras $R_G^\sim = G \times_H R_{G/H}^\sim$ on $G \times_H \mathfrak{g}^*$ and

$$R_G = \Gamma(G \times_H \mathfrak{g}^*, R_G^\sim) = \Gamma(G/H, R_{G/H}).$$

Equivalently, R_G can be described in the following way which will be useful in the next section. The map $\pi : \mathfrak{g}^* \to \mathfrak{h}^*$ induces a map $G \times_H \mathfrak{g}^* \longrightarrow G \times_H \mathfrak{h}^*$ which we still denote by π. There is an exact sequence of vector bundles

$$0 \longrightarrow T^*(G/H) \longrightarrow G \times_H \mathfrak{g}^* \xrightarrow{\pi} G \times_H \mathfrak{h}^* \longrightarrow 0$$

on G/H. Let $\mathcal{R}_H$ be the sheaf $G \times_H R_H^\sim$ on $G \times_H \mathfrak{h}^*$. Then we have an isomorphism of sheaves $R_G^\sim \cong \pi^* \mathcal{R}_H$ on $G \times_H \mathfrak{g}^*$ and

$$R_G \cong \Gamma(G \times_H \mathfrak{g}^*, \pi^* \mathcal{R}_H).$$

When R_H is the ring of functions on a coadjoint H-orbit, R_G is easy to identify.

Lemma 4.19 *Let $\mathcal{O}_H$ be a coadjoint orbit for H and let R_H be its ring of functions. Let H act on G by right multiplication. Then the induced orbit datum R_G is the ring of functions on the Hamiltonian reduction of T^*G at the coadjoint orbit $\mathcal{O}_H$.*

In some cases R_G will actually be the ring of functions on a coadjoint orbit (or on an orbit cover). An important example is parabolic induction of coadjoint orbits, as discussed in [17]. Here we will give another (rather special) case when this happens. The result will be used in Section 6.2.

Lemma 4.20 *Let $\mathcal{O}_H$ be a coadjoint orbit for H and let R_H be its ring of functions. Assume that $\pi^{-1}(\mathcal{O}_H)$ (which is an H-stable subvariety of $\mathfrak{g}^*$) is a single H-orbit. Then the induced orbit datum $R_G = \mathrm{Ind}_H^G R_H$ is the ring of functions on a cover of a coadjoint orbit for G.*

Proof: From Lemma 4.19 we know that R_G is the ring of functions on $G \times_H \pi^{-1}(\mathcal{O}_H)$, the Hamiltonian reduction of T^*G at $\mathcal{O}_H$. Since $\pi^{-1}(\mathcal{O}_H)$ is a single H-orbit, $G \times_H \pi^{-1}(\mathcal{O}_H)$ must be a single G-orbit. By a result of Kostant [14], the G-homogeneous Hamiltonian manifolds are precisely the covers of coadjoint orbits for G, which completes the proof. $\square$

5   Change of polarization

Let G be a complex algebraic group and let $\mathcal{O} = G \cdot \lambda$ be a coadjoint orbit for G. If the subgroups H_1 and H_2 of G give two polarizations of $\mathcal{O}$ at λ we expect that the two Dixmier algebras $\mathbb{D}(G/H_1, \lambda + \rho_{G/H_1})$ and $\mathbb{D}(G/H_2, \lambda + \rho_{G/H_2})$ are close to being isomorphic (here $\rho_{G/H} = \frac{1}{2}\operatorname{tr}_{(\mathfrak{g}/\mathfrak{h})^*}$). In this section we will prove that, for a certain type of orbit $\mathcal{O}$ and special polarizations H_1 and H_2 of $\mathcal{O}$, there is a homomorphism of Dixmier algebras

$$\mathbb{D}(G/H_1, \lambda + \rho_{G/H_1}) \longrightarrow \mathbb{D}(G/H_2, \lambda + \rho_{G/H_2})$$

which we call the *change of polarization map*. (This map gives the "near isomorphism" between the two algebras.) This result will be one of our main tools in the quantization of nilpotent orbits in Section 6.

The proof will use the most common change of polarization map – the Fourier transform – so we start by recalling the basic results in that case.

5.1   Fourier transform

Let V be a complex vector space, let V^* be its dual and let $\langle \cdot, \cdot \rangle$ denote the pairing between them. $W = V \oplus V^*$ is canonically a symplectic vector space with symplectic form ω given by

$$\omega(v, v^*) = \langle v, v^* \rangle, \quad \omega(v, v) = \omega(v^*, v^*) = 0.$$

Let

$$\mathcal{H}(V) = W \oplus \mathbb{C}$$

be the associated Heisenberg group. Recall that the multiplication in $\mathcal{H}(V)$ is given by

$$(w_1, z_1) \cdot (w_2, z_2) = (w_1 + w_2, z_1 + z_2 + \tfrac{1}{2}\omega(w_1, w_2)).$$

The Lie algebra $\mathfrak{h}(V)$ of $\mathcal{H}(V)$ can be identified with $W \oplus \mathbb{C}$ as well, and the bracket is given by

$$[(w_1, z_1), (w_2, z_2)] = (0, \omega(w_1, w_2)).$$

The symplectic group $Sp(W)$ acts on $\mathcal{H}(V)$ by group automorphisms, the action on $\mathbb{C}$ being trivial (in particular $GL(V)$ acts on $\mathcal{H}(V)$ by group automorphisms). Let c be the linear functional on $\mathfrak{h}(V)$ given by

$$c|_W = 0, \quad c|_{\mathbb{C}} = \operatorname{id}. \tag{5.1}$$

The $Sp(W)$-action leaves c fixed. The subgroups $V \oplus \mathbb{C}$ and $V^* \oplus \mathbb{C}$ of $\mathcal{H}(V)$ are polarizations of the $\mathcal{H}(V)$-coadjoint orbit through c. The independence of polarization for unipotent groups implies that there is an isomorphism of algebras of differential operators

$$\mathbb{D}(\mathcal{H}(V)/[V^* \oplus \mathbb{C}], c + \rho) \cong \mathbb{D}(\mathcal{H}(V)/[V \oplus \mathbb{C}], c + \rho^*). \tag{5.2}$$

If we make identifications

$$\mathcal{H}(V)/[V^* \oplus \mathbb{C}] \cong V, \quad \mathcal{H}(V)/[V \oplus \mathbb{C}] \cong V^*$$

the isomorphism (5.2) becomes the Fourier transform $\mathcal{F} : \mathbb{D}(V, c + \rho) \cong \mathbb{D}(V^*, c + \rho^*)$. We can describe it very explicitly. Fix coordinates $x_1, \ldots, x_n$ of V and dual coordinates $x_1^*, \ldots, x_n^*$ of V^* and put $\partial_i = \partial/\partial x_i$ and $\partial_i^* = \partial/\partial x_i^*$. Then the Fourier transform sends

$$x_i \mapsto \partial_i^*, \quad \partial_i \mapsto -x_i^*. \tag{5.3}$$

Actually, the map (5.2) is an isomorphism of Dixmier algebras for $Sp(W) \ltimes \mathcal{H}(V)$. A priori it is not clear even why $\mathbb{D}(\mathcal{H}(V)/[V^* \oplus \mathbb{C}], c + \rho)$ and $\mathbb{D}(\mathcal{H}(V)/[V \oplus \mathbb{C}], c + \rho^*)$ are Dixmier algebras for $Sp(W)$. Denote by $P(V^*)$ the parabolic in $Sp(W)$ which stabilizes V^*. Then $P(V^*) \ltimes \mathcal{H}(V)$ acts on $\mathcal{H}(V)/[V^* \oplus \mathbb{C}]$ and $\mathbb{D}(\mathcal{H}(V)/[V^* \oplus \mathbb{C}], c + \rho^*)$ is a Dixmier algebra for this group. Explicitly, the images of $\mathfrak{h}$ and $\mathfrak{p}(V^*)$ in $\mathbb{D}(V, c + \rho)$ are spanned by

$$x_i, \ \partial_i, \ 1 \text{ and } x_i \partial_j + \tfrac{1}{2}\delta_{ij}, \ x_i x_j \qquad (1 \leq i, j \leq n).$$

The span of

$$x_i \partial_j + \tfrac{1}{2}\delta_{ij}, \ x_i x_j, \ \partial_i \partial_j \qquad (1 \leq i, j \leq n)$$

is a copy of $\mathfrak{sp}(W)$. This gives a Lie algebra homomorphism $\mathfrak{sp}(W) \ltimes \mathfrak{h}(V) \to \mathbb{D}(V, c + \rho)$. The adjoint action of $\mathfrak{sp}(W) \ltimes \mathfrak{h}(V)$ on $\mathbb{D}(V, c + \rho)$ is locally finite, so it exponentiates to an action of $Sp(W) \ltimes \mathcal{H}(V)$. In this way, $\mathbb{D}(V, c + \rho)$ becomes a Dixmier algebra for $Sp(W) \ltimes \mathcal{H}(V)$. Similarly, $\mathbb{D}(V^*, c + \rho^*)$ can be shown to be a Dixmier algebra for $Sp(W) \ltimes \mathcal{H}(V)$ and it is clear from (5.3) that the Fourier transform (5.2) is an isomorphism of Dixmier algebras.

5.2 The change of polarization map

Suppose U is an abelian unipotent group, H acts on U by group automorphisms, $\mathsf{G} = \mathsf{H} \ltimes \mathsf{U}$ and $\mu \in \mathfrak{g}^*$ is such that

a) $\mathsf{G}_\mu \subseteq \mathsf{H}$, and

b) the restriction of μ to $\mathfrak{h}$ is a character.

Then both H and $\mathsf{G}_\mu \mathsf{U}$ are polarizations of $\mathsf{G} \cdot \mu$ at μ. To see this first notice that the restriction of μ to $\mathrm{Lie}(\mathsf{G}_\mu \mathsf{U})$ is a character. If a subgroup K of G contains G_μ and if the restriction of μ to its Lie algebra is a character then $\dim \mathsf{K}/\mathsf{G}_\mu \leq \tfrac{1}{2} \dim \mathsf{G}/\mathsf{G}_\mu$ with equality if and only if K is a polarization of $\mathsf{G} \cdot \mu$. When we apply this to the subgroups H and $\mathsf{G}_\mu \mathsf{U}$ of G we see that

$$\dim \mathsf{H}/\mathsf{G}_\mu \leq \tfrac{1}{2} \dim \mathsf{G} \cdot \mu, \quad \dim[\mathsf{G}_\mu \mathsf{U}]/\mathsf{G}_\mu \leq \tfrac{1}{2} \dim \mathsf{G} \cdot \mu. \tag{5.4}$$

Since

$$\dim \mathsf{G}/\mathsf{G}_\mu = \dim \mathsf{H}/\mathsf{G}_\mu + \dim \mathsf{U} = \dim \mathsf{H}/\mathsf{G}_\mu + \dim \mathsf{G}_\mu\mathsf{U}/\mathsf{G}_\mu \le \dim \mathsf{G}/\mathsf{G}_\mu$$

we see that equalities in (5.4) do hold, so both H and $\mathsf{G}_\mu\mathsf{U}$ are polarizations of $\mathsf{G} \cdot \mu$. As a by-product we see that $\dim \mathsf{G} \cdot \mu = 2 \dim \mathsf{U}$.

We will construct a change of polarization map for these two polarizations. To do it we will embed $\mathsf{G} \cdot \mu$ as a dense open subset of the coadjoint orbit $\mathcal{H}(\mathsf{U}) \cdot c$. The polarizations of $\mathcal{H}(\mathsf{U}) \cdot c$ given by $V^* + \mathbb{C}$ and $V + \mathbb{C}$ will restrict to the polarizations of $\mathsf{G} \cdot \mu$ given by H and $\mathsf{G}_\mu\mathsf{U}$. The Fourier transform (5.2) will produce the change of polarization map for $\mathsf{G} \cdot \mu$ that we want.

First notice that the H-action on U and U^* induces an H-action on $\mathcal{H}(\mathsf{U})$ (such that H acts trivially on the center). Combined with the left multiplication action of U on $\mathcal{H}(\mathsf{U})$, this gives a G-action on $\mathcal{H}(\mathsf{U})$. Consider the map $i : \mathsf{G} \longrightarrow \mathcal{H}(\mathsf{U})$ defined by

$$i : uh \mapsto (u, h\mu|_{\mathfrak{u}}, \langle h\mu, u \rangle). \tag{5.5}$$

We will show that this is a G-equivariant embedding of the orbit $\mathsf{G} \cdot \mu$ in $\mathcal{H}(\mathsf{U})$, where G acts on the left-hand side by left multiplication. It suffices to check its H-equivariance and U-equivariance separately. The following shows the U-equivariance:

$$i(u_1 u_2 h) = (u_1 + u_2, h\mu, \langle h\mu, u_1 + u_2 \rangle)$$

$$u_1 \cdot i(u_2 h) = (u_1, 0, 0) \cdot (u_2, h\mu, \langle h\mu, u_2 \rangle) = (u_1 + u_2, h\mu, \langle h\mu, u_1 \rangle + \langle h\mu, u_2 \rangle).$$

Denote the adjoint H-action on U by $(h, u) \mapsto h \circ u$. The H-equivariance follows from

$$i(h_1 u h_2) = i((h_1 \circ u)h_1 h_2) = (h_1 \circ u, h_1 h_2 \mu, \langle h_1 h_2 \mu, h_1 \circ u \rangle)$$

$$h_1 \cdot i(uh_2) = h_1 \cdot (u, h_2\mu, \langle h_2\mu, u \rangle) = (h_1 \circ u, h_1 h_2 \mu, \langle h_2\mu, u \rangle)$$

since the pairing $\mathsf{U} \times \mathsf{U}^* \longrightarrow \mathbb{C}$ is $GL(\mathsf{U})$-equivariant, hence H-equivariant. This implies that i is G-equivariant. To calculate the stabilizer in G of a point on $i(\mathsf{G})$ notice that by (5.5) $i(uh) = i(1)$ implies $u = 1$ and $h(\mu|_{\mathfrak{u}}) = \mu|_{\mathfrak{u}}$. Since $\mu = \mu|_{\mathfrak{u}} + \mu|_{\mathfrak{h}}$ and H fixes $\mu|_{\mathfrak{h}}$ (H is a polarization of $\mathsf{G} \cdot \mu$ at μ) $h(\mu|_{\mathfrak{u}}) = \mu|_{\mathfrak{u}}$ implies $h \in \mathsf{H}_\mu = \mathsf{G}_\mu$ and we see that the image of G in $\mathcal{H}(\mathsf{U})$ is $\mathsf{G}/\mathsf{G}_\mu$, i.e. it is isomorphic to the orbit $\mathsf{G} \cdot \mu$. Now the map $\mathcal{H}(\mathsf{U}) \to \mathcal{H}(\mathsf{U}) \cdot c$ embeds $\mathsf{G} \cdot \mu$ into $\mathcal{H}(\mathsf{U}) \cdot c$. Since $\dim \mathsf{G} \cdot \mu = \dim \mathcal{H}(\mathsf{U}) \cdot c = 2 \dim \mathsf{U}$ we see that this actually is an open embedding.

Next we consider the polarizations of $\mathcal{H}(\mathsf{U}) \cdot c$ given by $V^* + \mathbb{C}$ and $V + \mathbb{C}$. Their spaces of leaves are $\mathcal{H}(\mathsf{U})/[V^* + \mathbb{C}]$ and $\mathcal{H}(\mathsf{U})/[V + \mathbb{C}]$ respectively. The induced polarizations of $\mathsf{G} \cdot \mu$ are given by H and $\mathsf{G}_\mu\mathsf{U}$. The corresponding spaces of leaves are

$$\mathsf{G}/\mathsf{H} \cong \mathsf{U} \cong \mathcal{H}(\mathsf{U})/[\mathsf{U}^* + \mathbb{C}] \quad \text{and} \quad \mathsf{G}/\mathsf{G}_\mu\mathsf{U} \subseteq \mathsf{U}^* \cong \mathcal{H}(\mathsf{U})/[\mathsf{U} + \mathbb{C}] \tag{5.6}$$

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(the second space is the G-orbit of $\mu|_{\mathfrak{u}}$ in U^*, which is a dense open subset of U^*).

Finally, consider the Fourier transform map

$$\mathbb{D}(\mathcal{H}(U)/[U^* + \mathbb{C}], c + \rho) \longrightarrow \mathbb{D}(\mathcal{H}(U)/[U + \mathbb{C}], c + \rho^*). \tag{5.7}$$

We know from Section 5.1 that this is an isomorphism of Dixmier algebras for $Sp(U + U^*) \ltimes \mathcal{H}(U)$. Since the $G = H \ltimes U$-action on these algebras comes via the map $G = H \ltimes U \longrightarrow Sp(U + U^*) \ltimes \mathcal{H}(U)$, it must be an isomorphism of Dixmier algebras for G. So we have a commutative diagram of morphisms of Dixmier algebras for G

$$
\begin{array}{ccc}
\mathbb{D}(\mathcal{H}(U)/[U^* + \mathbb{C}], c + \rho) & \overset{\mathcal{F}}{\longrightarrow} & \mathbb{D}(\mathcal{H}(U)/[U + \mathbb{C}], c + \rho^*) \\
\cong \uparrow & & \downarrow{\scriptstyle restr} \\
\mathbb{D}(G/H, \mu + \rho_{G/H}) & & \mathbb{D}(G/G_\mu U, \mu + \rho_{G/G_\mu U})
\end{array}
$$

The composition of the three maps above gives the desired change of polarization map

$$\mathbb{D}(G/H, \mu + \rho_{G/H}) \longrightarrow \mathbb{D}(G/G_\mu U, \mu + \rho_{G/G_\mu U}).$$

So we have proved

Proposition 5.1 *Suppose* U *is an abelian unipotent group,* H *acts on* U *by group automorphisms,* $G = H \ltimes U$, *and* $\mu \in \mathfrak{g}^*$ *is such that* $G_\mu \subseteq H$ *and the restriction of* μ *to* $\mathfrak{h}$ *is a character. Then both* H *and* $G_\mu U$ *are polarizations of* $G \cdot \mu$ *at* μ *and there is a change of polarization map*

$$\mathbb{D}(G/H, \mu + \rho_{G/H}) \longrightarrow \mathbb{D}(G/G_\mu U, \mu + \rho_{G/G_\mu U}).$$

Here are a couple of examples that illustrate the construction of the change of polarization map.

Example 5.2 Let $G = GL_n \ltimes M_n$ where M_n is the abelian group of $n \times n$ matrices and GL_n acts on M_n by multiplication on the left. Let $\mu = \mathrm{tr}_{M_n}$. Then G_μ is trivial, so $G \cdot \mu$ is dense in $\mathfrak{g}^*$. Both GL_n and M_n are polarizations of $G \cdot \mu$ at μ. G embeds into $\mathcal{H}(M_n)$ via

$$mg \mapsto (m, {}^t g^{-1}, \mathrm{tr}(g^{-1}m)).$$

The change of polarization map

$$\mathbb{D}(G/GL_n, \mu + \rho) \longrightarrow \mathbb{D}(G/M_n, \mu + \rho)$$

is the composition

$$\mathbb{D}(G/GL_n, \mu + \rho) \cong \mathbb{D}(M_n, c + \rho) \overset{\mathcal{F}}{\longrightarrow}$$
$$\overset{\mathcal{F}}{\longrightarrow} \mathbb{D}(M_n^*, c + \rho) \overset{restr}{\longrightarrow} \mathbb{D}(G/M_n, \mu + \rho).$$

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Example 5.3 Suppose W is a vector space carrying a non-degenerate symmetric bilinear form, $\mathsf{H} = GL(W)$, $\mathsf{U} = S^2 W$ (U is an abelian unipotent group) and $\mathsf{G} = GL(W) \ltimes S^2 W$. The bilinear form on W gives a linear functional $\mu \in \mathfrak{u}^*$ which can be extended to $\mathfrak{g}$ by setting $\mu|_{\mathfrak{h}} = 0$. Then

$$\mathsf{G}_\mu = O(W) \subseteq \mathsf{H},$$

and both H and $\mathsf{G}_\mu \mathsf{U}$ are polarizations of $\mathsf{G} \cdot \mu$ at μ. We get a change of polarization map

$$\mathbb{D}(S^2 W, \mu + \rho) \longrightarrow \mathbb{D}(GL(W)/O(W), \mu + \rho).$$

Actually, in Section 6 we will need to use a change of polarization map in the following slightly more general situation.

Proposition 5.4 *Suppose U is a unipotent group, H acts on U by group automorphisms, $\mathsf{G} = \mathsf{H} \ltimes \mathsf{U}$, and $\mu \in \mathfrak{g}^*$ is such that $\mathsf{G}_\mu \mathsf{H}$ is a subgroup of G and the restrictions of μ to both of $\mathsf{G}_\mu \mathsf{H}$ and $\mathsf{G}_\mu \mathsf{U}$ are characters. Assume that $\mathsf{N} = \mathsf{G}_\mu \cap \mathsf{U}$ is an ideal of G, $\mathsf{V} = \mathsf{U}/\mathsf{N}$ is abelian (we will identify it with its Lie algebra via the exponential map) and that $\mu|_{\mathfrak{n}} = 0$. Then $\mathsf{G}_\mu \mathsf{H}$ and $\mathsf{G}_\mu \mathsf{U}$ are polarizations of $\mathsf{G} \cdot \mu$ and there is a change of polarization map*

$$\mathbb{D}(\mathsf{G}/\mathsf{G}_\mu \mathsf{H}, \mu + \rho) \longrightarrow \mathbb{D}(\mathsf{G}/\mathsf{G}_\mu \mathsf{U}, \mu + \rho).$$

Proof: When $\mathsf{N} = 0$ we are in the setting of Proposition 5.1. Using the fact that $\mu|_{\mathfrak{n}} = 0$ and the obvious Lemma 5.5 we see that the proof actually reduces to that case. $\square$

Lemma 5.5 *Let $\lambda \in \mathfrak{g}^*$ be a linear functional and let N be a normal subgroup of G such that $\lambda|_{\mathfrak{n}} = 0$. Let $\mathsf{G}' = \mathsf{G}/\mathsf{N}$ and let λ' be the extension of λ to a functional on $\mathfrak{g}'$. Then*

- $\mathsf{N} \subseteq \mathsf{G}_\lambda$.

- *Let P' be a polarization of $\mathsf{G}' \cdot \lambda'$ at λ and let P be the preimage of P' in G. Then P is a polarization of $\mathsf{G} \cdot \lambda$ and*

$$\mathbb{D}(\mathsf{G}'/\mathsf{P}', \lambda' + \rho) \cong \mathbb{D}(\mathsf{G}/\mathsf{P}, \lambda + \rho)$$

with N acting trivially on both algebras.

6 Quantization of nilpotent orbits

Suppose G is a connected complex reductive algebraic group and $\mathcal{O} \subseteq \mathfrak{g}^*$ is a nilpotent coadjoint orbit. Our goal is to construct a Dixmier algebra A_G for G which quantizes the ring of functions on $\mathcal{O}$ (or on some cover of $\mathcal{O}$) in the sense of Definition 2.3. We will be able to do this if the orbit satisfies certain extra conditions.

The construction of A_G follows the outline given in Section 3.6. The first ingredient is the quantization of a dense subvariety $\mathcal{O}_{\overline{P}}$ of $\mathcal{O}$ obtained as follows. Let $\lambda \in \mathcal{O}$, let $X \in \mathfrak{g}$ be its image under the Killing isomorphism, and let $\{X, Y, H\}$ be an $\mathfrak{sl}_2$-triple with nilpositive element X. Let P be the Jacobson-Morozov parabolic associated to $\{X, Y, H\}$ and let $\overline{P}$ be the opposite parabolic (definitions are given in Section 6.1). We call the orbit $\mathcal{O}$ *small* if $\mathcal{O}_{\overline{P}} = \overline{P} \cdot \lambda$ is a dense subvariety of $\mathcal{O}$. If $\mathcal{O}$ is a small orbit and if $\{X, Y, H\}$ is an $\mathfrak{sl}_2$-triple with $X \in \mathcal{O}$, we construct the quantization $A_{\overline{P}}$ of $\mathcal{O}_{\overline{P}} \subseteq \mathcal{O}$. A_G will be defined as a subalgebra of $A_{\overline{P}}$.

To construct A_G inside $A_{\overline{P}}$ we will use a Lagrangian covering of $\mathcal{O}$, i.e. a family of Lagrangian subvarieties of $\mathcal{O}$ such that each point of $\mathcal{O}$ belongs to at least one subvariety (see Section 3.5). Actually, the families we use will be the set of G-translates of a Lagrangian subvariety L passing through λ. A choice of L completely determines such a family. The following construction gives several candidates for L. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{n} + \overline{\mathfrak{n}}$, put $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$, and let B be the corresponding Borel subgroup of G. In [9] Ginsburg has shown that each irreducible component L of $\mathcal{O} \cap \mathfrak{n}$ is a Lagrangian subvariety of $\mathcal{O}$. If Q denotes the stabilizer of L in G (a parabolic subgroup, since all components of $\mathcal{O} \cap \mathfrak{n}$ are stable under B), the space of G-translates of L is G/Q. We will construct A_G as a quotient (or, possibly, a finite extension of a quotient) of an algebra of twisted differential operators $\mathbb{D}(G/Q, \chi \otimes \rho_{G/Q})$ for some appropriately chosen character χ of $\mathfrak{q}$. More specifically, we will need to choose an $\mathfrak{sl}_2$-triple $\{X, Y, H\}$ with $X \in L$ so that we can construct a map

$$\mathbb{D}(G/Q, \chi \otimes \rho_{G/Q}) \to A_{\overline{P}}. \tag{6.1}$$

A_G will be the subalgebra of elements of $A_{\overline{P}}$ transforming finitely under the adjoint action of $\mathfrak{g} \subseteq \mathbb{D}(G/Q, \chi \otimes \rho_{G/Q})$. We can construct the map (6.1) only if the following conditions hold:

a) The Lie algebra $\mathfrak{q}$ decomposes into a direct sum of eigenspaces under the ad H-action. This means that H must normalize $\mathfrak{q}$. Since each parabolic is its own normalizer, H must be contained in $\mathfrak{q}$.

b) Suppose we have fixed an $\mathfrak{sl}_2$-triple with $X \in L$, $H \in \mathfrak{q}$, obtaining $\mathcal{O}_{\overline{P}}$ and $A_{\overline{P}}$. The first step in the construction of the map (6.1) is a restriction map

$$\mathbb{D}(G/Q, \chi \otimes \rho_{G/Q}) \to \mathbb{D}(\overline{P}/[\overline{P} \cap Q], \chi \otimes \rho_{G/Q})$$

(see (3.3)). For this we need that $\overline{P}/[\overline{P} \cap Q]$ is the dense $\overline{P}$-cell on G/Q. We will show that this is equivalent to being able to choose $\{X, Y, H\}$ so that $X \in L$ and H is dominant in $\mathfrak{h}$. We call a Lagrangian $L \subseteq \mathcal{O} \cap \mathfrak{n}$ *dominant* if we can choose such an $\mathfrak{sl}_2$-triple. In Section 6.3 we will describe the set of dominant Lagrangians and will show that it is not empty.

c) The second step is a change of polarization map (similar to (3.4)). In order to get such a map we need $\mathfrak{q}/\mathfrak{q}_\lambda$ to be a Lagrangian subspace of $\mathfrak{g}/\mathfrak{g}_\lambda$ (we know that $\mathfrak{q}/\mathfrak{q}_\lambda$ is an isotropic subspace of $\mathfrak{g}/\mathfrak{g}_\lambda$ since $Q \cdot \lambda \subseteq L$). This means that $Q \cdot \lambda$ must be dense in L. Finally, the last step is a quotient map (similar to (3.6)) whose construction requires that G_λ/Q_λ is a projective variety and that the character $\rho_\lambda = \frac{1}{2} \operatorname{tr}_{(\mathfrak{g}_\lambda/\mathfrak{q}_\lambda)^*}$ of $\mathfrak{q}_\lambda$ lifts to a character of $\mathfrak{q}$. The projectivity of G_λ/Q_λ, combined with the density of $Q \cdot \lambda$ in L, implies that $L = Q \cdot \lambda$, i.e. L is a homogeneous variety for Q. We call such Lagrangians *homogeneous*.

We call a Lagrangian L *good* if it is dominant, homogeneous, and if for some (or equivalently, for all) $\lambda \in L$, ρ_λ lifts to a character of $\mathfrak{q}$. In Section 6.4 we will construct a Dixmier algebra A_G associated to a SMALL orbit $\mathcal{O}$ (or to a cover of $\mathcal{O}$) from a GOOD Lagrangian $L \subseteq \mathcal{O} \cap \mathfrak{n}$. In Section 7 we will discuss how restrictive these conditions are and will give examples of coadjoint orbits which can be quantized using this method.

We start by recalling some results about $\mathfrak{sl}_2$-triples in reductive Lie algebras and about equivariant covers of nilpotent orbits that we will need. Then we proceed with the construction of A_G, following the outline given above.

6.1 Review of the theory of $\mathfrak{sl}_2$-triples and of equivariant orbit covers

The theory of $\mathfrak{sl}_2$-triples. We recall the main results of the theory of $\mathfrak{sl}_2$-triples. Our basic reference is [6].

Let G be a complex connected reductive algebraic group with Lie algebra $\mathfrak{g}$ and let $X \in \mathfrak{g}$ be a nilpotent element. Then there exist $H, Y \in \mathfrak{g}$ such that $\{X, Y, H\}$ is an $\mathfrak{sl}_2$-triple. Conversely, let $\phi \colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$ be an embedding of Lie algebras and let X be the nilpositive element of $\mathfrak{sl}_2$. Then X is a nilpotent element of $\mathfrak{g}$. This sets up a bijection between nilpotent adjoint orbits and orbits of $\mathfrak{sl}_2$-triples in $\mathfrak{g}$. More precise result is Theorem 6.1.

Next, fix an $\mathfrak{sl}_2$-triple $\{X, Y, H\}$. The H-action on $\mathfrak{g}$ gives a decomposition of $\mathfrak{g}$ into H-eigenspaces

$$\mathfrak{g} \cong \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i. \tag{6.2}$$

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We have $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ and $X \in \mathfrak{g}_2$. The Killing form establishes non-degenerate pairings $\mathfrak{g}_i \times \mathfrak{g}_{-i} \to \mathbb{C}$ which shows that $\dim \mathfrak{g}_i = \dim \mathfrak{g}_{-i}$. To the $\mathfrak{sl}_2$-triple $\{X, Y, H\}$ we can associate the subalgebra $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}_i$ of $\mathfrak{g}$. It actually is a parabolic subalgebra ([6], Lemma 3.8.4) called the *Jacobson-Morozov parabolic* attached to the triple. It has Levi factor $\mathfrak{g}_0$ and nilradical $\mathfrak{u} = \bigoplus_{i > 0} \mathfrak{g}_i$. We will denote by $\bar{\mathfrak{p}} = \bigoplus_{i \leq 0} \mathfrak{g}_i$ the opposite parabolic, with nilradical $\bar{\mathfrak{u}} = \bigoplus_{i < 0} \mathfrak{g}_i$.

Let $\mathfrak{g}_X$ denote the centralizer of X in $\mathfrak{g}$ and let $\mathfrak{g}_\phi$ be the centralizer of the $\mathfrak{sl}_2$-triple. Since X is an eigenvector for H, $\mathfrak{g}_X$ will be graded, and it is not hard to see that $\mathfrak{g}_X$ has components only in non-negative degrees ([6], Lemma 3.4.3):

$$\mathfrak{g}_X \cong \bigoplus_{i \geq 0} \mathfrak{g}_{X,i}. \tag{6.3}$$

It follows that $\mathfrak{g}_X \subseteq \mathfrak{p}$. Furthermore, $\mathfrak{g}_{X,0} = \mathfrak{g}_\phi$, $\mathfrak{u}_X = \bigoplus_{i \geq 1} \mathfrak{g}_{X,i}$ is the nilradical of $\mathfrak{g}_X$ and $\mathfrak{g}_X \cong \mathfrak{g}_\phi \ltimes \mathfrak{u}_X$ is the Levi decomposition of $\mathfrak{g}_X$. Similarly, if G_X is the centralizer of X in G and G_ϕ is the centralizer of the $\mathfrak{sl}_2$-triple, we have Levi decomposition $G_X \cong G_\phi \ltimes U_X$.

Now we can state the relation between conjugacy classes of $\mathfrak{sl}_2$-triples and nilpotent adjoint orbits more precisely.

Theorem 6.1 ([6], Theorems 3.4.10, 3.4.12) *Let $\mathfrak{g}$ be a reductive complex Lie algebra.*

a) *Any two $\mathfrak{sl}_2$-triples with the same nilpositive element X are conjugate under the action of U_X, the unipotent radical of G_X.*

b) *Any two $\mathfrak{sl}_2$-triples with the same semisimple element H are conjugate under the action of $G_0 = G_H$. The set of nilpositive elements X of such $\mathfrak{sl}_2$-triples form a dense G_0-orbit in $\mathfrak{g}_2$.*

Since $\dim \mathfrak{g}_i = \dim \mathfrak{g}_{-i}$ we get

Corollary 6.2 *Let $\{X, Y, H\}$ be an $\mathfrak{sl}_2$-triple, let $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ be the grading on $\mathfrak{g}$ given by the H-action, and let $\mathfrak{g}_\phi \subseteq \mathfrak{g}_0$ be the centralizer of the $\mathfrak{sl}_2$-triple. Then $\dim \mathfrak{g}_0 - \dim \mathfrak{g}_\phi = \dim \mathfrak{g}_{-2}$.*

Let $\lambda \in \mathfrak{g}^*$ be the image of X under the Killing isomorphism. The bilinear form $B_\lambda = \lambda \circ [\cdot, \cdot]$ makes $\mathfrak{g}/\mathfrak{g}_\lambda$ into a symplectic vector space. Using (6.3) we obtain non-degenerate pairings

$$\mathfrak{g}_i / \mathfrak{g}_{\lambda,i} \times \mathfrak{g}_{-2-i} \longrightarrow \mathbb{C} \text{ for } i \geq -1. \tag{6.4}$$

In particular, since $\mathfrak{g}_{\lambda,-1} = 0$, we see that $\mathfrak{g}_{-1}$ is a symplectic vector space.

Equivariant orbit covers. We keep the same notation as in the previous section. A reference for the results we recall is [6], Section 6.1. We would like to study the G-equivariant covers of $\mathcal{O}$. Define the G-equivariant fundamental group of $\mathcal{O}$ to be

$$\pi_1^G(\mathcal{O}) = G_\lambda/G_\lambda^o$$

for $\lambda \in \mathcal{O}$. This is the same as the usual fundamental group of $\mathcal{O}$ if G is simply connected. Then $\widetilde{\mathcal{O}} \cong G/G_\lambda^o$ is the universal G-equivariant cover of $\mathcal{O}$ and any other G-equivariant cover is a quotient of $\widetilde{\mathcal{O}}$ by a subgroup of $\pi_1^G(\mathcal{O})$.

Since $G_\lambda \cong G_\phi \ltimes U_\lambda$ and U_λ is connected, we see that

$$\pi_1^G(\mathcal{O}) \cong G_\phi/G_\phi^o.$$

6.2 Quantization of $\mathcal{O}_{\overline{P}}$ and its covers

Let $\mathcal{O} = G \cdot \lambda$ be a coadjoint orbit, and let $X \in \mathfrak{g}$ be the image of λ under the Killing isomorphism. Pick an $\mathfrak{sl}_2$-triple $\{X, Y, H\}$ with nilpositive element X. As in Section 6.1 we obtain a grading on $\mathfrak{g}$ and define $\overline{\mathfrak{p}} = \bigoplus_{i \leq 0} \mathfrak{g}_i$ to be the opposite Jacobson-Morozov parabolic associated to this grading. In this section we will first give a characterization of the orbits $\mathcal{O}$ such that $\mathcal{O}_{\overline{P}} = \overline{P} \cdot \lambda$ is dense in $\mathcal{O}$ and then will construct the quantization $A_{\overline{P}}$ of the ring of functions on $\mathcal{O}_{\overline{P}}$.

First we have

Lemma 6.3 *Let the notation be as above. $\mathcal{O}_{\overline{P}}$ is a dense open subset of the orbit $\mathcal{O}$ if and only if H has no non-zero eigenspaces $\mathfrak{g}_i$ with $|i| \geq 3$.*

Proof: Let $\overline{\mathfrak{p}}_\lambda = \mathfrak{g}_\lambda \cap \overline{\mathfrak{p}}$ be the stabilizer of λ in $\overline{\mathfrak{p}}$. Then $\overline{P} \cdot \lambda$ is dense in $\mathcal{O}$ if and only if the inclusion $\overline{\mathfrak{p}} \to \mathfrak{g}$ induces an isomorphism

$$\overline{\mathfrak{p}}/\overline{\mathfrak{p}}_\lambda \cong \mathfrak{g}/\mathfrak{g}_\lambda.$$

From (6.3) we know that
$$\mathfrak{g}/\mathfrak{g}_\lambda \cong \overline{\mathfrak{p}}/\overline{\mathfrak{p}}_\lambda \oplus \mathfrak{u}/\mathfrak{u}_\lambda.$$

Therefore $\overline{P} \cdot \lambda$ is dense in $\mathcal{O}$ if and only if $\mathfrak{u} = \mathfrak{u}_\lambda$, i.e. if $\mathfrak{g}_i = \mathfrak{g}_{\lambda,i}$ for all $i \geq 1$. From (6.4) we see that this is equivalent to

$$\mathfrak{g}_i = 0 \text{ for } i \leq -3$$

which finishes the proof. $\qquad\qquad\square$

This shows that the condition that $\mathcal{O}_{\overline{P}}$ is dense in $\mathcal{O}$ is independent of the choice of an $\mathfrak{sl}_2$-triple with $X \in \mathcal{O}$. We shall call orbits for which this holds *small*. Next, suppose $\mathcal{O}$ is a small orbit, $\{X, Y, H\}$ is an $\mathfrak{sl}_2$-triple with $X \in \mathcal{O}$, and $\mathcal{O}_{\overline{P}}$ is the corresponding dense open subset of $\mathcal{O}$. We will construct the Dixmier algebra $A_{\overline{P}}$ associated to $\mathcal{O}_{\overline{P}}$.

From (6.3) we see that $\overline{P}_\lambda = G_{\lambda,0} = G_\phi$. Therefore

$$\mathcal{O}_{\overline{P}} \cong \overline{P}/\overline{P}_\lambda \cong G_0 G_{-1} G_{-2}/G_\phi.$$

The quantization of $\mathcal{O}_{\overline{P}}$ is easiest to construct when $\mathfrak{g}_{-1} = 0$, i.e. when the orbit $\mathcal{O}$ is even. In that case the subgroup $G_\phi G_{-2}$ of $\overline{P}$ is a polarization of $\mathcal{O}_{\overline{P}}$ at λ. To see this first observe that $\lambda|_{\mathfrak{g}_\phi + \mathfrak{g}_{-2}}$ is a character. Moreover, $G_\phi G_{-2}$ contains $\overline{P}_\lambda$ and

$$\dim G_\phi G_{-2}/G_\phi = \dim G_{-2} = \tfrac{1}{2} \dim \mathcal{O}.$$

Therefore the Dixmier algebra associated to $\mathcal{O}_{\overline{P}}$ is $A_{\overline{P}} = \mathbb{D}(\overline{P}/G_\phi G_{-2}, \lambda + \rho_{\overline{P}/G_\phi G_{-2}})$.

If $\mathfrak{g}_{-1} \neq 0$ the orbit $\mathcal{O}_{\overline{P}}$ does not admit an easy polarization. However in the general case $\mathcal{O}_{\overline{P}}$ is an induced orbit, as shown in the following Lemma.

Lemma 6.4 *Let $\mathcal{O}_{G_\phi \overline{U}}$ be the coadjoint orbit*

$$G_\phi \overline{U} \cdot \lambda|_{\mathfrak{g}_\phi + \overline{\mathfrak{u}}} \cong \overline{U} \cdot \lambda|_{\overline{\mathfrak{u}}}.$$

Then $\mathcal{O}_{\overline{P}}$ is an induced orbit, i.e. $\mathcal{O}_{\overline{P}} \cong \mathrm{Ind}_{G_\phi \overline{U}}^{\overline{P}}(\mathcal{O}_{G_\phi \overline{U}})$.

Proof: First notice that the centralizer of $\lambda|_{\mathfrak{g}_\phi + \overline{\mathfrak{u}}}$ in $G_\phi \overline{U}$ is $G_\phi G_{-2}$, so

$$\mathcal{O}_{G_\phi \overline{U}} \cong G_\phi \overline{U}/G_\phi G_{-2} \cong \overline{U}/G_{-2}.$$

Denote the map $\overline{\mathfrak{p}}^* \to (\mathfrak{g}_\phi + \overline{\mathfrak{u}})^*$ by π. We will show in a moment that

$$\pi^{-1}(\mathcal{O}_{G_\phi \overline{U}}) = G_\phi \overline{U} \cdot \lambda \cong G_\phi \overline{U}/G_\phi \qquad (6.5)$$

is a single $G_\phi \overline{U}$-orbit. Using Lemma 4.20 we then obtain

$$\mathrm{Ind}_{G_\phi \overline{U}}^{\overline{P}}(\mathcal{O}_{G_\phi \overline{U}}) \cong \overline{P} \times_{G_\phi \overline{U}} \pi^{-1}(\mathcal{O}_{G_\phi \overline{U}}) \cong \overline{P}/G_\phi \cong \mathcal{O}_{\overline{P}}$$

which proves the Lemma.

To prove (6.5), first observe that $G_\phi \overline{U} \cdot \lambda \subseteq \pi^{-1}(\mathcal{O}_{G_\phi \overline{U}})$. The fiber of the map

$$G_\phi \overline{U} \cdot \lambda \to G_\phi \overline{U} \cdot \lambda|_{\mathfrak{g}_\phi + \overline{\mathfrak{u}}}$$

over $\lambda|_{\mathfrak{g}_\phi + \overline{\mathfrak{u}}}$ is $G_{-2} \cdot \lambda$, i.e. it is an orbit of the unipotent group G_{-2} acting (algebraically, by affine transformations) on the affine space $\pi^{-1}(\lambda|_{\mathfrak{g}_\phi + \overline{\mathfrak{u}}})$. All such orbits of unipotent groups are closed, hence $G_{-2} \cdot \lambda$ must be closed in $\pi^{-1}(\lambda|_{\mathfrak{g}_\phi + \overline{\mathfrak{u}}})$. But by Corollary 6.2

$$\dim G_{-2} \cdot \lambda = \dim \mathfrak{g}_{-2} = \dim \mathfrak{g}_0 - \dim \mathfrak{g}_\phi = \dim \pi^{-1}(\lambda|_{\mathfrak{g}_\phi + \overline{\mathfrak{u}}}).$$

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Therefore $G_{-2} \cdot \lambda = \pi^{-1}(\lambda|_{\mathfrak{g}_\phi + \overline{\mathfrak{u}}})$ and $\pi^{-1}(\mathcal{O}_{G_\phi \overline{U}}) = G_\phi \overline{U} \cdot \lambda$, which proves (6.5). $\qquad\square$

We will find the Dixmier algebra $A_{G_\phi \overline{U}}$ associated to $\mathcal{O}_{G_\phi \overline{U}}$ and then define $A_{\overline{P}} = \mathrm{Ind}_{G_\phi \overline{U}}^{\overline{P}} A_{G_\phi \overline{U}}$. To find $A_{G_\phi \overline{U}}$, we will show that $\mathcal{O}_{G_\phi \overline{U}}$ is isomorphic to a coadjoint orbit for a Heisenberg group. From (6.4) we know that the bilinear form B_λ makes $\mathfrak{g}_{-1}$ into a symplectic vector space. Let $\mathfrak{m} = \{Z \in \mathfrak{g}_{-2} \mid \lambda(Z) = 0\}$, an abelian ideal of $\overline{\mathfrak{u}}$, and let M be the corresponding normal subgroup of $\overline{U}$. Then $\mathfrak{z} = \mathfrak{g}_{-2}/\mathfrak{m} \cong \mathbb{C}$, $\overline{\mathfrak{u}}/\mathfrak{m} \cong \mathfrak{g}_{-1} + \mathfrak{z}$ is isomorphic to the Heisenberg Lie algebra $\mathfrak{h}(\mathfrak{g}_{-1})$, and we can identify the Heisenberg group $\mathcal{H}(\mathfrak{g}_{-1})$ with $\overline{U}/M$. Denote the restriction of λ to $\overline{\mathfrak{u}}/\mathfrak{m}$ by c. The coadjoint orbit $\mathcal{H}(\mathfrak{g}_{-1}) \cdot c$ is isomorphic to

$$\mathcal{H}(\mathfrak{g}_{-1})/\mathfrak{z} \cong \overline{U}/G_{-2} \cong G_\phi \overline{U}/G_\phi G_{-2} \cong \mathcal{O}_{G_\phi \overline{U}}.$$

Let $\mathfrak{l}_{-1}$ be a Lagrangian subspace of $\mathfrak{g}_{-1}$ and let $L_{-1} = \exp(\mathfrak{l}_{-1})$ be the corresponding abelian subgroup of $\mathcal{H}(\mathfrak{g}_{-1})$. We saw in Section 5.1 that the algebra of differential operators

$$A_{G_\phi \overline{U}} = \mathbb{D}(\mathcal{H}(\mathfrak{g}_{-1})/[L_{-1}\mathfrak{z}], \lambda + \rho) \cong \mathbb{D}(\overline{U}/L_{-1}G_{-2}, \lambda + \rho) \qquad (6.6)$$

is the quantization of $\mathcal{H}(\mathfrak{g}_{-1}) \cdot c$, i.e. of the $\overline{U}$-coadjoint orbit $\mathcal{O}_{G_\phi \overline{U}}$. We also saw that it actually is a Dixmier algebra for $Sp(\mathfrak{g}_{-1}) \ltimes \mathcal{H}(\mathfrak{g}_{-1})$. Since G_ϕ is in the centralizer of λ, it acts on $\mathfrak{g}_{-1}$ by symplectomorphisms. Therefore we have a group homomorphism

$$G_\phi \ltimes \overline{U} \to Sp(\mathfrak{g}_{-1}) \ltimes \mathcal{H}(\mathfrak{g}_{-1})$$

extending the group homomorphism

$$\overline{U} \to \mathcal{H}(\mathfrak{g}_{-1}).$$

This shows that $A_{G_\phi \overline{U}}$ actually is a Dixmier algebra for $G_\phi \overline{U}$ which quantizes the $G_\phi \overline{U}$-coadjoint orbit $\mathcal{O}_{\overline{P}}$. We have proven

Lemma 6.5 *The quantization of the $G_\phi \overline{U}$-coadjoint orbit $\mathcal{O}_{G_\phi \overline{U}} = G_\phi \overline{U} \cdot \lambda|_{\mathfrak{g}_\phi + \overline{\mathfrak{u}}}$ is given by the algebra $A_{G_\phi \overline{U}}$ defined in (6.6).*

Finally, combining Lemma 6.4 and Lemma 6.5 we see that the Dixmier algebra associated to $\mathcal{O}_{\overline{P}}$ is $A_{\overline{P}}$.

Remark. When $\mathfrak{g}_{-1} = 0$ the orbit $\mathcal{O}_{G_\phi \overline{U}}$ reduces to the single point $\lambda|_{\mathfrak{g}_\phi + \overline{\mathfrak{u}}}$, so the claim that $\mathcal{O}_{\overline{P}} = \mathrm{Ind}_{G_\phi \overline{U}}^{\overline{P}}(\mathcal{O}_{G_\phi \overline{U}})$ reduces to the fact we observed earlier, that (when $\mathfrak{g}_{-1} = 0$) the subgroup $G_\phi \overline{U}$ gives a polarization of $\mathcal{O}_{\overline{P}}$ at λ.

We will also need the Dixmier algebras $\widetilde{A}_{\overline{P}}$ associated to $\overline{P}$-equivariant orbit covers of $\mathcal{O}_{\overline{P}}$. Recall from Section 6.1 that G-equivariant orbit covers of $\mathcal{O}$ correspond to subgroups of the component group $G_\lambda/G_\lambda^o \cong G_\phi/G_\phi^o$ of G_λ. If $\widetilde{\mathcal{O}} \to \mathcal{O}$ is one such cover then the pre-image $\widetilde{\mathcal{O}}_{\overline{P}}$ of $\mathcal{O}_{\overline{P}}$ is a dense open subset of $\widetilde{\mathcal{O}}$ (and a $\overline{P}$-equivariant cover of $\mathcal{O}_{\overline{P}}$). The construction of $A_{\overline{P}}$ can be easily modified to produce the Dixmier algebra $\widetilde{A}_{\overline{P}}$ associated to $\widetilde{\mathcal{O}}_{\overline{P}}$.

In more detail, let Γ be a subgroup of G_ϕ/G_ϕ^o. Denote its pre-image in G_ϕ under the map $G_\phi \to G_\phi/G_\phi^o$ by ΓG_ϕ^o. The corresponding covers of $\mathcal{O}$ and $\mathcal{O}_{\overline{P}}$ are

$$\widetilde{\mathcal{O}} \cong G/\Gamma G_\phi^o U_\lambda, \quad \widetilde{\mathcal{O}}_{\overline{P}} \cong \overline{P}/\Gamma G_\phi^o.$$

As in Lemma 6.4 we see that

$$\widetilde{\mathcal{O}}_{\overline{P}} \cong \operatorname{Ind}_{\Gamma G_\phi^o \overline{U}}^{\overline{P}}(\mathcal{O}_{G_\phi \overline{U}})$$

and as before we conclude that the Dixmier algebra associated to $\widetilde{\mathcal{O}}_{\overline{P}}$ is

$$\widetilde{A}_{\overline{P}} \cong \operatorname{Ind}_{\Gamma G_\phi^o \overline{U}}^{\overline{P}}(A_{G_\phi \overline{U}}).$$

6.3 B-stable Lagrangian subvarieties of $\mathcal{O}$

Let $\mathcal{O}$ be a nilpotent (co-)adjoint orbit. Fix a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$. We get a triangular decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{n} + \overline{\mathfrak{n}}.$$

The B-stable Lagrangians in $\mathcal{O}$ are precisely the irreducible components of $\mathcal{O} \cap \mathfrak{n}$. These components which are *dominant* and *homogeneous* are going to be used in the construction of a Dixmier algebra associated to $\mathcal{O}$. In this section we will define dominant homogeneous Lagrangians in $\mathcal{O}$ and discuss their properties.

Let L be a component of $\mathcal{O} \cap \mathfrak{n}$ and let $Q = \{g \in G \mid g \cdot L = L\}$ be the stabilizer of L in G. Since B is a connected group it preserves each irreducible component of $\mathcal{O} \cap \mathfrak{n}$, therefore $B \subseteq Q$. Suppose $\lambda \in L$, $X \in \mathfrak{g}$ is its image under the Killing isomorphism, and $\{X, Y, H\}$ is an $\mathfrak{sl}_2$-triple with nilpositive element X. Let G_ϕ be the centralizer of this $\mathfrak{sl}_2$, G_λ be the stabilizer of λ in G, and U_λ be its unipotent radical. Put $Q_\lambda = Q \cap G_\lambda$, $Q_\phi = Q \cap G_\phi$. As we saw in Section 6.1, G_λ has Levi decomposition $G_\lambda = G_\phi \ltimes U_\lambda$.

Homogeneous Lagrangians in $\mathcal{O}$. Let $M \cong G/Q$ be the space of G-translates of L in $\mathcal{O}$, and let

$$Z = \{(\lambda, L) \mid L \in M, \lambda \in L \subseteq \mathcal{O}\} \cong G \times_Q L.$$

The maps $p : (\lambda, L) \mapsto \lambda$ and $\pi : (\lambda, L) \mapsto L$ give rise to a double fibration

$$
\begin{array}{ccc}
 & Z & \\
{}^{p}\swarrow & & \searrow{}^{\pi} \\
\mathcal{O} & & M
\end{array}
\qquad (6.7)
$$

such that p has projective fibers and the fibers of π are isomorphic to L.

The Lagrangian L is called *homogeneous* if $L = Q \cdot \lambda$, i.e. L is a homogeneous variety for Q. Such Lagrangians have the following important properties.

Lemma 6.6 *If L is a homogeneous Lagrangian then*

    a) G_λ/Q_λ is a projective variety.

    b) $U_\lambda \subseteq Q$ and Q_λ/U_λ is a parabolic subgroup of G_ϕ.

Proof: If $L = Q \cdot \lambda$ is a single Q-orbit we have $Z \cong G \times_Q Q \cdot \lambda \cong G/Q_\lambda$, so the fibers of p are isomorphic to G_λ/Q_λ. Since p has projective fibers, we see that Q_λ is a parabolic subgroup of G_λ, i.e. Q_λ contains the unipotent radical U_λ of G_λ and Q_λ/U_λ is a parabolic subgroup of the reductive group G_ϕ. $\qquad \square$

The following Lemma is sometimes useful for proving that a Lagrangian is homogeneous. We will have a chance to use it in Section 7.3.

Lemma 6.7 *Let $Q \cdot \lambda$ be a Lagrangian subvariety of $\mathcal{O}$ (hence a dense open subset of some component L of $\mathcal{O} \cap \mathfrak{n}$). Then L is a homogeneous Lagrangian (i.e. $L = Q \cdot \lambda$) if and only if Q_λ is a parabolic subgroup of G_λ.*

Proof: We will use the fibration (6.7). Since $Q \cdot \lambda$ is a dense open subset of L, $G \times_Q Q \cdot \lambda \cong G/Q_\lambda$ must be a dense open subset of Z. The composition

$$
G/Q_\lambda \hookrightarrow Z \longrightarrow \mathcal{O} \cong G/G_\lambda
$$

is a proper map since its fibers are isomorphic to G_λ/Q_λ, so they are projective. By [11], Corollary 4.8, the open embedding $G/Q_\lambda \hookrightarrow Z$ must also be a proper map, which is only possible if $G/Q_\lambda \cong Z$, i.e. $L \cong Q \cdot \lambda$. $\qquad \square$

Lemma 6.8 *If $L \subseteq \mathcal{O}$ is a homogeneous Lagrangian then any two $\mathfrak{sl}_2$-triples $\{X, Y, H\}$ such that $X \in L$ are conjugate under the adjoint action of Q.*

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Proof: Since L is a single Q-orbit it is sufficient to prove that any two $\mathfrak{sl}_2$-triples having the same nilpositive element X are conjugate under Q. By Theorem 6.1 any two $\mathfrak{sl}_2$-triples with the same nilpositive element are conjugate under the unipotent radical U_X of the centralizer of X. Since L is a homogeneous Lagrangian, Lemma 6.6 shows that U_X is contained in Q, which completes the proof. $\qquad\square$

Dominant homogeneous Lagrangians in $\mathcal{O} \cap \mathfrak{n}$. The Lagrangian L is called *dominant* if there exists an $\mathfrak{sl}_2$-triple $\{X, Y, H\}$ such that $X \in L$ and H is a dominant element of $\mathfrak{h}$.

Now suppose that $L \subseteq \mathcal{O} \cap \mathfrak{n}$ is a dominant homogeneous Lagrangian and that $\{X, Y, H\}$ is an $\mathfrak{sl}_2$-triple such that $X \in L$ (with no restriction on H). Since there exists such a triple with $H \in \mathfrak{h} \subseteq \mathfrak{q}$ and since, by Lemma 6.8, any two such triples are Q-conjugate, we see that H lies in $\mathfrak{q}$. As in (6.2) we obtain decompositions $\mathfrak{g} \cong \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ and $\mathfrak{q} \cong \bigoplus_{i \in \mathbb{Z}} \mathfrak{q}_i$ by H-eigenspaces. Since $L = Q \cdot \lambda$ is a Lagrangian subvariety of $G \cdot \lambda$, $\mathfrak{q}/\mathfrak{q}_\lambda$ must be a maximal isotropic subspace of $\mathfrak{g}/\mathfrak{g}_\lambda$. Then (6.4) shows that

$$\mathfrak{q}_i/\mathfrak{q}_{\lambda,i} \subseteq \mathfrak{g}_i/\mathfrak{g}_{\lambda,i} \text{ is the annihilator of } \mathfrak{q}_{-2-i} \text{ for } i \geq -1. \tag{6.8}$$

In particular $\mathfrak{q}_{-1}$ is a maximal isotropic subspace of $\mathfrak{g}_{-1}$.

Since $\mathfrak{q}/\mathfrak{q}_\lambda$ is an isotropic subspace of $\mathfrak{g}/\mathfrak{g}_\lambda$, it follows that $\lambda|_{[\mathfrak{q}_0, \mathfrak{q}_{-2}]} = 0$. From $H \in \mathfrak{q}_0$ we see that $[\mathfrak{q}_0, \mathfrak{q}_{-2}] = \mathfrak{q}_{-2}$, hence

$$\lambda|_{\mathfrak{q}_{-2}} = 0. \tag{6.9}$$

Let $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}_i$ be the Jacobson-Morozov parabolic associated to $\{X, Y, H\}$ and let $\bar{\mathfrak{p}}$ be its opposite. Dominant Lagrangians have the following important property.

Lemma 6.9 $\overline{P}/\overline{P} \cap Q$ *is the open* $\overline{P}$*-cell in* G/Q.

Proof: Since L is a dominant Lagrangian, we can conjugate the $\mathfrak{sl}_2$-triple by an element of Q so that H becomes a dominant element of $\mathfrak{h}$ (conjugating the $\mathfrak{sl}_2$-triple by Q doesn't change the $\overline{P}$-cell in G/Q). If H is dominant in $\mathfrak{h}$ we have $\mathfrak{b} \subseteq \mathfrak{p}$ and $\mathfrak{u}_P \subseteq \mathfrak{n}$. Since $\mathfrak{b} \subseteq \mathfrak{q}$ we have $\mathfrak{u}_P \subseteq \mathfrak{q}$. This implies

$$\begin{aligned}
\mathfrak{q}_i &= \mathfrak{g}_i \text{ for } i \geq 1, \\
\bar{\mathfrak{p}}/\bar{\mathfrak{p}} \cap \mathfrak{q} &\cong \mathfrak{g}/\mathfrak{q}.
\end{aligned} \tag{6.10}$$

The last property shows that $\overline{P}/\overline{P} \cap Q$ is the open $\overline{P}$-cell in G/Q. $\qquad\square$

Next we will give a better description of the set of dominant Lagrangians in $\mathcal{O} \cap \mathfrak{n}$. All $\mathfrak{sl}_2$-triples $\{X, Y, H\}$ with $X \in \mathcal{O}$ and dominant $H \in \mathfrak{h}$ actually share the same H. By 6.1, the set of elements X in such triples is a single G_0-orbit, namely the dense G_0-orbit in $\mathfrak{g}_2 \subseteq \mathfrak{n}$. This shows that the dominant Lagrangians are precisely these components of $\mathcal{O} \cap \mathfrak{n}$ which intersect $\mathfrak{g}_2$. In particular, the set of dominant Lagrangians is non-empty.
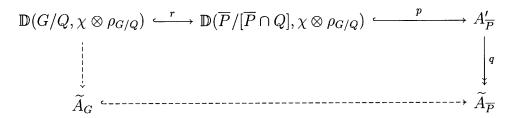
6.4 Construction of A_G

As in Section 6.3, fix a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$. We get a decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{n} + \overline{\mathfrak{n}}.$$

Let $\mathcal{O}$ be a small nilpotent coadjoint orbit and let $L \subseteq \mathcal{O} \cap \mathfrak{n}$ be a dominant homogeneous Lagrangian. Let $\lambda \in L$, let $X \in \mathfrak{g}$ be its image under the Killing isomorphism, and let $\{X, Y, H\}$ be an $\mathfrak{sl}_2$-triple with nilpositive element X. The Lagrangian L determines a certain cover $\widetilde{\mathcal{O}}$ of $\mathcal{O}$ and we will construct the Dixmier algebra $\widetilde{A}_G$ associated to it. To describe $\widetilde{\mathcal{O}}$, denote by Q the stabilizer of L in G and by Γ the component group of Q_λ. Since $Q_\lambda \cong Q_\phi \ltimes U_\lambda$ and U_λ is connected, Γ is isomorphic to the component group of Q_ϕ. The assumption that L is homogeneous implies that Q_ϕ is a parabolic subgroup of G_ϕ (Lemma 6.6). Since parabolic subgroups of connected reductive groups are connected, we see that Q_ϕ^o is a parabolic subgroup of G_ϕ^o and that $\Gamma \cong Q_\phi/Q_\phi^o$ is a subgroup of G_ϕ/G_ϕ^o, the equivariant fundamental group of $\mathcal{O}$. Let $\widetilde{\mathcal{O}}$ be the corresponding cover of $\mathcal{O}$ (i.e. $\widetilde{\mathcal{O}}$ is the quotient of the universal G-equivariant cover of $\mathcal{O}$ by Γ).

As in Section 6.2, if $\overline{P}$ denotes the opposite Jacobson-Morozov parabolic, we obtain a dense open subset $\mathcal{O}_{\overline{P}}$ of $\mathcal{O}$, its pre-image $\widetilde{\mathcal{O}}_{\overline{P}}$ in $\widetilde{\mathcal{O}}$, and the Dixmier algebra $\widetilde{A}_{\overline{P}}$ associated to $\widetilde{\mathcal{O}}_{\overline{P}}$. We will construct $\widetilde{A}_G$ as a subalgebra of $\widetilde{A}_{\overline{P}}$. To see how to do this, let χ be a character of $\mathfrak{q}$ (arbitrary for now). We will show that, for an appropriate choice of χ, there will be a map $\mathbb{D}(G/Q, \chi \otimes \rho_{G/Q}) \to \widetilde{A}_{\overline{P}}$. The map will be obtained in several steps, as shown in the following diagram.

$$
\begin{array}{ccccc}
\mathbb{D}(G/Q, \chi \otimes \rho_{G/Q}) & \xrightarrow{\ r\ } & \mathbb{D}(\overline{P}/[\overline{P} \cap Q], \chi \otimes \rho_{G/Q}) & \xrightarrow{\ \ \ p\ \ \ } & A'_{\overline{P}} \\
\big\downarrow & & & & \big\downarrow{\scriptstyle q} \\
\widetilde{A}_G & \xleftarrow{\hspace{6cm}} & & & \widetilde{A}_{\overline{P}}
\end{array}
$$

The map r will be a restriction map, p will be a change of polarization map (constructed using results from Section 5), and q will be a quotient map. $\widetilde{A}_G$ will be defined as the algebra of all G-finite elements in $\widetilde{A}_{\overline{P}}$, i.e. the ones finite under the adjoint action of $\mathfrak{g} \subseteq \mathbb{D}(G/Q, \chi \otimes \rho_{G/Q})$ on which the $\mathfrak{g}$-action exponentiates to a G-action. $\widetilde{A}_G$ will be a finite extension of the image of $\mathbb{D}(G/Q, \chi \otimes \rho_{G/Q})$.

Change of polarization. The first step is easy. Since L is a dominant Lagrangian, Lemma 6.9 shows that $\overline{P}/\overline{P} \cap Q$ is a dense open subset of G/Q. Therefore there is a restriction map

$$\mathbb{D}(G/Q, \chi \otimes \rho_{G/Q}) \hookrightarrow \mathbb{D}(\overline{P}/[\overline{P} \cap Q], \chi \otimes \rho_{G/Q}). \tag{6.11}$$

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Notice that $\overline{P} = G_0 G_{-1} G_{-2}$ and $\overline{P} \cap Q = Q_0 Q_{-1} Q_{-2}$ (slightly abusing notation, we denote the subgroup of G associated to the subalgebra $\mathfrak{g}_0 + \mathfrak{g}_{-1} + \mathfrak{g}_{-2}$ by $G_0 G_{-1} G_{-2}$). We will construct a change of polarization map

$$\mathbb{D}(\overline{P}/[\overline{P} \cap Q], \chi \otimes \rho_{G/Q}) \longrightarrow \mathbb{D}(\overline{P}/[Q_\phi Q_{-1} G_{-2}], \chi \otimes \lambda \otimes \rho_{\overline{P}/[Q_\phi Q_{-1} G_{-2}]}) \overset{def}{=} A'_{\overline{P}}.$$
(6.12)

To understand the geometry underlying the existence of such a map, notice that the algebra $\mathbb{D}(G/Q, \chi \otimes \rho_{G/Q})$ is the quantization of some coadjoint orbit $\mathcal{O}'$ (namely, the one dense in $G \cdot (\chi + \mathfrak{n}_Q)$), and $\mathbb{D}(\overline{P}/[\overline{P} \cap Q], \chi \otimes \rho_{G/Q})$ is the quantization of a dense open subset $\mathcal{O}'_{\overline{P}}$ of $\mathcal{O}'$ (namely, the intersection of $\mathcal{O}'$ with $\overline{P} \cdot (\chi + \mathfrak{n}_Q)$). Notice that usually $\mathcal{O}'_{\overline{P}}$ will not be a single $\overline{P}$-orbit. Now $\mathcal{O}'_{\overline{P}}$ admits two polarizations – one coming from $\mathcal{O}'$, and another one, whose space of leaves is $\overline{P}/[Q_\phi Q_{-1} G_{-2}]$. The change of polarization map (6.12) is a map between the Dixmier algebras obtained by using these polarizations.

We will obtain this map in a slightly roundabout way, using induction of Dixmier algebras. The Ind notation for twisted induction (introduced in Section 4) will be used (for example, the algebra $\mathbb{D}(G/Q, \chi \otimes \rho_{G/Q})$ will be denoted $\operatorname{Ind}_Q^G(\chi)$). We will use extensively the two basic properties of induction of Dixmier algebras – transitivity and Mackey isomorphism. The trivial character of a Lie algebra will be denoted by o.

First, by transitivity of induction and Mackey isomorphism (using the fact that, by restriction, χ is a character of $\mathfrak{q}_0$, hence a character of $\mathfrak{q}_0 + \mathfrak{q}_{-1} + \mathfrak{g}_{-2}$), we see that

$$\begin{aligned}
\mathbb{D}(\overline{P}/[\overline{P} \cap Q], \chi \otimes \rho_{G/Q}) &\cong \operatorname{Ind}_{Q_0 Q_{-1} Q_{-2}}^{G_0 G_{-1} G_{-2}}(\chi) \\
&\cong \operatorname{Ind}_{Q_0 Q_{-1} G_{-2}}^{G_0 G_{-1} G_{-2}} \operatorname{Ind}_{Q_0 Q_{-1} Q_{-2}}^{Q_0 Q_{-1} G_{-2}}(\chi).
\end{aligned}$$
(6.13)

We will apply Proposition 5.4 (which constructs a change of polarization map) to the coadjoint orbit of $\lambda|_{\mathfrak{q}_0 + \mathfrak{q}_{-1} + \mathfrak{g}_{-2}}$ for the group $\mathsf{G} = Q_0 Q_{-1} G_{-2}$. Using (6.8) we see that $\mathsf{G}_\lambda = Q_\phi Q_{-1} Q_{-2}$ and if we put $\mathsf{H} = Q_0$ and $\mathsf{U} = Q_{-1} G_{-2}$, then both $\mathsf{G}_\lambda \mathsf{H}$ and $\mathsf{G}_\lambda \mathsf{U}$ give polarizations of $\mathsf{G} \cdot \lambda$ at λ. Applying Proposition 5.4 now gives a map of Dixmier algebras

$$\operatorname{Ind}_{Q_0 Q_{-1} Q_{-2}}^{Q_0 Q_{-1} G_{-2}}(\lambda) \longrightarrow \operatorname{Ind}_{Q_\phi Q_{-1} G_{-2}}^{Q_0 Q_{-1} G_{-2}}(\lambda)$$

which, after recalling from (6.9) that $\lambda|_{\mathfrak{q}_{-2}} = 0$ and, of course, that $\lambda|_{\mathfrak{g}_i} = 0$ for $i \neq -2$, becomes

$$\operatorname{Ind}_{Q_0 Q_{-1} Q_{-2}}^{Q_0 Q_{-1} G_{-2}}(o) \longrightarrow \operatorname{Ind}_{Q_\phi Q_{-1} G_{-2}}^{Q_0 Q_{-1} G_{-2}}(\lambda).$$

Tensoring both sides with χ and applying Mackey isomorphism gives

$$\operatorname{Ind}_{Q_0 Q_{-1} Q_{-2}}^{Q_0 Q_{-1} G_{-2}}(\chi) \longrightarrow \operatorname{Ind}_{Q_\phi Q_{-1} G_{-2}}^{Q_0 Q_{-1} G_{-2}}(\chi \otimes \lambda).$$
(6.14)

Now transitivity of induction combined with (6.13) gives a map

$$\mathbb{D}(\overline{P}/[\overline{P} \cap Q], \chi \otimes \rho_{G/Q}) \longrightarrow \operatorname{Ind}_{Q_\phi Q_{-1} G_{-2}}^{G_0 G_{-1} G_{-2}}(\chi \otimes \lambda)$$

which is the change of polarization map (6.12)we need.

The quotient map. We want to get a map

$$A'_{\overline{P}} = \operatorname{Ind}_{Q_\phi Q_{-1} G_{-2}}^{\overline{P}}(\chi \otimes \lambda) \longrightarrow \operatorname{Ind}_{\Gamma G_\phi^o \overline{U}}^{\overline{P}}(A_{G_\phi \overline{U}}) = \widetilde{A}_{\overline{P}} \qquad (6.15)$$

(recall from the end of Section 6.2 that Γ is the component group of Q_ϕ, which is a subgroup of the component group of G_ϕ; ΓG_ϕ^o is a subgroup of G_ϕ). We will rewrite $A'_{\overline{P}}$ in a way that makes obtaining such a map very easy. Notice that, by restriction, χ is a character of $\mathfrak{q}_\phi$, hence of $\mathfrak{q}_\phi + \mathfrak{g}_{-1} + \mathfrak{g}_{-2}$. By applying transitivity of induction followed by Mackey isomorphism for χ, we obtain

$$\begin{aligned}
A'_{\overline{P}} &= \operatorname{Ind}_{Q_\phi Q_{-1} G_{-2}}^{\overline{P}}(\chi \otimes \lambda) \\
&\cong \operatorname{Ind}_{Q_\phi G_{-1} G_{-2}}^{\overline{P}} \operatorname{Ind}_{Q_\phi Q_{-1} G_{-2}}^{Q_\phi G_{-1} G_{-2}}(\chi \otimes \lambda) \qquad (6.16) \\
&\cong \operatorname{Ind}_{Q_\phi G_{-1} G_{-2}}^{\overline{P}}[\chi \otimes_{\mathbb{C}} \operatorname{Ind}_{Q_\phi Q_{-1} G_{-2}}^{Q_\phi G_{-1} G_{-2}}(\lambda)].
\end{aligned}$$

Recall that $\mathfrak{q}_{-1}$ is a maximal isotropic subspace of the symplectic space $\mathfrak{g}_{-1}$. Invoking Lemma 6.5 we see that

$$\operatorname{Ind}_{Q_\phi Q_{-1} G_{-2}}^{Q_\phi G_{-1} G_{-2}}(\lambda) \cong A_{G_\phi \overline{U}}$$

hence the isomorphism (6.16) becomes

$$A'_{\overline{P}} \cong \operatorname{Ind}_{Q_\phi \overline{U}}^{\overline{P}}[\chi \otimes_{\mathbb{C}} A_{G_\phi \overline{U}}] \cong \operatorname{Ind}_{\Gamma Q_\phi^o \overline{U}}^{\overline{P}}[\chi \otimes_{\mathbb{C}} A_{G_\phi \overline{U}}]. \qquad (6.17)$$

Recall from Lemma 6.5 that $A_{G_\phi \overline{U}}$ is a Dixmier algebra for $G_\phi \overline{U}$, hence, by restriction, it is a Dixmier algebra for $\Gamma G_\phi^o \overline{U}$. Using once more transitivity of induction and Mackey isomorphism we obtain

$$\begin{aligned}
A'_{\overline{P}} &\cong \operatorname{Ind}_{\Gamma G_\phi^o \overline{U}}^{\overline{P}} \operatorname{Ind}_{\Gamma Q_\phi^o \overline{U}}^{\Gamma G_\phi^o \overline{U}}[\chi \otimes_{\mathbb{C}} A_{G_\phi \overline{U}}] \\
&\cong \operatorname{Ind}_{\Gamma G_\phi^o \overline{U}}^{\overline{P}}[A_{G_\phi \overline{U}} \otimes_{\mathbb{C}} \operatorname{Ind}_{\Gamma Q_\phi^o \overline{U}}^{\Gamma G_\phi^o \overline{U}}(\chi)].
\end{aligned} \qquad (6.18)$$

To construct the quotient map (6.15) we want to choose χ so that the algebra of differential operators

$$\operatorname{Ind}_{\Gamma Q_\phi^o \overline{U}}^{\Gamma G_\phi^o \overline{U}}(\chi) \cong \mathbb{D}(G_\phi^o/Q_\phi^o, \chi \otimes \rho_\phi) \qquad (6.19)$$

will have an obvious quotient (here $\rho_\phi = \rho_{G_\phi/Q_\phi}$). Since L is a good Lagrangian, we can choose χ so that $\chi|_{\mathfrak{q}_\phi} = -\rho_\phi$. Then the algebra in (6.19) becomes $\mathbb{D}(G_\phi^o/Q_\phi^o)$. Since L is a homogeneous Lagrangian, Lemma 6.6 shows that G_ϕ^o/Q_ϕ^o is a projective variety,

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so its space of global functions is $\mathbb{C}$. Global differential operators act on this space by multiplication by a constant, which gives a map

$$\mathbb{D}(G_\phi^o/Q_\phi^o) \longrightarrow \mathbb{C}. \qquad (6.20)$$

Using (6.18) we obtain the quotient map

$$A'_{\overline{P}} \longrightarrow \operatorname{Ind}_{\Gamma G_\phi^o \overline{U}}^{\overline{P}}(A_{G_\phi \overline{U}}) \cong \widetilde{A}_{\overline{P}}$$
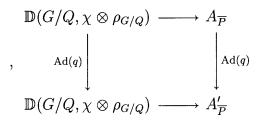
that we need.

The definition of $\widetilde{A}_G$. Now we have an algebra $\widetilde{A}_{\overline{P}}$ which quantizes the ring of functions on $\widetilde{\mathcal{O}}_{\overline{P}}$, and a map of algebras

$$\varphi : \mathbb{D}(G/Q, \chi \otimes \rho_{G/Q}) \longrightarrow \widetilde{A}_{\overline{P}}.$$

Since $\mathfrak{g} \subseteq \mathbb{D}(G/Q, \chi \otimes \rho_{G/Q})$, we can consider the adjoint $\mathfrak{g}$-action on $\widetilde{A}_{\overline{P}}$ given by $\operatorname{ad}_X(a) = \varphi(X)a - a\varphi(X)$ for $X \in \mathfrak{g}$ and $a \in \widetilde{A}_{\overline{P}}$. Let $\widetilde{A}_{\mathfrak{g}}$ denote the subalgebra of elements of $\widetilde{A}_{\overline{P}}$ transforming finitely under this action, and let $\widetilde{A}_G$ be the subalgebra of $\widetilde{A}_{\mathfrak{g}}$ on which the $\mathfrak{g}$-action exponentiates to a G-action. $\widetilde{A}_G$ is a G-algebra equipped with a G-equivariant map $U(\mathfrak{g}) \to \widetilde{A}_G$, i.e. it is a Dixmier algebra for G. Furthermore, Proposition 4.17 shows that $\widetilde{A}_{\overline{P}}$ is completely prime, therefore so is its subalgebra $\widetilde{A}_G$.

On the choice of χ. The construction of $\widetilde{A}_G$ depended on the choice of a character χ of $\mathfrak{q}$ such that $\chi|_{\mathfrak{q}_\phi} = -\rho_\phi$. It is not clear whether ρ_ϕ can always be extended to a character of $\mathfrak{q}$. Furthermore, in the cases when this is possible, there may be more than one way of doing it, and we have to choose the "correct" character χ. We will not concern ourselves with this problem for two reasons. First, in the cases when there is more than one way of extending $\mathfrak{q}_\phi$ to $\mathfrak{q}$, the ring of functions on $\widetilde{\mathcal{O}}$ will be induced (as an orbit datum, in the sense of Definition 4.18) from some proper parabolic of G. We are primarily concerned with associating Dixmier algebras to rigid (i.e. non-induced) orbit data, and in the case of such orbit covers there cannot be an ambiguity in the choice of χ. Second, in all applications of this construction that we will give in Section 7, there will be a unique choice of χ.

$\widetilde{A}_G$ depends only on the choice of a Lagrangian in $\mathcal{O}$. Finally we will show that the Dixmier algebra $\widetilde{A}_G$ we obtained is independent of the choice of the $\mathfrak{sl}_2$-triple $\{X, Y, H\}$. Indeed, let $\{X', Y', H'\}$ be another $\mathfrak{sl}_2$-triple such that $X \in L$. By Lemma 6.8

there exists an element $q \in Q$ such that $\mathrm{Ad}(q) \cdot \{X, Y, H\} = \{X', Y', H'\}$. Then $\mathrm{Ad}(q)$ takes $\overline{P}$ to $\overline{P}'$ and it is easy to see that there is a commutative diagram

$$
, \qquad
\begin{array}{ccc}
\mathbb{D}(G/Q, \chi \otimes \rho_{G/Q}) & \longrightarrow & A_{\overline{P}} \\
\Big\downarrow{\scriptstyle \mathrm{Ad}(q)} & & \Big\downarrow{\scriptstyle \mathrm{Ad}(q)} \\
\mathbb{D}(G/Q, \chi \otimes \rho_{G/Q}) & \longrightarrow & A'_{\overline{P}}
\end{array}
$$

which shows that A_G and A'_G (the G-finite parts of $A_{\overline{P}}$ and $A'_{\overline{P}}$) are isomorphic as Dixmier algebras for G.

7 Examples

We saw in Section 6 that we can construct a Dixmier algebra from a Lagrangian subvariety L in a nilpotent coadjoint orbit $\mathcal{O}$ if several conditions are satisfied.

a) $\mathcal{O}$ must be a small orbit.

b) L must be a dominant homogeneous B-stable Lagrangian subvariety of $\mathcal{O}$.

c) There must be a character χ of $\mathfrak{q}$ such that $\chi|_{\mathfrak{q}_\phi} = -\rho_\phi$ (see (6.19)).

If $\mathcal{O}$ is an arbitrary (not necessarily small) nilpotent orbit, $\lambda \in \mathcal{O}$, and P and $\overline{P}$ are the Jacobson-Morozov parabolic and its opposite, associated to $\mathcal{O}$ as in Section 6.1, then $\mathcal{O}_{\overline{P}} = \overline{P} \cdot U_P \cdot \lambda$ is a dense $\overline{P}$-invariant subvariety of $\mathcal{O}$. Examples suggest that it should be possible to construct the Dixmier algebra $A_{\overline{P}}$ associated to $\mathcal{O}_{\overline{P}}$ using the methods employed in Section 6.2 for small orbits, but I have not been able to do this in such generality.

Using Lemma 6.3 it is easy to determine which nilpotent orbits for simple groups are small (for example the minimal orbit is always small). To apply the construction of the Dixmier algebra to one of these orbits we need a good Lagrangian contained in the orbit. In the following sections we will show that there are good Lagrangians in many of these small orbits so we can construct the corresponding Dixmier algebras.

7.1 The spherical orbits in $Sp(2n, \mathbb{C})$

Suppose $(\mathbb{C}^{2n}, \omega)$ is a symplectic vector space with standard basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$, and let $G = Sp(2n, \mathbb{C})$. Then

$$\mathfrak{sp}(2n, \mathbb{C}) \cong \left\{ \begin{pmatrix} A & S \\ T & -{}^tA \end{pmatrix} \mid A \in \mathfrak{gl}(n, \mathbb{C}),\ S \text{ and } T \text{ are symmetric} \right\}.$$

Let $\mathfrak{q} = \left\{ \begin{pmatrix} A & S \\ 0 & -{}^tA \end{pmatrix} \right\}$ and let Q be the corresponding parabolic subgroup of G. If W is the span of $e_1, \ldots, e_n$, a maximal isotropic subspace of $\mathbb{C}^{2n}$, then Q is the stabilizer of W in G. We will show that each small nilpotent coadjoint orbit for G contains a good Q-stable Lagrangian. This will allow us to apply the construction from Section 6 to each of these orbits and obtain a corresponding Dixmier algebra.

The symplectic form ω identifies the span of $f_1, \ldots, f_n$ with the dual of W and we will denote that span by W^*. Now let $T \in \mathfrak{u}_Q$, the nilradical of $\mathfrak{q}$. Then $T : W \mapsto 0$, $W^* \mapsto W$, so T gives a bilinear form $(\cdot, \cdot)$ on W^*. The form can be explicitly written as

$$(w_1, w_2) = \omega(w_1, Tw_2)$$

for $w_1, w_2 \in W^*$, and the fact that $T \in \mathfrak{sp}(2n, \mathbb{C})$ implies that it is symmetric:

$$(w_2, w_1) = \omega(w_2, Tw_1) = -\omega(Tw_2, w_1) = \omega(w_1, Tw_2) = (w_1, w_2).$$

This establishes an isomorphism of Q-modules

$$\mathfrak{u}_Q \cong S^2 W. \tag{7.1}$$

The set $L_k \subseteq \mathfrak{u}_Q$ of symmetric matrices of rank k is a single Q-orbit of dimension

$$\dim L_k = nk - \tfrac{1}{2}k(k-1). \tag{7.2}$$

This shows that there are $n + 1$ Q-orbits in $\mathfrak{u}_Q$, namely $L_0, \ldots, L_n$.

Let $\mathcal{O}_k = G \cdot L_k$. The $\mathcal{O}_k$ is a nilpotent coadjoint orbit and $L_k = \mathcal{O}_k \cap \mathfrak{u}_Q$ is a closed isotropic subvariety of $\mathcal{O}_k$. Using [6], Corollary 6.1.4, we see that

$$\dim \mathcal{O}_k = 2nk - k(k-1).$$

This, along with (7.2), shows that L_k is a homogeneous Lagrangian in $\mathcal{O}_k$ whose stabilizer is Q.

Put

$$X_k = \begin{pmatrix} 0 & I_{k,n} \\ 0 & 0 \end{pmatrix}, \quad Y_k = \begin{pmatrix} 0 & 0 \\ I_{k,n} & 0 \end{pmatrix}, \quad H_k = \begin{pmatrix} I_{k,n} & 0 \\ 0 & -I_{k,n} \end{pmatrix},$$

where $I_{k,n}$ denotes the $n \times n$ matrix $\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$. Then it is easy to check that $\{X_k, Y_k, H_k\}$ is an $\mathfrak{sl}_2$-triple with $X_k \in L_k$ and H_k is a dominant element of the standard Cartan subalgebra of $\mathfrak{sp}(2n, \mathbb{C})$. Therefore L_k is a dominant Lagrangian.

It is clear that the eigenvalues of $\mathrm{ad}(H_k)$ on $\mathfrak{g}$ are not greater than 2. Using Proposition 6.3 we see that all orbits $\mathcal{O}_k$ are small. Actually, these are all the small orbits of $Sp(2n, \mathbb{C})$. The partition of $2n$ corresponding to $\mathcal{O}_k$ (as described in [6], Theorem 5.1.3) is $[2^k, 1^{2n-2k}]$. This shows that the small orbits for $Sp(2n, \mathbb{C})$ are precisely the spherical orbits, i.e. those in which some Borel subgroup has a dense orbit (since it is known that the spherical orbits for $Sp(2n, \mathbb{C})$ are precisely those whose partitions contain only 1's and 2's).

Let $\mathfrak{g}_\phi$ be the centralizer of the $\mathfrak{sl}_2$-triple $\{X_k, Y_k, H_k\}$. Then, if we write the matrices in basis $e_1, \ldots, e_k, e_{k+1}, \ldots, e_n, f_{k+1}, \ldots, f_n, f_1, \ldots, f_k$, we have

$$\mathfrak{g}_\phi = \begin{pmatrix} \mathfrak{so}_k & 0 & 0 \\ 0 & \mathfrak{sp}_{2n-2k} & 0 \\ 0 & 0 & -{}^t\mathfrak{so}_k \end{pmatrix}, \quad \mathfrak{q}_\phi = \mathfrak{g}_\phi \cap \mathfrak{q} = \begin{pmatrix} \mathfrak{so}_k & 0 & 0 & 0 \\ 0 & \mathfrak{gl}_{n-k} & * & 0 \\ 0 & 0 & -{}^t\mathfrak{gl}_{n-k} & 0 \\ 0 & 0 & 0 & -{}^t\mathfrak{so}_k \end{pmatrix}.$$

Since

$$\mathfrak{q} = \begin{pmatrix} \mathfrak{gl}_n & * \\ 0 & -{}^t\mathfrak{gl}_n \end{pmatrix},$$

the restriction of the character $\mathrm{tr}_{\mathfrak{gl}_n}$ of $\mathfrak{q}$ to $\mathfrak{q}_\phi$ is $\mathrm{tr}_{\mathfrak{gl}_{n-k}}$. It is easy to see that $\rho_\phi = \frac{n-k}{2}\,\mathrm{tr}_{\mathfrak{gl}_{n-k}}$. Hence the unique character χ of $\mathfrak{q}$ such that $\chi|_{\mathfrak{q}_\phi} = -\rho_\phi$ is $\chi = -\frac{n-k}{2}\,\mathrm{tr}_{\mathfrak{gl}_n}$.

All these properties imply that L_k is a good Lagrangian in $\mathcal{O}_k$. Applying the construction from Section 6 to L_k we obtain a Dixmier algebra A_k which should be associated to $\mathcal{O}_k$. (As a matter of fact, the algebra A_k should be associated to a certain cover of $\mathcal{O}_k$. We can be more specific: in the notation of Section 6.2, A_k should be associated to the quotient of the universal G-equivariant cover of $\mathcal{O}_k$ by the component group Γ of Q_ϕ. In this case, the G-equivariant fundamental group of $\mathcal{O}_k$ is $\mathbb{Z}/2$ (since G_ϕ has two components) and $\Gamma \cong \mathbb{Z}/2$, hence A_k is associated to $\mathcal{O}_k$.) A_k is obtained as a quotient (or a finite extension of a quotient) of $\mathbb{D}(G/Q, -\frac{n-k}{2}\,\mathrm{tr}_{\mathfrak{gl}_n} + \rho_{G/Q}) = \mathbb{D}(G/Q, \frac{k}{2}\,\mathrm{tr}_{\mathfrak{gl}_n})$. In particular, the infinitesimal character of A_k is $[n - \frac{k}{2}, \ldots, 1 - \frac{k}{2}]$.

7.2 Certain spherical orbits in $SO(2n, \mathbb{C})$

In this section we will show how the construction from Section 7.1 can be adapted to obtain the Dixmier algebras associated to certain spherical orbits for $SO(2n, \mathbb{C})$.

Suppose $(\mathbb{C}^{2n}, (\cdot, \cdot))$ is a vector space with a non-degenerate symmetric bilinear form and let $G = SO(2n, \mathbb{C})$. Denote the standard basis by $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$, so that $(e_i, e_j) = (f_i, f_j) = 0$ and $(e_i, f_j) = \delta_{ij}$. Then

$$\mathfrak{so}(2n, \mathbb{C}) \cong \left\{ \begin{pmatrix} A & S \\ T & -{}^tA \end{pmatrix} \mid A \in \mathfrak{gl}(n, \mathbb{C}),\ S \text{ and } T \text{ are skew-symmetric} \right\}.$$

Let $\mathfrak{q} = \left\{ \begin{pmatrix} A & S \\ 0 & -{}^tA \end{pmatrix} \right\}$ and let Q be the corresponding parabolic subgroup of G. If W is the span of $e_1, \ldots, e_n$, a maximal isotropic subspace of $\mathbb{C}^{2n}$, then Q is the stabilizer of W in G. As in Section 7.1, we will show that each Q-orbit L in the nilradical $\mathfrak{u}_Q$ of $\mathfrak{q}$ is a good Lagrangian in the corresponding G-orbit $\mathcal{O} = G \cdot L$. About a half of all spherical orbits of $SO(2n, \mathbb{C})$ are obtained in this way. We will apply the construction from Section 6 to each of these orbits and obtain a corresponding Dixmier algebra.

The symmetric form $(\cdot, \cdot)$ identifies the span of $f_1, \ldots, f_n$ with the dual of W and we will denote that span by W^*. Let $T \in \mathfrak{u}_Q$. Then $T : W \mapsto 0$, $W^* \mapsto W$, so T gives a bilinear form $\omega(\cdot, \cdot)$ on W^*. The form can be explicitly written as

$$\omega(w_1, w_2) = (w_1, Tw_2)$$

for $w_1, w_2 \in W^*$, and the fact that $T \in \mathfrak{so}(2n, \mathbb{C})$ implies that it is skew-symmetric:

$$\omega(w_2, w_1) = (w_2, Tw_1) = -(Tw_2, w_1) = -(w_1, Tw_2) = -\omega(w_1, w_2).$$

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This establishes an isomorphism of Q-modules

$$\mathfrak{u}_Q \cong \Lambda^2 W.$$

The set $L_{2k} \subseteq \mathfrak{u}_Q$ of skew-symmetric matrices of rank $2k$ is a single Q-orbit of dimension

$$\dim L_{2k} = 2k(n-k) - k.$$

This shows that there are $\left[\frac{n}{2}\right] + 1$ Q-orbits in $\mathfrak{u}_Q$, namely $L_0, L_2, \ldots, L_{2\left[\frac{n}{2}\right]}$.

Let $\mathcal{O}_{2k} = G \cdot L_{2k}$. The $\mathcal{O}_{2k}$ is a nilpotent coadjoint orbit and $L_{2k} = \mathcal{O}_{2k} \cap \mathfrak{u}_Q$ is a closed isotropic subvariety of $\mathcal{O}_{2k}$. Using [6], Corollary 6.1.4, we see that

$$\dim \mathcal{O}_{2k} = 4k(n-k) - 2k = 2 \dim L_{2k}.$$

This shows that L_{2k} is a homogeneous Lagrangian in $\mathcal{O}_{2k}$ whose stabilizer is Q.

Put

$$X_{2k} = \begin{pmatrix} 0 & J_{2k,n} \\ 0 & 0 \end{pmatrix}, \quad Y_{2k} = \begin{pmatrix} 0 & 0 \\ -J_{2k,n} & 0 \end{pmatrix}, \quad H_{2k} = \begin{pmatrix} I_{2k,n} & 0 \\ 0 & -I_{2k,n} \end{pmatrix},$$

where $I_{2k,n}$ denotes the $n \times n$ matrix $\begin{pmatrix} I_{2k} & 0 \\ 0 & 0 \end{pmatrix}$ and $J_{2k,n}$ denotes the $n \times n$ matrix $\begin{pmatrix} 0 & I_k & 0 \\ -I_k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then it is easy to check that $\{X_{2k}, Y_{2k}, H_{2k}\}$ is an $\mathfrak{sl}_2$-triple with $X_{2k} \in L_{2k}$ and H_{2k} is a dominant element of the standard Cartan subalgebra of $\mathfrak{so}(2n, \mathbb{C})$. Therefore L_{2k} is a dominant Lagrangian.

It is clear that the eigenvalues of $\mathrm{ad}(H_{2k})$ on $\mathfrak{g}$ are not greater than 2. Using Proposition 6.3 we see that all orbits $\mathcal{O}_{2k}$ are small. The partition of $2n$ corresponding to $\mathcal{O}_{2k}$ (as described in [6], Theorem 5.1.4) is $[2^{2k}, 1^{2n-4k}]$.

Let $\mathfrak{g}_\phi$ be the centralizer of the $\mathfrak{sl}_2$-triple $\{X_{2k}, Y_{2k}, H_{2k}\}$. Then, if we write the matrices in basis $e_1, \ldots, e_{2k}, e_{2k+1}, \ldots, e_n, f_{2k+1}, \ldots, f_n, f_1, \ldots, f_{2k}$, we have

$$\mathfrak{g}_\phi = \begin{pmatrix} \mathfrak{sp}_{2k} & 0 & 0 \\ 0 & \mathfrak{so}_{2n-4k} & 0 \\ 0 & 0 & -{}^t\mathfrak{sp}_{2k} \end{pmatrix}, \quad \mathfrak{q}_\phi = \mathfrak{g}_\phi \cap \mathfrak{q} = \begin{pmatrix} \mathfrak{sp}_{2k} & 0 & 0 & 0 \\ 0 & \mathfrak{gl}_{n-2k} & * & 0 \\ 0 & 0 & -{}^t\mathfrak{gl}_{n-2k} & 0 \\ 0 & 0 & 0 & -{}^t\mathfrak{sp}_{2k} \end{pmatrix}.$$

Since

$$\mathfrak{q} = \begin{pmatrix} \mathfrak{gl}_n & * \\ 0 & -{}^t\mathfrak{gl}_n \end{pmatrix},$$

the restriction of the character $\mathrm{tr}_{\mathfrak{gl}_n}$ of $\mathfrak{q}$ to $\mathfrak{q}_\phi$ is $\mathrm{tr}_{\mathfrak{gl}_{n-2k}}$. It is easy to see that $\rho_\phi = \frac{n-2k-1}{2} \mathrm{tr}_{\mathfrak{gl}_{n-2k}}$. Hence the unique character χ of $\mathfrak{q}$ such that $\chi|_{\mathfrak{q}_\phi} = -\rho_\phi$ is $\chi = -\frac{n-2k-1}{2} \mathrm{tr}_{\mathfrak{gl}_n}$.

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All these properties imply that L_{2k} is a good Lagrangian in $\mathcal{O}_{2k}$. Applying the construction from Section 6 to L_{2k} we obtain a Dixmier algebra A_{2k} which should be associated to $\mathcal{O}_{2k}$. (All of the orbits $\mathcal{O}_{2k}$ are simply connected, so A_{2k} cannot be associated to a cover of $\mathcal{O}_{2k}$.) A_{2k} is obtained as a quotient of $\mathbb{D}(G/Q, -\frac{n-2k-1}{2}\operatorname{tr}_{\mathfrak{gl}_n} + \rho_{G/Q}) = \mathbb{D}(G/Q, k\operatorname{tr}_{\mathfrak{gl}_n})$. In particular, the infinitesimal character of A_{2k} is $[n - k - 1, n - k - 2, \ldots, -k]$.

7.3 The double cover of the maximal spherical orbit of $Sp(4n, \mathbb{C})$

In this section we will work out the example that we used in Section 3 as a motivation for the construction of Dixmier algebras.

Suppose $(\mathbb{C}^{4n}, \omega)$ is a symplectic vector space and $G = Sp(4n, \mathbb{C})$. As in Section 7.1, put $W = \langle e_1, \ldots, e_{2n} \rangle$, $W^* = \langle f_1, \ldots, f_{2n} \rangle$. Let P be the stabilizer of W, $P = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$, and let $\mathfrak{u}_P$ be the nilradical of $\mathfrak{p}$, the Lie algebra of P. As we saw in (7.1), $\mathfrak{u}_P \cong S^2 W$. Let

$$\lambda = \begin{pmatrix} 0 & 0 & 0 & I_n \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using the Killing form) and let $\mathcal{O} = G \cdot \lambda$.

We already know from Section 7.1 that $\mathcal{O}$ is a small orbit (actually $\mathcal{O}$ is the maximal spherical orbit of $Sp(4n, \mathbb{C})$, with corresponding partition $[2^{2n}]$) and that $\mathcal{O}$ contains a good P-stable Lagrangian from which we obtained the Dixmier algebra associated to $\mathcal{O}$. However $\mathcal{O}$ has a double cover $\widetilde{\mathcal{O}}$ (see Section 3) and we will use a different good Lagrangian in $\mathcal{O}$ to obtain the Dixmier algebra associated to $\widetilde{\mathcal{O}}$.

All matrices will be written in basis $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}, f_{n+1}, \ldots, f_{2n}, f_1, \ldots, f_n$ from now on. In this basis, $\lambda = \begin{pmatrix} 0 & I_{2n} \\ 0 & 0 \end{pmatrix}$. Let Q be the parabolic subgroup

$$Q = \begin{pmatrix} GL_n & * & * \\ 0 & Sp_{2n} & * \\ 0 & 0 & {}^tGL_n^{-1} \end{pmatrix}. \tag{7.3}$$

Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{n} + \bar{\mathfrak{n}}$ be the standard triangular decomposition of $\mathfrak{g}$. Since $\lambda \in \mathfrak{u}_Q$, $Q \cdot \lambda$ is contained in $\mathfrak{n} \cap \mathcal{O}$, hence it is an isotropic subvariety of $\mathcal{O}$. To show that L is a homogeneous Lagrangian in $\mathcal{O}$, we will use an approach different from that in Section 7.1. Since the centralizer of λ is

$$G_\lambda = \begin{pmatrix} O_{2n} & * \\ 0 & {}^tO_{2n}^{-1} \end{pmatrix}, \tag{7.4}$$

we have

$$
Q_\lambda = \begin{pmatrix} A & * & * & * \\ 0 & {}^tA^{-1} & * & * \\ 0 & 0 & A & * \\ 0 & 0 & 0 & {}^tA^{-1} \end{pmatrix}
\tag{7.5}
$$

with $A \in GL_n$. It is easy to compute that $\dim Q \cdot \lambda = 2n^2 + n = \frac{1}{2} \dim \mathcal{O}$. This shows that $L = Q \cdot \lambda$ is a (not necessarily closed) B-stable Lagrangian subvariety of $\mathcal{O}$, hence a dense open subset of some component of $\mathcal{O} \cap \mathfrak{n}$. From (7.4) and (7.5) we see that Q_λ is a parabolic subgroup of G_λ, so by Proposition 6.7 we see that $L = Q \cdot \lambda$ is a homogeneous Lagrangian.

Put

$$
X = \begin{pmatrix} 0 & I_{2n} \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ I_{2n} & 0 \end{pmatrix}, \quad H = \begin{pmatrix} I_{2n} & 0 \\ 0 & -I_{2n} \end{pmatrix}.
$$

Then $\{X, Y, H\}$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ with $X \in L$ and H a dominant element of $\mathfrak{h}$, so L is a dominant Lagrangian. Finally, the centralizer of the $\mathfrak{sl}_2$-triple $\{X, Y, H\}$ in $\mathfrak{g}$ is

$$
\mathfrak{g}_\phi = \begin{pmatrix} \mathfrak{so}_{2n} & 0 \\ 0 & -{}^t\mathfrak{so}_{2n} \end{pmatrix}, \text{ so}
$$

$$
\mathfrak{q}_\phi = \begin{pmatrix} A & * & 0 & 0 \\ 0 & -{}^tA & 0 & 0 \\ 0 & 0 & A & * \\ 0 & 0 & 0 & -{}^tA \end{pmatrix}
$$

with $A \in \mathfrak{gl}_n$. From (7.3) we see that the restriction map establishes a bijection between characters of $\mathfrak{q}$ and characters of $\mathfrak{q}_\phi$. In particular, the unique character χ of $\mathfrak{q}$ such that $\chi|_{\mathfrak{q}_\phi} = -\rho_\phi$ (notation as in Section 6.4) is $\chi = -\frac{n-1}{2} \operatorname{tr}_{\mathfrak{gl}_n}$. Therefore $L = Q \cdot \lambda$ is a good Lagrangian. By applying the construction from Section nilpotent orbit quantization to it, we obtain a Dixmier algebra A. To see that A should be associated to the double cover of $\mathcal{O}$, notice that the G-equivariant fundamental group of $\mathcal{O}$ is $\mathbb{Z}/2$ (since $G_\phi \cong O(2n)$ has two components) and that Q_ϕ is connected, so $\Gamma \cong \{1\}$.

The Dixmier algebra A is obtained as a quotient of $\mathbb{D}(G/Q, -\frac{n-1}{2} \operatorname{tr}_{\mathfrak{gl}_n} + \rho_{G/Q})$. This shows that its infinitesimal character is $[n, n-1, n-1, n-2, \ldots, 1, 1, 0]$. Notice that the quantization of $\mathcal{O}$ is the algebra of differential operators $\mathbb{D}(G/P, \rho_{G/P})$, which has infinitesimal character $[n - \frac{1}{2}, n - \frac{3}{2}, \ldots, -n + \frac{1}{2}]$, so the infinitesimal characters of the Dixmier algebras associated to $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ are different.

7.4 The Clifford orbit datum in $Sp(4n, \mathbb{C})$

Finally, we will construct a non-commutative completely prime orbit datum R_G (the Clifford orbit datum) and the Dixmier algebra A_G which should be associated to it. This

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example shows that certain completely prime Dixmier algebras cannot be associated to any commutative orbit datum and should be associated to non-commutative ones.

The model Dixmier algebra. In his paper [15] on model Dixmier algebras McGovern has constructed a model Dixmier algebra A_G for G, i.e. an algebra such that each finite dimensional representation of G appears exactly once in A_G. A_G is a 2-fold extension of the quotient of $U(\mathfrak{g})$ sitting in it and has infinitesimal character $[n, n - \frac{1}{2}, n - 1, \ldots, 1, \frac{1}{2}]$. Also, the kernel of the map $U(\mathfrak{g}) \to A_G$ is the maximal ideal of this infinitesimal character, hence A_G has the smallest Gelfand-Kirillov dimension among all Dixmier algebras of this infinitesimal character. It is easy to see (by comparing the G-module structures) that neither A_G nor the quotient of $U(\mathfrak{g})$ that it contains can be associated to any orbit cover.

The Clifford orbit datum. The Clifford orbit datum will be supported on the maximal spherical orbit $\mathcal{O}$ of $Sp(4n, \mathbb{C})$ considered in Section 7.3. Define λ, P, Q, $\{X, Y, H\}$, etc. , as in that section. Then $G_\lambda \cong O(2n, \mathbb{C}) \ltimes U_P$. Let $C(2n)$ be the Clifford algebra of $\mathbb{C}^{2n}$ and let $\mathcal{C}$ be the associated sheaf of algebras on $\mathcal{O} \cong G/G_\lambda$. The CLIFFORD ORBIT DATUM is the algebra R_G of global sections of $\mathcal{C}$.

R_G has a natural $\mathbb{Z}$-grading coming from the $\mathbb{C}^\times$-action on $\mathcal{O}$. We will prove in Proposition 7.4 that it is completely prime, so it is a natural candidate for quantization. Moreover, Proposition 7.6 shows that R_G is a model representation of G, i.e. each representation of G appears in R_G with multiplicity 1. These properties suggest that the Clifford orbit datum should be associated to the model Dixmier algebra.

To prove the complete primality we will use the following easy lemmas.

Lemma 7.1 *Let $\mathcal{A}$ be a sheaf of algebras on a connected algebraic variety X. Then $\Gamma(X, \mathcal{A})$ is completely prime if and only if $\Gamma(U, \mathcal{A})$ is completely prime for some open subset U of X.*

Lemma 7.2 *Let H be a closed subgroup of the connected algebraic group G, let A be an H-algebra, and let $\mathcal{A}$ be the sheaf of sections of the associated bundle of algebras $G \times_H A$ on G/H. If A is completely prime then so is the induced algebra $\Gamma(G/H, \mathcal{A})$. (The induction considered here is ordinary induction of modules.)*

Corollary 7.3 *If $K \subseteq H$ are closed subgroups of a connected algebraic group G, A_K is a K-algebra and the induced H-algebra A_H is completely prime, then the induced G-algebra A_G is completely prime as well.*

Proposition 7.4 *The Clifford orbit datum R_G is completely prime.*

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Proof: To show the complete primality of $R_G = \Gamma(\mathcal{O}_G, \mathcal{C})$, it suffices (by Lemma 7.1, since $\mathcal{O}_{\overline{P}} = \overline{P}/O(2n)$ is dense in $\mathcal{O}_G$) to check the complete primality of $\Gamma(\overline{P}/O(2n), \mathcal{C})$. By Corollary 7.3 it is sufficient to check that $\Gamma(GL(2n)/O(2n), \mathcal{C})$ is completely prime. Applying Lemma 7.1 again we see that it suffices to check the complete primality of $\Gamma(B/\mathbb{Z}_2^{2n}, \mathcal{C})$ (since $B/\mathbb{Z}_2^{2n} \subseteq GL(2n)/O(2n)$ is a dense open subset, B denoting the group of upper triangular matrices). Using Corollary 7.3 we see that it is sufficient to show that $\Gamma(\mathbb{C}^{\times 2n}/\mathbb{Z}_2^{2n}, \mathcal{C})$ is completely prime. This is done in Lemma 7.5. $\qquad\square$

Lemma 7.5 *The algebra* $\Gamma(\mathbb{C}^{\times 2n}/\mathbb{Z}_2^{2n}, \mathcal{C})$ *is completely prime.*

Proof: Write the ring of functions on $\mathbb{C}^{\times 2n}$ as $\mathbb{C}[z_1, z_1^{-1}, \ldots, z_{2n}, z_{2n}^{-1}]$. Then the ring of functions on $\mathbb{C}^{\times 2n}/\mathbb{Z}_2^{2n}$ is $\mathbb{C}[z_1^2, z_1^{-2}, \ldots, z_{2n}^2, z_{2n}^{-2}]$. Let $v_1, \ldots, v_{2n}$ be the orthonormal basis of $\mathbb{C}^{2n}$ on which $\mathbb{Z}_2^{2n}$ acts by $\varepsilon_i(v_i) = -v_i$, $\varepsilon_i(v_j) = v_j$ for $j \neq i$. We can extend this action to an action of $\mathbb{C}^{\times 2n}$ by $z(v_i) = z_i v_i$, where $z = (z_1, \ldots, z_{2n})$. Then the $\mathbb{Z}_2^{2n}$-action on the Clifford algebra $C(2n)$ extends to a $\mathbb{C}^{\times 2n}$-action (which does not respect the algebra structure). This shows that $\mathcal{C}$ is a free sheaf on $\mathbb{C}^{\times 2n}/\mathbb{Z}_2^{2n}$, so its space of global sections is a free module over $\mathbb{C}[z_1^2, z_1^{-2}, \ldots, z_{2n}^2, z_{2n}^{-2}]$ on generators $f_I : \mathbb{C}^{\times 2n} \to C(2n)$, $f_I(z) = z_{i_1} \ldots z_{i_k} v_{i_1} \ldots v_{i_k}$ (where $z = (z_1, \ldots, z_{2n})$ and $I = (i_1, \ldots, i_k)$). The Clifford algebra relations $v_i^2 = 1$ and $v_i v_j + v_j v_i = 0$ transform into $f_i^2 = z_i^2$ and $f_i f_j + f_j f_i = 0$. Notice that $f_i z_i^{-2}$ is an inverse to f_i, which we denote by f_i^{-1}. Then it is clear that $\Gamma(\mathbb{C}^{\times 2n}/\mathbb{Z}_2^{2n}, \mathcal{C})$ is just the algebra $\mathbb{C}[f_1, f_1^{-1}, \ldots, f_{2n}, f_{2n}^{-1}]$ of Laurent polynomials in $2n$ anti-commuting variables. This algebra does not have zero divisors, which finishes the proof. $\qquad\square$

Proposition 7.6 *The Clifford orbit datum is a model representation of G, i.e. every finite-dimensional representation of G appears in it with multiplicity one.*

Proof: Since the orbit $\mathcal{O}$ is isomorphic to the homogeneous variety $G/[O(2n) \ltimes U_P]$, we need to show that, for any finite-dimensional representation V of G,

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(V, \operatorname{Ind}_{O(2n) \ltimes U_P}^G C(2n)) = 1. \tag{7.6}$$

(In this proof Ind denotes usual induction of modules, not induction of Dixmier algebras.) Using Frobenius reciprocity, we see that

$$\operatorname{Hom}_{O(2n) \ltimes U_P}(V, C(2n)) = \operatorname{Hom}_G(V, \operatorname{Ind}_{O(2n) \ltimes U_P}^G C(2n)).$$

Next we use another version of Frobenius reciprocity: if W is an H-module, then

$$\operatorname{Hom}_{H \ltimes U}(V, W) = \operatorname{Hom}_H(V_U, W), \tag{7.7}$$

where $V_U = V/\langle u \cdot v - v \rangle$ denotes the space of co-invariants of U. After applying (7.7) with $H = O(2n, \mathbb{C})$, $U = U_P$, and $W = C(2n)$, we get

$$\mathrm{Hom}_{O(2n) \ltimes U_P}(V, C(2n)) = \mathrm{Hom}_{O(2n)}(V_{U_P}, C(2n)).$$

Next, observe that if V is a G-module, then V_{U_P} is a module for the Levi factor $L_P \cong GL(2n, \mathbb{C})$ of the parabolic P, and the correspondence $V \mapsto V_{U_P}$ establishes a bijection between G-modules and $GL(2n, \mathbb{C})$-modules. Applying Frobenius reciprocity once more, we see that

$$\mathrm{Hom}_{O(2n)}(V_{U_P}, C(2n)) = \mathrm{Hom}_{GL(2n)}(V_{U_P}, \mathrm{Ind}_{O(2n)}^{GL(2n)} C(2n)).$$

So we have reduced the proof of (7.6) to showing that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{GL(2n)}(V_{U_P}, \mathrm{Ind}_{O(2n)}^{GL(2n)} C(2n)) = 1$$

for every $GL(2n, \mathbb{C})$-module V_{U_P}, i.e. we need to show that $\mathrm{Ind}_{O(2n)}^{GL(2n)} C(2n)$ is a model representation for $GL(2n, \mathbb{C})$. This is shown in the paper [3] on model representations. $\square$

The Dixmier algebra associated to the Clifford orbit datum. We will show that the construction from Section 6 can be used to obtain the Dixmier algebra A_G which should be associated to the Clifford orbit datum R_G. A_G and the model Dixmier algebra constructed by McGovern contain the same quotient of $U(\mathfrak{g})$ but I have not proven that the two algebras are isomorphic.

To construct A_G we will use the Q-stable good Lagrangian L used in Section 7.3. The only difference with the construction in that section is the choice if χ – rather than choosing $\chi = -\frac{n-1}{2} \mathrm{tr}_{\mathfrak{gl}_n}$ (which gives the Dixmier algebra associated to the double cover of $\mathcal{O}$) we choose $\chi = -\frac{n}{2} \mathrm{tr}_{\mathfrak{gl}_n}$.

We will trace the steps of the construction of A_G, as described in Section 6.4, to see where the Clifford algebra will make its appearance.

The first step is the restriction map

$$\mathbb{D}(G/Q, -\tfrac{n}{2} \mathrm{tr}_{\mathfrak{gl}_n} \otimes \rho_{G/Q}) \longrightarrow \mathbb{D}(\overline{P}/[\overline{P} \cap Q], -\tfrac{n}{2} \mathrm{tr}_{\mathfrak{gl}_n} \otimes \rho_{G/Q}) \tag{7.8}$$

from (6.11). Next, we have the change of polarization map

$$\mathbb{D}(\overline{P}/[\overline{P} \cap Q], -\tfrac{n}{2} \mathrm{tr}_{\mathfrak{gl}_n} \otimes \rho_{G/Q}) \longrightarrow A'_{\overline{P}}$$
$$\overset{def}{=} \mathrm{Ind}_{\Gamma G^o_\phi \overline{U}}^{\overline{P}}[\lambda \otimes \mathrm{Ind}_{\Gamma Q^o_\phi \overline{U}}^{\Gamma G^o_\phi \overline{U}}(-\tfrac{n}{2} \mathrm{tr}_{\mathfrak{gl}_n})] \tag{7.9}$$
$$\cong \mathrm{Ind}_{G_\phi \overline{U}}^{\overline{P}}[\lambda \otimes \mathrm{Ind}_{\Gamma Q^o_\phi \overline{U}}^{G_\phi \overline{U}}(-\tfrac{n}{2} \mathrm{tr}_{\mathfrak{gl}_n})],$$

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defined in (6.12) and (6.18). ($A_{G_\phi \overline{U}} \cong \mathrm{End}(\mathbb{C}_\lambda)$ since the orbit it even, so $\mathfrak{g}_{-1} = 0$, see (6.6).)

In this case, $G_\phi \cong O(2n, \mathbb{C})$, $\Gamma \cong \{1\}$ as noticed in Section 7.3, and $Q_\phi = Q_\phi^o \cong \mathbb{Q}_{SO}$ (notation as in Example 4.14) is the parabolic in $O(2n, \mathbb{C})$ which stabilizes a maximal isotropic subspace of $\mathbb{C}^{2n}$.

The next step is constructing a quotient map of $A'_{\overline{P}}$. To obtain such a map, recall from (6.19) that there is an isomorphism of Dixmier algebras

$$\mathrm{Ind}_{\Gamma Q_\phi^o \overline{U}}^{G_\phi \overline{U}}(-\tfrac{n}{2} \, \mathrm{tr}_{\mathfrak{gl}_n}) \cong \mathbb{D}(G_\phi / \Gamma Q_\phi^o, -\tfrac{n}{2} \, \mathrm{tr}_{\mathfrak{gl}_n} \otimes \rho_\phi)$$

$$\cong \mathbb{D}(O(2n)/\mathbb{Q}_{SO}, -\tfrac{1}{2} \, \mathrm{tr}_{\mathfrak{gl}_n}) \tag{7.10}$$

since $\rho_\phi = \frac{n-1}{2} \, \mathrm{tr}_{\mathfrak{gl}_n}$, as is easy to see. As observed in Example 4.14, there is an algebra homomorphism

$$\mathbb{D}(O(2n)/\mathbb{Q}_{SO}, -\tfrac{1}{2} \, \mathrm{tr}_{\mathfrak{gl}_n}) \longrightarrow C(2n) \tag{7.11}$$

and the Clifford algebra $C(2n)$ is a 2-fold extension of the image of $\mathbb{D}(O(2n)/\mathbb{Q}_{SO}, -\tfrac{1}{2} \, \mathrm{tr}_{\mathfrak{gl}_n})$. Combining (7.9), (7.10), (7.11), and using the usual machinery of transitivity of induction and Mackey isomorphism, we get a map

$$A'_{\overline{P}} \longrightarrow \mathrm{Ind}_{G_\phi \overline{U}}^{\overline{P} \, \overline{U}}[\lambda \otimes C(2n)] \overset{def}{=} A_{\overline{P}}.$$

$A_{\overline{P}}$ should be the Dixmier algebra associated to $R_{\overline{P}} = \Gamma(\mathcal{O}_{\overline{P}}, \mathcal{C})$. The map (7.8), together with the algebra homomorphism $U(\mathfrak{g}) \to \mathbb{D}(G/Q, -\tfrac{n}{2} \, \mathrm{tr}_{\mathfrak{gl}_n} \otimes \rho_{G/Q})$, provides an algebra homomorphism $U(\mathfrak{g}) \to A_{\overline{P}}$, and we define A_G to be the G-finite part of $A_{\overline{P}}$. The infinitesimal character of A_G is the same as that of $\mathbb{D}(G/Q, -\tfrac{n}{2} \, \mathrm{tr}_{\mathfrak{gl}_n} \otimes \rho_{G/Q})$, which is $[n, n - \tfrac{1}{2}, \ldots, 1, \tfrac{1}{2}]$. We know that the model Dixmier algebra is the algebra of smallest possible Gelfand–Kirillov dimension with this infinitesimal character. By comparing the Gelfand–Kirillov dimensions of A_G and the model algebra, we see that the quotients of $U(\mathfrak{g})$ that they contain are the same.

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