Algebraic Combinatorics of Graph Spectra, Subspace Arrangements and Tutte Polynomials

by

Christos A. Athanasiadis

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Mathematics

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 1996

Author ....
Department of Mathematics
May 3, 1996

Certified by ..
Richard P. Stanley
Professor of Applied Mathematics
Thesis Supervisor

Accepted by ... .
David A. Vogan
Chairman, Departmental Committee on Graduate Students

Algebraic Combinatorics of Graph Spectra, Subspace Arrangements and Tutte Polynomials

by

Christos A. Athanasiadis

Submitted to the Department of Mathematics
on May 3, 1996, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy in Mathematics

Abstract

The present thesis consists of three independent parts. In the first part we employ an elementary counting method to study the eigenvalues of the adjacency matrices of some special families of graphs. The main example is provided by the directed graph $D(G)$, constructed by Propp on the vertex set of oriented spanning rooted trees of a given directed graph G. We describe the eigenvalues of $D(G)$ in terms of information about G, proving in particular a conjecture of Propp. We also report on some work on Hanlon's eigenvalue conjecture. The second part deals with the theory of subspace arrangements. Specifically, we give a combinatorial interpretation to the characteristic polynomial of a rational subspace arrangement in real Euclidean space, valid for sufficiently large prime values of the argument. This observation, which generalizes a theorem of Blass and Sagan, reduces the computation of the characteristic polynomial of such an arrangement to a counting problem and provides an explanation for the wealth of combinatorial results discovered in the theory of hyperplane arrangements in recent years. The basic idea has its origins in work of Crapo and Rota. As a consequence, we find new classes of hyperplane arrangements whose characteristic polynomials have simple form and very often factor completely over the nonnegative integers. Applications include simple derivations of the characteristic polynomials of the Shi arrangements and various generalizations, and another proof of a conjecture of Stanley about the number of regions of the Linial arrangement. We also extend our method to the computation of the face numbers of a rational hyperplane arrangement. In the third part we extend the definition of the Tutte polynomial of a matroid to a more general class of objects, which we call hypermatroids. We carry out much of the classical theory of the Tutte polynomial in this more general setting. We extend some concepts and results about geometric lattices, like Rota's NBC theorem and the basis theorem for the Orlik-Solomon algebra, to more general atomic lattices. A hypergraph defines naturally a hypermatroid and thus, we get a convenient way to define the Tutte polynomial of a hypergraph.

Thesis Supervisor: Richard P. Stanley
Title: Professor of Applied Mathematics

Acknowledgments

There is not much a graduate student can write to express in full his gratitude to his thesis advisor. I will just say that I got to know a little about the work of Richard Stanley in combinatorics when I seemed to be losing some of my enthusiasm for other areas of mathematics. A glance at one of his papers and his book entitled "Enumerative Combinatorics I," as well as a couple of his lectures at MIT in the theory of symmetric functions in the Spring of 1993, were enough to convince me that algebraic combinatorics was the "right" area for me. His philosophy about mathematics turned out to be exactly what I was looking for.

I would also like to thank Gian-Carlo Rota, Jim Propp, Sergey Fomin and Daniel Kleitman for everything I have learnt and the ways I have benefited from them during my student years at MIT. I feel especially indebted to Professor Rota for having modernized the field of combinatorics and having created a whole generation of mathematicians to expand and enrich it, for being such an inspiring, influential and friendly personality and for some encouraging discussions on parts II and III of the present work. Jim Propp motivated the first part of this thesis when he explained his conjecture to me and provided numerous helpful comments on an earlier version of Chapters 1 and 2. Sergey Fomin was always willing to discuss with me specific combinatorial problems and occasionally gave me invaluable advice on how to go about doing research in mathematics.

The second part of this thesis would have never existed had Bruce Sagan not sent me his survey paper on the characteristic polynomial. Bruce Sagan and Günter Ziegler provided useful comments on earlier versions of parts of this thesis and Phil Hanlon kindly discussed with me his eigenvalue conjecture in a DIMACS meeting. Special thanks go to Serafim Batzoglou for his generous help with the computation of $R(4)$ and $R(5)$, mentioned in the last section.

I would like to take this opportunity and mention David Jerison, Richard Melrose and Victor Guillemin who encouraged me to pursue my studies at MIT, together with my apologies that I have tried to become a combinatorial analyst, as opposed to a "mere analyst." Jim Munkres has always been a person to talk to and seek advice. Maureen Lynch has been very helpful with job applications and as the secretary for the calculus class I was a TA for in the Fall of 1994, under Professor Kleitman. Overall, I would like to thank the MIT Mathematics Department for its hospitality, except maybe for the basement offices, and the many opportunities it has offered me.

I greatly acknowledge my parents Andreas and Ekaterini for all the support throughout my student years and dedicate the present thesis to them. I also thank my sister Sofia, who is now almost a grown up, for her funny letters, my brother Gabriel for teaching me some guitar songs last summer and my Washburn D-13 acoustic guitar for being very often such a nice company and for giving me a chance to explore the combinatorics of this instrument.

Lastly, I acknowledge the MIT Coffee House as an excellent place to proofread this thesis and, in general, hang around to relax, talk to friends and waste time.

"The road of excess leads to the palace of wisdom."

William Blake

Contents

List of Figures

Part I

GRAPH SPECTRA

Chapter 1

Eigenvalues and counting

Spectral graph theory is a well developed area of mathematics which studies the eigenvalues of certain matrices associated to graphs. The motivation and applications of the theory lie very often within areas outside combinatorics, such as linear algebra, probability theory and geometry. The first part of the present thesis touches upon some combinatorial aspects of the theory. Classical expositions in spectral graph theory can be found in [21] and [29].

In the first part of this thesis we use an elementary enumerative method, that of counting closed walks, to obtain information about the adjacency eigenvalues associated to special families of graphs. The spectral properties of such families of graphs seem to have puzzled mathematicians in recent years. When this counting method succeeds, it explains certain phenomena about eigenvalues of combinatorial matrices in a particularly elegant way. Traditionally, it has been used to obtain combinatorial information about a graph, given its adjacency eigenvalues. Here we proceed in the opposite direction.

Applied on different situations, this method often leads to various interesting combinatorial problems. These problems may or may not be easier to solve than the original problems which are usually stated in the language of linear algebra.

Overview of Part I. The rest of the introduction contains basic background from graph theory. We first give the necessary preliminaries about notation and terminology. Then we describe the basic enumerative method to compute eigenvalues and give a few examples. These include a proof of a classical theorem of linear algebra about the characteristic polynomials of AB and BA, where A and B are $n \times m$ and $m \times n$ matrices respectively.

In Chapter 2 we use the counting method to prove and generalize a conjecture of Propp, which was the motivation for the first part of this thesis. The basic result describes the adjacency eigenvalues of a directed graph $D(G)$ associated to a given directed graph G, in terms of the eigenvalues of the induced subgraphs and the Laplacian matrix of G. The vertices of the directed graph $D(G)$ are the oriented rooted spanning trees of G and its edges are constructed using the so called re-rooting moves.

In Chapter 3 we consider another family of graphs which are conjectured to have remarkable spectral properties. These graphs were introduced in [31] by Hanlon and arose naturally from problems in Lie algebra homology. Our method reduces an important special case of Hanlon's conjecture, which remains open, to a purely combinatorial statement.

It would be interesting to find other graphs for which the method of counting closed walks gives nontrivial spectral information. Such an example appears in [18, Thm. 1] which implies that the even and odd "Aztec diamonds" AD_n and OD_n have identical nonzero eigenvalues.

1.1 Graphs and matrices

We think of a graph as a slightly more general object than a matrix with complex entries. We consider what is usually called a "weighted directed graph," to treat uniformly all kinds of graphs that will be of interest in this first part of the thesis.

The vertex set of a graph G is a finite, totally ordered set $\{v_1, v_2, \ldots, v_n\}$, denoted $V(G)$. For each ordered pair (v_i, v_j), there is a finite set of edges with initial vertex v_i and terminal vertex v_j. We denote such an edge by $e = v_i v_j$ when this creates no ambiguity. We also say that e is directed from v_i to v_j. Each edge e has a nonzero complex number assigned to it, called the weight of e and denoted by $\mathrm{wt}_G(e)$, or simply by $\mathrm{wt}(e)$. We use the notation $E(G)$ for the set of weighted edges of G. We write $G = (V, E)$ to denote a graph with vertex set V and edge set E.

To each complex matrix $A = (a_{ij})$ we can assign a graph on the vertex set $[n] = \{1, 2, \ldots, n\}$ with edges corresponding to the nonzero entries of A. For $i, j \in [n]$, there is an edge with initial vertex i and terminal vertex j with weight a_{ij} if $a_{ij} \neq 0$ and no such edge if $a_{ij} = 0$. Conversely, let $G = (V, E)$ be a graph on the vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* of G, denoted $A(G)$, is the $n \times n$ matrix whose (i, j) entry is the sum of the weights of all edges of G with initial vertex v_i and terminal vertex v_j. For many purposes, one can replace the edges directed from v_i to v_j by a single edge with weight the sum of the weights of these edges, without affecting definitions or results. The complication of multiple edges will be useful for technical reasons in Chapter 2, as well as in the proof of Theorem 1.4.1.

Ordinary, or else undirected graphs differ from the ones we have described in the fact that their edges have no orientation. That is, they have two endpoints which

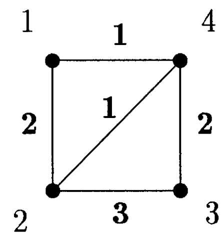

Figure 1.1: An ordinary graph G

may coincide, with no distinction into initial and terminal vertex. However, we can consider ordinary graphs to be graphs as described above by replacing each edge e with a pair of edges with the same endpoints as e but of opposite orientation and weight the weight of e. Figure 1.1 shows an ordinary graph G on four vertices and the weights attached to the edges. Its adjacency matrix is the symmetric matrix

$$A(G) = \begin{pmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 3 & 1 \\ 0 & 3 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}.$$

Walks and trees. Let l be a nonnegative integer. By an l-walk or an l-path W in a graph $G = (V, E)$ we mean an alternating sequence $(u_0, e_1, u_1, \ldots, e_l, u_l)$ of vertices and edges of G such that for each $1 \leq i \leq l$, the edge e_i has initial vertex u_{i-1} and terminal vertex u_i. We call l the *length* of W and the vertices u_0 and u_l its initial and terminal vertices, respectively. We call $u_0, u_1, \ldots, u_l$ the vertices *visited by* W. The walk is said to be *closed* if $u_0 = u_l$. The weight of a walk is the product of the weights of its edges. More generally, by the weight of any object K associated to a graph, like a tree or a forest, we always mean the product of the weights of the edges of K. If $\mathcal{K}$ is a finite family of such objects, we usually refer to the sum of the weights of the objects in $\mathcal{K}$ as simply the "weighted number of objects" in $\mathcal{K}$. We sometimes refer to the cardinality of $\mathcal{K}$ as the "unweighted number of objects" in $\mathcal{K}$, to emphasize the distinction.

If S is a nonempty subset of the vertex set V, we denote by G_S the induced subgraph of G on the vertex set S, that is the graph obtained from G by deleting the vertices not in S and all edges incident to them. The weights of the edges remaining are the same as in the original graph G. In the matrix language, this just means that we delete the rows and columns of $A(G)$ which correspond to the vertices not in S. A subgraph is an induced subgraph with some of its edges deleted. It is called *spanning* if its vertex set is V, the whole vertex set of G.

An *oriented rooted spanning tree* on G, or simply a *rooted spanning tree* on G

is a spanning subgraph T of G having a distinguished vertex r, called the *root*, such that for every $v \in V$ there is a unique path in T with initial vertex v and terminal vertex r. A *rooted forest* on G with root set S is a spanning subgraph which is a union of vertex-disjoint rooted spanning trees on subgraphs of G. The roots of these trees are the elements of S. We call G *strongly connected* if for any two vertices u, v of G, there exists a walk in G with initial vertex u and terminal vertex v.

Homomorphisms. Now let $G' = (V', E')$ be another graph. A *graph homomorphism* $f : G' \longrightarrow G$ consists of two maps $f : V' \longrightarrow V$ and $f : E' \longrightarrow E$ which preserve weights and incidence relations. This means that if $e' \in E'$ is directed from u' to v' in G', then $f(e')$ is directed from $f(u')$ to $f(v')$ in G and $\mathrm{wt}_G(f(e')) = \mathrm{wt}_{G'}(e')$. Clearly, any walk in G' maps under f to a walk in G with the same weight.

1.2 Laplacians and the Matrix-Tree theorem

Let G be a graph with n vertices and let $u \in V(G)$. The *outdegree* of u is the sum of the weights of the edges of G having initial vertex u, i.e. the sum of the entries in the row of $A(G)$ corresponding to u. We denote by $O(G)$ the diagonal $n \times n$ matrix with diagonal entries the vetrex outdegrees. The matrix $L(G) = O(G) - A(G)$ is called the *Laplacian* matrix of G. Note that it is independent of the loops of G, i.e. the edges whose initial and terminal vertices coincide. The Laplacian of the graph of Figure 1.1 is

$$L(G) = \begin{pmatrix} 3 & -2 & 0 & -1 \\ -2 & 6 & -3 & -1 \\ 0 & -3 & 5 & -2 \\ -1 & -1 & -2 & 4 \end{pmatrix}.$$

As it will be aparent from the theorem below, the Laplacian matrix gives valuable information about the graph. Its eigenvalues play an important role in spectral graph theory. For an exposition of many of the known results about Laplacians of ordinary graphs see [38].

The main result we will use about Laplacians is the following version of the Matrix-Tree theorem. A proof and a generalization can be found in [17]. Recall that by the weighted number of rooted forests below we mean the sum of their weights.

Theorem 1.2.1 *For $S \subseteq G$ we denote by $L(G)|_S$ the submatrix of $L(G)$ obtained by deleting the rows and columns corresponding to the vertices in S. Then $\det(L(G)|_S)$ is the weighted number of rooted forests on G with root set S.*

1.3 Eigenvalues and closed walks

Given a graph G, we call the eigenvalues of its adjacency matrix $A(G)$ the *adjacency eigenvalues*, or simply the *eigenvalues* of G. We denote by $\mu_G(\lambda)$ the multiplicity of λ as an eigenvalue of G. We call the eigenvalues of $L(G)$ the *Laplacian eigenvalues* of G. We now relate the eigenvalues with the closed walks in G.

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. It is well known that the (i,j) entry of the matrix $A(G)^l$ equals the weighted number of l-walks in G with initial vertex v_i and terminal vertex v_j. Hence, the weighted number of closed l-walks in G, which we denote by $w(G,l)$, equals the trace of $A(G)^l$ and hence the sum of the l^{th} powers of the eigenvalues of G.

Thus a common technique to count walks in G is to compute its eigenvalues. Our approach here will be the opposite. We will count the weighted number of closed walks in G combinatorially and then read off its eigenvalues from the answer. This will be possible thanks to the following elementary and well known fact.

Lemma 1.3.1 *Suppose that for some nonzero complex numbers a_i, b_j, where $1 \leq i \leq r$ and $1 \leq j \leq s$, we have*

$$\sum_{i=1}^{r} a_i^l = \sum_{j=1}^{s} b_j^l \tag{1.1}$$

for all positive integers l. Then $r = s$ and the a_i are a permutation of the b_j.

Proof: Multiplying (1.1) by x^l and summing over l gives

$$\sum_{i=1}^{r} \frac{a_i x}{1 - a_i x} = \sum_{j=1}^{s} \frac{b_j x}{1 - b_j x},$$

for small x. Since clearing denominators we obtain a polynomial equation, this has to hold for all x. If γ is any nonzero complex number, we can multiply the equation above by $1 - \gamma x$ and set $x = 1/\gamma$ to conclude that γ appears among the a_i as many times as among the b_j. $\square$

Example 1.3.2 To illustrate the idea, let r be a positive integer and consider the graph $M_n(r)$ on the vertex set $[n]$ with r edges of weight 1 directed from vertex i to vertex j, for any $i, j \in [n]$. Its adjacency matrix is easily seen to have nr as its only nonzero eigenvalue, its multiplicity being one. Alternatively, there are $(nr)^l$ ways to choose a closed l walk $(u_0, e_1, u_1, \ldots, e_l, u_l)$ in $M_n(r)$. Indeed, we have n^l choices for the vertices $u_0, \ldots, u_{l-1}$ and r choices for each edge e_i. Hence

$$w(M_n(r), l) = (nr)^l,$$

yielding the unique nonzero eigenvalue nr.

Example 1.3.3 Consider now the complete bipartite graph $H_{r,s}$, where r, s are positive integers. Its vertex set is the disjoint union of two sets $[r] = \{1, \ldots, r\}$ and $[s]' = \{1', \ldots, s'\}$ with r and s elements respectively. For each pair of vertices (a, b) one of which is in $[r]$ and the other in $[s]'$, there is an edge with weight 1 directed from a to b.

It is not difficult to show directly that the nonzero eigenvalues of $H_{r,s}$ are $\sqrt{rs}$ and $-\sqrt{rs}$, each with multiplicity one. To use the counting method instead, consider a closed l-walk $(u_0, e_1, u_1, \ldots, e_l, u_l)$ in $H_{r,s}$. Clearly, no such walk exists if l is odd. There are $2(rs)^l$ such walks if l is even, each with weight 1. Indeed, if $u_0 \in [r]$, there are r ways to choose each u_i for even values of i and s ways to choose each u_i for odd values of i, where $0 \leq i \leq l-1$ and similarly for the case $u_0 \in [s]'$. Thus,

$$w(H_{r,s}, l) = (\sqrt{rs})^l + (-\sqrt{rs})^l$$

and Lemma 1.3.1 gives the eigenvalues of $H_{r,s}$ as promised.

1.4 Matrix products and eigenvalues

We now give a less trivial application of the counting method described in the previous section. The following theorem is a classical and useful result of linear algebra, rarely mentioned in introductory courses in the subject.

Theorem 1.4.1 *If A and B are $n \times m$ and $m \times n$ matrices respectively having complex entries, then the matrices AB and BA have the same nonzero eigenvalues, including multiplicities. In other words,*

$$\lambda^n \operatorname{ch}(BA, \lambda) = \lambda^m \operatorname{ch}(AB, \lambda), \tag{1.2}$$

where $\operatorname{ch}(T, \lambda) = \det(\lambda I - T)$ is the characteristic polynomial of the square matrix T.

Simple linear algebra proofs appeared in [42][52][66] for the case where A and B are square matrices. A proof of (1.2) for general A and B with entries from an arbitrary commutative ring with a unit can be found in [51]. Of course, this more general statement follows immediately from the complex case. We believe that the combinatorial approach in the proof below provides a better understanding of this elementary fact.

Proof of Theorem 1.4.1: Consider a bipartite graph H on the vertex set $[m] \cup [n]'$, where the notation is as in Example 1.3.3, with edges constructed as follows: For each nonzero entry a_{ij} of A, include an edge directed from i' to j with weight a_{ij} and similarly, for each nonzero entry b_{ij} of B, include an edge directed from i to j' with weight b_{ij}.

Now construct two new graphs G_1 and G_2 on the vertex sets $[m]$ and $[n]'$ respectively. The edges of G_1 correspond to the walks of length 2 in H with initial (and hence terminal) vertex in $[m]$. Specifically, for any two edges uv and vw in H with $u, w \in [m]$, we construct an edge of G_1 with initial vertex u and terminal vertex w and define its weight as the product of the weights in H of the edges uv and vw. Similarly, the edges of G_2 correspond to the walks of length 2 in H with initial vertex in $[n]'$.

It is easy to check that the adjacency matrices of the graphs G_1 and G_2 are the martices BA and AB respectively. To prove the theorem, it is enough to prove that the sum of the weights of the closed walks of a given positive length l is the same for G_1 and G_2. Hence, it suffices to establish a length and weight preserving bijection between the closed walks of positive length of the two graphs. But, by construction, the closed walks of G_1 are the closed walks $(u_1, v_1, u_2, v_2, \ldots, u_k, v_k, u_1)$ of H with $u_1 \in [m]$ and the closed walks of G_2 are the closed walks $(v_1, u_1, v_2, u_2, \ldots, v_k, u_k, v_1)$ of H with $v_1 \in [n]'$, where we have suppressed the edges from the notation. The map sending

$$(u_1, v_1, u_2, v_2, \ldots, u_k, v_k, u_1) \longmapsto (v_1, u_2, v_2, u_3, \ldots, v_k, u_1, v_1)$$

provides an obvious example of such a bijection. $\square$

Chapter 2

Spectra of the Propp Graphs

2.1 The graph $D(G)$ and Propp's conjecture

The graph $D(G)$ was defined by Propp for "unweighted" directed graphs G, having no loops or multiple edges. We give here the definition more generally for an arbitrary graph G. This is possible since the concept of orientation for the edges is already inherent in our notion of a graph.

We first define what we mean by a *re-rooting move* on a graph $G = (V, E)$. Let T be a rooted spanning tree of G with root r. Let $e \in E$ be directed from r to $u \in V$ and denote by $T(e)$ the tree obtained from T by adding the edge e and deleting the edge of T with initial vertex u. Note that $T(e)$ is another rooted spanning tree of G with root u. We say that $T(e)$ is obtained from T by a re-rooting move on G with respect to e. Figure 2.1 shows a re-rooting move on the graph of Figure 1-1 with respect to the edge directed from vertex 4 to vertex 1. The roots of the trees are drawn as unfilled circles and determine the orientation of the edges of the trees.

We can now give the definition of $D(G)$.

Definition 2.1.1 (Propp) *Let $G = (V, E)$ be a graph. The new graph $D(G)$ has as vertices the rooted spanning trees of G. Its edges are constructed as follows: Let T be a rooted spanning tree on G with root r. Let e be an edge in G with initial vertex r and weight* $\mathrm{wt}(e)$ *and let $T(e)$ be the tree obtained from T by a re-rooting move on G with respect to e. Then add an edge in $D(G)$ of weight* $\mathrm{wt}(e)$*, directed from T to $T(e)$.*

The idea of the construction of $D(G)$ appeared for the first time implicitly in the proof of the Markov chain tree theorem by Anantharam and Tsoucas [2]. In this paper the authors needed to lift a random walk in a directed graph G to a random

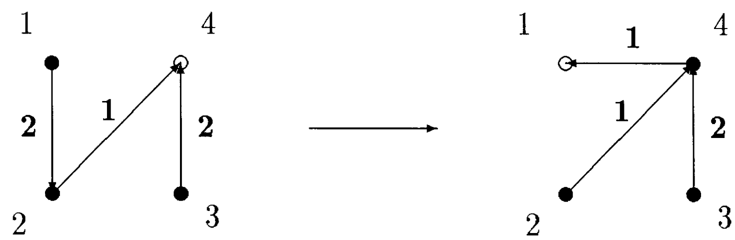

Figure 2.1: A re-rooting move on G

walk in the set of arborescences of G, which coincides with the set of rooted spanning trees if G is strongly connected. On the other hand, Propp's motivation for defining $D(G)$ came from problems related to domino tilings of regions. The re-rooting move on G is analogous to a certain operation on domino tilings, called an "elementary move" in [25]. In fact, under an appropriate coding, the elementary moves can be viewed as a special case of a type of move very similar to the re-rooting move. An even more general operation is described in [47]. Proposition 2.2.5, stated in §2.2, is the analogue of the fact that any domino tiling of a simply connected region can be obtained from any other tiling of the same region by a sequence of elementary moves. Thus $D(G)$ encodes the ways one can reach any rooted spanning tree on G from any other, assuming that G is strongly connected, by performing re-rooting moves.

The main problem we pose here and answer in the following section is to describe the eigenvalues of $D(G)$ in terms of information contained in the original graph G. The motivation for posing this question comes from a conjecture of Propp [46] concerning an interesting special case. We denote by H_n the complete graph on the vertex set $[n]$ without loops. This means that for each pair (i, j) of vertices with $i \neq j$, there is an edge in H_n with weight 1, directed from i to j. The Table 2.1 was constructed by Propp. The entry in the row labeled with i and column labeled with j is the sum of the multiplicities of $0, 1, \ldots, j$ as eigenvalues of the Laplacian of $D(H_i)$. The data provided by this table led Propp to formulate the following conjecture.

Conjecture 2.1.2 (Propp) *The Laplacian eigenvalues of $D(H_n)$ are all integers ranging from 0 to n. The multiplicities of 0, 1, $n-1$ and n are 1, $n^2 - 2n$, 0 and $n^{n-1} - (n-1)^{n-1}$ respectively.*

In §2.3 we will prove Propp's conjecture and we will find the multiplicities of other eigenvalues.

	0	1	2	3	4	5
2	1	1	2	2	2	2
3	1	4	4	9	9	9
4	1	9	27	27	64	64
5	1	16	96	256	256	625

Table 2.1: Sums of multiplicities of Laplacian eigenvalues of $D(H_n)$

2.2 Covering spaces of graphs and eigenvalues

We start with a few more useful definitions. Let $G = (V, E)$ and $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ be two graphs. We say that $\widetilde{G}$ is a *covering space* of G if there exists a graph homomorphism $p : \widetilde{G} \longrightarrow G$ with the following property: If $p(\tilde{u}) = u$ and $e \in E$ is any edge in G with initial vertex u, then there is a unique edge $\tilde{e} \in \widetilde{E}$ with initial vertex $\tilde{u}$ such that $p(\tilde{e}) = e$. Such a homomorphism is called a covering map of graphs.

It follows immediately that given u, $\tilde{u}$ as above and any walk W in G with initial vertex u, there is a unique walk $\widetilde{W}$ in $\widetilde{G}$ with initial vertex $\tilde{u}$ which projects to W under p. We call $\widetilde{W}$ the *lift* of W under p with initial vertex $\tilde{u}$. We call the set $p^{-1}(u) = \{\tilde{u} \in \widetilde{V} \mid p(\tilde{u}) = u\}$ the *fiber* above u. Note that, by construction, the Propp graph $D(G)$ is a covering space of the graph G. Indeed, the covering map p maps each vertex T of $D(G)$ to its root and each edge constructed by a re-rooting move on G with respect to e to the edge e in G. This observation will be fundamental in determining the eigenvalues of $D(G)$.

To relate the weighted number of closed walks on G and $\widetilde{G}$, note that any closed walk in $\widetilde{G}$ projects under p to a closed walk in G. However, if the closed walk W in G has initial vertex u, then the lift of W under p with initial vertex $\tilde{u} \in p^{-1}(u)$ may or may not be closed. Let $t_p(W)$ be the number of $\tilde{u}$ in the fiber $p^{-1}(u)$ which yield a closed walk in $\widetilde{G}$.

The following theorem gives an expression for $w(\widetilde{G}, l)$ in a crucial special case. For any G and nonnegative integer l, we will denote by $g(G, l)$ the sum of the weights of the closed l-walks which visit all vertices of G.

Theorem 2.2.1 *Let $p : \widetilde{G} \longrightarrow G$ be a covering map of graphs. Suppose that the quantity $t_p(W)$ defined above, depends only on the set U of vertices visited by the closed walk W. We denote this quantity by $t_p(U)$. Then*

$$w(\widetilde{G}, l) = \sum_{S \subseteq V} r_p(S)\ w(G_S, l), \tag{2.1}$$

where

$$r_p(S) = \sum_{S\subseteq U\subseteq V} (-1)^{\#(U-S)}\ t_p(U)$$

and V is the vertex set of G.

Proof: Let's fix a subset U of the vertex set V of G. Let W be a closed l-walk in G such that U is the set of vertices visited by W. As we have remarked before, any closed l-walk in $\tilde{G}$ projects under p to such a walk W, for some $U \subseteq V$. The unweighted number of closed walks in $\tilde{G}$ which project to W is, by assumption, $t_p(U)$. Since p is weight preserving, it follows that

$$w(\tilde{G}, l) = \sum_{U\subseteq V} t_p(U)\ g(G_U, l). \tag{2.2}$$

The inclusion-exclusion principle gives

$$g(G_U, l) = \sum_{S\subseteq U} (-1)^{\#(U-S)}\ w(G_S, l). \tag{2.3}$$

Hence, after using (2.3) to compute $g(G_U, l)$ and changing the order of sumation, (2.2) becomes

$$w(\tilde{G}, l) = \sum_{S\subseteq V} w(G_S, l) \sum_{S\subseteq U\subseteq V} (-1)^{\#(U-S)} t_p(U),$$

which is equivalent to (2.1). □

From Theorem 2.2.1 and the discussion in §1.3 we get immediately the following corollary.

Corollary 2.2.2 *Under the assumptions of Theorem 2.2.1, the nonzero eigenvalues of $\tilde{G}$ are included in the nonzero eigenvalues of the induced subgraphs of G. Moreover, if $\gamma \neq 0$ then the multiplicity of γ as an eigenvalue of $\tilde{G}$ is*

$$\sum_{S\subseteq V} r_p(S)\ \mu_{G_S}(\gamma),$$

where $\mu_{G_S}(\gamma)$ stands for the multiplicity of γ as an eigenvalue of G_S. □

Corollary 2.2.3 *Under the assumptions of Theorem 2.2.1, for all complex numbers $\gamma \neq 0$ we have*

$$\sum_{S\subseteq V} r_p(S)\ \mu_{G_S}(\gamma) \geq 0.$$

□

We now specialize the above results to describe the eigenvalues of the Propp graphs and their multiplicities.

Corollary 2.2.4 *For a graph G on the vertex set V we have*

$$w(D(G),l) = \sum_{S \subseteq V} \det(L(G_0)|_S - I)\ w(G_S, l),$$

where G_0 is the graph obtained from G by replacing all weights of its edges with 1 and I denotes the identity matrix of the appropriate size. The nonzero eigenvalues of $D(G)$ are included in the nonzero eigenvalues of the induced subgraphs of G. Moreover, if $\gamma \neq 0$ then the multiplicity of γ as an eigenvalue of $D(G)$ is

$$\sum_{S \subseteq V} \det(L(G_0)|_S - I)\ \mu_{G_S}(\gamma). \tag{2.4}$$

In particular, for all complex numbers $\gamma \neq 0$ we have

$$\sum_{S \subseteq V} \det(L(G_0)|_S - I)\ \mu_{G_S}(\gamma) \geq 0.$$

Proof: Let $p : D(G) \longrightarrow G$ be the covering map defining $D(G)$. We will show that it satisfies the assumption of Theorem 2.2.1 and that $r_p(S) = \det(L(G_0)|_S - I)$. The assertions follow from Theorem 2.2.1 and its corollaries.

Let $W = (u_0, e_1, u_1, \dots, e_l, u_l)$ be a closed l-walk in G with initial vertex $u_0 = r$. Let U be the set of vertices visited by W. Consider the lift $\widetilde{W} = (T_0, \tilde{e}_1, T_1, \dots, \tilde{e}_l, T_l)$ of W in $D(G)$ with initial vertex the spanning tree $T = T_0$ on G, rooted at r. We want to compute the unweighted number $t_p(W)$ of all such T for which the walk $\widetilde{W}$ satisfies $T_l = T$.

At each step of the walk $\widetilde{W}$ from T_{i-1} to T_i we add the edge e_i, which is directed from u_{i-1} to u_i and delete the edge with initial vertex u_i. Therefore, at the end of our walk T_l, a vertex of G appearing for the last time in W as u_{i-1}, where $2 \leq i \leq l$, will be directed to u_i with e_i and the vertices of G not visited by W will be directed as in T_0. To construct a rooted tree $T = T_0$ with the desired property, we have to choose an edge with initial vertex u for any vertex of G other than r. The edges are prescribed by W for the vertices in U and we are free to choose the rest to produce a spanning tree, rooted at r. It follows that $t_p(W)$ is the number $\tau(G_0, U)$ of rooted forests on G_0 with root set U. This indeed depends only on U. The expression $\det(L(G_0)|_S - I)$ for the alternating sum

$$r_p(S) = \sum_{S \subseteq U \subseteq V} (-1)^{\#(U-S)}\ \tau(G_0, U)$$

follows from the Matrix-Tree theorem (Theorem 1.2.1) and some elementary linear algebra. $\square$

Zero as a Laplacian eigenvalue. At this point we digress to show directly that zero is a simple eigenvalue of the Laplacian of $D(H_n)$, meaning that its multiplicity is 1. In the remaining of this section we assume that all edges of G have weight 1. Suppose that G is a disjoint union of strongly connected graphs. Then the multiplicity of 0 as an eigenvalue of the Laplacian of G is the number of connected components of G. Indeed, a basis of the corresponding eigenspace is the set of vectors with entry 1 on the vertices of G belonging to a given connected component and 0 on the rest. Hence in particular, if $D(G)$ is strongly connected then $L(D(G))$ has zero as a simple eigenvalue. In general, $D(G)$ might be disconnected even though G is connected. The same is not true, however, with strong connectedness. The following proposition, proved independently for the first time by Propp, shows that $D(H_n)$ is indeed strongly connected.

Proposition 2.2.5 *If $G = (V, E)$ is strongly connected then so is $D(G)$. In particular, the Laplacian of $D(G)$ has zero as a simple eigenvalue.*

Proof: Given two oriented spanning trees T_0 and T_1 on G with roots r_0 and r_1, we want to find a walk in $D(G)$ from T_0 to T_1. Such a walk is the lift of the walk in G directed from r_0 to r_1, constructed as follows: We start at r_0 and follow the unique path in T_1 from r_0 to r_1. Then we pick the furthest vertex v in G away from r_1 and follow the shortest walk in G (with respect to length) from r_1 to v. This can be done by strong connectedness of G. Now we follow the unique path in T_1 from v to r_1 and continue in the same way with the second furthest vertex in G away from r_1 until only r_1 remains. At this point we stop.

This walk has the property that the last time a vertex u other than r_1 is visited by our walk, it is followed by its successor in T_1, and hence it induces a walk in $D(G)$ from T_0 to T_1. $\square$

2.3 Applications and the proof of Propp's conjecture

Recall that the complete graph H_n is the graph on the vertex set $[n] = \{1, 2, \ldots, n\}$ with exactly one edge of weight 1 directed from i to j for each $i \neq j$ and $i, j \in [n]$. The number of vertices of $D(H_n)$ is the number of rooted spanning trees on $[n]$, which is well known to equal n^{n-1}. We now apply the results of the previous section to give an extension and proof of Conjecture 2.1.2.

Proposition 2.3.1 *The adjacency eigenvalues of $D(H_n)$, where $n \geq 2$, are -1, 1, ..., $n-1$. The multiplicity of i is*

$$\mu_{D(H_n)}(i) = \begin{cases} i \binom{n}{i+1} (n-1)^{n-i-2} & \text{if } 1 \leq i \leq n-1; \\ n^{n-1} - (n-1)^{n-1} & \text{if } i = -1. \end{cases}$$

Proof: The induced subgraphs of H_n are isomorphic to H_m for some $1 \leq m \leq n$. It is easy to show directly that the eigenvalues of H_m are $m-1$ with multiplicity 1 and -1 with multiplicity $m-1$. Hence, by Corollary 2.2.4, the nonzero eigenvalues of $D(H_n)$ are included in the set $\{-1, 1, \ldots, n-1\}$. Moreover the eigenvalue $m-1$, for $2 \leq m \leq n$, has multiplicity

$$\binom{n}{m} \det(L(H_n)|_{[m]} - I) = \binom{n}{m}(m-1)(n-1)^{n-m-1},$$

while -1 has multiplicity

$$\sum_{m=2}^{n} \binom{n}{m}(m-1)\det(L(H_n)|_{[m]} - I)$$

$$= \sum_{m=1}^{n} (m-1)^2 \binom{n}{m} (n-1)^{n-m-1} = n^{n-1} - (n-1)^{n-1}.$$

We have evaluated the last sum by classical elementary methods. The multiplicities we have so far add up to n^{n-1}, so 0 is not an eigenvalue of $D(H_n)$ and the proposition follows. □

Corollary 2.3.2 *The Laplacian eigenvalues of $D(H_n)$, where $n \geq 2$, are $0, 1, \ldots, n-2, n$. The multiplicity of i is*

$$\begin{cases} (n-i-1)\ \binom{n}{i}\ (n-1)^{i-1} & \text{if } 0 \leq i \leq n-1; \\ n^{n-1} - (n-1)^{n-1} & \text{if } i = n. \end{cases}$$

Proof: It suffices to use Proposition 2.3.1 and the fact that

$$L(D(H_n)) = (n-1)I - A(D(H_n)).$$

□

As a variation of the above result we give the following proposition. Its proof consists of a similar computation and is omitted.

Proposition 2.3.3 *Let $M_n(r)$ be the graph on the vertex set $[n]$ of Example 1.3.2, having r edges of weight 1 directed from i to j, for all $i, j \in [n]$. Then the nonzero eigenvalues of $D(M_n(r))$ are $r, 2r, \ldots, nr$ with*

$$\mu_{D(M_n(r))}(ir) = \begin{cases} (ir-1)\ \binom{n}{i}\ (nr-1)^{n-i-1} & \text{if } 1 \leq i \leq n; \\ (nr)^{n-1} - (nr-1)^{n-1} & \text{if } i = 0. \end{cases}$$

□

As a final specialization of Corollary 2.3, we consider the complete bipartite graph $H_{r,s}$ of Example 1.3.3.

Proposition 2.3.4 *The nonzero eigenvalues of $D(H_{r,s})$ are $\sqrt{pq}$ and $-\sqrt{pq}$ for $1 \leq p \leq r$, $1 \leq q \leq s$. The characteristic polynomial of its adjacency matrix is*

$$x^t \prod_{p=1}^{r} \prod_{q=1}^{s} (x^2 - pq)^{m(p,q)},$$

where t is a nonnegative integer depending on r, s and

$$m(p,q) = \binom{r}{p}\binom{s}{q}(r-1)^{s-q-1}(s-1)^{r-p-1}(qr + ps - pq - r - s + 1).$$

Note that $m(p,q)$ is to be interpreted as 1 *if $r = p = 1$, $q = s$,* 0 *if $r = p = 1$, $q < s$ and similarly for the case $s = q = 1$.*

Proof: There are $\binom{r}{p}\binom{s}{q}$ subgraphs of $H_{r,s}$ isomorphic to $H_{p,q}$ for $0 \leq p \leq r$, $0 \leq q \leq s$. The ones with $p = 0$ or $q = 0$ have only zero eigenvalues. We proved in Example 1.3.3 that the nonzero eigenvalues of $H_{p,q}$, where p and q are positive, are $\sqrt{pq}$ and $-\sqrt{pq}$, each with multiplicity one. Therefore Corollary 2.2.4 gives the set of eigenvalues proposed as the nonzero eigenvalues of $D(H_{r,s})$. Morover, the multiplicity of $\sqrt{pq}$ and $-\sqrt{pq}$ contributed by the $H_{p,q}$ induced subgraphs is

$$m(p,q) = \binom{r}{p}\binom{s}{q}\det(L(H_{r,s})|_{[p]\cup[q]'} - I).$$

The above determinant equals

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A = (s-1)I_{(r-p)\times(r-p)}$, $D = (r-1)I_{(s-q)\times(s-q)}$, $B = -J_{(r-p)\times(s-q)}$, $C = -J_{(s-q)\times(r-p)}$ and J denotes a matrix with all entries equal to 1. This determinant can easily be shown to equal

$$(r-1)^{s-q-1}(s-1)^{r-p-1}(qr + ps - pq - r - s + 1)$$

using elementary row and column operations. This yields the suggested value of $m(p,q)$ and completes the proof of the proposition. □

Some further questions. The method of counting closed walks was successful in determining the adjacency eigenvalues of the Propp graphs $D(G)$. It does not seem to be strong enough to give other information about these matrices, such as their eigenvectors or the structure of their Jordan canonical forms. Since these matrices are not necessarily symmetric, in general they are not diagonalizable and hence they have nontrivial Jordan canonical forms. We would thus like to pose the problem of describing the eigenspaces of these matrices, as it was possible to do for the 0-eigenspace of a Laplacian, and their Jordan canonical forms, in terms of information about the original graph G. We have no indication that these questions have elegant answers.

The Jordan block structure of the Propp matrices $L(D(H_n))$ has been computed for $n = 2, 3, 4$ by A. Edelman [23]. For $n = 3$ the eigenvalue 3 has one 1×1 and two 2×2 Jordan blocks and for $n = 4$ the eigenvalue 4 has four 1×1, twelve 2×2 and three 3×3 Jordan blocks. The rest of the eigenvalues for these values of n were found to be semisimple.

Chapter 3

The Hanlon Graphs

In this chapter we are concerned with another interesting family of graphs, defined by Hanlon in [31]. These graphs, which we call the *Hanlon graphs*, arose in the study of the Laplacian operator on certain complexes associated to the Heisenberg Lie algebra. We consider the Hanlon graphs here because they are conjectured in [31, §1] to have very interesting spectral properties. Unlike the situation with the Propp graphs, Hanlon's eigenvalue conjectures remains largely untouched.

In this chapter we apply the method of counting closed walks to reduce an important special case of Hanlon's conjecture to a purely combinatorial statement. We believe that our arguments, although incomplete, help to some extent to understand why the Hanlon conjectures might be true.

3.1 Hanlon's eigenvalue conjecture

We begin with the relevant notation and background from [31].

The Heisenberg and related Lie algebras. Let $\mathcal{H}$ be the 3-dimensional Heisenberg Lie algebra. As a complex vector space, $\mathcal{H}$ has a basis $\{e, f, x\}$ with Lie brackets

$$[e, f] = x, \quad [e, x] = [f, x] = 0.$$

Now fix a nonnegative integer k and let $\mathcal{H}_k$ be the Lie algebra

$$\mathcal{H}_k = \mathcal{H} \otimes (\mathbb{C}[t] \,/\, (t^{k+1})),$$

where (t^{k+1}) is the principal ideal in $\mathbb{C}[t]$ generated by t^{k+1}. The Lie bracket in $\mathcal{H}_k$ is given by

$$[g \otimes p(t), h \otimes q(t)] = [g, h] \otimes p(t)\ q(t).$$

For $i \in [0,k] = \{0,1,\ldots,k\}$, let e_i, f_i and x_i denote $e \otimes t^i$, $f \otimes t^i$ and $x \otimes t^i$, respectively. These elements form the standard basis of $\mathcal{H}_k$, with the only nonzero brackets among them having the form

$$[e_i, f_j] = x_{i+j},$$

where $i + j \leq k$. Denote by E, F and X the subspaces spanned by the e_i, f_i and x_i, respectively. We then have

$$\mathcal{H}_k = E \oplus F \oplus X$$

and also

$$\Lambda\mathcal{H}_k = (\Lambda E) \wedge (\Lambda F) \wedge (\Lambda X),$$

where Λ stands for the exterior algebra. For a set of indices $I = \{i_1, i_2, \ldots, i_r\}$ with $0 \leq i_1 < i_2 < \cdots < i_r \leq k$, written shortly as $I = \{i_1, i_2, \ldots, i_r\}_<$, set $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r}$, and similarly for f_I and x_I. The elements

$$e_A \wedge f_B \wedge x_C,$$

where A, B and C range over all subsets of $[0,k]$, form a basis of $\Lambda\mathcal{H}_k$, which we call the *standard basis*. Denote by $V_k(a,b,c)$ the subspace $(\Lambda^a E) \wedge (\Lambda^b F) \wedge (\Lambda^c X)$ of $\Lambda\mathcal{H}_k$. Its standard basis consists of the elements $e_A \wedge f_B \wedge x_C$ satisfying $\#A = a$, $\#B = b$ and $\#C = c$.

Lie algebra homology and the Laplacian operator. Let $\partial : \Lambda\mathcal{H}_k \longrightarrow \Lambda\mathcal{H}_k$ be the *boundary map* defining the Koszul complex of $\mathcal{H}_k$. Thus ∂ is a linear map defined on elements of the standard basis by

$$\partial(m_1 \wedge m_2 \wedge \cdots \wedge m_r) = \sum_{1 \leq i < j \leq r} (-1)^{i+j-1} [m_i, m_j] m_1 \wedge \cdots \wedge \hat{m}_i \wedge \cdots \wedge \hat{m}_j \wedge \cdots \wedge m_r,$$

where $m \in \{e, f, x\}$. Then

$$\partial\partial = 0$$

and

$$H_*(\mathcal{H}_k) = \ker \partial \,/\, \operatorname{Im} \partial$$

is the (Lie algebra) homology of $\mathcal{H}_k$. Define the *Laplacian operator* $L : \Lambda\mathcal{H}_k \longrightarrow \Lambda\mathcal{H}_k$ to be

$$L = \partial\partial^* + \partial^*\partial,$$

where the adjoint of ∂ (the *coboundary map*) is taken with respect to the Hermitian form for which the standard basis of $\Lambda\mathcal{H}_k$ is orthonormal. This operator is not to be confused with the Laplacian matrix of a graph. It is proved in [36] that $\ker L$ and $H_*(\mathcal{H}_k; \mathbb{C})$ are isomorphic as graded vector spaces, so that the (graded) multiplicity of zero as an eigenvalue of L gives the dimensions of homology groups of $\mathcal{H}_k$. The grading on $\Lambda\mathcal{H}_k$ is obtained by assigning degree 1 to each nonzero element of $\mathcal{H}_k$, so that nonzero elements of $V_k(a,b,c)$ have degree $a+b+c$. The maps ∂ and ∂^* do not preserve this grading since

$$\partial : V_k(a,b,c) \longrightarrow V_k(a-1,b-1,c+1)$$

and

$$\partial^* : V_k(a-1,b-1,c+1) \longrightarrow V_k(a,b,c),$$

although L does. Each subspace $V_k(a,b,c)$ is therefore invariant under L. It is also clear that ∂ and ∂^* do preserve another grading of $\Lambda\mathcal{H}_k$, defined by assigning degree i to e_i, f_i and x_i for each i. With this grading, an element $e_A \wedge f_B \wedge x_C$ of $\Lambda\mathcal{H}_k$ has degree $\|A\| + \|B\| + \|C\|$, where $\|S\|$ stands for the sum of the elements of S. This quantity is called the *weight* of the triple (A,B,C).

The matrix representing the restriction of the Laplacian L to $V_k(a,b,c)$ with respect to the standard basis is the main object of study in [31]. This matrix is symmetric, since the Laplacian is a self-adjoint operator. The graph which corresponds to this matrix (see §1.1) is denoted in [31] by $G_k(a,b,c)$. The component $G_k(a,b,c;w)$ of $G_k(a,b,c)$ corresponds to the matrix representing the restriction of L to the homogeneous component of $V_k(a,b,c)$ of total weight w. Thus $G_k(a,b,c)$ is the disjoint union of the graphs $G_k(a,b,c;w)$, for all possible values of w.

In [31], Hanlon is primarily concerned with the families of graphs $G_k(a,b,0)$ and $G_k(a,b,0;w)$, which are denoted by $G_k(a,b)$ and $G_k(a,b;w)$ respectively, for simplicity. Let $\mu_k(a,b;r)$ be the multiplicity of r as an eigenvalue of $G_k(a,b)$. Let also

$$M_k(x,y,\lambda) = \sum_{a,b,r} \mu_k(a,b;r)\; x^a y^b \lambda^r$$

be the generating function for these multiplicities. Hanlon's remarkable conjecture can be stated as follows. We refer the interested reader to [31] for more information.

Conjecture 3.1.1 (Hanlon [31]) *The eigenvalues of $G_k(a,b)$ are all nonnegative integers. Moreover,*

$$M_k(x,y,\lambda) = \prod_{i=0}^{k} (1 + x + y + \lambda^{i+1} xy).$$

Hanlon determined explicitly the eigenvalues of $G_k(a,b;w)$ under certain restrictions on the parameters, the so called *stable case* [31, Thm. 2.5]. Since for most values of a, b and k, some values of w are not stable, the above conjecture remained unsettled for almost all cases. The note [1] contains a proof of Hanlon's conjecture in the following cases:

(*i*) $a = 1$, $b = 2$, arbitrary k (nonzero eigenvalues).

(*ii*) $a = 1$, arbitrary b and k (the zero eigenvalue).

A conjecture for more general nilpotent Lie algebras is stated in [31, §3]. In the next section we describe a slightly more general version of Conjecture 3.1.1 and its interpretation in terms of closed walks in a graph.

3.2 Closed walks in the Hanlon graphs

The graphs $G_k(a, b; w)$ are described in [31, §1]. We will consider here only the graphs $G_k(a, b)$ which are the relevant to Conjecture 3.1.1. This conjecture implies that their connected components $G_k(a, b; w)$ also have nonnegative integers as eigenvalues.

Combinatorial description of the Hanlon graphs. Note that $\partial^* = 0$ on $V_k(a, b, 0)$. Hence the restriction of the Laplacian L on $V_k(a, b, 0)$ has the form $\partial^*\partial$, where

$$\partial : V_k(a, b, 0) \longrightarrow V_k(a-1, b-1, 1)$$

and

$$\partial^* : V_k(a-1, b-1, 1) \longrightarrow V_k(a, b, 0).$$

The vertex set of $G_k(a, b)$ consists of all pairs (A, B) of subsets of $[0, k]$, satisfying $\#A = a$ and $\#B = b$. These pairs correspond to the elements $e_A \wedge f_B$ of the standard basis of $V_k(a, b, 0)$. The $G_k(a, b)$ are ordinary graphs with no multiple edges. Let (U, V) and (X, Y) be vertices of $G_k(a, b)$. There is an edge between these two vertices if there exist $u \in U$, $v \in V$ and $z \in \mathbb{Z}$ such that

(*i*) $X = (U - \{u\}) \cup \{u + z\}$,

(*ii*) $Y = (V - \{v\}) \cup \{v - z\}$,

(*iii*) $u + v \leq k$.

Clearly, this relation is symmetric in (U, V) and (X, Y). Let $U = \{u_1, u_2, \ldots, u_a\}_<$ and $V = \{v_1, v_2, \ldots, v_b\}_<$ and suppose that $u = u_i$ and $v = v_j$. To each triple (u, v, z), as above, corresponds naturally the sign $\epsilon_1\epsilon_2$, where

$$e_{u_1} \wedge \cdots \wedge e_{u_i+z} \wedge \cdots \wedge e_{u_a} = \epsilon_1 \; e_X$$

and

$$e_{v_1} \wedge \cdots \wedge e_{v_j-z} \wedge \cdots \wedge e_{v_b} = \epsilon_2 \; e_Y$$

in the exterior algebra. The weight of the edge between (U, V) and (X, Y) is the sum of the signs corresponding to all such triples (u, v, z). If $(U, V) \neq (X, Y)$ there is at most one such triple and the weight of a possible edge is ± 1. On the other hand we have $z = 0$, $\epsilon_1 = \epsilon_2 = 1$ if $X = U$, $Y = V$ and hence there is a loop of weight at most ab attached to some of the vertices of $G_k(a,b)$.

For example, the component $G_2(2,2;4)$ of the graph $G_2(2,2)$ has vertices $(01,12)$, $(02,02)$ and $(12,01)$. Its adjacency matrix is

$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

and its eigenvalues are 2, 2 and 5.

A generalization. We consider here a generalization of the graphs $G_k(a,b)$ and of Conjecture 3.1.1, due to Hanlon. Let's fix two variables η, θ. We can assume that they are complex variables. The new graphs depend also on the additional parameters a, b, k. We denote them by $\Gamma_k(a,b)$, supressing the dependence on η, θ. The vertex set of $\Gamma_k(a,b)$ is the same as that of $G_k(a,b)$, i.e. the set of pairs (A,B) of subsets of $[0,k]$, satisfying $\#A = a$ and $\#B = b$. There is an edge in $\Gamma_k(a,b)$ between (U,V) and (X,Y) if there exist $u \in U$, $v \in V$, $I \in X$, $J \in Y$ with

$$I + J \equiv u + v \pmod{k+1}$$

such that

(*i*) $X = (U - \{u\}) \cup \{I\}$,

(*ii*) $Y = (V - \{v\}) \cup \{J\}$.

The weight corresponding to such a quadruple (u,v,I,J) is the sign $\epsilon_1\epsilon_2$, calculated as before, possibly muliplied by η or θ or both. The sign is multiplied by η if $u+v > k$ and by θ if $I + J > k$. The weight of the edge between (U,V) and (X,Y) is the sum of the weights corresponding to all such (u,v,I,J).

As an example, consider the component $\Gamma_3(2,3;6)$ of $\Gamma_3(2,3)$, with vertices of weight 6. These vertices are $(01,023)$, $(02,013)$, $(03,012)$ and $(12,012)$. The adjacency matrix of $\Gamma_3(2,3;6)$, with the given ordering of the vertices, is

$$\begin{pmatrix} 5+\eta\theta & 1 & -\eta\theta & 1 \\ 1 & 5+\eta\theta & \eta\theta & 1 \\ -\eta\theta & \eta\theta & 4+2\eta\theta & 0 \\ 1 & 1 & 0 & 5+\eta\theta \end{pmatrix}.$$

Note that $\Gamma_k(a,b)$ reduces to $G_k(a,b)$ if $\eta = \theta = 0$. Let $\nu_k(a,b;r)$ be the multiplicity of r as an eigenvalue of $\Gamma_k(a,b)$ and also

$$N_k(x,y,\lambda,\rho) = \sum_{a,b,n,m} \nu_k(a,b;n+m\eta\theta)\; x^a y^b \lambda^n \rho^m.$$

The generalization of Hanlon's eigenvalue conjecture is as follows.

Conjecture 3.2.1 (Hanlon) *The eigenvalues of $\Gamma_k(a,b)$ are all of the form $n+m\eta\theta$, where n,m are nonnegative integers. Moreover,*

$$N_k(x,y,\lambda,\rho) = \prod_{i=0}^{k}(1+x+y+\lambda^{k+1-i}\rho^i\; xy). \tag{3.1}$$

Closed walks. We now consider the counting method of §1.3. The weighted number of closed l-walks in $\Gamma_k(a,b)$ has a purely combinatorial description. Thus, it is a combinatorial challenge to show that $w(\Gamma_k(a,b),l)$ equals the sum of the l powers of the eigenvalues $n+m\eta\theta$ with the multiplicities predicted by (3.1). To state this more precisely, let

$$F_{a,b,k}(t) = \sum_{l\geq 0} w(\Gamma_k(a,b),l)\; \frac{t^l}{l!}$$

be the exponential generating function of the weighted number of closed l-walks in $\Gamma_k(a,b)$. Since $w(\Gamma_k(a,b),l)$ is the sum of the l^{th} powers of the eigenvalues of $\Gamma_k(a,b)$, $F_{a,b,k}(t)$ is a sum of exponentials of the form $e^{\gamma t}$, where γ is such an eigenvalue. Conjecture 3.2.1 translates into the equation

$$\sum_{a,b=0}^{k} F_{a,b,k}(t)\; x^a y^b = \prod_{i=0}^{k}(1+x+y+e^{t(k+1-i+i\eta\theta)}\; xy), \tag{3.2}$$

where we have set $\rho = \lambda^{\eta\theta}$. The weights of the closed walks in $\Gamma_k(a,b)$ carry a $\pm$ sign. The general idea to approach (3.2) is to use an involution principle argument to cancel most of these weights and enumerate easily the rest. Let Γ_k be the disjoint union of the graphs $\Gamma_k(a,b)$ for all $a,b \in [0,k]$. The first part of Conjecture 3.2.1 would follow from the specialization $x=y=1$ of (3.2):

$$\sum_{l\geq 0} w(\Gamma_k,l)\; \frac{t^l}{l!} = \prod_{i=0}^{k}(3+e^{t(k+1-i+i\eta\theta)}). \tag{3.3}$$

Let us describe first in detail the closed l-walks in $\Gamma_k(a,b)$. It is easier to do this with an example. The following is a closed 4-walk in $\Gamma_3(2,3)$:

$$(0\hat{2},01\hat{3}) \longrightarrow (\hat{0}3,\hat{0}12) \longrightarrow (1\hat{3},3\hat{1}2) \longrightarrow (\hat{1}0,30\hat{2}) \longrightarrow (20,301). \tag{3.4}$$

If an edge has initial vertex (U, V), the $\hat{}$ sign is used to indicate the element of U or V which is subject to change and not a missing element. We think of the vertices of these walks as pairs of sequences of length a and b respectively. Each sequence has nonrepeated elements and the ones in the initial vertex are strictly increasing. The weight of the walk (3.4) has sign -1 since the signs of the permutation maps $02 \longrightarrow 20$ and $013 \longrightarrow 301$ are -1 and $+1$ respectively. The total weight of the walk is $-(\eta\theta)^2$ since the four edges are weighted, except for sign, by $\eta\theta$, θ, η and 1 respectively. As an immediate product of the counting method we get the following proposition.

Proposition 3.2.2 *Given k, a, b, the eigenvalues of $\Gamma_k(a, b)$ depend only on the product $\eta\theta$.*

Proof: In view of the discussion in §1.3, it suffices to show that any closed walk W in $\Gamma_k(a, b)$ has weight of the form $\pm(\eta\theta)^s$, for some nonnegative integer s. Consider the sum of the elements of A and B, where (A, B) is a vertex of W. An edge with weight ± 1 or $\pm\eta\theta$ leaves this sum unchanged when we move from the initial to the terminal vertex. An edge with weight $\pm\eta$ decreases the sum by $k+1$, while an edge with weight $\pm\theta$ increases the sum by $k+1$. Since the walk is closed, there are as many edges with weight $\pm\eta$ as edges with weight $\pm\theta$. $\square$

Nonreduced walks. We now consider a variation of the closed walks in $\Gamma_k(a, b)$. A *nonreduced* l-walk in $\Gamma_k(a, b)$ is an l-walk in $\Gamma_k(a, b)$ except that we allow multiple elements in the sets A, B of vertices (A, B) other than the initial vertex. A nonreduced walk is *closed* if its initial and terminal vertices coincide, except for the order within each coordinate. An example of a nonreduced closed 4-walk in $\Gamma_3(2, 3)$ is

$$(0\hat{2}, \hat{0}13) \longrightarrow (\hat{0}1, 1\hat{1}3) \longrightarrow (2\hat{1}, 13\hat{3}) \longrightarrow (2\hat{3}, \hat{1}31) \longrightarrow (20, 031). \tag{3.5}$$

Its weight is $(\eta\theta)^2$. Again, we think of the vertices of nonreduced walks as pairs of sequences with elements from $[0, k]$ and length a, b respectively. The sequences in the initial vertex have distinct elements, written in increasing order, but not necessarily the rest.

Nonreduced walks are somewhat simpler to work with, since we don't have to guarantee nonrepeated elements at each step. Let $\bar{w}(\Gamma_k(a, b), l)$ be the sum of the weights of all nonreduced closed l-walks in $\Gamma_k(a, b)$. For our purposes, we can replace closed walks in $\Gamma_k(a, b)$ with nonreduced closed walks, as the next proposition shows. We refer to U and V as the first and second coordinate respectively of a vertex (U, V).

Proposition 3.2.3 *For each $l \geq 0$ we have*

$$\bar{w}(\Gamma_k(a, b), l) = w(\Gamma_k(a, b), l).$$

Proof: We prove this by constructing a sign reversing involution ω on the set of nonreduced closed l-walks in $\Gamma_k(a,b)$. We can assume that $l > 0$. If W is a closed l-walk in $\Gamma_k(a,b)$ let $\omega(W) = W$. Suppose now that W is a nonreduced closed l-walk in $\Gamma_k(a,b)$ which is not a walk in the usual sense. Consider the first vertex (U, V) of W with an occurrence of repeated elements in U or in V. Suppose that repeated elements occur in $U = \{u_1, u_2, \ldots, u_a\}$, otherwise we work with V. Let i be the smallest index such that u_i reappears in U and let j be the smallest of the indices $s > i$ such that $u_s = u_i$. Let $\omega(W)$ be the nonreduced walk obtained by interchanging the i^{th} and j^{th} entries in the first coordinate of the vertices of W, starting with (U, V). This includes the ˆ sign. For instance, the new nonreduced walk produced from (3.5) is

$$(0\hat{2}, \hat{0}13) \longrightarrow (\hat{0}1, \hat{1}13) \longrightarrow (2\hat{1}, 31\hat{3}) \longrightarrow (2\hat{3}, 3\hat{1}1) \longrightarrow (20, 301).$$

Note that $\mathrm{wt}(\omega(W)) = -\mathrm{wt}(W)$. It is easy to see that ω defines an involution on the set of closed nonreduced l-walks in $\Gamma_k(a,b)$, fixing the ordinary walks. The result follows. $\square$

3.3 Special cases

We now consider the special case $a = 1$ and $b \geq 1$. The coefficient of xy^b in the product of (3.1) is

$$\binom{k}{b-1} \sum_{i=0}^{k} \lambda^{k+1-i} \rho^i + (k+1)\binom{k}{b}.$$

This means that the conjectured nonzero eigenvalues of $\Gamma_k(1,b)$ are $k + 1 - i + i\eta\theta$ for $i \in [0,k]$, each with multiplicity $\binom{k}{b-1}$. Equivalently, for $l > 0$ we should have

$$\bar{w}(\Gamma_k(1,b), l) = \binom{k}{b-1} \sum_{i=0}^{k} (k + 1 - i + i\eta\theta)^l. \tag{3.6}$$

This is quite easy to see for $b = 1$. Let W be a nonreduced l-walk in $\Gamma_k(1,1)$. Vertices are now pairs (u, v) of elements of $[0,k]$. The sum $j \equiv u+v \bmod k+1$, where $j \in [0,k]$, remains unchanged throughout W. The weight of an edge with initial vertex (u, v) is multiplied by η if $u > j$. By Proposition 3.2.2, the weight of W is $(\eta\theta)^s$ if s of the first l vertices (u, v) visited by W (not counting the terminal vertex) satisfy $u > j$. Since the independent choices $u \in [0,k]$ for the first l vertices uniquely determine W, it easily follows that

$$\bar{w}(\Gamma_k(1,1), l) = \sum_{j=0}^{k} (j + 1 + (k-j)\eta\theta)^l,$$

in agreement with (3.6).

Now let $b \geq 2$. Let W be a nonreduced walk in $\Gamma_k(1,b)$. Its vertices are pairs (u,V) where $u \in [0,k]$ and V is a sequence of elements of $[0,k]$ of length b. We call W *essential* if the $\hat{}$ sign appears in a single place in the second coordinate of the vertices of W. In other words, the vertices visited by W should have the form $(u, v_1 \ldots \hat{v}_i \ldots v_b)$, for some fixed $1 \leq i \leq b$. This implies that the entries other than the i^{th} one in the second coordinate of the vertices remain unchanged throughout W. We call the rest of the nonreduced walks in $\Gamma_k(1,b)$ *inessential*.

The essential nonreduced closed walks in $\Gamma_k(1,b)$ are the ones which are easy to count. The weight of such a walk W is the weight of the corresponding walk in $\Gamma_k(1,1)$, obtained by ignoring the entries in which the $\hat{}$ sign does not appear. Each closed nonreduced l-walk in $\Gamma_k(1,1)$ gives rise to $\binom{k}{b-1}$ such walks in $\Gamma_k(1,b)$. This is because by definition, the initial vertex (u,V) of a nonreduced walk has all the elements of V distinct.

It follows that, for $l > 0$, the weighted number of essential nonreduced closed l-walks in $\Gamma_k(1,b)$ is

$$\binom{k}{b-1} \sum_{i=0}^{k} (k+1-i+i\eta\theta)^l = \bar{w}(\Gamma_k(1,b),l).$$

Hence, Conjecture 3.2.1 in the case $a = 1$, $b \geq 2$ is equivalent to the following conjecture.

Conjecture 3.3.1 *For $l > 0$, the sum of the weights of all inessential nonreduced closed l-walks in $\Gamma_k(1,b)$ is zero.*

Conjecture 3.3.1 differs from 3.2.1 in the fact that it is stated in purely combinatorial terms. It seems to be suited for an involution principle argument, like Proposition 3.2.3. Unfortunately, we haven't been able to find this argument yet.

Remark. The involution which would prove Conjecture 3.3.1 is easy to find if $\eta\theta = 1$, which we assume for the rest of this section. Indeed, in this case, by Proposition 3.2.2, the weight of any closed nonreduced walk is ± 1. Let W be an inessential nonreduced closed walk. Consider the last two $\hat{}$ signs placed on the i^{th} and j^{th} entries in the second coordinate, for some $i \neq j$. Interchange the i^{th} and j^{th} entries after the last time the $\hat{}$ sign appears on them. This gives another walk with weight $-\mathrm{wt}(W)$ and defines the desired involution. For instance, if $b = k = 3$, the walk corresponding to

$$(\hat{2}, \hat{0}12) \longrightarrow (\hat{3}, 31\hat{2}) \longrightarrow (\hat{0}, 3\hat{1}1) \longrightarrow (\hat{1}, \hat{3}01) \longrightarrow (2, 201)$$

is

$$(\hat{2}, \hat{0}12) \longrightarrow (\hat{3}, 31\hat{2}) \longrightarrow (\hat{0}, 3\hat{1}1) \longrightarrow (\hat{3}, \hat{3}21) \longrightarrow (2, 021).$$

More generally, consider a nonreduced walk W in $\Gamma_k(a,b)$. Let $i \in [a]$ and $j \in [b]$. We say that i is *W-related* to j if for some vertex (U,V) of W, the $\hat{}$ sign appears on the i^{th} entry of U and the j^{th} entry of V. We say that W is essential if no $i \in [a]$ is W-related to more than one $j \in [b]$ and also if no two distinct $i_1, i_2 \in [a]$ are W-related to the same $j \in [b]$. Otherwise W is inessential.

It is easy to see that the involution described above extends easily and shows that the weights of the inessential nonreduced closed l-walks in $\Gamma_k(a,b)$ sum up to zero. This can be used to prove that, in the case $\eta\theta = 1$, the eigenvalues of $\Gamma_k(a,b)$ are all integers of the form $m(k+1)$, where $0 \leq m \leq k+1$. We will not give the details of this argument here since Hanlon's conjecture for $\eta = \theta = 1$ follows from the work of Kostant [36].

"There are no coincidences in Mathematics."

Gian-Carlo Rota

Part II

SUBSPACE ARRANGEMENTS

Chapter 4

Introduction to Arrangements

The work in this second part of the thesis was motivated by Sagan's expository paper [50]. In [50] the author surveys three methods that have been used in the past to show that the characteristic polynomials of certain graded lattices factor completely over the nonnegative integers. The first method considers the characteristic polynomial of certain subspace arrangements. The author gives the most general class of subspace arrangements known for which a combinatorial interpretation of the characteristic polynomial exists.

Our objective is to give a much more general theorem which provides such a combinatorial interpretation that is missing from [50]. The key idea to the problem is quite old. It is contained in a theorem of Crapo and Rota, related to the famous critical problem. We quote from [20, §16], where V_n stands for a vector space of dimension n over the finite field with q elements and S is a set of points in V_n not including the origin.

> Theorem 1. The number of linearly ordered sequences $(L_1, L_2, \ldots, L_k)$ of k linear functionals in V_n which distinguish the set S is given by $p(q^k)$, where $p(\nu)$ is the characteristic polynomial of the geometric lattice spanned by the set S.

The sequence $(L_1, L_2, \ldots, L_k)$ is said to distinguish the set S if for each $s \in S$ there exists at least one i for which $L_i(s) \neq 0$. Specialized to $k = 1$, this theorem expresses $p(q)$ as the number of linear functionals L satisfying $L(s) \neq 0$ for all $s \in S$. Thinking dually, we can replace each point of S with the hyperplane H_s which is "orthogonal" to s and passes through the origin. The geometric lattice spanned by the set S is the intersection lattice of the resulting central hyperplane arrangement. The previous statement is equivalent to saying that $p(q)$ counts the number of points in V_n, not in any of the hyperplanes H_s.

This is, in a special case, the theorem we will present. Specifically, let $\mathcal{A}$ be any subspace arrangement in $\mathbb{R}^n$ defined over the integers and let $\mathbb{F}_q$ denote the finite field with q elements. If q is a large prime, we prove that the characteristic polynomial $\chi(\mathcal{A}, q)$ of $\mathcal{A}$ counts the number of n-tuples $(x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n$ that do not lie in any of the subspaces of $\mathcal{A}$, viewed as subsets of $\mathbb{F}_q^n$.

Unfortunately, the Crapo-Rota theorem was overlooked for quite a long time in the later developement of the theory of hyperplane arrangements. For example, some of its immediate enumerative consequences for real central hyperplane arrangements were only derived much later and independently by Zaslavsky [71].

The purpose of the second part of this thesis is twofold. In the first place we generalize the theorem to affine subspace arrangements, which have appeared in the meantime, as stated above. Secondly, we show that even when restricted to the hyperplane case, this theorem provides a powerful enumerative tool which simplifies and extends enormously much of the work done in the past decade on the combinatorics of special classes of real hyperplane arrangements. As far as we know, this tool has not been of use in the past, althought it is stated for hyperplane arrangements over finite fields in the standard reference [41]. It is possible to transfer the Crapo-Rota idea to the real case simply because real arrangements defined by linear equations with integer coefficients can be reduced to arrangements over finite fields having the same intersection semilattice. A refinement of the method appears in Chapter 8. We will sometimes refer to this approach on the characteristic polynomial as the "finite field method."

Overview of Part II. In the rest of this chapter we include basic background and terminology about subspace arrangements. The main point of reference for Chapters 6-8 is Zaslavsky's theorem about the face numbers of a real hyperplane arrangement.

In Chapter 5 we briefly present previous work to explain our motivation. We prove the main theorem of this thesis about the characteristic polynomial of a rational subspace arrangement and give a few examples.

In Chapter 6 we provide simple proofs for the formulas giving the characteristic polynomials of the Shi arrangements and several generalizations. We also give a simple formula for the characteristic polynomial of the Linial arrangement, thus providing another proof of a theorem of Postnikov recently conjectured by Stanley, for its number of regions.

In Chapter 7 we give further applications to related arrangements.

In Chapter 8 we use the finite field method to extend our main result to the Whitney polynomial, a two variable generalization of the characteristic polynomial. Our approach gives a method to compute all face numbers of a rational hyperplane arrangement, or equivalently the f-vector of its dual complex. As an application, we compute the face numbers of some of the arrangements considered before, including the Shi arrangement of type A. We close with some directions for further research.

4.1 Subspace arrangements

We refer the reader to Stanley's book [60, §3] for any undefined terminology about posets.

A *subspace arrangement* $\mathcal{A}$ in $\mathbb{R}^n$ is a finite collection of proper affine subspaces of $\mathbb{R}^n$. If all the affine subspaces in $\mathcal{A}$ are hyperplanes, i.e. they have dimension $n-1$, then $\mathcal{A}$ is called a *hyperplane arrangement.* The theory of hyperplane arrangements has deep connections with areas of mathematics other than combinatorics, see for example [41]. A nice exposition for the more modern theory of subspace arrangements can be found in [5].

Here we will be concerned with subspace arrangements $\mathcal{A}$ in $\mathbb{R}^n$ that can be defined over the integers. This just means that every affine subspace of $\mathcal{A}$ is an intersection of hyperplanes of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = d, \tag{4.1}$$

where the a_i's and d are integers. We will also call these arrangements *rational* for obvious reasons. A subspace arrangement $\mathcal{A}$ is called *central* if all its subspaces are linear subspaces, i.e. they pass through the origin. We call $\mathcal{A}$ *centered* if its subspaces have nonvoid intersection and *centerless* otherwise. We will mostly focus on centerless hyperplane arrangements.

Examples. Figure 4.1 shows a hyperplane arrangement in $\mathbb{R}^2$, consisting of four lines in the plane. Classical examples of rational hyperplane arrangements in $\mathbb{R}^n$ are the arrangements $\mathcal{A}_n$, $\mathcal{B}_n$, $\mathcal{D}_n$, defined as

$$\begin{aligned}
\mathcal{A}_n &= \{x_i = x_j \mid 1 \le i < j \le n\},\\
\mathcal{D}_n &= \mathcal{A}_n \cup \{x_i = -x_j \mid 1 \le i < j \le n\},\\
\mathcal{B}_n &= \mathcal{D}_n \cup \{x_i = 0 \mid 1 \le i \le n\}.
\end{aligned}$$

They are the arrangements of reflecting hyperplanes corresponding to the finite Coxeter groups of type A_n, B_n and D_n respectively. A subspace arrangement that has received a lot of attention recently (see for example [5, §3] and [8, §6]) is the *k-equal* arrangement

$$\mathcal{A}_{n,k} = \{x_{i_1} = x_{i_2} = \cdots = x_{i_k} \mid 1 \le i_1 < i_2 < \cdots < i_k \le n\}.$$

The intersection semilattice. A fundamental combinatorial object associated to $\mathcal{A}$ is its intersection semilattice $L_{\mathcal{A}}$. It consists of all subspaces of $\mathbb{R}^n$ that can be written as the intersection of some of the subspaces of $\mathcal{A}$. The partial order on $L_{\mathcal{A}}$ is reverse inclusion. Thus the empty intersection, which is $\mathbb{R}^n$ itself, is the unique minimal element $\hat{0}$ of $L_{\mathcal{A}}$. The poset $L_{\mathcal{A}}$ is a meet-semilattice, i.e. any two elements

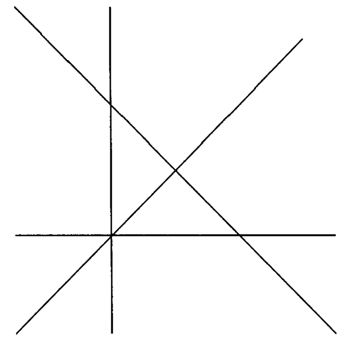

Figure 4.1: An arrangement of 4 lines in the plane

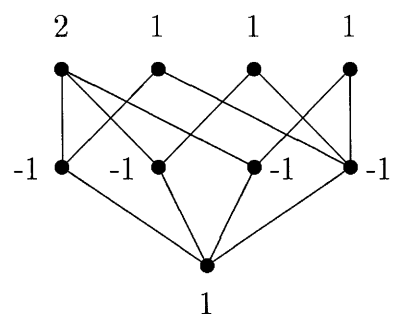

Figure 4.2: The intersection semilattice

have a greatest lower bound (meet). If $\mathcal{A}$ is centered, the intersection of its subspaces is the unique maximal element and $L_{\mathcal{A}}$ is actually a lattice.

For example, with the notation introduced above, $L_{\mathcal{A}_n}$ is the partition lattice Π_n and $L_{\mathcal{A}_{n,k}}$ is the lattice of set partitions of an n-element set with no blocks of size $2, \ldots, k-1$, ordered by refinement. If $\mathcal{A}$ and $\mathcal{B}$ are subspace arrangements in $\mathbb{R}^n$, we say that $\mathcal{A}$ is embedded in $\mathcal{B}$ if every subspace of $\mathcal{A}$ is the intersection of some of the subspaces of $\mathcal{B}$, i.e. $\mathcal{A} \subseteq L_{\mathcal{B}}$.

Figure 4.2 shows the intersection semilattice of the arrangement of Figure 4.1 and the values $\mu(\hat{0}, x)$ of its Möbius function, which we introduce next.

The Möbius function. Let P be any finite poset with a unique minimal element $\hat{0}$. The finiteness assumption can be weakened for the following definition, but we include it since we will only be interested in finite posets. The *Möbius function* μ of

P was defined recursively by Rota [48] for $x, y \in P$ with $x \leq y$ as follows:

$$\mu(x,y) = \begin{cases} 1 & \text{if } \ y = x; \\ -\sum_{x \leq z < y} \mu(x,z) & \text{if } \ y > x. \end{cases}$$

The following theorem is the famous Möbius inversion theorem of Rota [48] (see also [60, Thm. 3.7.1]).

Theorem 4.1.1 *Let $f, g : P \longrightarrow \mathbb{C}$ be functions. Then*

$$f(x) = \sum_{y \geq x} g(y) \quad \text{for all} \quad x \in P$$

if and only if

$$g(x) = \sum_{y \geq x} \mu(x,y) f(y) \quad \text{for all} \quad x \in P.$$

□

The characteristic polynomial. Our main object of study in the second part of the thesis is the *characteristic polynomial* $\chi(\mathcal{A}, q)$. The characteristic polynomial plays an important role in the combinatorial and topological aspects of the theory of arrangements. It is defined by

$$\chi(\mathcal{A}, q) = \sum_{x \in L_{\mathcal{A}}} \mu(\hat{0}, x) \, q^{\dim x},$$

where μ stands for the Möbius function of $L_{\mathcal{A}}$. The arrangement of Figure 4.1 has characteristic polynomial $q^2 - 4q + 5$. For the arrangements $\mathcal{A}_n$, $\mathcal{B}_n$ and $\mathcal{D}_n$ the characteristic polynomial is given by the well known formulas

$$\begin{aligned} \chi(\mathcal{A}_n, q) &= q(q-1)(q-2)\cdots(q-n+1), \\ \chi(\mathcal{B}_n, q) &= (q-1)(q-3)\cdots(q-2n+1), \\ \chi(\mathcal{D}_n, q) &= (q-1)(q-3)\cdots(q-2n+3)(q-n+1). \end{aligned}$$

When $\chi(\mathcal{A}, q)$ has q as a factor, we will use the notation

$$\tilde{\chi}(\mathcal{A}, q) = \frac{1}{q} \, \chi(\mathcal{A}, q).$$

For example, $\tilde{\chi}(\mathcal{A}_n, q) = (q-1)(q-2)\cdots(q-n+1)$. This is the characteristic polynomial of the arrangement formed by the intersections of the hyperplanes of $\mathcal{A}_n$ with the plane $H = \{x_1 + x_2 + \cdots + x_n = 0\}$. The ambient space of this arrangement is the $(n-1)$-dimensional space H.

Restriction of arrangements. Let $\mathcal{A}$ be any subspace arrangement in $\mathbb{R}^n$. For any $x \in L_{\mathcal{A}}$ we denote by $\mathcal{A}^x$ the arrangement whose elements are the proper subspaces of x obtained by intersecting the subspaces of $\mathcal{A}$ with x. This arrangement is called the *restriction* of $\mathcal{A}$ to x. Thus, the ambient Euclidean space for $\mathcal{A}^x$ is the space x and

$$\mathcal{A}^x = \{x \cap y \mid y \in \mathcal{A} \text{ and } x \cap y \neq \emptyset, x\}.$$

The intersection semilattice of $\mathcal{A}^x$ is the dual order ideal $\{z \in L_{\mathcal{A}} \mid z \geq x\}$ of $L_{\mathcal{A}}$ corresponding to x. Its characteristic polynomial is given by

$$\chi(\mathcal{A}^x, q) = \sum_{\substack{z \in L_{\mathcal{A}} \\ z \geq x}} \mu(x, z)\, q^{\dim z}.$$

4.2 Zaslavsky's theorem

We write $\bigcup \mathcal{A}$ for the set theoretic union of the elements of the collection $\mathcal{A}$. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$. We denote by $r(\mathcal{A})$ the number of regions of $\mathcal{A}$, that is the number of connected components of the space $M_{\mathcal{A}} = \mathbb{R}^n - \bigcup \mathcal{A}$. Similarly, we denote by $b(\mathcal{A})$ the number of *bounded* regions of $\mathcal{A}$. For the arrangement of Figure 4.1 we have $r = 10$ and $b = 2$. It will be evident in the following chapters that the quantities $r(\mathcal{A})$ and $b(\mathcal{A})$ are very often of great combinatorial significance. The following theorem, discovered by Zaslavsky [69, §2] provides a good reason to study the characteristic polynomial for hyperplane arrangements.

Theorem 4.2.1 **(Zaslavsky)** *For any hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$ we have*

$$r(\mathcal{A}) = (-1)^n \chi(\mathcal{A}, -1) = \sum_{x \in L_{\mathcal{A}}} |\mu(\hat{0}, x)|$$

and

$$b(\mathcal{A}) = (-1)^{\rho(L_{\mathcal{A}})} \chi(\mathcal{A}, 1) = \left| \sum_{x \in L_{\mathcal{A}}} \mu(\hat{0}, x) \right|,$$

where $\rho(L_{\mathcal{A}})$ denotes the rank (one less than the number of levels) of the intersection semilattice $L_{\mathcal{A}}$. □

For example, it is easy to see that $r(\mathcal{A}_n) = n!$, as predicted by Theorem 4.2.1 and the formula for $\chi(\mathcal{A}_n, q)$. Note also that the the characteristic polynomial $q^2 - 4q + 5$ of the arrangement of Figure 4.1 takes the "right" values 10 and 2 for $q = -1$ and $q = 1$ respectively. We will give many nontrivial applications of Zaslavsky's theorem in the following chapters. Zaslavsky gave a generalization of the first statement of Theorem 4.2.1 which we discuss next.

Face numbers and the Whitney polynomial. Let again $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$. The arrangement $\mathcal{A}$ defines a cellular decomposition of $\mathbb{R}^n$. These cells are the regions of the arrangements $\mathcal{A}^x$, where x ranges over the subspaces in $L_{\mathcal{A}}$. Let $0 \leq k \leq n$ be an integer. The cells of dimension k, which are the regions of $\mathcal{A}^x$ corresponding to all $x \in L_{\mathcal{A}}$ of dimension k, were called by Zaslavsky the *faces of dimension* k of the arrangement $\mathcal{A}$. Following [69, §2], we denote by $f_k(\mathcal{A})$ the number the k-dimensional faces of $\mathcal{A}$. Thus, $f_n(\mathcal{A}) = r(\mathcal{A})$. For the arrangement of Figure 4.1 we have $f_0 = 4$, $f_1 = 13$ and $f_2 = 10$. By Theorem 4.2.1, the number of k-dimensional faces of $\mathcal{A}$ is given by

$$f_k(\mathcal{A}) = \sum_{\dim x = k} (-1)^k \chi(\mathcal{A}^x, -1) = \sum_{\substack{\dim x = k \\ x \leq_{L_{\mathcal{A}}} z}} |\mu(x,z)|. \tag{4.2}$$

The formulas in (4.2) can be written in an elegant way in terms of the *Whitney polynomial* of $\mathcal{A}$, denoted by $w(\mathcal{A}, t, q)$. This polynomial was defined by Zaslavsky in [69, §1] for hyperplane arrangements and called the Möbius polynomial. It was further investigated by the same author in [71, §2] for hyperplane arrangements defined by signed graphs, i.e. arrangements contained in $\mathcal{B}_n$, and named the Whitney polynomial of the signed graph. We give the definition for an arbitrary subspace arrangement $\mathcal{A}$.

Definition 4.2.2 *The Whitney polynomial of $\mathcal{A}$ is the two variable polynomial*

$$\begin{aligned} w(\mathcal{A}, t, q) &= \sum_{x \leq_{L_{\mathcal{A}}} z} \mu(x,z)\, t^{n-\dim x} q^{\dim z} \\ &= \sum_{x \in L_{\mathcal{A}}} t^{n-\dim x} \chi(\mathcal{A}^x, q). \end{aligned}$$

Since $\dim x = n$ if and only if $x = \hat{0} = \mathbb{R}^n$, the Whitney polynomial $w(\mathcal{A}, t, q)$ specializes to the characteristic polynomial $\chi(\mathcal{A}, q)$ for $t = 0$. We refer to the polynomial

$$\sum_{i=0}^{n} f_i(\mathcal{A})\, t^{n-i}$$

as the *f-polynomial of* $\mathcal{A}$. The following theorem is also due to Zaslavsky [69, §2]. It is an immediate consequence of (4.2) and Definition 4.2.2.

Theorem 4.2.3 **(Zaslavsky)** *Let $\mathcal{A}$ be any hyperplane arrangement in $\mathbb{R}^n$. The f-polynomial of $\mathcal{A}$ satisfies*

$$\sum_{i=0}^{n} f_i(\mathcal{A})\, t^{n-i} = (-1)^n w(\mathcal{A}, -t, -1).$$

□

The arrangement of Figure 4.1 has Whitney polynomial

$$w(t,q) = 4t^2 + (4q-9)t + q^2 - 4q + 5$$

and $(-1)^2 w(-t,-1) = 4t^2 + 13t + 10$, as expected. We will use Theorem 4.2.3 in Chapter 8 to evaluate the face numbers of some interesting arrangements.

The dual complex. It is also common in the literature to consider the f-vector of the dual complex of $\mathcal{A}$, instead of its face numbers. Suppose first that $\mathcal{A}$ is central, with hyperplanes $a_i \cdot x = 0$ for $1 \le i \le N$. The *dual complex* of $\mathcal{A}$ is the zonotope

$$Z[\mathcal{A}] = \sum_{i=1}^{N} S_i,$$

where $S_i = \operatorname{conv}\{\pm a_i\}$ is the segment joining a_i and $-a_i$. For an exposition of the theory of zonotopes see [37]. The dual complex was considered by Zaslavsky in [69, §6] and also in [74] for the arrangements that correspond to a signed graph Σ. The vertices of the zonotope $Z[\mathcal{A}]$ correspond to the regions of the complement $\mathbb{R}^n - \bigcup \mathcal{A}$. In general, the $(n-k)$-dimensional faces of $Z[\mathcal{A}]$ correspond to the k-dimensional faces of $\mathcal{A}$.

If $\mathcal{A}$ is any hyperplane arrangement in $\mathbb{R}^n$, the dual complex $Z[\mathcal{A}]$ of $\mathcal{A}$ is a zonotopal complex, i.e. a polyhedral complex all of whose facets are zonotopes. Each facet of $Z[\mathcal{A}]$ is the zonotope corresponding to a maximal centered arrangement contained in $\mathcal{A}$. The correspondence of the faces of $Z[\mathcal{A}]$ and $\mathcal{A}$ described above carries through. In fact it is inclusion reversing, so that the face poset of $Z[\mathcal{A}]$ is the dual of the poset of closures of the faces of $\mathcal{A}$, ordered by inclusion. The number $f_{n-k}(Z[\mathcal{A}])$ of $(n-k)$-dimensional faces of $Z[\mathcal{A}]$ satisfies

$$f_{n-k}(Z[\mathcal{A}]) = f_k(\mathcal{A}) \tag{4.3}$$

and hence is given by (4.2). The equation in Theorem 4.2.3 can be restated as

$$\sum_{i=0}^{n} f_i(Z[\mathcal{A}])\, t^i = (-1)^n w(\mathcal{A}, -t, -1). \tag{4.4}$$

We also refer to f-polynomial of $\mathcal{A}$, which appears on the left, as the *f-polynomial of* $Z[\mathcal{A}]$ and to the vector

$$(f_0(Z[\mathcal{A}]), f_1(Z[\mathcal{A}]), \ldots, f_n(Z[\mathcal{A}]))$$

as the *f-vector of* $Z[\mathcal{A}]$.

A little topology. We note here that the numbers $r(\mathcal{A})$ and $b(\mathcal{A})$ also give information about the topology of the complexified hyperplane arrangement $\mathcal{A}^{\mathbb{C}}$. Indeed, let $M_{\mathcal{A}^{\mathbb{C}}} = \mathbb{C}^n - \bigcup \mathcal{A}$ be the complement in $\mathbb{C}^n$ of the union of the hyperplanes of $\mathcal{A}$, now viewed as subsets of $\mathbb{C}^n$. Let $\beta^i(M_{\mathcal{A}^{\mathbb{C}}})$ be the Betti numbers of $M_{\mathcal{A}^{\mathbb{C}}}$, i.e the ranks of the singular cohomology groups $H^i(M_{\mathcal{A}^{\mathbb{C}}})$. It follows from the work of Orlik and Solomon [40] and Theorem 4.2.1 that

$$r(\mathcal{A}) = \sum_{i \ge 0} \beta^i(M_{\mathcal{A}^{\mathbb{C}}})$$

and

$$b(\mathcal{A}) = \left| \sum_{i \geq 0} (-1)^i \, \beta^i(M_{\mathcal{A}^{\mathbb{C}}}) \right|.$$

We refer the reader to [5, §1] for more details. The following result generalizes the first statement of Theorem 4.2.1 and is much more difficult to prove (see [5, Thm. 7.3.1]).

Theorem 4.2.4 *For any subspace arrangement $\mathcal{A}$ in $\mathbb{R}^n$, the space $M_{\mathcal{A}} = \mathbb{R}^n - \bigcup \mathcal{A}$ has Euler characteristic*

$$(-1)^n \chi(\mathcal{A}, -1).$$

□

Chapter 5

Rational Arrangements

5.1 Previous work

At this point we briefly present previous work on the combinatorics of the characteristic polynomial of rational subspace arrangements. As noted before, it is unfortunate that this work proceeded independently of the "finite field" point of view of Crapo and Rota.

We commented in the previous chapter that there are nice product formulas for the characteristic polynomial of the Coxeter arrangements $\mathcal{A}_n$, $\mathcal{B}_n$, $\mathcal{D}_n$. In general, the characteristic polynomial of an arrangement seems to factor over the nonnegative integers much more often than random polynomials do. It was thus natural to ask if there exists a combinatorial interpretation of the characteristic polynomial that can explain this factorization phenomenon. Results in this direction constitute the first method described in [50]. The other two methods are the theory of free hyperplane arrangements [41, Ch. 4][65] and Stanley's factorization theorem for supersolvable lattices [57, Thm. 4.1] and its generalizations. Stanley's method is also combinatorial.

Graphical arrangements. The first combinatorial interpretation known was for the class of hyperplane arrangements contained in $\mathcal{A}_n$. Such an arrangement $\mathcal{G}$ is called *graphical* because its set of hyperplanes $x_i = x_j$ may be identified with the set of edges ij of a graph G with vertices $1, 2, \ldots, n$. The characteristic polynomial of such an arrangement is the chromatic polynomial of the corresponding graph. The value of this polynomial on a nonnegative integer q in the number of proper colorings of the vertices of G with the colors $1, 2, \ldots, q$, where a coloring is *proper* if no two vertices of the same color are joined with an edge in G. This explains the nice product formula for the characteristic polynomial of $\mathcal{A}_n$.

Signed graphs and the $\mathcal{B}_n$ arrangement. Zaslavsky generalized this fact about the chromatic polynomial with his theory of signed graph coloring [70][71][72].

He proved that the characteristic polynomial of any hyperplane arrangement $\mathcal{A} \subseteq \mathcal{B}_n$ is the "signed chromatic polynomial" of a certain "signed graph" associated to $\mathcal{A}$. This explains the product formulas for $\chi(\mathcal{B}_n, q)$ and $\chi(\mathcal{D}_n, q)$, mentioned in the previous chapter. Another interpretation as the chromatic polynomial of a certain hypergraph is implicit in [61, Thm. 3.4]. This theorem applies to subspace arrangements embedded in $\mathcal{A}_n$. It gives a simple proof of a theorem, first obtained by Björner and Lovaśz [7], which computes the characteristic polynomial of the k-equal arrangement in an exponential generating function form.

Blass and Sagan [11, Thm. 2.1][50, Thm. 2.2] generalized all previous results by giving a combinatorial interpretation to $\chi(\mathcal{A}, q)$ for any subspace arrangement $\mathcal{A}$ embedded in $\mathcal{B}_n$. They proved their result using a Möbius inversion argument by interpreting the quantity $t^{\dim x}$ as the cardinality of a set. As usual, we denote by $[a, b]$ the set of integers $\{a, a+1, \ldots, b\}$, for any integers a, b with $a \leq b$. We also use the notation $\#S$ for the cardinality of the finite set S, to avoid confusion with the absolute value symbol.

Theorem 5.1.1 ([11, Thm. 2.1][50, Thm. 2.2]) *If $\mathcal{A}$ is any subspace arrangement embedded in $\mathcal{B}_n$, then for any $q = 2s + 1$,*

$$\chi(\mathcal{A}, q) = \#([-s, s]^n - \bigcup \mathcal{A}).$$

□

To see why this result implies the previous ones, it suffices to think of a point $p \in [-s, s]^n$ as a coloring of the vertices $1, 2, \ldots, n$ with the q colors $-s, -s+1, \ldots, s$. The i^{th} coordinate of p is the color of vertex i. For a graphical arrangement $\mathcal{A}$, the condition $p \in [-s, s]^n - \bigcup \mathcal{A}$ is equivalent to the statement that p is proper.

In [11] the authors comment that Theorem 5.1.1 is the *only* combinatorial interpretation known for the characteristic polynomial of a class of subspace (as opposed to hyperplane) arrangements. It is our objective in the following section to show that the ideas of Crapo and Rota extend naturally and give a similar simple combinatorial interpretation for the whole class of rational subspace arrangements.

5.2 The finite field method

To motivate our result, we consider the arrangement

$$\mathcal{A}'_n = \mathcal{A}_n \cup \{x_1 - x_n = 1\},$$

obtained from $\mathcal{A}_n$ by adding the hyperplane $x_1 - x_n = 1$. This is a reflecting hyperplane, one corresponding to the highest root $e_1 - e_n$, of the infinite arrangement associated to the affine Weyl group of type A_{n-1}. A few computations suffice to conjecture that

$$\chi(\mathcal{A}'_n, q) = q\,(q-2)\prod_{i=2}^{n-1}(q-i). \tag{5.1}$$

However, Theorem 2.1 is not general enough to prove this innocent looking formula. Thus it is conceivable that a generalization of the result of Blass and Sagan exists in which the assumption that $\mathcal{A}$ is embedded in $\mathcal{B}_n$ is dropped. This is achieved by replacing the cube $[-s,s]^n$ with $\mathbb{F}_q^n$, where $\mathbb{F}_q$ stands for the finite field with q elements.

Convention. We point out here that we will only need q to be a prime number and not any power of a prime. For reasons of simplicity and to avoid any ambiguity with the notation "mod q," from now and on we restrict our attention to finite fields $\mathbb{F}_q$ for which q is a prime number.

Note that a subspace arrangement $\mathcal{A}$ in $\mathbb{R}^n$, defined over the integers, gives rise to an arrangement over the finite field $\mathbb{F}_q$. Indeed, any subspace K in $\mathcal{A}$ is the intersection of hyperplanes of the form (4.1). The corresponding subspace in $\mathbb{F}_q^n$ consists of all n-tuples $(x_1, x_2, \ldots, x_n)$ which satisfy the defining equations of K in $\mathbb{F}_q$. We will denote the arrangement in $\mathbb{F}_q^n$ corresponding to $\mathcal{A}$ by the same symbol, hoping that it will be clear from the context which of the two we are actually considering. Thus, the set $\mathbb{F}_q^n - \bigcup\mathcal{A}$ in the next theorem is the set of all $(x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n$ that do not satisfy in $\mathbb{F}_q$ the defining equations of any of the subspaces in $\mathcal{A}$. The Möbius inversion argument in the proof below is very similar to the one used originally by Crapo and Rota [20, §16] and later by Blass and Sagan [11]. We include it for the sake of completeness.

Theorem 5.2.1 *Let $\mathcal{A}$ be any subspace arrangement in $\mathbb{R}^n$ defined over the integers and q be a large enough prime number. Then*

$$\chi(\mathcal{A}, q) = \#(\mathbb{F}_q^n - \bigcup\mathcal{A}). \tag{5.2}$$

Equivalently, identifying $\mathbb{F}_q^n$ with $\{0, 1, \ldots, q-1\} = [0, q-1]^n$, $\chi(\mathcal{A}, q)$ is the number of points in $[0, q-1]^n$ which do not satisfy mod q the defining equations of any of the subspaces in $\mathcal{A}$.

Proof: Let $x \in L_{\mathcal{A}}$, the intersection semilattice of the real arrangement and let $\dim x$ be the dimension of x as an affine subspace of $\mathbb{R}^n$. The proof is based on the fact that $\#x = q^{\dim x}$, viewing x as a subspace of $\mathbb{F}_q^n$, provided that we choose the prime q to be sufficiently large. This is because the usual Gaussian elimination algorithm to solve a given linear system with integer coefficients works also over $\mathbb{F}_q$, if q is large. If not, x might reduce, for example, to the empty set or the whole space $\mathbb{F}_q^n$. We construct two functions $f, g : L_{\mathcal{A}} \longrightarrow \mathbb{Z}$ by

$$f(x) = \#x$$
$$g(x) = \#(x - \textstyle\bigcup_{y>x} y),$$

where all cardinalities are taken in $\mathbb{F}_q^n$. Thus $g(x)$ is the number of elements of x, not in any further intersection strictly contained in x. In particular, $g(\mathbb{R}^n) = \#(\mathbb{F}_q^n - \bigcup \mathcal{A})$. By our first remark above we have $f(x) = q^{\dim x}$. It is clear that $f(x) = \sum_{y \geq x} g(y)$, so by the Möbius inversion theorem [48][60, Thm. 3.7.1]

$$\#(\mathbb{F}_q^n - \bigcup \mathcal{A}) = g(\hat{0}) = \sum_{y \in L_{\mathcal{A}}} \mu(\hat{0}, y) f(y)$$

$$= \sum_{y \in L_{\mathcal{A}}} \mu(\hat{0}, y)\, q^{\dim y} = \chi(\mathcal{A}, q),$$

as desired. □

Zaslavsky notes in [71] the interpretability of his more general chromatic polynomial only for odd arguments. This corresponds to our assumption that q is a large enough prime. Of course, it suffices for Theorem 5.2.1 that q is a positive integer relatively prime to an integer depending only on the arrangement, once the field $\mathbb{F}_q$ is replaced by the abelian group of integers mod q.

Theoretically, Theorem 5.2.1 computes the characteristic polynomial only for large prime values of q. For specific arrangements though, when computed for such q, the right hand side of (5.2) will be a polynomial in q. Since $\chi(\mathcal{A}, q)$ is also a polynomial, the two polynomials will have to agree for all q. It is clear that our theorem is equivalent to the result of Blass and Sagan if $\mathcal{A}$ is embedded in $\mathcal{B}_n$ and hence that it implies all the specializations of Theorem 5.2.1 mentioned ealrier.

It was pointed out to us by Richard Stanley that in the special case of hyperplane arrangements, Theorem 2.2 also appeared as Theorem 2.69 in [41], stated again for hyperplane arrangements over finite fields. No consequences of the theorem for real arrangements seem to have been derived in [41] either. The generalization to subspace arrangements was obtained independently by Björner and Ekedahl in their recent work [6].

We also mention that a summation formula for the characteristic polynomial of an arbitrary hyperplane arrangement was recently found by Postnikov [45] (see also the comments in [62, §1]). This formula generalizes that of Whitney [68] which concerns $\chi(\mathcal{G}, q)$, where $\mathcal{G}$ is a graphical arrangement with associated graph G. Whitney's theorem states that

$$\chi(\mathcal{G}, q) = \sum_{S \subseteq E(G)} (-1)^{\#S} q^{c(S)},$$

where $E(G)$ denotes the set of edges of G, and $c(S)$ is the number of connected components of the spanning subgraph G_S of G with edge set S. Postnikov and

Stanley use this generalization to study classes of hyperplane arrangements, called *deformations of* $\mathcal{A}_n$. A deformation of $\mathcal{A}_n$ is an arrangement of the form

$$x_i - x_j = \alpha_{ij}^{(m)},\ 1 \leq m \leq m_{ij}, \tag{5.3}$$

where $\alpha_{ij}^{(m)}$ are arbitrary real numbers. As it turns out, Postnikov's generalization of the Whitney formula (at least for rational hyperplane arrangements) is related to Theorem 5.2.1 by the famous principle of inclusion-exclusion. Thus, more generally, one can derive from Theorem 5.2.1 an analogue of Whitney's formula, very often hard to work with, for any rational subspace (as opposed to hyperplane) arrangement.

It is easy to see that for deformations of $\mathcal{A}_n$, the interpretation of Theorem 5.2.1 is valid for all large enough positive integers q, once the field $\mathbb{F}_q$ is replaced by the abelian group of integers mod q. We will not need to make use of this fact at all.

An example. To get a first feeling of the applicability of Theorem 5.2.1 consider the hyperplane arrangement $\mathcal{A}_n'$ introduced in the beginning of this section. Theorem 5.2.1 is saying that for large prime numbers q, $\chi(\mathcal{A}_n', q)$ counts the number of n-tuples $(x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n$ for which

$$\begin{aligned} &x_i \neq x_j,\ 1 \leq i < j \leq n, \\ &x_1 - x_n \neq 1. \end{aligned}$$

There are q ways to choose x_1, then $q-2$ ways to choose x_n so that $x_n \neq x_1, x_1 - 1$, $q-2$ ways to choose x_2 so that $x_2 \neq x_1, x_n$ etc, and finally $q-n+1$ ways to choose x_{n-1}. This simple count proves (5.1). It follows from Theorem 1.1 that $r(\mathcal{A}_n') = \frac{3}{2}n!$ and $b(\mathcal{A}_n') = (n-2)!$, which can be easily seen otherwise. Note that the normal vectors to the hyperplanes of $\mathcal{A}_n'$ span an $(n-1)$-dimensional linear subspace of $\mathbb{R}^n$ and hence the term *bounded* regions has to be interpreted in an $(n-1)$-dimensional sense.

In Chapters 6-7 we will give a large number of examples of more complicated arrangements for which the finite field method is particularly elegant. We reserve for Chapter 8 a generalization of Theorem 5.2.1 which gives an interpretation to the Whitney polynomial of a rational subspace arrangement.

We also note that in view of the remark at the end of the introduction, Theorem 5.2.1 gives a combinatorial method to compute the Euler characteristic of the space $M_{\mathcal{A}} = \mathbb{R}^n - \bigcup \mathcal{A}$ for any rational subspace arrangement $\mathcal{A}$.

5.3 Coxeter hyperplane arrangements

We finally describe a less straightforward example which seems to deserve some mention. We observe that a simple and universal proof of a theorem about the characteristic polynomial of a Coxeter hyperplane arrangement can be derived from Theorem 5.2.1. This theorem is due to Blass and Sagan [11][50] and independently, in an equivalent form, due to Haiman [30].

We take this opportunity to introduce briefly terminology and notation related to Coxeter arrangements, to be kept throughout all of Part II. We follow Humphreys' exposition [34] and rely on it for basic background on Coxeter groups.

Let W be a finite Coxeter group, determined by an irreducible crystallographic root system Φ spanning $\mathbb{R}^n$. The hyperplanes which pass through the origin and are orthogonal to the roots define the *Coxeter arrangement* $\mathcal{W}$, associated to W. The reflections in these hyperplanes generate the group W. Let $Z(\Phi)$ be the *coweight lattice* associated to Φ, i.e. the set of vectors $x \in \mathbb{R}^n$ satisfying $(\alpha, x) \in \mathbb{Z}$ for all roots $\alpha \in \Phi$. By the term "lattice" here we mean a discrete subgroup of $\mathbb{R}^n$, not to be confused with a poset whose finite subsets have joins and meets.

For any positive real t we define

$$P_t(\Phi) = \{x \in Z(\Phi) \mid (\alpha, x) < t \text{ for all } \alpha \in \Phi\}.$$

We now fix a simple system

$$\Delta = \{\sigma_1, \ldots, \sigma_n\}$$

of Φ. This means that Δ is a basis of $\mathbb{R}^n$ and that any root $\alpha \in \Phi$ can be expressed as an integer linear combination

$$\alpha = \sum_{i=1}^{n} c_i(\alpha)\sigma_i,$$

where the coefficients $c_i(\alpha)$ are either all ≥ 0 or all ≤ 0. The *highest* root $\tilde{\alpha}$ is characterized by the conditions $c_i(\tilde{\alpha}) \geq c_i(\alpha)$ for all $\alpha \in \Phi$ and $1 \leq i \leq n$.

Finally, let $L(\Phi^\vee)$ denote the coroot lattice of Φ, i.e. the $\mathbb{Z}$-span of

$$\Phi^\vee = \{\frac{2\alpha}{(\alpha, \alpha)} \mid \alpha \in \Phi\}$$

in $\mathbb{R}^n$. Then the index of $L(\Phi^\vee)$ as a subgroup of $Z(\Phi)$ is called the *index of connection* of Φ and is denoted by f. We are now able to state the result in the language used by Blass and Sagan.

Theorem 5.3.1 ([11, Thm. 4.1][30, Thm. 7.4.2][50, Thm. 2.3]) *Let Φ be an irreducible crystallographic root system for a Weyl group W with associated Coxeter arrangement $\mathcal{W}$. Let t be a positive integer relatively prime to all the coefficients $c_i = c_i(\tilde{\alpha})$. Then*

$$\chi(\mathcal{W}, t) = \frac{1}{f}\ \#(P_t(\Phi) - \bigcup \mathcal{W}).$$

The outlined proof which appears in [11] is a case by case verification, which uses the classification theorem of finite Coxeter groups and some computer calculations for the case of the root systems E_6, E_7 and E_8. In both papers [11] and [50], the authors raise the question of finding a simpler, more conceptual proof of the theorem which works simultaneously for all root systems. We will show below how Theorem 5.2.1 can be used to obtain such a proof, after we introduce some more useful notation. This proof turns out to be closely related to an argument outlined by Haiman in [30] (see the remark following Theorem 7.4.2 in [30]).

Let W_a be the affine Weyl group associated to Φ and let $\mathcal{W}_a$ be the associated infinite hyperplane arrangement. Thus $\mathcal{W}_a$ is the set of hyperplanes of the form

$$(\alpha, x) = k,$$

where $\alpha \in \Phi$, $k \in \mathbb{Z}$, and W_a is the group generated by the reflections in these hyperplanes.

For $x \in \mathbb{R}^n$, let $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ be defined by

$$x_i^* = (x, \sigma_i).$$

In other words, x^* is the n-tuple of coordinates of x in the dual basis

$$\{\varpi_1^\vee, \varpi_2^\vee, \ldots, \varpi_n^\vee\}$$

to Δ, with respect to the standard inner product $(\ ,\ )$. Note that $x \in Z(\Phi)$ if and only if $x^* \in \mathbb{Z}^n$. Hence the map $x \longrightarrow x^*$ defines a vector space isomorphism of $\mathbb{R}^n$ under which the lattice $Z(\Phi)$ corresponds to $\mathbb{Z}^n$.

Proof of Theorem 5.3.1: Since

$$(\alpha, x) = (\sum_{i=1}^n c_i(\alpha)\sigma_i, x) = \sum_{i=1}^n c_i(\alpha) x_i^*,$$

the arrangement $\mathcal{W}$ corresponds, under the isomorphism, to an arrangement $\mathcal{W}'$ defined over the integers. Let t be a large prime. Theorem 5.2.1 implies that

$$\begin{aligned}\chi(\mathcal{W}, t) &= \#\{x^* \in [0, t-1]^n \ \mid \ (\alpha, x) \neq 0, \pm t, \pm 2t, \ldots \text{ for all } \alpha \in \Phi\} \\ &= \#\{x^* \in [0, t-1]^n \ \mid \ \frac{x}{t} \text{ is not in } \bigcup \mathcal{W}_a\} = \#(R_t - \bigcup \mathcal{W}_a),\end{aligned}$$

where

$$R_t = R \cap \frac{1}{t} Z(\Phi)$$

and R is the parallelepiped

$$\{\sum_{i=1}^n y_i \varpi_i^\vee \ \mid \ 0 \leq y_i \leq 1\}.$$

It is known that $R - \bigcup \mathcal{W}_a$ has $(\#W)/f$ connected components [34, pp. 99] and that W_a acts transitively on them. It is also clear that W_a preserves the points of R_t. It follows that each connected component of $R - \bigcup \mathcal{W}_a$ has the same number of points belonging to R_t. Hence, if

$$A_\circ = \{x \in \mathbb{R}^n \mid 0 < (\alpha, x) < 1 \text{ for all } \alpha \in \Phi\}$$

is the fundamental alcove of the affine Coxeter arrangement $\mathcal{W}_a$, then

$$\chi(\mathcal{W}, t) = \frac{\#W}{f} \#(A_\circ \cap \frac{1}{t} Z(\Phi))$$

$$= \frac{\#W}{f} \#\{x \in Z(\Phi) \mid 0 < (\alpha, x) < t \text{ for all } \alpha \in \Phi\}$$

$$= \frac{1}{f} \#(P_t(\Phi) - \bigcup \mathcal{W}).$$

The last equality follows from the fact that

$$\{x \in \mathbb{R}^n \mid (\alpha, x) < t \text{ for all } \alpha \in \Phi\} - \bigcup \mathcal{W}$$

has $\#W$ connected components and W acts simply transitively on them and preserves $Z(\Phi)$. This proves the result, at least for large primes t. Let $\mathbb{P}$ denote the set of positive integers. Note that, by the defining property of the highest root we have

$$\#\{x \in Z(\Phi) \mid 0 < (\alpha, x) < t \text{ for all } \alpha \in \Phi\} =$$

$$= \#\{x \in Z(\Phi) \mid 0 < (\alpha, x) \text{ for all } \alpha \in \Phi \text{ and } (\tilde{\alpha}, x) < t\} =$$

$$= \#\{x^* \in \mathbb{P}^n \mid \sum_{i=1}^{n} c_i(\alpha) x_i^* < t\},$$

which is the *Ehrhart quasi-polynomial* of the open simplex bounded by the coordinate hyperplanes and the hyperplane $\sum c_i x_i^* = 1$. This quasi-polynomial is a polynomial in t for t relatively prime to the c_i. Hence the expressions $\chi(\mathcal{W}, t)$ and $\frac{1}{f}\# \, (P_t(\Phi) - \bigcup \mathcal{W})$ agree for all such t. $\square$

Chapter 6

The Shi and Linial Arrangements

In this chapter we focus on two types of hyperplane arrangements, first introduced by Shi and Linial respectively, related to the affine Weyl groups [34, Ch. 4], as well as several variations. Theorem 5.2.1 applied to these arrangements leads to some interesting elementary counting problems. The solutions to these problems give several new results, easy proofs of results obtained in the past by much more complicated methods and suggest some vast generalizations.

Let Φ be an irreducible crystallographic root system spanning $\mathbb{R}^l$ with associated Weyl group W. We use the letter l instead of n for the rank of Φ, i.e. the dimension of the linear span of Φ, for reasons which will be apparent below. For any $\alpha \in \mathbb{R}^l$ and $k \in \mathbb{R}$, we denote by $H_{\alpha,k}$ the hyperplane defined by the linear equation

$$(\alpha, x) = k.$$

We recall from §5.3 that the arrangement $\mathcal{W}_a$, corresponding to the affine Weyl group W_a, is the collection of all $H_{\alpha,k}$ for $\alpha \in \Phi$ and integers k. In this and the next chapter we will be primarily concerned with finite arrangements contained in $\mathcal{W}_a$ for various Coxeter groups W.

6.1 The Shi arrangements

Fix a set of positive roots $\Phi^+ \subset \Phi$ once and for all. We define the Shi arrangement corresponding to Φ as the collection of hyperplanes

$$\{H_{\alpha,k} \mid \alpha \in \Phi^+ \text{ and } k = 0, 1\}.$$

We denote it by $\widehat{\mathcal{W}}$ except that, in order to be consistent with our earlier notation, we denote the Shi arrangement

$$x_i - x_j = 0,\ 1 \text{ for } 1 \leq i < j \leq n,$$

corresponding to the root system A_{n-1}, by $\widehat{\mathcal{A}}_n$, rather than $\widehat{\mathcal{A}}_{n-1}$. Thus, for the arrangement $\widehat{\mathcal{A}}_n$ we should keep in mind that $l = n-1$, although we consider it to be an arrangement in $\mathbb{R}^n$. This arrangement was denoted by $\mathcal{S}_n$ in [62]. We choose the notation $\widehat{\mathcal{A}}_n$ here to be consistent with other root systems. Figure 6.1 shows the intersection of $\widehat{\mathcal{A}}_3$ with the plane $x_1 + x_2 + x_3 = 0$. The hyperplanes in any deformation of $\mathcal{A}_n$ are orthogonal to $x_1 + x_2 + \cdots + x_n = 0$ and hence we lose nothing by restricting on this hyperplane.

We note that $\widehat{\mathcal{W}}$ depends on Φ and not only on the isomorphism class of the group W, as the notation might suggest. Thus $\widehat{\mathcal{B}}_n$ and $\widehat{\mathcal{C}}_n$ are different arrangements.

Shi [53] proved that the number of regions of $\widehat{\mathcal{A}}_n$ is

$$r(\widehat{\mathcal{A}}_n) = (n+1)^{n-1} \tag{6.1}$$

using group-theoretic techniques and later [54] generalized his result to show that

$$r(\widehat{\mathcal{W}}) = (h+1)^l,$$

where h is the *Coxeter number* of W [34, pp. 75]. Shi's proof is universal for all root systems, but still quite complicated. For an outline of a bijective proof of (6.1) and a refinement see the discussion in [62, §5]. Assuming Shi's result, Headley computed the characteristic polynomial of $\widehat{\mathcal{W}}$ as follows.

Theorem 6.1.1 ([32] [33, Ch. VI]) *Let Φ, W, $\widehat{\mathcal{W}}$ and l be as above. Let h be the Coxeter number of W. Then*

$$\chi(\widehat{\mathcal{W}}, q) = (q-h)^l.$$

□

Since $h = n$ for the Coxeter group A_{n-1}, it follows that the characteristic polynomial of $\widehat{\mathcal{A}}_n$ is $q(q-n)^{n-1}$. The extra factor of q corresponds to the fact that the dimension of the ambient Euclidean space of $\widehat{\mathcal{A}}_n$ exceeds by one the dimension of the corresponding space in Theorem 6.1.1.

Headley's proof was done case by case. Stanley [62, §5] noted for the arrangement $\widehat{\mathcal{A}}_n$ that Headley's proof can be simplified using an exponential generating function argument. This approach though, still has the disadvantage that it relies on Shi's result, whose proof is quite involved. Note that Theorem 5.2.1 implies the following for a general root system. The reasoning is the same with that in the proof of Theorem 5.3.1.

Corollary 6.1.2 *Let Φ and $\widehat{\mathcal{W}}$ be given, as before. For large primes q we have*

$$\chi(\widehat{\mathcal{W}}, q) =$$

$$= \#\{x^* \in [0, q-1]^n \mid (\alpha, x) \neq 0, 1 \pmod q \text{ for all } \alpha \in \Phi^+\},$$

where the notation is from Chapter 5. □

It would be highly desirable to obtain Theorem 6.1.1 directly from Corollary 6.1.2 (compare also with the remarks before Theorem 7.2.1). We believe that this should be possible to do, but we haven't found the correct argument yet.

In the following two sections we give a simple proof of Theorem 6.1.1, as well as several generalizations, for the case of the four infinite families of root systems. This enables us to obtain Shi's result as a corollary, via Theorem 1.1.

6.2 The Shi arrangement of type A

We first discuss the interpretation for $\chi(\mathcal{A}, q)$ given by Theorem 5.2.1 when $\mathcal{A}$ is a deformation of $\mathcal{A}_n$. Suppose that $\mathcal{A}$ is as in (5.3) and that the $\alpha_{ij}^{(m)}$ are integers. For large primes q, $\chi(\widehat{\mathcal{A}}_n, q)$ counts the number of n-tuples $(x_1, x_2, \dots, x_n) \in \mathbb{F}_q^n$ which satisfy conditions of the form

$$x_i - x_j \neq \alpha \tag{6.2}$$

in $\mathbb{F}_q$. We think of such an n-tuple as a map from $[n] = \{1, 2, \dots, n\}$ to $\mathbb{F}_q$, sending i to the class $x_i \in \mathbb{F}_q$. We think of the elements of $\mathbb{F}_q$ as boxes arranged and labeled cyclically with the classes mod q. The top box is labeled with the zero class, the clockwise next box is labeled with the class 1 mod q etc. The n-tuples in $\mathbb{F}_q^n$ become placements of the integers from 1 to n in the q boxes and $\chi(\mathcal{A}, q)$ counts the number of placements which satisfy certain restrictions prescribed by conditions (6.2). The restriction prescribed by $x_i - x_j \neq 0$ is that i and j are not allowed to be placed in the same box. In general, the restriction prescribed by (6.2) is that i cannot be placed in the box labeled with $x_j + \alpha$, where x_j is the label of the box that j occupies. In other words, i cannot follow j clockwise "by α boxes," if $\alpha > 0$ and cannot precede j clockwise by $-\alpha$ boxes," if $\alpha < 0$.

Since it is only the relative positions of the integers from 1 to n that matters, we can remove the labels from the boxes. We refer to the boxes in this case as "unlabeled," implying that they are indistinguishable. The placements that satisfy the restrictions prescribed by (6.2) are counted by $\tilde{\chi}(\mathcal{A}, q)$, where we have used the $\tilde{\chi}$ notation of §4.1. For each such placement we have q choices to decide where the zero class mod q will be, so that $\chi(\mathcal{A}, q) = q\tilde{\chi}(\mathcal{A}, q)$.

If $\mathcal{A}$ contains $\mathcal{A}_n$, then no two distinct integers are allowed to be placed in the same box, so we are counting placements without repetitions. When dealing with unlabeled boxes, we can disregard the occupied boxes in the placement. Thus $\tilde{\chi}(\mathcal{A}, q)$ simply counts the number of appropriate circular placements of the integers from 1 to n and

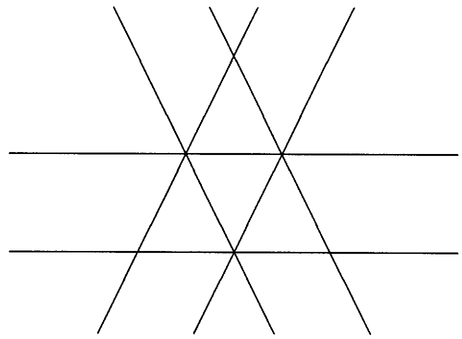

Figure 6.1: The Shi arrangement of type A_2

$q-n$ unlabelled boxes. The restriction imposed by the condition $x_i - x_j \neq \alpha$ is that i cannot follow j clockwise by α objects, where an object is either an integer or an unlabelled box.

We now explain some useful terminology about placements that we will be using in what follows whenever we consider deformations of $\mathcal{A}_n$. The boxes can be labeled or unlabeled. We say that two integers i and j are *consecutive* if one occupies the class a mod q and the other the class $a+1$ mod q for some a, i.e. if one follows the other clockwise with no other objects in between. We call the pair (i,j) a *descent* of the placement if $i<j$ and i,j are consecutive with i following j clockwise. This corresponds to the condition $x_i - x_j = 1$. We define ascents similarly. We say that i and j are *weakly consecutive* if one follows the other clockwise, except for unoccupied boxes in between. We define weak descents and weak ascents in an analogous way.

As a first application of these ideas we consider the Shi arrangement $\widehat{\mathcal{A}}_n$.

Theorem 6.2.1 *The characteristic polynomial of $\widehat{\mathcal{A}}_n$ is*

$$\chi(\widehat{\mathcal{A}}_n, q) = q(q-n)^{n-1}.$$

In particular, $r(\widehat{\mathcal{A}}_n) = (n+1)^{n-1}$ *and* $b(\widehat{\mathcal{A}}_n) = (n-1)^{n-1}$.

Proof: By the previous discussion, for large primes q, $\tilde{\chi}(\widehat{\mathcal{A}}_n, q)$ counts the number of circular placements of the integers from 1 to n and $q-n$ unlabeled boxes, such that no descent occurs. Equivalently, any string of consecutive integers must be clockwise increasing.

To count these placements, let's consider first the $q-n$ unlabeled boxes, placed around a circle. There are $(q-n)^{n-1}$ ways to place the elements of $[n]$ in the $q-n$

spaces between the boxes. Here we consider that there is only one way to place the first element of $[n]$ because of the cyclic symmetry of the arrangement of the $q-n$ unlabeled boxes. There is one way to order the elements placed in each space in clockwise increasing order. This gives the desired value for $\tilde{\chi}(\widehat{\mathcal{A}}_n, q)$. $\square$

A generalization. To generalize our previous result, let S be any subset of the edge set

$$E_n = \{ij \mid 1 \leq i < j \leq n\}$$

of the complete graph on n vertices. Here and for the rest of Part II, we denote the two element set $\{i, j\}$, where $i < j$, by ij for simplicity. Thus such a set S defines a simple graph on the vertex set $[n]$. To every $S \subseteq E_n$ we assign the hyperplane arrangement

$$\begin{aligned} x_i - x_j &= 0 \quad \text{for } 1 \leq i < j \leq n, \\ x_i - x_j &= 1 \quad \text{for } 1 \leq i < j \leq n,\ ij \in S \end{aligned}$$

and denote it by $\widehat{\mathcal{A}}_{n,S}$. The arrangement $\widehat{\mathcal{A}}_n$ corresponds to the complete graph and the arrangement $\mathcal{A}_n$ to the empty graph. In general, $\widehat{\mathcal{A}}_{n,S}$ interpolates between the two arrangements. The following generalization of Theorem 6.2.1 produces a new large class of hyperplane arrangements whose characteristic polynomials factor completely over the nonnegative integers. Compare also with the definition and corresponding property of chordal graphs [57, Example 4.6]. Recall that the notation $ij \in S$ implies that $i < j$.

Theorem 6.2.2 *Suppose that the set $S \subseteq E_n$ has the following property: if $ij \in S$, then $ik \in S$ for all $j < k \leq n$. Then*

$$\chi(\widehat{\mathcal{A}}_{n,S}, q) = q \prod_{1<j\leq n} (q - n + j - a_j - 1),$$

where $a_j = \#\{i < j \mid ij \in S\}$. In particular,

$$r(\widehat{\mathcal{A}}_{n,S}) = \prod_{1<j\leq n} (n - j + a_j + 2)$$

and

$$b(\widehat{\mathcal{A}}_{n,S}) = \prod_{1<j\leq n} (n - j + a_j).$$

Proof: The idea is similar to that in the proof of Theorem 6.2.1. By Theorem 5.2.1, $\tilde{\chi}(\widehat{\mathcal{A}}_{n,S}, q)$ counts the number of circular placements of the integers from 1 to n and $q-n$ unlabeled boxes, such that no "S-descent" occurs. An S-descent is a descent (i, j) of the placement with $ij \in S$ (hence $1 \leq i < j \leq n$).

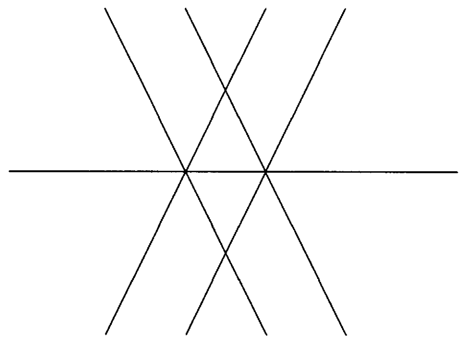

Figure 6.2: The arrangement of type A_2 for $S = \{12, 13\}$

We consider again the $q-n$ unlabeled boxes, placed around a circle. We will now enter the integers $1, 2, \ldots, n$ into the $q-n$ spaces between the boxes one by one, in the order indicated. We claim that, for each $j \geq 2$, after having inserted $1, \ldots, j-1$ to obtain a circular placement of the $q-n$ boxes and the first $j-1$ positive integers, there are $q-n+j-a_j-1$ ways to insert j. This is because there are $q-n+j-1$ spaces in all and the a_j spaces immediately before the a_j positive integers $i < j$ for which $ij \in S$ are forbidden. Indeed, if j is placed immediately before i, where $ij \in S$, then by construction, the element immediately preceding i in the final placement will be some $k > i$. This will produce an S-descent, since by the assumption on S, $ik \in S$. $\square$

As a simple example, let $n = 3$ and $S = \{12, 13\}$, which satisfies the condition of the previous theorem. The arrangement $\widehat{\mathcal{A}}_{n,S}$ has the hyperplanes

$$\begin{aligned} &x_i - x_j = 0 \quad \text{for } 1 \leq i < j \leq 3, \\ &x_1 - x_2 = 1, \\ &x_1 - x_3 = 1. \end{aligned}$$

Figure 6.2 shows the intersection of these hyperplanes with $x_1 + x_2 + x_3 = 0$. The integers a_2, a_3 both have the value 1. The characteristic polynomial factors as $q(q-2)(q-3)$ and yields 12 regions, 2 of which are bounded.

We now mention some corollaries of Theorem 6.2.2 which demonstrate its wide applicability.

Corollary 6.2.3 *Let $1 \leq k \leq n$ be an integer. The arrangement $\widehat{\mathcal{A}}_{n,S}$*

$$\begin{aligned} &x_i - x_j = 0 \quad \text{for } 1 \leq i < j \leq n, \\ &x_i - x_j = 1 \quad \text{for } 1 \leq i < j \leq k, \end{aligned}$$

corresponding to $S = \{ij \mid 1 \le i < j \le k\}$, has characteristic polynomial

$$\chi(\widehat{\mathcal{A}}_{n,S}, q) = q(q-k)^{k-1} \prod_{k \le j \le n-1} (q-j).$$

In particular, $r(\widehat{\mathcal{A}}_{n,S}) = \frac{n!}{k!}(k+1)^{k-1}$ *and* $b(\widehat{\mathcal{A}}_{n,S}) = \frac{(n-2)!}{(k-1)!}(k-1)^k$.

Proof: Clearly, the characteristic polynomial is the same with that of $\widehat{\mathcal{A}}_{n,T}$, where $T = \{ij \mid n-k+1 \le i < j \le n\}$. This choice of T satisfies the hypothesis of Theorem 6.2.2. We have $a_j = 0$ for $2 \le j \le n-k$ and $a_{n-k+j} = j-1$ for $1 \le j \le k$. The result follows from Theorem 6.2.2. □

For $k = 1$ and $k = n$ we obtain again the characteristic polynomials of $\mathcal{A}_n$ and $\widehat{\mathcal{A}}_n$ respectively.

Corollary 6.2.4 *Let $0 \le k \le n-1$ and $0 \le l \le n-k-1$ be integers. Let $S \subseteq E_n$ be*

$$S = \{ij \mid i < j,\ 1 \le i \le k\} \cup \{k+1j \mid n-l+1 \le j \le n\}.$$

Then

$$\chi(\widehat{\mathcal{A}}_{n,S}, q) = q\ (q-n)^{k-1}(q-k-l-1) \prod_{k+1 < j \le n} (q-j).$$

In particular,

$$r(\widehat{\mathcal{A}}_{n,S}) = \frac{(n+1)!}{(k+2)!}\ (k+l+2)\ (n+1)^{k-1}$$

and

$$b(\widehat{\mathcal{A}}_{n,S}) = \frac{(n-1)!}{k!}\ (k+l)\ (n-1)^{k-1}.$$

Proof: It follows directly from Theorem 6.2.2 since, for the given S, $a_j = j-1$ for $2 \le j \le k$, $a_{k+1} = \cdots = a_{n-l} = k$ and $a_{n-l+1} = \cdots = a_n = k+1$. □

For $k = n-1$, $l = 0$ the formulas check with Theorem 6.2.1 once more. For $k = 0$, $l = 1$ we get the result for the arrangement $\mathcal{A}'_n$ of Chapter 2. More generally, for $k = 0$ and any $0 \le l \le n-1$ we get the following specialization of Corollary 6.2.4.

Corollary 6.2.5 *The arrangement*

$$\begin{aligned} x_i - x_j &= 0 \quad \text{for } 1 \le i < j \le n, \\ x_1 - x_j &= 1 \quad \text{for } n-l+1 \le j \le n \end{aligned}$$

has characteristic polynomial

$$q\ (q-l-1) \prod_{1 < j \le n-1} (q-j).$$

In particular, for this arrangement $r = \frac{1}{2}(l+2)n!$ *and* $b = l(n-2)!$. $\square$

Finally, we mention separately the special case $l = 0$ of Corollary 6.2.4.

Corollary 6.2.6 *The arrangement*

$$\begin{aligned} x_i - x_j &= 0 \quad \text{for } 1 \le i < j \le n, \\ x_i - x_j &= 1 \quad \text{for } i < j,\ 1 \le i \le k \end{aligned}$$

has characteristic polynomial

$$q\,(q-n)^{k-1} \prod_{k<j\le n} (q-j).$$

In particular, for this arrangement

$$r = \frac{(n+1)!}{(k+1)!}\,(n+1)^{k-1}$$

and

$$b = \frac{(n-1)!}{(k-1)!}\,(n-1)^{k-1}.$$

$\square$

Some further generalizations. Considering also affine hyperplanes which correspond to negative roots, we obtain the following generalization of Theorem 6.2.2. Here, S is a subset of

$$\mathcal{E}_n = \{(i,j) \in [n] \times [n] \mid i \ne j\},$$

the edge set of the complete directed graph on n vertices, having no loops.

Theorem 6.2.7 *Suppose that the set* $S \subseteq \mathcal{E}_n$ *has the following properties:*

(i) *If* $i, j < k$, $i \ne j$ *and* $(i,j) \in S$, *then* $(i,k) \in S$ *or* $(k,j) \in S$.

(ii) *If* $i, j < k$, $i \ne j$ *and* $(i,k) \in S$, $(k,j) \in S$, *then* $(i,j) \in S$.

Then the characteristic polynomial of the arrangement

$$\begin{aligned} x_i - x_j &= 0 \quad \text{for } 1 \le i < j \le n, \\ x_i - x_j &= 1 \quad \text{for } (j,i) \in S \end{aligned}$$

factors as in Theorem 6.2.2, where

$$a_j = \#\{i < j \mid (j,i) \in S\} + \#\{i < j \mid (i,j) \in S\}.$$

Proof: The proof is similar to the one of Theorem 6.2.2. We want to compute the number of circular placements of the integers in $[n]$ and $q-n$ unlabeled boxes such that no two integers i,j are consecutive with j following i clockwise, if $(i,j) \in S$. We insert the integers $1,2,\ldots,n$ into the $q-n$ spaces between the $q-n$ boxes one by one, in the order indicated. Condition (i) guarantees that once a forbidden pattern ij is created when inserting i or j, at least one (possibly different) forbidden pattern will exist after inserting the rest of the integers. Condition (ii) guarantees that the a_j forbidden spaces to insert k are all distinct. □

In the spirit of Corollary 6.2.3 we give the following specialization of the previous theorem.

Corollary 6.2.8 *Let $1 \le k \le n$ be an integer. The arrangement*

$$\begin{aligned} x_i - x_j &= 0 \quad \text{for } 1 \le i < j \le n, \\ x_i - x_j &= 1 \quad \text{for } i \ne j,\ 1 \le i,j \le k, \end{aligned}$$

corresponding to $S = \{(i,j) \in \mathcal{E}_n \mid 1 \le i,j \le k\}$, has characteristic polynomial

$$q \prod_{j=k}^{n-1} (q-j) \prod_{k+1 \le j < 2k} (q-j).$$

In particular, for this arrangement we have

$$r = \frac{n!}{k+1} \binom{2k}{k}$$

and

$$b = (n-2)!(k-1)\binom{2k-2}{k-1}.$$

Proof: Clearly, the characteristic polynomial is the same with that of the arrangement corrsponding to the set $S = \{(i,j) \in \mathcal{E}_n \mid n-k+1 \le i,j \le n\}$. This choice of S satisfies the hypothesis of Theorem 6.2.7. We have $a_j = 0$ for $2 \le j \le n-k+1$ and $a_{n-k+j} = 2(j-1)$ for $2 \le j \le k$. The result follows from Theorem 6.2.7. □

Setting $k = 1$, we get $\chi(\mathcal{A}_n, q)$ once more. Setting $k = n$, or $S = \mathcal{E}_n$ in Theorem 6.2.7 we obtain the following corollary. A different generalization appears in Corollary 7.1.3.

Corollary 6.2.9 *The arrangement*

$$x_i - x_j = 0, 1 \text{ for } i \neq j,\ 1 \leq i, j \leq n,$$

has characteristic polynomial

$$q \prod_{j=n+1}^{2n-1} (q - j).$$

□

Lastly, the following generalization of Theorem 6.2.7. provides an even larger class of deformations of $\mathcal{A}_n$ whose characteristic polynomials factor completely over the nonnegative integers. The proof is as in Theorem 6.2.7 and is omitted.

Theorem 6.2.10 *Suppose that $S_1, S_2, \ldots$ are subsets of $\mathcal{E}_n$ with $S_m = \emptyset$ for sufficiently large m. We assume the following:*

(i) If $i, j < k$, $i \neq j$, $(i, j) \in S_m$ and $1 \leq r \leq m$, then $(i, k) \in S_r$ or $(k, j) \in S_{m-r+1}$.

(ii) If $i, j < k$, $i \neq j$, $(i, k) \in S_m$ and $(k, j) \in S_p$, then $(i, j) \in S_{m+p-1}$.

(iii) If $i, j < k$, $i \neq j$, $(k, i) \in S_m$ and $(k, j) \in S_p$ for some $m > p$, then $(j, i) \in S_{m-p}$.

(iv) If $i, j < k$, $i \neq j$, $(i, k) \in S_m$ and $(j, k) \in S_p$ for some $m > p$, then $(i, j) \in S_{m-p}$.

Then the characteristic polynomial of the arrangement

$$\begin{aligned} x_i - x_j &= 0 \text{ for } 1 \leq i < j \leq n, \\ x_i - x_j &= m \text{ for } (j, i) \in S_m \end{aligned}$$

factors as in Theorem 6.2.2, where

$$a_j = \sum_{m=1}^{\infty} \#\{i < j \mid (j, i) \in S_m\} + \sum_{m=1}^{\infty} \#\{i < j \mid (i, j) \in S_m\}.$$

□

We give again an example. Let $n = 3$, $S_1 = \{(2, 1), (3, 1), (2, 3)\}$, $S_2 = \{(2, 1)\}$ and $S_m = \emptyset$ for $m > 2$, which satisfy the conditions of the previous theorem. The resulting deformation of $\mathcal{A}_n$ has the hyperplanes

$$\begin{aligned} x_i - x_j &= 0 \text{ for } 1 \leq i < j \leq 3, \\ x_1 - x_2 &= 1, 2 \\ x_1 - x_3 &= 1, \\ x_3 - x_2 &= 1. \end{aligned}$$

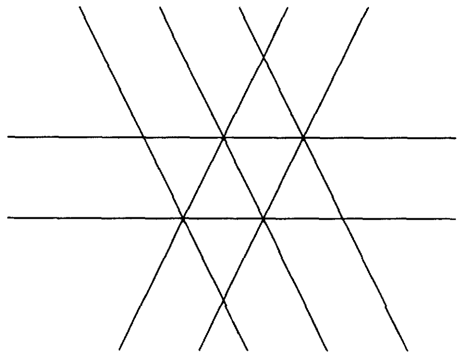

Figure 6.3: Another deformation of $\mathcal{A}_3$

Figure 6.3 shows this arrangement restricted on the plane $x_1 + x_2 + x_3 = 0$. The conditions of the Theorem 6.2.10 are satisfied and $a_2 = a_3 = 2$. The characteristic polynomial factors as $q(q-3)(q-4)$ and yields 20 regions, 6 of which are bounded.

Clearly, one can consider many interesting specializations of Theorem 6.2.10. We give one such below.

Corollary 6.2.11 *Fix an integer* $1 \le k \le n$. *The arrangement*

$$\begin{aligned} x_i - x_j &= 0, 1 \quad \text{for } i \neq j,\ 1 \le i, j \le n, \\ x_i - x_k &= 2 \quad \text{for } i \neq k, 1 \le i \le n \end{aligned}$$

has characteristic polynomial

$$q \prod_{j=n+2}^{2n} (q-j).$$

In particular, for this arrangement we have $r = \frac{(2n+1)!}{(n+2)!}$ *and* $b = \frac{(2n-1)!}{n!}$.

Proof: Clearly, we can assume $k = 1$ without changing the characteristic polynomial. The arrangement obtained for $k = 1$ corresponds to the choice $S_1 = \mathcal{E}_n$, $S_2 = \{(1,i) \mid i \in [2,n]\}$ and $S_m = \emptyset$ for $m > 2$, which satisfies the conditions of the Theorem 6.2.10. To get the result, note that $a_j = 2j - 1$ for all $2 \le j \le n$. $\square$

6.3 Shi arrangements for other root systems

Here we consider $\widehat{\mathcal{B}}_n$, $\widehat{\mathcal{C}}_n$, $\widehat{\mathcal{D}}_n$ and related arrangements. $\widehat{\mathcal{D}}_n$ is the arrangement

$$x_i - x_j = 0, 1 \quad \text{for } 1 \leq i < j \leq n,$$
$$x_i + x_j = 0, 1 \quad \text{for } 1 \leq i < j \leq n.$$

We first describe our general approach to solve the counting problem suggested by Theorem 5.2.1 for deformations of $\mathcal{B}_n$. The hyperplanes of such an arrangement can be one of

$$\begin{aligned} x_i &= \alpha, \\ x_i - x_j &= \beta, \\ x_i + x_j &= \gamma, \end{aligned}$$

where $1 \leq i \leq n$ and $1 \leq i < j \leq n$ respectively. Here α, β, γ will be rational numbers, usually integers. For sufficiently large primes q, we want to count the number of n-tuples $(x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n$ which satisfy the corresponding conditions

$$\begin{aligned} x_i &\neq \alpha, \\ x_i - x_j &\neq \beta, \\ x_i + x_j &\neq \gamma \end{aligned} \tag{6.3}$$

in $\mathbb{F}_q$. As with the deformations of $\mathcal{A}_n$, we think of the elements of $\mathbb{F}_q$ as boxes arranged and labeled cyclically with the classes mod q. The top box is labeled with the zero class, the clockwise next box is labeled with the class 1 mod q etc. We find it more convenient now to think of the n-tuple $(x_1, x_2, \ldots, x_n)$ as a map from $\pm[n] = \{\pm 1, \pm 2, \ldots, \pm n\}$ to $\mathbb{F}_q$, sending i to the class $x_i \in \mathbb{F}_q$ and $-i$ to the class $-x_i$. We call the elements of $\pm[n]$ the signed integers from 1 to n.

We think of these maps as placements of the signed integers from 1 to n in the q labeled boxes, as in the $\mathcal{A}_n$ case. When dealing with signed integers we always assume that i and $-i$ are placed "symmetrically," as described above. The characteristic polynomial counts the number of placements which satisfy certain restrictions prescribed by the conditions (6.3). We will use the following total order on $\pm[n]$:

$$1 \lhd 2 \lhd \cdots \lhd n \lhd -n \lhd \cdots \lhd -1. \tag{6.4}$$

Let k be a positive integer. A k-descent of a placement will be a pair of signed integers (i, j) with $i \neq \pm j$ and $i \lhd j$, such that j occupies the class a mod q and i occupies the class $a + k$, for some a. If $k = 1$, i.e. i occupies the class $a + 1$, we simply call (i, j) a descent of the placement.

A variation. For simplicity, we first consider the arrangement obtained from $\widehat{\mathcal{D}}_n$ by adding the hyperplanes $x_i = 0$. We denote this arrangement by $\widehat{\mathcal{D}}_n^\circ$. Figure 6.4 shows this arrangement for $n = 2$.

Theorem 6.3.1 *The characteristic polynomial of $\widehat{\mathcal{D}}_n^\circ$ is*

$$\chi(\widehat{\mathcal{D}}_n^\circ, q) = (q - 2n + 1)^n.$$

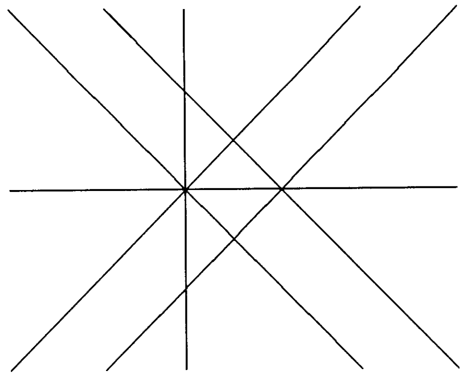

Figure 6.4: The arrangement $\widehat{\mathcal{D}}_2^\circ$

In particular, $r(\widehat{\mathcal{D}}_n^\circ) = (2n)^n$ *and* $b(\widehat{\mathcal{D}}_n^\circ) = (2n-2)^n$.

Proof: By Theorem 5.2.1 and the discussion above, for large primes q, $\chi(\widehat{\mathcal{D}}_n^\circ, q)$ counts the number of placements of the signed integers from 1 to n in the q labeled boxes, subject to the restrictions imposed by the conditions

$$\begin{aligned} &x_i \neq 0 \quad \text{for } 1 \leq i \leq n, \\ &x_i - x_j \neq 0, 1 \quad \text{for } 1 \leq i < j \leq n, \\ &x_i + x_j \neq 0, 1 \quad \text{for } 1 \leq i < j \leq n. \end{aligned}$$

The conditions $x_i \neq 0$, $x_i \pm x_j \neq 0$ impose the restriction that no signed integer is sent to the zero class and no two distinct signed integers are sent to the same class. The conditions $x_i - x_j \neq 1$ for $1 \leq i < j \leq n$ imply that no $i < j$ can be placed to the class $x_j + 1$, i.e. immediately after j clockwise and that no $-i$, where $i < j$, can be placed to the class $-x_j - 1$, i.e. immediately before $-j$. The condition $x_i + x_j \neq 1$ for $i \neq j$ implies that no $-i$, with $i \neq j$, can be placed to the class $x_j - 1$, i.e. immediately before j. In other words, no negative integer $-i$ can immediately precede a positive one j with $i \neq j$. By the definition (6.4) of the order $\lhd$, overall we require that our placement has no repetitions, no descents and that the top box, labeled with the zero class, is unoccupied.

We can now concentrate on what happens only on the right half of the circle, i.e. the classes from 0 to $\frac{1}{2}(q-1)$, included. Indeed, if a signed integer is placed in one of these classes, say a, then its negative is placed in the class $-a$ and an allowable placement on the right half gives an allowable placement on the left half. For each pair $(i, -i)$, where $i \in [n]$, exactly one of $i, -i$ should appear in the right semicircle. So we are looking for the number of placements of the elements of $[n]$ in the $\frac{1}{2}(q-1)$ boxes on this semicircle, each element with a $\pm$ sign, subject to the restrictions in the previous paragraph.

We note again that these restrictions prescribe the clockwise order in any string of consecutive signed integers placed on the semicircle. This order should be compatible with $\lhd$, for example $2, 5, 7, -6, -3$. Now there are $\frac{1}{2}(q+1) - n$ boxes which will be unoccupied in the end, starting with the top box labeled with the zero class, and $\frac{1}{2}(q+1) - n$ spaces, starting with the space to the right of the top box. There are $q - 2n + 1$ choices to place each element of $[n]$, which is twice the number of possible spaces, accounting for the freedom to choose one of two possible signs. Hence, there are $(q - 2n + 1)^n$ placements in all. $\square$

As in the case with $\widehat{\mathcal{A}}_n$, we can extend the previous argument to get a vast generalization of Theorem 6.3.1. We denote by $\widehat{\mathcal{D}}^\circ_{n,S,T}$ the arrangement

$$\begin{aligned} &x_i = 0 \ \text{ for } 1 \leq i \leq n, \\ &x_i \pm x_j = 0 \ \text{ for } 1 \leq i < j \leq n, \\ &x_i - x_j = 1 \ \text{ for } 1 \leq i < j \leq n, \ ij \in S, \\ &x_i + x_j = 1 \ \text{ for } 1 \leq i < j \leq n, \ ij \in T, \end{aligned}$$

where $S, T \subseteq E_n$, the edge set of the complete graph on the vertex set $[n]$. This arrangement interpolates between $\mathcal{B}_n$ and $\widehat{\mathcal{D}}^\circ_n$.

Theorem 6.3.2 *Suppose that the sets $S, T \subseteq E_n$ have the following properties:*

(i) If $ij \in S$ and $i < j < k$, then $ik \in S \cap T$.

(ii) If $ij \in T$ and $i < j < k$, then $ik \in S$ or $jk \in T$ and also, $ik \in T$ or $jk \in S$.

(iii) If $ik \in S$, $jk \in T$ and $i < j < k$, then $ij \in T$ and similarly, if $jk \in S$, $ik \in T$ and $i < j < k$, then $ij \in T$.

Then

$$\chi(\widehat{\mathcal{D}}^\circ_{n,S,T}, q) = \prod_{j=1}^{n} (q - 2n + 2j - 1 - a_j - b_j),$$

where $a_j = \#\{i < j \mid ij \in S\}$ and $b_j = \#\{i < j \mid ij \in T\}$. In particular,

$$r(\widehat{\mathcal{D}}^\circ_{n,S,T}) = \prod_{j=1}^{n} (2n - 2j + a_j + b_j + 2)$$

and

$$b(\widehat{\mathcal{D}}^\circ_{n,S,T}) = \prod_{j=1}^{n} (2n - 2j + a_j + b_j).$$

Proof: We follow the argument in the proof of Theorem 6.3.1. The conditions $x_i \pm x_j \neq 1$ are now replaced by $x_i - x_j \neq 1$ if $ij \in S$ and $x_i + x_j \neq 1$ if $ij \in T$.

To count the corresponding number of placements of signed integers and unlabeled boxes on a semicircle, we insert the integers $1, 2, \ldots, n$ in this order, each with a sign, in the $\frac{1}{2}(q+1) - n$ spaces between the boxes. These spaces include the one to the right of the last box. We claim that we have $q - 2n + 2j - 1 - a_j - b_j$ choices to insert j. This is because there are $\frac{1}{2}(q+1) - n + j - 1$ spaces between the empty boxes and the first $j - 1$ integers already inserted, and we have two choices for the sign of j, giving a total of $q - 2n + 2j - 1$ choices. Some of these choices though are forbidden. For each $i < j$ with $ij \in S$, we should avoid the patterns ji and $-i-j$, due to the restriction imposed by $x_i - x_j \neq 1$. This accounts for a_j choices. The condition $x_i + x_j \neq 1$ implies that we should avoid the patterns $-ji$ and $-ij$ and excludes b_j more possibilities. Assumption (iii) guarantees that the forbidden choices to insert j are distinct. Assumptions (i) and (ii) ensure that, once we create a forbidden pattern when inserting j, a (possibly different) forbidden pattern will still exist after we have inserted the rest of the integers. □

For $S = T = E_n$, the previous theorem reduces to Theorem 6.3.1. Note that for $S = \emptyset$, $T = \{n-1n\}$, $\widehat{\mathcal{D}}^{\circ}_{n,S,T}$ has the same intersection lattice with the arrangement obtained from $\mathcal{B}_n$ by adding the hyperplane $x_1 + x_2 = 1$. This is the B_n analogue of the arrangement $\mathcal{A}'_n$, considered early in Chapter 2, since $e_1 + e_2$ is the highest root in B_n. Theorem 6.3.2 implies that the characteristic polynomial of this arrangement is $(q-2)(q-3)(q-5)\cdots(q-2n+1)$. To give a more general example, we mention the following two specializations. The proof of the second is completely analogous to the proof of the first.

Corollary 6.3.3 *Let $1 \leq k \leq n$ be an integer. The arrangement*

$$\begin{aligned} &x_i = 0 \ \text{ for } 1 \leq i \leq n, \\ &x_i \pm x_j = 0 \ \text{ for } 1 \leq i < j \leq n, \\ &x_i + x_j = 1 \ \text{ for } 1 \leq i < j \leq k \end{aligned}$$

has characteristic polynomial

$$\prod_{j=k}^{n-1}(q - 2j - 1) \prod_{j=k}^{2k-1}(q - j).$$

In particular, for this arrangement

$$r = 2^{n-k} n! \binom{2k}{k}$$

and

$$b = 2^{n-k}(n-1)!\,(k-1)\binom{2k-2}{k-1}.$$

Proof: This arrangement has the same characteristic polynomial with $\widehat{\mathcal{D}}^{\circ}_{n,S,T}$ for $S = \emptyset$, $T = \{ij \mid n-k+1 \leq i < j \leq n\}$. These sets satisfy the conditions of Theorem 6.3.2 and yield the values $a_j = 0$ for $1 \leq j \leq n$, $b_1 = \cdots = b_{n-k} = 0$, $b_{n-k+j} = j-1$ for $1 \leq j \leq k$. The result follows. □

Corollary 6.3.4 *Let $1 \leq k \leq n$ be an integer. The arrangement*

$$\begin{aligned} &x_i = 0 \ \text{ for } 1 \leq i \leq n, \\ &x_i \pm x_j = 0 \ \text{ for } 1 \leq i < j \leq n, \\ &x_i \pm x_j = 1 \ \text{ for } 1 \leq i < j \leq k \end{aligned}$$

has characteristic polynomial

$$(q-2k+1)^k \prod_{j=k}^{n-1} (q-2j-1).$$

In particular, for this arrangement

$$r = 2^n \, \frac{n!}{k!} \, k^k$$

and

$$b = 2^n \, \frac{(n-1)!}{(k-1)!} \, (k-1)^k.$$

□

The other infinite families of root systems. Note that $\mathcal{B}_n$ and $\mathcal{C}_n$ coincide over a field of characteristic different from 2. This does not happen for $\widehat{\mathcal{B}}_n$ and $\widehat{\mathcal{C}}_n$, so we consider them seperately. The arrangements $\widehat{\mathcal{B}}_n$, $\widehat{\mathcal{C}}_n$ are obtained from $\widehat{\mathcal{D}}_n$, mentioned before Theorem 6.3.1, by adding the hyperplanes $x_i = 0, 1$ for $1 \leq i \leq n$ and $2x_i = 0, 1$ for $1 \leq i \leq n$, respectively. We also consider the Shi arrangement $\widehat{\mathcal{BC}}_n = \widehat{\mathcal{B}}_n \cup \widehat{\mathcal{C}}_n$, which corresponds to the *nonreduced* system $BC_n = B_n \cup C_n$. For $n = 2$ these four arrangements are shown in Figures 6.5 and 6.6. The following theorem computes the characteristic polynomials of $\widehat{\mathcal{B}}_n$, $\widehat{\mathcal{C}}_n$, $\widehat{\mathcal{D}}_n$ and $\widehat{\mathcal{BC}}_n$. It verifies Theorem 6.2.1, since the Weyl groups B_n, C_n and D_n have Coxeter numbers $2n$, $2n$ and $2n-2$ respectively and suggests that the Coxeter number of BC_n should be $2n+1$.

Theorem 6.3.5 *We have*

$$\chi(\widehat{\mathcal{W}}, q) = \begin{cases} (q-2n)^n, & \text{if } \Phi = B_n \text{ or } C_n; \\ (q-2n+2)^n, & \text{if } \Phi = D_n \end{cases}$$

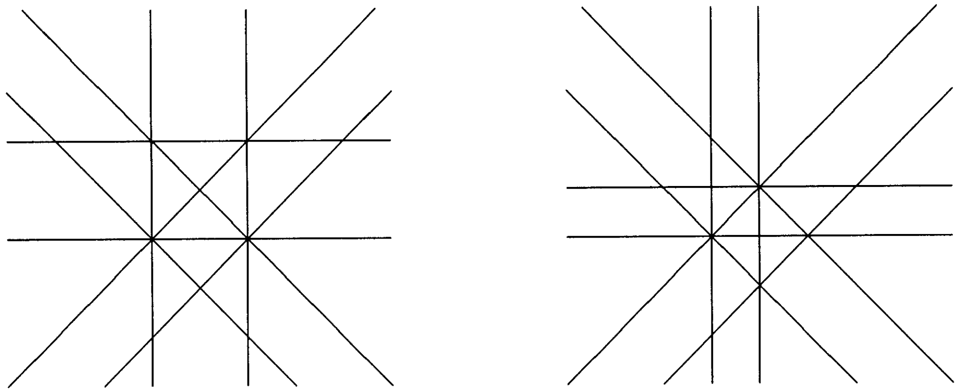

Figure 6.5: The arrangements $\widehat{\mathcal{B}}_2$ and $\widehat{\mathcal{C}}_2$

and also

$$\chi(\widehat{\mathcal{BC}}_n, q) = (q-2n-1)^n.$$

In particular, $r(\widehat{\mathcal{B}}_n) = r(\widehat{\mathcal{C}}_n) = (2n+1)^n$, $b(\widehat{\mathcal{B}}_n) = b(\widehat{\mathcal{C}}_n) = r(\widehat{\mathcal{D}}_n) = (2n-1)^n$, $b(\widehat{\mathcal{D}}_n) = (2n-3)^n$, $r(\widehat{\mathcal{BC}}_n) = (2n+2)^n$ *and* $b(\widehat{\mathcal{BC}}_n) = (2n)^n$.

Proof: We modify the argument in the proof of Theorem 6.3.1. If $\Phi = B_n$, we have the extra conditions $x_i \neq 1$. This means that the class 1 mod q should either be empty or occupied by a negative integer $-i$, in case some $x_i = -1$. Hence, by the restrictions on the order of consecutive integers, the space immediately following the zero class should contain only negative integers and their order is prescribed. The choice of sign is arbitrary for the remaining $\frac{1}{2}(q-1)-n$ spaces. Thus, for each $i \in [n]$, we have $1 + 2\left(\frac{q-1}{2} - n\right) = q - 2n$ choices to place i and hence $(q-2n)^n$ placements in all.

If $\Phi = C_n$, we have the extra conditions $2x_i \neq 1$, i.e. $x_i \neq \frac{1}{2}(q+1)$, or $-x_i \neq \frac{1}{2}(q-1)$. So now the last class $\frac{1}{2}(q-1)$ in the right semicircle should either be empty or occupied by a positive integer, which implies that we are forced to choose the positive sign when inserting integers in the last space. The rest of the reasoning is as before.

If $\Phi = BC_n$, we have both type of conditions $x_i \neq 1$ and $2x_i \neq 1$. We can only insert negative integers in the first space and positive in the last. Thus we have $2\left(\frac{q-1}{2} - n\right) = q - 2n - 1$ choices for each $i \in [n]$.

If $\Phi = D_n$, the conditions $x_i \neq 0$ are missing. We have now one more allowable space between the $\frac{1}{2}(q+1)-n$ unlabeled boxes, namely the one immediately to the left of the first box. This space will be nonempty if $x_i = 0$ for some i. In this case $-x_i = 0$, so i and $-i$ are both placed in the zero class and the rest of the integers (if any) in the first space should be negative. Thus, sign and order are prescribed for placing

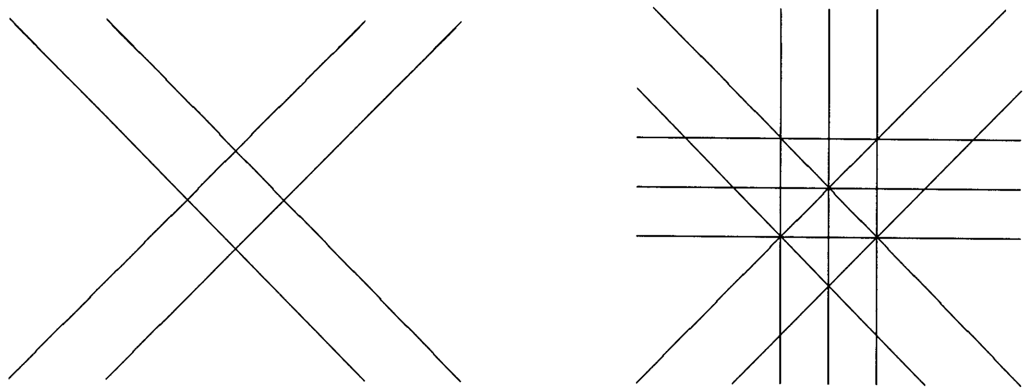

Figure 6.6: The arrangements $\widehat{\mathcal{D}}_2$ and $\widehat{\mathcal{BC}}_2$

integers in the first space. Hence, for each $i \in [n]$ we have $1+2\left(\frac{q+1}{2} - n\right) = q-2n+2$ placement choices, giving again the desired result. $\square$

Related arrangements. There are some further variations of the results in this chapter which can be obtained using the same reasoning. We give some of them below and leave it to the reader's imagination to construct other generalizations or specializations.

Theorem 6.3.6 *Let $S \subseteq E_n$ be as in Theorem 6.2.2. In other words, if $ij \in S$, then $ik \in S$ for all $j < k \leq n$. Let Φ be one of the systems B_n, C_n, BC_n or D_n. Consider the arrangement obtained from the corresponding Shi arrangement $\widehat{\mathcal{W}}$ by removing the hyperplanes $x_i - x_j = 1$ for all $i < j$ for which ij is not in S. If $\Phi = B_n$ for example, this arrangement has hyperplanes*

$$\begin{aligned}
&x_i = 0, 1 \ \text{ for } 1 \leq i \leq n, \\
&x_i \pm x_j = 0 \ \text{ for } 1 \leq i < j \leq n, \\
&x_i - x_j = 1 \ \text{ for } 1 \leq i < j \leq n, \ ij \in S, \\
&x_i + x_j = 1 \ \text{ for } 1 \leq i < j \leq n.
\end{aligned}$$

The characteristic polynomial of this new arrangement associated to Φ factors as

$$\prod_{j=1}^{n} (q - h + j - a_j - 1),$$

where $a_j = \#\{i < j \mid ij \in S\}$ and h is the corresponding Coxeter number, which has the value $2n+1$ for the nonreduced system BC_n. In particular, for this arrangement we have

$$r = \prod_{j=1}^{n} (h - j + a_j + 2)$$

and

$$b = \prod_{j=1}^{n} (h - j + a_j).$$

Proof: We need only to modify the proof of the previous theorem, based on the argument in the proof of Theorem 6.2.2. We treat the B_n case. We have $q - 2n$ choices to place 1 in the $\frac{1}{2}(q+1) - n$ spaces defined by the boxes, as we can only place 1 with a minus sign in the first space, i.e. the one to the left of the first box. At each stage, we can only place a negative integer in the first space, due to the conditions $x_i \neq 1$ and once the pattern $-ji$ is forbidden for all $j \neq i$. We leave it to the reader to check that this gives $q - 2n + j - a_j - 1$ possibilities to place the integer j for all $1 \leq j \leq n$.

The other cases are treated similarly, as in Theorem 6.3.5. □

Note that the previous theorem provides an analogue of Theorem 6.3.2 specialized to $T = E_n$. A full analogue cannot be obtained. The arrangement

$$\begin{aligned} &x_i = 0, 1 \ \text{ for } i = 1, 2, \\ &x_1 \pm x_2 = 0, \\ &x_1 - x_2 = 1, \end{aligned}$$

for example, has characteristic polynomial $q^2 - 7q + 13$, which does not factor over the integers.

The following special case of Theorem 6.3.6 is obtained as usual. We give the B_n version.

Corollary 6.3.7 *Let $1 \leq k \leq n$ be an integer. The arrangement*

$$\begin{aligned} &x_i = 0, 1 \ \text{ for } 1 \leq i \leq n, \\ &x_i \pm x_j = 0 \ \text{ for } 1 \leq i < j \leq n, \\ &x_i - x_j = 1 \ \text{ for } 1 \leq i < j \leq k, \\ &x_i + x_j = 1 \ \text{ for } 1 \leq i < j \leq n \end{aligned}$$

has characteristic polynomial

$$(q - 2n)^{k-1} \prod_{j=n+k}^{2n} (q - j).$$

In particular, for this arrangement we have

$$r = (2n + 1)^{k-1} \frac{(2n + 1)!}{(n + k)!}$$

and

$$b = (2n-1)^{k-1}\frac{(2n-1)!}{(n+k-2)!}.$$

□

Theorem 6.3.8 *Let N be a nonnegative integer. Suppose that the sets $S, T \subseteq E_n$ satisfy the conditions of Theorem 6.3.2. Then the arrangement*

$$\begin{aligned} &x_i = 0, \pm 1, \ldots, \pm N \ \text{ for } 1 \le i \le n, \\ &x_i \pm x_j = 0 \ \text{ for } 1 \le i < j \le n, \\ &x_i - x_j = 1 \ \text{ for } 1 \le i < j \le n, \ ij \in S, \\ &x_i + x_j = 1 \ \text{ for } 1 \le i < j \le n, \ ij \in T \end{aligned}$$

has characteristic polynomial

$$\prod_{j=1}^{n}(q - 2n - 2N + 2j - 1 - a_j - b_j),$$

where $a_j = \#\{i < j \mid ij \in S\}$ and $b_j = \#\{i < j \mid ij \in T\}$. In particular,

$$r = \prod_{j=1}^{n}(2n + 2N - 2j + a_j + b_j + 2)$$

and

$$b = \prod_{j=1}^{n}(2n + 2N - 2j + a_j + b_j).$$

Proof: The argument is a direct extension of that in the proof of Theorem 6.3.2.
□

6.4 The Linial arrangement

In this section we will be primarily concerned with the *Linial* arrangement

$$x_i - x_j = 1 \ \text{ for } 1 \le i < j \le n.$$

Following [62], we denote this arrangement by $\mathcal{L}_n$ and let $r(\mathcal{L}_n) = g_n$. Figure 6.7 shows $\mathcal{L}_3$, restricted in two dimensions as usual. It has 7 regions. In general, the number g_n has a surprising combinatorial interpretation, initially conjectured by Stanley on the basis of data provided by Linial and Ravid, and recently proved by Postnikov [45][62, §4]. We will give another proof of Stanley's conjecture based on Theorem

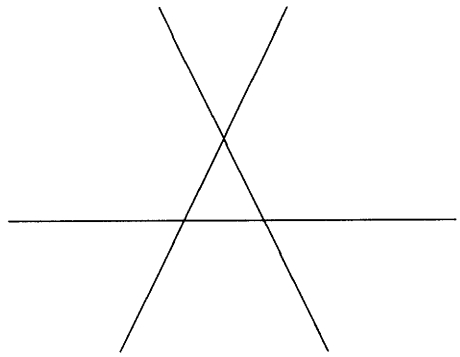

Figure 6.7: The arrangement $\mathcal{L}_3$

5.2.1. We start with the necessary definitions and refer the reader to [62, §4] for more information and other combinatorial interpretations of g_n.

An *alternating tree* or *intransitive tree* on $n+1$ vertices is a labeled tree with vertices $0, 1, \ldots, n$, such that no i, j, k with $i < j < k$ are consecutive vertices of a path in the tree. In other words, for any path in the tree with consecutive vertices $a_0, a_1, \ldots, a_l$ we have $a_0 < a_1 > a_2 < a_3 \cdots a_l$ or $a_0 > a_1 < a_2 > a_3 \cdots a_l$. For an example see Figure 6.8. Alternating trees first arose in the context of hypergeometric functions [28]. In the paper [44], Postnikov proved that if f_n denotes the number of alternating trees on $n+1$ vertices and if

$$y = \sum_{n \geq 0} f_n \frac{x^n}{n!},$$

then

$$y = e^{\frac{x}{2}(y+1)}$$

and

$$f_{n-1} = \frac{1}{n 2^{n-1}} \sum_{k=1}^{n} \binom{n}{k} k^{n-1}.$$

Postnikov's theorem (initially conjectured by Stanley) can be stated as follows.

Theorem 6.4.1 **([45][62, Theorem 4.1])** *For all $n \geq 0$ we have $f_n = g_n$.*

In the following theorem we give a new, explicit formula for the characteristic polynomial of $\mathcal{L}_n$ which implies Theorem 6.4.1, via Theorem 4.2.1.

Theorem 6.4.2 *For all $n \geq 1$ we have*

$$\chi(\mathcal{L}_n, q) = \frac{q}{2^n} \sum_{j=0}^{n} \binom{n}{j} (q-j)^{n-1}. \tag{6.5}$$

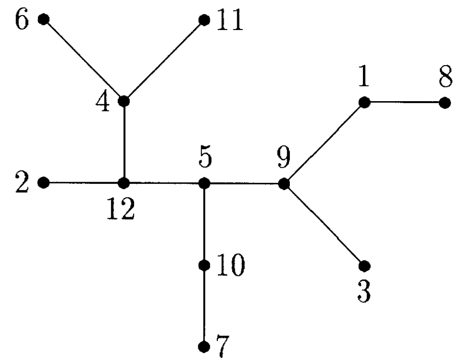

Figure 6.8: An alternating tree on 12 vertices

In particular,

$$g_n = \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j} (j+1)^{n-1} = f_n$$

and

$$b(\mathcal{L}_n) = \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j} (j-1)^{n-1}.$$

We postpone the proof of Theorem 6.4.2 until later in this section. We remark here that it would be interesting to find a combinatorial interpretation for the numbers $b(\mathcal{L}_n)$, similar to the one that Theorem 6.4.1 gives for g_n.

A generalization of $\widehat{\mathcal{A}}_n$ and $\mathcal{L}_n$. For any nonnegative integer k we consider the arrangement with hyperplanes

$$x_i - x_j = 1, 2, \ldots, k \quad \text{for } 1 \leq i < j \leq n.$$

We denote this arrangement by $\widehat{\mathcal{A}}_n^{[k]}$. Note that for $k = 0$ it reduces to the empty arrangement in $\mathbb{R}^n$, with characteristic polynomial q^n, and for $k = 1$ to the Linial arrangement $\mathcal{L}_n$. We also denote by $\widehat{\mathcal{A}}_n^{[0,k]}$ the arrangement

$$x_i - x_j = 0, 1, \ldots, k \quad \text{for } 1 \leq i < j \leq n,$$

obtained from $\widehat{\mathcal{A}}_n^{[k]}$ by adding the hyperplanes $x_i - x_j = 0$ for $1 \leq i < j \leq n$. This arrangement provides another generalization of the Shi arrangement $\widehat{\mathcal{A}}_n$. The notation introduced above will be naturally generalized in §7.1.

We now show that the characteristic polynomials of $\widehat{\mathcal{A}}_n^{[k]}$ and $\widehat{\mathcal{A}}_n^{[0,k]}$ are closely related. We use the terminology about placements introduced in the beginning of §6.2, in particular the definition of a weak descent. We refer to the empty boxes between i and j in the clockwise direction from j to i as the boxes that seperate the two integers.

Theorem 6.4.3 *For all $n, k \geq 1$ we have*

$$\tilde{\chi}(\widehat{\mathcal{A}}_n^{[0,k]}, q) = \tilde{\chi}(\widehat{\mathcal{A}}_n^{[k-1]}, q-n).$$

Proof: Let q be a large prime. Using Theorem 5.2.1 as in the proof of Theorem 6.2.1, $\tilde{\chi}(\widehat{\mathcal{A}}_n^{[0,k]}, q)$ counts the number of circular placements of the integers from 1 to n and $q-n$ unlabeled boxes, such that at least k boxes seperate two integers $i < j$ which form a weak descent (i,j) of the placement. We call these placements of type α. It follows easily (see (6.6) in the proof of the next theorem) that the number of placements of type α is a polynomial in q, so the previous interpretation for $\tilde{\chi}(\widehat{\mathcal{A}}_n^{[0,k]}, q)$ is true for all $q > n$.

Using Theorem 5.2.1 again, for large primes q, $\tilde{\chi}(\widehat{\mathcal{A}}_n^{[k-1]}, q-n)$ counts the number of placements of the integers from 1 to n into $q-n$ cyclically arranged unlabeled boxes, such that at least $k-1$ empty boxes seperate two integers $i < j$ which form a weak descent (i,j) of the placement. Note that we are now allowed to place many integers in the same box. We linearly order the integers in any occupied box to be increasing in the clockwise direction. We call these placements of type β.

To prove the result, it suffices to establish a bijection between the placements of type α and those of type β. Starting with a placement of type α, remove a box from each maximal string of consecutive unlabeled boxes. If a single box forms such a string by itself, i.e. is preceded and followed by integers (not defining a weak descent), then place a bar between these integers after removing the box. The maximal clockwise increasing strings of consecutive integers, with no bars (or boxes) in between, define the occupied boxes of the placement of type β thus produced. This placement has now $q-n$ boxes, since the correspondence described reduces the number of objects by one between each of the n pairs of weakly consecutive integers of the placement of type α, and we had q objects to start with. For example, if i and j formed an ascent in the placement of type α, then they will be placed in the same box in the placement of type β, decreasing the total number of objects by one. It is easy to see that this correspondence is indeed a bijection. $\square$

Note that for $k=1$, Theorem 6.4.3 reduces to Theorem 6.2.1. We now give some formulas for the characteristic polynomial and number of regions of $\widehat{\mathcal{A}}_n^{[0,k]}$. Here and in what follows, we denote by $[x^n]f(x)$ the coefficient of x^n in a formal series $f(x)$ of the form $f(x) = \sum_{n \geq n_0} c_n x^n$, where $n_0 \in \mathbb{Z}$.

Theorem 6.4.4 *For all $n, k \geq 1$, $q > n$ we have*

$$\chi(\widehat{\mathcal{A}}_n^{[0,k]}, q) = q\ [y^{q-n}]\ (1 + y + y^2 + \cdots + y^{k-1})^n \sum_{j=0}^{\infty} j^{n-1} y^{jk}$$

and

$$r(\widehat{\mathcal{A}}_n^{[0,k]}) = [y^{kn+1}]\ (1+y+y^2+\cdots+y^{k-1})^n \sum_{j=0}^{\infty} j^{n-1} y^{jk}.$$

In particular,

$$r(\widehat{\mathcal{A}}_n^{[0,2]}) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2j+1} (n-j)^{n-1}.$$

Proof: To count the placements of type α described in the proof of Theorem 6.4.3, we first choose a cyclic placement w of the integers from 1 to n. Then we distribute the $q-n$ unlabeled boxes in the n spaces between the integers, placing at least k boxes in each space with a descent. A descent of w is a pattern ji with $i<j$, i.e. a pair (i,j) with $i<j$, such that i immediately follows j clockwise in w. Let $d(w)$ be the number of descents of w. Given w, there are $\binom{q-kd(w)-1}{n-1}$ ways to distribute the $q-n$ boxes according to the above restriction. Hence

$$\tilde{\chi}(\widehat{\mathcal{A}}_n^{[0,k]}, q) = \sum_{w \in P_n} \binom{q-kd(w)-1}{n-1}, \tag{6.6}$$

where P_n stands for the set of $(n-1)!$ cyclic placements of the elements of $[n]$. Note that the cyclic placements of $[n]$ with j descents correspond to permutations of $[n-1]$ with $j-1$ descents. Indeed, we can remove the largest entry n of the placement and unfold to get a linear permutation with one less descent than before. Thus,

$$\begin{aligned}\tilde{\chi}(\widehat{\mathcal{A}}_n^{[0,k]}, q) = \sum_{w \in S_{n-1}} \binom{q-kd(w)-k-1}{n-1} &= [y^{q-n}]\ \frac{\sum_{w\in S_{n-1}} y^{k+kd(w)}}{(1-y)^n} \\ &= [y^{q-n}]\ (1+y+y^2+\cdots+y^{k-1})^n\ \frac{\sum_{w\in S_{n-1}} y^{k(1+d(w))}}{(1-y^k)^n}.\end{aligned}$$

The proposed formula for $\chi(\widehat{\mathcal{A}}_n^{[0,k]}, q)$ follows from the well known identity

$$\frac{\sum_{w\in S_{n-1}} \lambda^{1+d(w)}}{(1-\lambda)^n} = \sum_{j=0}^{\infty} j^{n-1}\lambda^j.$$

A proof and generalization of this identity is provided by Stanley's theory of P-partitions [55] (see also [60, Thm. 4.5.14]).

To obtain the formula for $r(\widehat{\mathcal{A}}_n^{[0,k]})$ we use Theorem 1.1 in the first summation formula for $\tilde{\chi}(\widehat{\mathcal{A}}_n^{[0,k]}, q)$ after (6.6). Here $a(w) = n-2-d(w)$ stands for the number of ascents of w.

$$r(\widehat{\mathcal{A}}_n^{[0,k]}) = (-1)^{n-1} \sum_{w\in S_{n-1}} \binom{-kd(w)-k-2}{n-1} = \sum_{w\in S_{n-1}} \binom{n+kd(w)+k}{n-1}$$

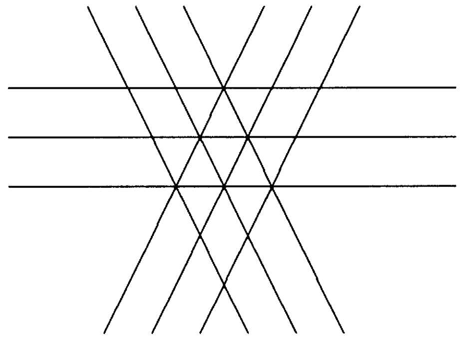

Figure 6.9: The arrangement $\widehat{\mathcal{A}}_n^{[0,2]}$

$$= \ [y^{kn+1}] \ \frac{\sum_{w \in S_{n-1}} y^{k+ka(w)}}{(1-y)^n} \ = \ [y^{kn+1}] \ (1+y+y^2+\cdots+y^{k-1})^n \sum_{j=0}^{\infty} j^{n-1} y^{jk}.$$

The specialization for $k=2$ mentioned at the end of the theorem is an immediate consequence of the result above. The arrangement corresponding to $n=3$ has 31 regions and is drawn in Figure 6.9. $\square$

We are now able to prove Theorem 6.4.2.

Proof of Theorem 6.4.2: Theorems 6.4.3 and 6.4.4 for $k=2$ yield

$$\begin{aligned} \tilde{\chi}(\mathcal{L}_n, q) \ &= \ [y^q] \ (1+y)^n \sum_{j=0}^{\infty} j^{n-1} y^{2j} \ = \ \frac{1}{2^{n-1}} \ [y^q] \ (1+y)^n \sum_{j=0}^{\infty} (2j)^{n-1} y^{2j} \\ &= \ \frac{1}{2^{n-1}} \sum_{\substack{j=0 \\ j \equiv q \pmod 2}}^{n} \binom{n}{j} (q-j)^{n-1} \ = \ \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j} (q-j)^{n-1}, \end{aligned}$$

as desired. The last equality follows from the fact that the quantity

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} (q-j)^{n-1}$$

is identically zero, as an n^{th} finite difference of a polynomial of degree $n-1$. $\square$

A messy formula. Finally we observe that a different counting argument gives a more complicated formula for the characteristic polynomial of $\mathcal{L}_n$. We don't know how to prove directly that the resulting formula for g_n gives the number of alternating trees f_n. Here and in what follows, $S(n,k)$ denotes a Stirling number of the second kind.

Theorem 6.4.5 *For all $n \geq 1$ we have*

$$\chi(\mathcal{L}_n, q) = q \sum_{1 \leq k \leq p \leq n} (k-1)!\ S(n,k) \binom{n-k}{p-k} \binom{q-p-1}{k-1}.$$

In particular,

$$g_n = \sum_{1 \leq k \leq p \leq n} (-1)^{n-k}\ (k-1)!\ S(n,k) \binom{n-k}{p-k} \binom{p+k}{k-1}$$

and

$$b(\mathcal{L}_n) = \sum_{1 \leq k \leq p \leq n} (-1)^{n-k}\ (k-1)!\ S(n,k) \binom{n-k}{p-k} \binom{p+k-2}{k-1}.$$

Proof: We want to count the placements of type β, defined in the proof of Theorem 6.4.3, for $k = 2$ and with q replacing $q - n$. Let's list the integers in any occupied box in increasing order, so that the integers placed on any string of consecutive occupied boxes, call it a part of the placement, strictly increase. Let's place a bar to separate the integers placed in two consecutive occupied boxes, for every such pair of boxes. Let k denote the number of parts of the placement and p the number of occupied boxes. There are $S(n,k)$ ways to form the parts, each with its elements in increasing order, $(k-1)!$ ways to permute them cyclically and $\binom{n-k}{p-k}$ ways to insert $p-k$ bars in the $n-k$ possible spaces within the parts. Lastly there are $\binom{q-p-1}{k-1}$ ways to insert the remaining $q-p$ boxes in the spaces among the parts and q ways to choose the box with label the zero class of $\mathbb{F}_q$. $\square$

Chapter 7

Other Hyperplane Arrangements

In this chapter we give a few other examples of hyperplane arrangements, related to the ones we have discussed so far, on which Theorem 5.2.1 sheds light. We will focus mainly on deformations of $\mathcal{A}_n$, as defined in (5.3).

7.1 Other deformations of $\mathcal{A}_n$

Let ℓ be a finite set of integers. We denote by $\widehat{\mathcal{A}}_n^{\ell}$ the following deformation of $\mathcal{A}_n$:

$$x_i - x_j = s \text{ for all } 1 \leq i < j \leq n, \ s \in \ell.$$

In particular, if a, b be integers with $a \leq b$, we denote by $\widehat{\mathcal{A}}_n^{[a,b]}$ the arrangement with hyperplanes

$$x_i - x_j = a, a+1, \ldots, b \text{ for all } 1 \leq i < j \leq n.$$

This agrees with the notation for $\widehat{\mathcal{A}}_n^{[k]}$ and $\widehat{\mathcal{A}}_n^{[0,k]}$, whose characteristic polynomials were computed in §6.4. To simplify the notation, if ℓ consists of nonnegative integers we denote by $\mathcal{A}_n^{\ell}$ the arrangement of hyperplanes in $\mathbb{R}^n$

$$x_i - x_j = s \text{ for all } i \neq j, \ s \in \ell.$$

The following theorem is an easy consequence of Theorem 5.2.1.

Theorem 7.1.1 *Let a, b be integers satisfying $0 \leq a \leq b$. Then*

$$\tilde{\chi}(\widehat{\mathcal{A}}_n^{[-a,b]}, q) = \tilde{\chi}(\widehat{\mathcal{A}}_n^{[0,b-a]}, q - an).$$

Proof: We use terminology from §6.2. For a large prime q, $\tilde{\chi}(\widehat{\mathcal{A}}_n^{[-a,b]}, q)$ counts the number of circular placements of the integers from 1 to n and $q - n$ unlabeled boxes, such that at least b boxes seperate two integers forming a weak descent and at least a boxes seperate two integers forming a weak ascent. We remove a from the boxes between each of the n pairs of weakly consecutive integers to get a bijection with the placements counted by $\tilde{\chi}(\widehat{\mathcal{A}}_n^{[0,b-a]}, q-an)$. $\square$

We give two applications. The first one gives another generalization of Theorem 6.2.1. It is an immediate consequence of this theorem and Theorem 7.1.1, applied for $b = a+1$.

Corollary 7.1.2 *Let $a \geq 1$ be an integer. For the arrangement*

$$x_i - x_j = -a+1, -a+2, \dots, a \quad \text{for } 1 \leq i < j \leq n$$

we have

$$\tilde{\chi}(\widehat{\mathcal{A}}_n^{[-(a-1),a]}, q) = q(q-an)^{n-1}.$$

In particular,

$$r(\widehat{\mathcal{A}}_n^{[-(a-1),a]}) = (an+1)^{n-1}$$

and

$$b(\widehat{\mathcal{A}}_n^{[-(a-1),a]}) = (an-1)^{n-1}.$$

$\square$

We now let a be any nonnegative integer. As a second application of Theorem 7.1.1 we consider the arrangement $\widehat{\mathcal{A}}_n^{[-a,a]}$. It has hyperplanes

$$x_i - x_j = 0, 1, \dots, a \quad \text{for all } i \neq j.$$

It was noted by Stanley [62, §2] that the number of regions of this arrangement can be shown to equal

$$\frac{n!}{an+1}\binom{(a+1)n}{n}.$$

Stanley states this result in the context of generalized interval orders. See [62] for special cases that have appeared earlier in the literature. A simple explicit product formula for the characteristic polynomial, which implies the above formula via Theorem 4.2.1, follows from the formula for $\chi(\mathcal{A}_n, q)$ and Theorem 7.1.1 by setting $a = b$. It generalizes Corollary 6.2.9. A stronger version of the next corollary was obtained by Edelman and Reiner (see Theorem 3.2 in [24]).

Corollary 7.1.3 *The arrangement $\mathcal{A}_n^{[0,a]}$, denoted also by $\widehat{\mathcal{A}}_n^{[-a,a]}$, has characteristic polynomial*

$$q\prod_{j=1}^{n-1}(q-an-j).$$

In particular,

$$r(\mathcal{A}_n^{[0,a]}) = \frac{(an+n)!}{(an+1)!}$$

and

$$b(\mathcal{A}_n^{[0,a]}) = \frac{(an+n-2)!}{(an-1)!}.$$

□

More generally, let $\ell = \{\ell_1, \ell_2, \ldots, \ell_m\}$ be a set of m positive integers satisfying $\ell_1 < \ell_2 < \cdots < \ell_m$. By our earlier notation, we denote by $\mathcal{A}_n^{\ell\cup 0}$ the following deformation of $\mathcal{A}_n$:

$$x_i - x_j = 0,\ \ell_1,\ \ell_2, \ldots, \ell_m \ \text{ for all } i \neq j. \tag{7.1}$$

If ℓ is empty ($m = 0$), this arrangement reduces to $\mathcal{A}_n$ and if $\ell = \{1, 2, \ldots, k\}$, we get the arrangement $\mathcal{A}_n^{[0,k]}$. The following theorem gives an expression for the characteristic polynomial and number of regions of $\mathcal{A}_n^{\ell\cup 0}$, under a certain assumption on ℓ.

Theorem 7.1.4 *Suppose that the set of positive integers not in ℓ is closed under addition. Let $p_\ell(x) = \sum_{j=1}^{m} x^{\ell_j - 1}$. Then, for all integers $q > n\ell_m$,*

$$\chi(\mathcal{A}_n^{\ell\cup 0}, q) = q\ (n-1)!\ [x^{q-n}] \left(\frac{1 + (x-1)p_\ell(x)}{1-x} \right)^n.$$

Moreover, let

$$(1 + (x-1)p_\ell(x))^n = (1-x)^n Q_n(x) + R_n(x),$$

where Q_n and R_n are polynomials with $\deg R_n < n$. *Then*

$$r(\mathcal{A}_n^{\ell\cup 0}) = (n-1)!\ [x^n]\ \frac{x^{n-1} R_n(1/x)}{(1-x)^n}$$

and

$$b(\mathcal{A}_n^{\ell\cup 0}) = (n-1)!\ [x^{n-2}]\ \frac{x^{n-1} R_n(1/x)}{(1-x)^n}.$$

Proof: To construct the placements counted by $\tilde{\chi}(\mathcal{A}_n^{\ell\cup 0}, q)$, we first cyclically permute the integers from 1 to n in $(n-1)!$ ways. We want to insert $q - n$ unlabeled boxes between them, so that no two integers are seperated by $\ell_j - 1$ objects, where $1 \leq j \leq m$. Because of the assumption on ℓ, it suffices not to insert $\ell_j - 1$ boxes between any two consecutive integers. Hence by Theorem 5.2.1, for large primes q we have

$$\tilde{\chi}(\mathcal{A}_n^{\ell\cup 0}, q) = (n-1)!\ [x^{q-n}] \left(\sum_k x^k \right)^n,$$

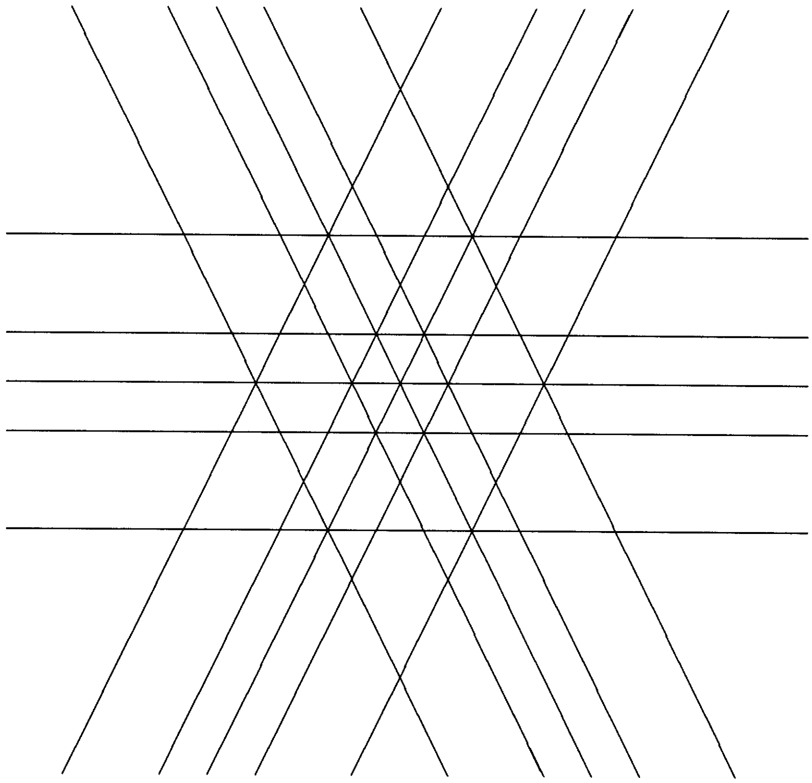

Figure 7.1: The arrangement $\mathcal{A}_3^{\{0,1,3\}}$

where, in the sum, k ranges over all nonnegative integers different from $\ell_j - 1$ for $1 \leq j \leq m$. This is equivalent to the proposed formula for the characteristic polynomial. The result holds for all $q > n\ell_m$ since the coefficient of x^k in the rational function on the right is a polynomial in k, say $P(k)$, for $k > n\ell_m - n$ (see [60, Prop. 4.2.2, Cor. 4.3.1]).

To obtain the value of $\chi(\mathcal{A}_n^{\ell \cup 0}, q)$ at -1, we need to evaluate $P(k)$ at $k = -n-1$. By construction,

$$\sum_{k=0}^{\infty} P(k)x^k = \frac{R_n(x)}{(1-x)^n}.$$

Proposition 4.2.3 in [60] implies that

$$\sum_{k=0}^{\infty} P(-k)x^k = -\frac{R_n(1/x)}{(1-\frac{1}{x})^n}.$$

Since

$$r(\mathcal{A}_n^{\ell \cup 0}) = (-1)^n \chi(\mathcal{A}_n^{\ell \cup 0}, -1) = (-1)^{n+1}(n-1)!\; P(-n-1),$$

the result for the number of regions follows. Similarly we get the formula for the number of bounded regions by evaluating $P(k)$ at $k = -n+1$. $\square$

If $\ell = (1, 2, \ldots, a)$ then the condition in Theorem 7.1.4 is trivially satisfied, $1 + (x-1)p_\ell(x) = x^a$ and one can easily deduce the formula for the number of regions obtained in Theorem 7.1.3 directly from Theorem 7.1.4.

To give another example, let $n = 3$, $\ell = \{1, 3\}$, which also satisfies the assumption of Theorem 7.1.4. Then $1 + (x-1)p_\ell(x) = 1 + (x-1)(1+x^2) = x - x^2 + x^3$ and the remainder of $(x - x^2 + x^3)^3$ upon division with $(1-x)^3$ is $R_3(x) = 13 - 30x + 18x^2$. Theorem 7.1.4 predicts that

$$r(\mathcal{A}_3^{\{0,1,3\}}) = 2\ [x^3]\ (13x^2 - 30x + 18)(1 + 3x + 6x^2 + 10x^3 + \cdots) = 78$$

and

$$b(\mathcal{A}_3^{\{0,1,3\}}) = 2\ [x]\ (13x^2 - 30x + 18)(1 + 3x + 6x^2 + 10x^3 + \cdots) = 48.$$

The regions of $\mathcal{A}_3^{\{0,1,3\}}$ are shown in Figure 7.1. The reader is invited to count them and observe that 48 of them are bounded.

The Shi arrangement, paths and cycles. We digress here to consider the arrangements $\widehat{\mathcal{A}}_{n,S}$, introduced in §6.2, where S is a disjoint union of paths or a directed cycle.

First, let S be a disjoint union of paths, having a total of $m-1$ edges for some $1 \leq m \leq n$. We can assume that the edges are taken from the path $\{12, 23, \ldots, n-1n\}$.

Theorem 7.1.5 *Suppose $I \subseteq [n-1]$ has cardinality $m-1$ and let $S_I = \{ii+1 \mid i \in I\}$. The characteristic polynomial of the arrangement*

$$\begin{aligned} x_i - x_j &= 0 \quad \text{for } 1 \leq i < j \leq n, \\ x_i - x_{i+1} &= 1 \quad \text{for } i \in I \end{aligned}$$

depends only on m and is given by the formula

$$\chi(\widehat{\mathcal{A}}_{n,S_I}, q) = q \sum_{k=1}^{m} (-1)^{k-1} \binom{m-1}{k-1} (q-k)(q-k-1)\cdots(q-n+1). \tag{7.2}$$

In particular,

$$r(\widehat{\mathcal{A}}_{n,S_I}) = \sum_{k=1}^{m} \frac{n!}{k!} \binom{m-1}{k-1}$$

and

$$b(\widehat{\mathcal{A}}_{n,S_I}) = \sum_{k=2}^{m} \frac{(n-2)!}{(k-2)!} \binom{m-1}{k-1}.$$

Proof: We use Theorem 5.2.1 and the inclusion-exclusion principle. We will first count the number of n-tuples $(x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n$ satisfying $x_i - x_j \neq 0$ for

$1 \leq i < j \leq n$ and a given set of k of the $m-1$ conditions $x_i - x_{i+1} = 1$ for $i \in I$, where $0 \leq k \leq m-1$.

In terms of circular placements, imposing the condition $x_i - x_{i+1} = 1$ means that i has to be preceded by $i+1$. Imposing k of these conditions splits $[n]$ into blocks of the form $j, j-1, \ldots, i$, whose entries have to appear in order, with no boxes in between. Clearly, the number of these blocks is $n-k$. There are $(n-k-1)!$ ways to cyclically permute these blocks, $\binom{q-k-1}{n-k-1}$ ways to place $q-n$ unlabeled boxes in the $n-k$ spaces between the blocks and q ways to choose the box with label the zero class of $\mathbb{F}_q$. This gives a total of

$$q\ (q-k-1)\ (q-k-2)\cdots(q-n+1)$$

ways and (7.2) follows by inclusion-exclusion, once k is replaced by $k-1$. $\square$

For $m = n$ we obtain the following interesting corollary.

Corollary 7.1.6 *Let S be the path $\{12, 23, \ldots, n-1n\}$. For the arrangement*

$$\begin{aligned} &x_i - x_j = 0 \ \text{ for } 1 \leq i < j \leq n, \\ &x_i - x_{i+1} = 1 \ \text{ for } 1 \leq i \leq n-1, \end{aligned}$$

we have

$$\chi(\widehat{\mathcal{A}}_{n,S}, q) = q \sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1}\ (q-k)(q-k-1)\cdots(q-n+1).$$

In particular, the number of regions

$$r(\widehat{\mathcal{A}}_{n,S}) = \sum_{k=1}^{n} \frac{n!}{k!} \binom{n-1}{k-1}$$

is the number of ways to partition a set with n elements and linearly order each block.
$\square$

The same reasoning gives an expression for the characteristic polynomial of the arrangement of Theorem 6.2.7 when S is a disjoint union of cycles. We give this formula in the case of one cycle, for simplicity.

Corollary 7.1.7 *Let $1 \leq m \leq n$ be an integer. The arrangement*

$$\begin{aligned} &x_i - x_j = 0 \ \text{ for } 1 \leq i < j \leq n, \\ &x_i - x_{i+1} = 1 \ \text{ for } 1 \leq i \leq m-1, \\ &x_m - x_1 = 1 \end{aligned}$$

has characteristic polynomial

$$q\sum_{k=1}^{m}(-1)^{k-1}\binom{m}{k-1}(q-k)(q-k-1)\cdots(q-n+1).$$

In particular, for this arrangement

$$r=\sum_{k=1}^{m}\frac{n!}{k!}\binom{m}{k-1}$$

and

$$b=\sum_{k=2}^{m}\frac{(n-2)!}{(k-2)!}\binom{m}{k-1}.$$

□

7.2 Deformations in other root systems

First we generalize some of the notation in the previous section to other root systems.

Let Φ be an irreducible crystallographic root system spanning $\mathbb{R}^l$ with corresponding Weyl group W and set of positive roots Φ^+, as in Chapter 6. Let ℓ be a finite set of integers. We denote by $\widehat{\mathcal{W}}^\ell$ the arrangement of hyperplanes

$$\{H_{\alpha,k} \mid \alpha\in\Phi^+ \text{ and } k\in\ell\},$$

where $H_{\alpha,k}$ has the same meaning as in §6.1. If $\Phi=A_{n-1}$, $\widehat{\mathcal{W}}^\ell$ is the arrangement $\widehat{\mathcal{A}}_n^\ell$ restricted in an $(n-1)$-dimensional Euclidean space.

If $\Phi=B_n$, C_n or D_n, we denote $\widehat{\mathcal{W}}^\ell$ by $\widehat{\mathcal{B}}_n^\ell$, $\widehat{\mathcal{C}}_n^\ell$ and $\widehat{\mathcal{D}}_n^\ell$ respectively. Thus, $\widehat{\mathcal{D}}_n^\ell$ has hyperplanes

$$\begin{aligned} x_i-x_j=k \ \text{ for } 1\le i<j\le n,\ k\in\ell\\ x_i+x_j=k \ \text{ for } 1\le i<j\le n,\ k\in\ell. \end{aligned}$$

The arrangements $\widehat{\mathcal{B}}_n^\ell$, $\widehat{\mathcal{C}}_n^\ell$ can be obtained from $\widehat{\mathcal{D}}_n^\ell$ by adding the hyperplanes $x_i=k$ and $2x_i=k$ for $1\le i\le n$ and $k\in\ell$, respectively.

Similarly, if ℓ consists of nonnegative integers, we denote by $\mathcal{W}^\ell$ the hyperplane arrangement

$$\{H_{\alpha,k} \mid \alpha\in\Phi \text{ and } k\in\ell\}.$$

In both notations, for $\ell=\{0\}$ we get the Coxeter arrangement $\mathcal{W}$.

The following theorem extends Theorem 7.1.1. As with Headley's result (Theorem 6.1.1), it would be interesting to find a case-free, simple proof, based on Theorem 5.2.1.

Theorem 7.2.1 *Let Φ be a root system of type A, B, C or D and let h be the Coxeter number of the corresponding Coxeter group W. Let a, b be integers satisfying $0 \leq a \leq b$. Then*

$$\chi(\widehat{\mathcal{W}}^{[-a,b]}, q) = \chi(\widehat{\mathcal{W}}^{[0,b-a]}, q - ah).$$

Proof: The A_{n-1} case is covered by Theorem 7.1.1. We treat the other 3 families here in a similar way.

We use the reasoning and conventions explained in §6.3. Let $p = q - ah$. Both sides of the proposed equality count placements of the signed integers from 1 to n into boxes arranged and labeled cyclically with the classes mod q and those mod p respectively, subject to restrictions. We simply provide an easy bijection between the two type of placements, as in the proof of Theorem 7.1.1.

To describe the bijection, we consider first the B_n case. Since $\widehat{\mathcal{B}}_n^{[0,b-a]}$ contains $\mathcal{B}_n$, the placements of the signed integers counted by $\chi(\widehat{\mathcal{B}}_n^{[0,b-a]}, p)$ are without repetitions and the top box, i.e. the zero class, must be unoccupied. The other conditions state that no positive integer is placed in the first $b-a$ classes, starting with 1 mod p and that no two integers form a k-descent for any $1 \leq k \leq b-a$ (see §6.3). The promised bijection simply adds a extra boxes in each of the $2n$ spaces between the integers and the top box, to get a placement counted by $\chi(\widehat{\mathcal{B}}_n^{[-a,b]}, q)$. Two integers i, j define a space if $i \neq \pm j$ and there is no other integer placed in between. Hence, we do not add any boxes between the weakly consecutive integers i and $-i$ that appear at the bottom of the circle. This agrees with the fact that $h = 2n$ in the B_n case.

In the D_n case there are no conditions of the form $x_i \neq k$. The box labeled with the zero class can either be empty or occupied by two integers $\pm i$, for some $i \in [n]$. The bijection is obtained by adding a boxes in the $2n-2$ spaces defined by the signed integers from 1 to n, as before. Half of these spaces lie on the right semicircle, that is the classes from zero to $\frac{p-1}{2}$.

In the C_n case the conditions $x_i \neq k$ are replaced by $2x_i \neq k$. The proof proceeds as in the B_n case, except that some of the $2a$ extra boxes added immediately before and after the zero class now have to be added symmetrically between the classes $\frac{p-1}{2}$ and $\frac{p+1}{2}$ mod p.

It is easy to check that the maps described are indeed bijections. $\square$

To illustrate the bijections, let $a = 1$, $b = 3$, $n = 4$. We can represent a placement as a linear array if we "cut" the cicrle at the bottom and unfold. The symbol $\sqcup$ denotes an unoccupied box. The bijection described above transforms the placement

$$\sqcup\ 2\ -4\ \sqcup\ -1\ \sqcup\ \sqcup\ 3\ \sqcup\ -3\ \sqcup\ \sqcup\ 1\ \sqcup\ 4\ -2\ \sqcup \tag{7.3}$$

into

$$\sqcup\ 2\ \sqcup\ -4\ \sqcup\ \sqcup\ -1\ \sqcup\ \sqcup\ \sqcup\ 3\ \sqcup\ \sqcup\ \sqcup\ -3\ \sqcup\ \sqcup\ \sqcup\ 1\ \sqcup\ \sqcup\ 4\ \sqcup\ -2\ \sqcup$$

in the B case and into

$$\sqcup\, 2 \sqcup -4 \sqcup \sqcup -1 \sqcup \sqcup \sqcup\, 3 \sqcup -3 \sqcup \sqcup \sqcup\, 1 \sqcup \sqcup\, 4 \sqcup -2 \sqcup$$

in the D case.

As in the proof of Theorem 6.4.3, it is not difficult to see that the number of placements counted by each characteristic polynomial in the previous theorem is a polynomial in q for all large integers q (as opposed to only large primes). For this reason we don't need to assume that both q and $q - ah$ are primes when proving the proposed equality, which would cause serious problems with the argument. In fact we have a B_n analogue of Theorem 6.4.4. Details will appear elsewhere.

We now give the two analogues of Corollary 7.1.2 and Corollary 7.1.3. A stronger statement appears as Conjecture 3.3 in [24]. The first corollary below generalizes Headley's result (Theorem 6.1.1) for the cases of the four infinite families of root systems.

Corollary 7.2.2 *Let Φ be a root system of type A, B, C or D spanning $\mathbb{R}^l$ with associated Coxeter group W and Coxeter number h. If a is any integer satisfying $a \geq 1$, then*

$$\chi(\widehat{\mathcal{W}}^{[-(a-1),a]}, q) = (q - ah)^l.$$

□

Corollary 7.2.3 *Let Φ be a root system of type A, B, C or D spanning $\mathbb{R}^l$ with associated Coxeter group W and Coxeter number h. Let $\chi(\mathcal{W}, q)$ be the characteristic polynomial of the corresponding Coxeter arrangement $\mathcal{W}$ in $\mathbb{R}^l$. If a is any nonnegative integer, then*

$$\chi(\mathcal{W}^{[0,a]}, q) = \chi(\mathcal{W}, q - ah).$$

□

The next theorem is the analogue of Theorem 6.4.3 for other root systems. Unfortunately the argument given below fails when a is odd and $\Phi = C_n$, although numerical data suggests that the theorem is true in this case also.

Theorem 7.2.4 *Let Φ be a root system of type A, B, C or D spanning $\mathbb{R}^l$ with associated Coxeter group W and Coxeter number h. Let a be an even positive integer if Φ is of type C and any positive integer otherwise. Then*

$$\chi(\widehat{\mathcal{W}}^{[0,a]}, q) = \chi(\widehat{\mathcal{W}}^{[a-1]}, q - h).$$

Proof: Theorem 6.4.3 settles the A_{n-1} case. The argument for the other 3 families is almost identical. Starting with the placements counted by $\chi(\widehat{\mathcal{W}}^{[0,a]}, q)$, we remove an unoccupied box in each of h possible "spaces" to get a bijection with the placements counted by $\chi(\widehat{\mathcal{W}}^{[a-1]}, q-h)$. The spaces are defined as in the proof of Theorem 7.2.1 in the B_n and D_n case. In the C_n case with a even, the spaces are defined as for B_n. If there is no box to remove between two integers, we simply place the two integers in the same box of the new placement. It is easy to check that this map indeed gives a bijection between the two type of placements. □

Let $a = 2$. Under the bijection, the placement (7.3) transforms into

$$\sqcup\ (2,-4)\ \ -1\ \sqcup\ (3,-3)\ \sqcup\ 1\ (4,-2)\ \sqcup$$

in the B case, where parentheses are used to denote boxes occupied by more than one integer, and into

$$\sqcup\ (2,-4)\ \ -1\ \sqcup\ 3\ \sqcup\ -3\ \sqcup\ 1\ (4,-2)\ \sqcup$$

in the D case.

The previous theorems work almost as well for BC_n. For any finite set of integers ℓ we define $\widehat{\mathcal{BC}}_n^{\ell} = \widehat{\mathcal{B}}_n^{\ell} \cup \widehat{\mathcal{C}}_n^{\ell}$. We state the analogues of the two corollaries for simplicity. The proofs are obvious variations of the arguments in Theorem 6.3.5 and Theorem 7.2.1, once we distinguish the cases where a is even or odd to insert unoccupied boxes in the bottom space.

Theorem 7.2.5 *Let a be a positive integer. Then*

$$\chi(\widehat{\mathcal{BC}}_n^{[-(a-1),a]}, q) = (q-(2n+1)a)^n.$$

□

Theorem 7.2.6 *Let a be a nonnegative integer. Then*

$$\chi(\widehat{\mathcal{BC}}_n^{[-a,a]}, q) = \begin{cases} \chi(B_n, q-(2n+1)a), & \text{if } a \text{ is even;} \\ \chi(B_n, q-(2n+1)a-1), & \text{if } a \text{ is odd.} \end{cases}$$

□

The following theorem is obtained as the B_n and C_n cases of Theorem 7.2.4.

Theorem 7.2.7 *If a is an even positive integer, then*

$$\chi(\widehat{\mathcal{BC}}_n^{[0,a]}, q) = \chi(\widehat{\mathcal{BC}}_n^{[a-1]}, q-2n).$$

□

Conjecture 7.2.8 *If a is an odd positive integer, then*

$$\chi(\widehat{\mathcal{C}}_n^{[0,a]}, q) = \chi(\widehat{\mathcal{C}}_n^{[a-1]}, q-2n)$$

and

$$\chi(\widehat{\mathcal{BC}}_n^{[0,a]}, q) = \chi(\widehat{\mathcal{BC}}_n^{[a-1]}, q-2n-1).$$

7.3 Exponentially stable arrangements

For any set ℓ of positive integers, $\mathcal{A}_n^{\ell}$ is the arrangement obtained from $\mathcal{A}_n^{\ell\cup 0}$ by dropping the hyperplanes $x_i - x_j = 0$. The characteristic polynomials of $\mathcal{A}_n^{\ell}$ were related to those of $\mathcal{A}_n^{\ell\cup 0}$ by Postnikov and Stanley via an exponential generating function identity. The following theorem is equivalent (at least for integer ℓ_j's) to Theorem 2.3 in [62] (see also [62, Thm. 1.2]), which is proved in [45].

Theorem 7.3.1 *Let*

$$F_{\ell}(q,t) = \sum_{n=0}^{\infty} \chi(\mathcal{A}_n^{\ell}, q)\, \frac{t^n}{n!}$$

and

$$F_{\ell}^{0}(q,t) = \sum_{n=0}^{\infty} \chi(\mathcal{A}_n^{\ell\cup 0}, q)\, \frac{t^n}{n!}.$$

Then,

$$F_{\ell}(q,t) = F_{\ell}^{0}(q, e^t - 1).$$

□

We remark here that this theorem is quite easy to derive, once the characteristic polynomials are interpreted combinatorially as in the proof of Theorems 7.1.1 and 7.1.4. In fact, this reasoning can be applied to more general situations which we now describe.

Let $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2, \ldots)$ be a sequence of hyperplane arrangements defined over the integers, such that the ambient Euclidean space of $\mathcal{G}_n$ is $\mathbb{R}^n$. Let n, k be two positive integers satisfying $k < n$. The hyperplanes of $\mathcal{G}_n$ have the form (4.1), where the a_i's and d are integers. To avoid any ambiguity, we denote by $(z_1, z_2, \ldots, z_k)$ the standard coordinate system of $\mathbb{R}^k$. Let $y_i \in \{z_1, z_2, \ldots, z_k\}$ for $1 \le i \le n$ and suppose that for each $1 \le j \le k$ we have $z_j = y_i$ for at least one i. We say that the pair $(\mathcal{G}_k, \mathcal{G}_n)$ is *exponentially stable* if for each such choice of $y_1, \ldots, y_n$ the union of the sets obtained from the hyperplanes (4.1) of $\mathcal{G}_n$ by substituting y_i for x_i equals the union of the hyperplanes of $\mathcal{G}_k$. In particular, no hyperplane of $\mathcal{G}_n$ reduces to the trivial equation $0 = 0$ under such a substitution.

We say that $\mathcal{G}$ is *exponentially stable* if the pair $(\mathcal{G}_k, \mathcal{G}_n)$ is exponentially stable for all $k < n$. For a fixed finite set of positive integers ℓ, the $\mathcal{A}_n^{\ell}$ provide an example of an exponentially stable sequence of arrangements. For each n, consider the hyperplane arrangement

$$x_i + x_j = 1 \quad \text{for } 1 \le i < j \le n. \tag{7.4}$$

The resulting sequence is not exponentially stable. Indeed, if we replace both x_i, x_j in (7.4) by x_1 we get the hyperplane $2x_1 = 1$ which is missing from the arrangements that appear earlier in the sequence.

We now give our generalization to the Theorem 7.3.1.

Theorem 7.3.2 *Let $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2, \ldots)$ be an exponentially stable sequence of hyperplane arrangements. Let $\mathcal{G}_n^0$ be obtained from $\mathcal{G}_n$ by adding the hyperplanes $x_i = x_j$, i.e. $\mathcal{G}_n^0 = \mathcal{G}_n \cup \mathcal{A}_n$. If*

$$F(q,t) = \sum_{n=0}^{\infty} \chi(\mathcal{G}_n, q)\, \frac{t^n}{n!}$$

and

$$F^0(q,t) = \sum_{n=0}^{\infty} \chi(\mathcal{G}_n^0, q)\, \frac{t^n}{n!},$$

then

$$F(q,t) = F^0(q, e^t - 1). \tag{7.5}$$

Proof: We use the combinatorial interpretation of the characteristic polynomials given by Theorem 5.2.1. Consider $\chi(\mathcal{G}_n, q)$ and partition the set $\{x_1, x_2, \ldots, x_n\}$ into maximal blocks whose elements are equal to each other. Since $\mathcal{G}$ is exponentially stable, $\chi(\mathcal{G}_n, q)$ counts the number of ways to partition $[n]$ into blocks and place a structure on the set of blocks. If the number of blocks of the partition is k, then the number of possible structures is counted by $\chi(\mathcal{G}_k^0, q)$. Thus, the result follows from standard properties of exponential generating functions [63, §5.1]. □

As applications we give the following results.

Corollary 7.3.3 *Consider the following arrangement in $\mathbb{R}^n$, which we denote by $\mathcal{P}_n$:*

$$x_i + x_j = 0, 1 \quad \text{for } 1 \leq i \leq j \leq n.$$

It has characteristic polynomial

$$\chi(\mathcal{P}_n, q) = \sum_{k=1}^{n} S(n,k)\ (q-2k)\ (q-2k+1) \cdots (q-k-1).$$

In particular,

$$\sum_{n=0}^{\infty} r(\mathcal{P}_n)\, \frac{t^n}{n!} = \sum_{n=0}^{\infty} \binom{2n+1}{n} (1-e^{-t})^n.$$

Proof: Note that the sequence $(\mathcal{P}_n)_{n=1}^{\infty}$ is exponentially stable and that $\mathcal{P}_n \cup \mathcal{A}_n$ has hyperplanes

$$\begin{aligned} x_i - x_j &= 0 \quad \text{for } 1 \leq i < j \leq n, \\ x_i + x_j &= 0, 1 \quad \text{for } 1 \leq i \leq j \leq n. \end{aligned}$$

Setting $S = \emptyset$ in the C_n version of Theorem 6.3.6, we get that $\mathcal{P}_n \cup \mathcal{A}_n$ has characteristic polynomial

$$\prod_{j=n+1}^{2n} (q-j).$$

The result follows easily from (7.5). $\Box$

Corollary 7.3.4 *The arrangement*

$$\begin{aligned} x_i - x_j &= 1 \quad \text{for } i \neq j,\ 1 \leq i,j \leq n \\ x_i + x_j &= -1, 0, 1 \quad \text{for } 1 \leq i \leq j \leq n \end{aligned}$$

has characteristic polynomial

$$\sum_{k=1}^{n} S(n,k)\ (q-4k+1)\ (q-4k+3)\cdots(q-2k-1). \tag{7.6}$$

In particular, if r_n denotes the number of regions of this arrangement, we have

$$\begin{aligned} \sum_{n=0}^{\infty} r_n \frac{t^n}{n!} &= \sum_{n=0}^{\infty} 2^n \binom{2n}{n} (1-e^{-t})^n \\ &= \left(8e^{-t} - 7\right)^{-1/2}. \end{aligned}$$

Proof: We observe again that the given sequence of arrangements is exponentially stable. The arrangement obtained from the given one by adding the hyperplanes of $\mathcal{A}_n$ is $\mathcal{C}_n^{[0,1]}$ and satisfies

$$\chi(\mathcal{C}_n^{[0,1]}, q) = (q-4n+1)\ (q-4n+3)\cdots(q-2n-1)$$

by Corollary 7.2.3. The expression (7.6) follows again from (7.5). For the last equality in the result for r_n see Exercise 4a in [60, Ch. 1]. $\Box$

Chapter 8

The Whitney polynomial

In this chapter we consider the Whitney polynomial $w(\mathcal{A}, t, q)$ of a subspace arrangement $\mathcal{A}$, defined in §4.2. We give an interpretation of this polynomial for rational subspace arrangements, generalizing Theorem 5.2.1. We use this generalization to compute the face numbers of some of the arrangements considered in the previous two chapters.

8.1 Rational arrangements and the Whitney polynomial

Let $\mathcal{A}$ be any subspace arrangement in $\mathbb{R}^n$. Recall from §4.1 that for any $x \in L_{\mathcal{A}}$, $\mathcal{A}^x$ denotes the restricted arrangement with ambient space x. Theorem 2.2 can be stated more generally for $\mathcal{A}^x$ as follows.

Corollary 8.1.1 *If $\mathcal{A}$ is defined over the integers, $x \in L_{\mathcal{A}}$ and q is a large enough prime, then*

$$\chi(\mathcal{A}^x, q) = \#(\mathbb{F}_q^n \cap x - \bigcup \mathcal{A}^x).$$

Proof: With the notation in the proof of Theorem 2.2,

$$\#(\mathbb{F}_q^n \cap x - \bigcup \mathcal{A}^x) = g(x) = \sum_{z \geq x} \mu(x, z) f(z)$$

$$= \sum_{z \in L_{\mathcal{A}^x}} \mu(x, z)\, q^{\dim z} = \chi(\mathcal{A}^x, q).$$

□

The definition of the Whitney polynomial was given in §4.2. In the case that $\mathcal{A}$ is the central hyperplane arrangement corresponding to a signed graph Σ on n

vertices, Zaslavsky [71, §2] interpreted the Whitney polynomial as the generating function of all colorings of Σ, classified by the rank of the set of "impropriety". The following theorem extends Zaslavsky's observation to rational subspace arrangements. For $t = 0$ it reduces to Theorem 2.2.

Theorem 8.1.2 *Suppose that $\mathcal{A}$ is a subspace arrangement defined over the integers. For any point p, we denote by x_p the intersection of all elements of $\mathcal{A}$ which contain p. If q is a large enough prime, then*

$$w(\mathcal{A}, t, q) = \sum_{p \in \mathbb{F}_q^n} t^{n - \dim x_p}.$$

Proof: Given any $x \in L_{\mathcal{A}}$, we have $x_p = x$ if and only if p lies in x but in no further intersection $y > x$. Thus by Corollary 8.1.1, the number of $p \in \mathbb{F}_q^n$ which satisfy $x_p = x$ is $\chi(\mathcal{A}^x, q)$. Hence,

$$\sum_{p \in \mathbb{F}_q^n} t^{n-\dim x_p} = \sum_{x \in L_A} \sum_{\substack{p \in \mathbb{F}_q^n \\ x_p = x}} t^{n - \dim x}$$

$$= \sum_{x \in L_{\mathcal{A}}} t^{n-\dim x} \chi(\mathcal{A}^x, q) = w(\mathcal{A}, t, q).$$

□

Suppose $\mathcal{A}$ is a hyperplane arrangement defined over the integers. Then Theorem 8.1.2 gives a way to compute the face numbers of $\mathcal{A}$, or equivalently the f-vector of $Z[\mathcal{A}]$, discussed in §4.2, via Theorem 4.2.3. In the next sections we carry out the computations explicitly for some of the arrangements we have considered in Chapters 6 and 7. We state our results in terms of the f-vector of $Z[\mathcal{A}]$.

Theorem 8.1.2 extends Zaslavsky's interpretation [71, §2, Cor. 4.1″] which applies to hyperplane arrangements defined by signed graphs. Stanley used Zaslavsky's method [71, §2] to compute the f-vector of a zonotope related to graphical degree sequences [59, Thm. 4.2].

8.2 The Shi arrangement

As a first application of Theorem 8.1.2 we consider the Shi arrangement of type A_{n-1}. More generally, we compute the Whitney polynomial of the arrangement $\widehat{\mathcal{A}}_n^{[-(a-1),a]}$ with hyperplanes

$$x_i - x_j = -a+1, -a+2, \ldots, a \quad \text{for all } 1 \le i < j \le n,$$

mentioned in Corollary 7.1.2.

Theorem 8.2.1 *Let $a \geq 1$ be an integer. The coefficients in t of the Whitney polynomial of $\widehat{\mathcal{A}}_n^{[-(a-1),a]}$ are given by*

$$[t^k]\ w(\widehat{\mathcal{A}}_n^{[-(a-1),a]}, t, q) = \binom{n}{k}\ q \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} (q - an + ai)^{n-1}$$

for $0 \leq k \leq n$. In particular, the f-vector $(f_0, f_1, \ldots, f_n)$ of $Z[\widehat{\mathcal{A}}_n^{[-(a-1),a]}]$ is given by

$$f_k = \binom{n}{k} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (an - ai + 1)^{n-1}.$$

In other words,

$$f_k = \binom{n}{k}\ \#\{f : [n-1] \to [an+1] \mid \operatorname{Im} f \cap [(i-1)a+1, ia] \neq \emptyset \text{ for } 1 \leq i \leq k\}$$

for $0 \leq k \leq n-1$ and $f_n = 0$.

The constant term of $w(\widehat{\mathcal{A}}_n^{[-(a-1),a]}, t, q)$ in t agrees with the characteristic polynomial computed in Corollary 7.1.2, as expected. We give the proof of Theorem 8.2.1 after the following corollary. Recall that $\widehat{\mathcal{A}}_n^{[-(a-1),a]}$ reduces to the Shi arrangement $\widehat{\mathcal{A}}_n$ for $a = 1$.

Corollary 8.2.2 *The coefficients in t of the Whitney polynomial of $\widehat{\mathcal{A}}_n$ are given by*

$$[t^k]\ w(\widehat{\mathcal{A}}_n, t, q) = \binom{n}{k}\ q \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} (q - n + i)^{n-1}$$

for $0 \leq k \leq n$. In particular, the f-vector $(f_0, f_1, \ldots, f_n)$ of $Z[\widehat{\mathcal{A}}_n]$ is given by

$$f_k = \binom{n}{k} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (n - i + 1)^{n-1}.$$

In other words,

$$f_k = \binom{n}{k}\ \#\{f : [n-1] \to [n+1] \mid [k] \subseteq \operatorname{Im} f\}$$

for $0 \leq k \leq n-1$ and $f_n = 0$. $\square$

Proof of Theorem 8.2.1: We compute

$$[t^k]\ w(\widehat{\mathcal{A}}_n^{[-(a-1),a]}, t, q)$$

for a large prime q, using the interpretation of Theorem 8.1.2. As in §6.2, we think of a point $p = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n$ as a placement (with repetitions) of the elements of $[n]$ into q boxes arranged and labeled cyclically with the classes mod q. The integer i is placed in the box labeled with the class x_i. Consider the following relation on $[n]$: i and j are related if, say, $i < j$ and x_i, x_j satisfy one of the defining equations

$$x_i - x_j = -a+1, -a+2, \ldots, a$$

in $\mathbb{F}_q$. This means that either there are less than $a-1$ boxes between the boxes that i and j are placed into, including the possibility that i and j are placed in the same box, or that i is placed in the box labeled with $x_j + a$, where $i < j$ and x_j is the label of the box that j occupies. Call an equivalence class of the transitive closure of this relation a *block* of the placement. Then $\dim x_p$ is the number of blocks and we want to count all such placements with $n-k$ blocks.

We linearly order the elements in each occupied box to make them strictly increasing, clockwise. This defines a cyclic placement of the integers from 1 to n. To count the placements with $n-k$ blocks, we start with any cyclic placement w of the integers from 1 to n, insert $n-k$ bars in the spaces between them without repetitions to form the $n-k$ blocks and then insert empty boxes in the spaces between the integers to construct the placement. In the end, any string of consecutive integers in increasing order with no bars (or boxes) in between forms an occupied box.

Let $d(w)$ be the number of descents of w, as defined in the proof of Theorem 6.4.4. There are $\binom{n}{k}$ ways to insert the bars. Suppose that r of the bars are inserted in spaces where w has a descent, called descent cuts and the rest $s = n-k-r$ in spaces with ascents, called ascent cuts. To distinguish the blocks formed by the $n-k$ bars, we insert $a-1$ unlabeled boxes in each ascent cut and a unlabeled boxes in each descent cut.

The boxes that we have defined so far, that is strings of consecutive integers in increasing order with no bars in between and the boxes inserted, are $d(w) + s + ra + s(a-1) = d(w) + a(n-k)$. Hence we need to create $q - a(n-k) - d(w)$ more boxes. We claim that

$$[t^k]\ w(\widehat{\mathcal{A}}_n^{[-(a-1),a]}, t, q) =$$

$$= \binom{n}{k}\ q \sum_{w \in P_n} \left[y^{q-a(n-k)-d(w)}\right]\ (1-y)^{-(n-k)}(1+y+\cdots+y^{a-1})^k, \qquad (8.1)$$

where, as in the proof of Theorem 6.4.4, P_n stands for the set of $(n-1)!$ cyclic placements of the elements of $[n]$.

We can insert any number of boxes in the $n-k$ cuts and this accounts for the product $(1-y)^{-(n-k)}$. In the remaining k spaces within the blocks we may have any number of boxes less than $a-1$ in a space where an ascent occurs and any number of boxes less than a in a space where a descent occurs. We insert any number of boxes less than a in each of these k spaces, which accounts for the product $(1+y+\cdots+y^{a-1})^k$. The only restriction is that the total number of boxes inserted in the n possible spaces is $q-a(n-k)-d(w)$. In each of the spaces within the blocks with an ascent occuring we remove one of the j, say, boxes that we have inserted. If no box was inserted, i.e. $j=0$, we add a double bar. This means that the corresponding maximal string of consecutive increasing integers breaks into two parts and defines two occupied boxes instead of one. In general we have created j more boxes for our placement, as desired. As usual, q accounts for the number of ways to decide where the zero class mod q will be. This proves (8.1).

We now switch to $w \in S_{n-1}$, as in the proof of Theorem 6.4.4 and write the previous formula as

$$[t^k]\ w(\hat{\mathcal{A}}_n^{[-(a-1),a]}, t, q) = \binom{n}{k}\ q\ \left[y^{q-a(n-k)}\right]\ \frac{\sum_{w\in S_{n-1}} y^{1+d(w)}\ (1-y^a)^k}{(1-y)^n}$$

$$= \binom{n}{k}\ q\ \left[y^{q-a(n-k)}\right]\ (1-y^a)^k\ \sum_{j=0}^{\infty} j^{n-1} y^j$$

and the proposed formula follows.

The result about the f-vector follows from (4.4) and the formula obtained for the coefficients of the Whitney polynomial by setting $q=-1$. The combinatorial interpretation of f_k given in the end follows by inclusion-exclusion. □

8.3 Other applications

As another application of Theorem 8.1.2 we consider the arrangement $\mathcal{A}_n^{[0,a]}$ in $\mathbb{R}^n$ with hyperplanes

$$x_i - x_j = 0, 1, \ldots, a \quad \text{for all } i \neq j.$$

The characteristic polynomial was computed in Corollary 7.1.3.

Theorem 8.3.1 *For large q the Whitney polynomial of $\mathcal{A}_n^{[0,a]}$ is*

$$w(\mathcal{A}_n^{[0,a]}, t, q) = q\ \sum_{k=0}^{n}\sum_{r=k}^{n}\ (r-1)!\ S(n,r)\binom{r}{k}\ \left[y^{q-ak-r}\right]\frac{(1-y^a)^{r-k}}{(1-y)^r}\ t^{n-k}, \qquad (8.2)$$

or equivalently

$$w(\mathcal{A}_n^{[0,a]}, t, q) = q\ \sum_{r=0}^{n}\ (r-1)!\ S(n,r)\ t^{n-r}\ [y^q]\left(\frac{ty+(1-t)y^{a+1}}{1-y}\right)^r.$$

In particular, the f-polynomial of $Z[\mathcal{A}_n^{[0,a]}]$ satisfies

$$\sum_{i=0}^{n} f_i\ t^i = \sum_{r=0}^{n} (r-1)!\ S(n,r)\ c_{a,r}\ t^{n-r},$$

where $c_{a,r}$ is the leading coefficient, i.e. the coefficient of y^{r-1} in the remainder of the division in y of the polynomial $(ty+(1-t)y^{a+1})^r$ with the polynomial $(1-y)^r$.

Proof: We compute the coefficient of t^{n-k} in $w(\mathcal{A}_n^{[0,a]}, t, q)$ following the reasoning in the proof of Theorem 8.2.1.

We want to count placements of the integers from 1 to n in q boxes arranged and labeled cyclically with the classes mod q, according to the number of blocks. Again, we initially consider q to be a large enough prime. Now the elements i and j of $[n]$ are related if the labels x_i, x_j of the boxes they occupy satisfy one of the equations

$$x_i - x_j = -a, -a+1, \ldots, a$$

in $\mathbb{F}_q$. An equivalence class of the transitive closure of this relation forms a block. In other words, i and j are in different blocks if the boxes they occupy are seperated by some string of unoccupied boxes of length at least a (in both directions).

We want to count the placements with k blocks. We first choose the r occupied boxes by partitioning $[n]$ into r nonempty parts in $S(n,r)$ ways. We cyclically permute these boxes in $(r-1)!$ ways and insert k bars in the spaces between them in $\binom{r}{k}$ ways to form the k blocks. We now distribute $q-r$ more unlabeled boxes in the r spaces, inserting at least a boxes in each space with a bar and at most $a-1$ in the rest. The number of ways to do this is the coefficient of $[y^{q-r}]$ in the product

$$\left(\frac{y^a}{1-y}\right)^k (1+y+\cdots+y^{a-1})^{r-k}.$$

The expression proposed in (8.2) for the coefficient of t^{n-k} in $w(\mathcal{A}_n^{[0,a]}, t, q)$ follows. The second expression for $w(\mathcal{A}_n^{[0,a]}, t, q)$ follows from (8.2) by changing the order of summation and using the binomial theorem for $\sum_{k=0}^{r}\binom{r}{k}A^k$, where A is the appropriate function of y and t. This expression yields the last assertion for the f-polynomial via (4.4), in the same way we obtained the formula for the number of regions in Theorem 7.1.4. □

Corollary 8.3.2 *The Whitney polynomial of the hyperplane arrangement in $\mathbb{R}^n$*

$$x_i - x_j = 0, 1 \quad \text{for all } i \neq j$$

is

$$w(\mathcal{A}_n^{[0,1]}, t, q) = q\ \sum_{k=0}^{n}\sum_{r=k}^{n} (r-1)!\ S(n,r)\binom{r}{k}\binom{q-r-1}{k-1}\ t^{n-k}. \tag{8.3}$$

The f-vector $(f_0, f_1, \ldots, f_n)$ of $Z[\mathcal{A}_n^{[0,1]}]$ is given by

$$f_{n-k} = \sum_{r=k}^{n} (r-1)!\ S(n,r) \binom{r}{k} \binom{r+k}{k-1}$$

for $1 \le k \le n$ and $f_n = 0$. In particular, $f_0 = (n-1)!\ \binom{2n}{n-1}$,

$$f_1 = \frac{(4n-1)(2n-2)!}{2(n+1)(n-2)!}$$

and $f_{n-1} = \sum_{r=1}^{n} r!\ S(n,r)$.

Proof: The formula (8.3) for the Whitney polynomial follows from the formula (8.2) in Theorem 8.3.1 by setting $a = 1$. The result for the f-vector follows directly from (8.3) and (4.4). $\square$

8.4 Further directions

Free hyperplane arrangements were introduced by Terao in [64]. The basic result of Terao [65] (also [41, Thm. 4.137]) about free hyperplane arrangements implies that their characteristic polynomials factor completely over the nonnegative integers. The roots are the generalized "exponents" of the arrangement. We have already noted that most of the hyperplane arrangements considered in Chapters 4 and 6 have characteristic polynomials which factor completely over the nonnegative integers. One of the natural questions that the present work raises is the question of freeness for the centralizations of these arrangements. This question seems to be interesting in view of the algebraic structure associated to a free hyperplane arrangement [41, Ch. 4][50, §3][65, §2]. The centralization of $\mathcal{A}$ is obtained by homogenizing each hyperplane (4.1) of $\mathcal{A}$ to

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = dx_{n+1}$$

and adding the hyperplane $x_{n+1} = 0$. This operation makes the arrangement central and multiplies the characteristic polynomial by $q - 1$. Some results and conjectures in this direction have already appeared in [24].

The question of direct combinatorial proofs of our results for the number of regions and the number of bounded regions of the arrangements we have considered also arises naturally. For the number of regions of the Shi arrangement $\widehat{\mathcal{A}}_n$, such a proof can be obtained by combining a bijection due to Kreweras and one due to Pak and Stanley (see the discussion in [62, §5]). Combinatorial proofs of the results in this chapter about general face numbers would also be desirable.

We would also like to use Theorem 5.2.1 to study specific examples or classes of subspace (as opposed to hyperplane) arrangements.

"Never trust a god who doesn't dance."

Friedrich Nietzsche

Part III

TUTTE POLYNOMIALS

Chapter 9

Hypermatroids and the Tutte Polynomial

The Tutte polynomial was defined for graphs by Tutte in [67] and more generally for matroids by Crapo in [19]. It has been intensively studied in the past by many authors. We will refer the reader to [15] and [16] for general expositions of the theory of the Tutte polynomial. It was suggested by Stanley in [61, §3.3] that an interesting analogue of the Tutte polynomial of a graph might exist for hypergraphs. It is one of our objectives in this third part to define such a hypergraph analogue of the Tutte polynomial and study its main properties.

Our plan is to define the Tutte polynomial for more general objects than matroids, which we call hypermatroids. We introduce hypermatroids as a convenient way to model hypergraphs and define their Tutte polynomials. The concept of a hypermatroid is too general to be of independent interest, although it turns out to be of the "right" level of generality for the purpose of characterizing the affine relations that hold among the coefficients of the Tutte polynomial (see Theorem 9.3.3). Many of the results about the classical Tutte polynomial generalize to our theory, while others remain with no natural analogue.

Attempts to define the concept of a matroid for general posets that we are aware of can be found in [43], where a Tutte polynomial for distributive lattices is suggested, and in [3]. Our approach is completely different.

Overview of Part III. In the rest of this chapter we introduce hypermatroids and their Tutte invariants. We define the Tutte polynomial of a hypermatroid and show its universality as a Tutte invariant. Next we restrict our attention to hypermatroids defined by a simple construction on an arbitrary lattice. Matroids correspond to geometric lattices. We bring hypergraphs into the picture through their bond lattices and define the Tutte polynomial for hypergraphs and atomic lattices. We generalize most of the basic Tutte invariants of matroid theory to arbitrary hypermatroids. We also introduce the characteristic polynomial for the hypermatroids constructed from lattices and relate it to the Tutte polynomial.

In Chapter 10 we give a generalization of Rota's NBC Theorem and use it to prove that the coefficients of the characteristic polynomial "alternate in sign" for a class of hypergraphs that strictly contains the class of graphs.

In Chapter 11 we introduce a generalization of the Orlik-Solomon algebra over the integers of a geometric lattice. To achieve this we weaken the elimination axiom in one of the cryptomorphic characterizations of matroids. We extend the basis theorem by proving that the NBC sets induce a linear basis of this algebra. This enables us to define the Orlik-Solomon algebra for an arbitrary atomic lattice in such a way that the basis theorem is preserved under the same condition which makes the NBC theorem true. We conclude with a few remarks and some questions that arise from our considerations.

9.1 Definitions and basic properties

We begin by defining our notion of a hypermatroid. In what follows, 2^E denotes the Boolean algebra of subsets of E. We write $X \cup e$, $X - e$ instead of the more cumbersome $X \cup \{e\}$ and $X - \{e\}$.

Definition 9.1.1 *A hypermatroid $\mathcal{H}$ is a pair (E, r), where E is a finite set and r is a rank function on 2^E, that is a function*

$$r : 2^E \longrightarrow \{0, 1, 2, \ldots\}$$

satisfying the following two conditions:

(i) $r(\emptyset) = 0$;

(ii) $r(X \cup e) = r(X)$ *or* $r(X) + 1$ *for all* $X \subseteq E$, $e \in E$.

Thus a matroid is a hypermatroid satisfying the additional axiom of local semimodularity [39, §3]:

If $r(X \cup e_1) = r(X \cup e_2) = r(X)$ *then* $r(X \cup e_1 \cup e_2) = r(X)$.

We will often use the less ambiguous notation $\mathcal{H}(E)$ and $r_{\mathcal{H}}$ for a hypermatroid and its rank function respectively. Figure 9.1 shows a hypermatroid on the Boolean algebra on three elements.

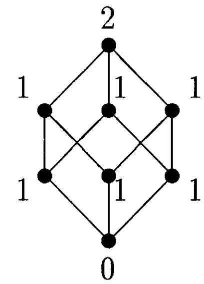

Figure 9.1: An example of a hypermatroid

The rank of a hypermatroid $\mathcal{H} = (E, r)$ is $r(E)$ and will also be denoted by $r(\mathcal{H})$. We will refer to the set E as the *edge set* of $\mathcal{H}$. The *restriction* of $\mathcal{H}$ on a subset F of E, or else the *deletion* of $E - F$ from $\mathcal{H}$ will be denoted by $\mathcal{H}(F)$. It is defined by restricting r on 2^F. We denote $\mathcal{H}(E - e)$ simply by $\mathcal{H} - e$. We call an element $e \in E$ an *isthmus* if $r(E - e) = r(E) - 1$ and a *loop* if $r(e) = 0$, as in the case of matroids. Note, however, that an element of E might be both an isthmus and a loop. The hypermatroid in Figure 9.1 has rank 2, no loops and three isthmuses.

We will denote by I, respectively by O, the hypermatroid with only one edge e satisfying $r(e) = 1$, respectively $r(e) = 0$. We use the letter O to denote the "loop" hypermatroid instead of the more common L, because L will stand for a lattice in the following sections. The *contraction* $\mathcal{H}/F$ is defined to be the hypermatroid on $E - F$ with rank function

$$r_{\mathcal{H}/F}(X) = r(X \cup F) - r(F)$$

for $X \subseteq E - F$. Thus,

$$r(E - e) = \begin{cases} r(E) - 1, & \text{if e is an isthmus;} \\ r(E), & \text{otherwise} \end{cases} \tag{9.1}$$

and

$$r_{\mathcal{H}/e}(X) = \begin{cases} r_{\mathcal{H}}(X \cup e), & \text{if e is a loop;} \\ r_{\mathcal{H}}(X \cup e) - 1, & \text{otherwise.} \end{cases} \tag{9.2}$$

A *hypermatroid isomorphism* between (E_1, r_1) and (E_2, r_2) is a bijection between E_1 and E_2 which induces a rank preserving isomorphism of the Boolean algebras 2^{E_1} and 2^{E_2}. An isomorphism f, defined on a class of hypermatroids closed under deletions and contractions, will be called a $Tutte-Grothendieck$ invariant, or simply a T-G invariant, if it satisfies the following conditions:

$$f(\mathcal{H}) = \begin{cases} f(\mathcal{H}-e)+f(\mathcal{H}/e), & \text{if } e \text{ is neither an isthmus nor a loop;} \\ (f(I)-1)f(\mathcal{H}-e)+f(\mathcal{H}/e), & \text{if } e \text{ is an isthmus but not a loop;} \\ f(\mathcal{H}-e)+(f(O)-1)f(\mathcal{H}/e), & \text{if } e \text{ is a loop but not an isthmus;} \\ (f(I)-1)f(\mathcal{H}-e)+(f(O)-1)f(\mathcal{H}/e), & \text{if } e \text{ is an isthmus and a loop.} \end{cases} \tag{9.3}$$

Here and in what follows, $e \in E$, the edge set of $\mathcal{H}$.

The rank generating polynomial. We define the two variable rank generating polynomial $S_{\mathcal{H}}$ as in [16, (6.6)]:

$$S_{\mathcal{H}}(x,y) = \sum_{X \subseteq E} x^{r(E)-r(X)} y^{\#X - r(X)}. \tag{9.4}$$

The following lemma can be proved exactly as in [16, Lemma 6.2.1]. We include the details for the sake of completeness.

Lemma 9.1.2 *$S_{\mathcal{H}}(x,y)$ is a T-G invariant for the class of all hypermatroids. Moreover,*

$$S_I(x,y) = x+1 \quad \text{and} \quad S_O(x,y) = y+1.$$

Proof: The last assertion is immediate. To prove the first, we break the sum in (9.4) into two parts.

$$S_{\mathcal{H}}(x,y) = \sum_{X \subseteq E-e} x^{r(E)-r(X)} y^{\#X-r(X)} + \sum_{\substack{X \subseteq E \\ e \in X}} x^{r(E)-r(X)} y^{\#X-r(X)}.$$

By (9.1), the first sum is $S_{\mathcal{H}-e}(x,y)$, multiplied by x in the case that e is an isthmus. Similarly, by (9.2), the second sum is $S_{\mathcal{H}/e}(x,y)$, multiplied by y in the case that e is a loop. This gives the result. □

The Tutte polynomial. We can now state the fundamental result of this section. It is the analogue of [16, Thm. 6.2.2] and shows that essentially, $S_{\mathcal{H}}(x,y)$ is the universal T-G invariant. It follows from Lemma 9.1.2 and straightforward induction arguments.

Theorem 9.1.3 *There is a unique function T from the class of all hypermatroids into the polynomial ring $\mathbb{Z}[x,y]$ having the following properties:*

(i) T is a T-G invariant.

(ii) $T_I(x, y) = x$ and $T_O(x, y) = y$.

In fact

$$T_{\mathcal{H}}(x, y) = S_{\mathcal{H}}(x-1, y-1). \tag{9.5}$$

Moreover, suppose that f is a function defined for any nonempty hypermatriod $\mathcal{H}(E)$ which takes values in a commutative ring R. Suppose also that f satisfies the T-G invariance conditions (9.3) for all hypermatroids with $\#E \geq 2$. Then for all nonempty hypermatroids $\mathcal{H}$,

$$f(\mathcal{H}) = T_{\mathcal{H}}(f(I), f(O)). \quad \square$$

We call $T_{\mathcal{H}}(x, y)$ the *Tutte polynomial* of $\mathcal{H}$. Thus, the Tutte polynomial of $\mathcal{H}(E)$ is given explicitly by the formula

$$T_{\mathcal{H}}(x, y) = \sum_{X \subseteq E} (x-1)^{r(E)-r(X)}(y-1)^{\#X-r(X)} \tag{9.6}$$

and satisfies the following conditions:

$$T_{\mathcal{H}}(x, y) = \begin{cases} T_{\mathcal{H}-e}(x, y) + T_{\mathcal{H}/e}(x, y), & \text{if } e \text{ is neither an isthmus nor a loop;} \\ (x-1)T_{\mathcal{H}-e}(x, y) + T_{\mathcal{H}/e}(x, y), & \text{if } e \text{ is an isthmus but not a loop;} \\ T_{\mathcal{H}-e}(x, y) + (y-1)T_{\mathcal{H}/e}(x, y), & \text{if } e \text{ is a loop but not an isthmus;} \\ (x-1)T_{\mathcal{H}-e}(x, y) + (y-1)T_{\mathcal{H}/e}(x, y), & \text{if } e \text{ is an isthmus and a loop.} \end{cases} \tag{9.7}$$

We will usually write $T_{\mathcal{H}}(x, y) = \sum_i \sum_j b_{ij} x^i y^j$ where the coefficients b_{ij} are integers, but not necessarily nonnegative, as is the case with the Tutte polynomial of a matroid. For example, for the hypermatroid in Figure 9.1 we have $T_{\mathcal{H}}(x, y) = x^2 + 3xy - 2x - 2y$.

Duality and direct sum. The notions of duality and direct sum for matroids generalize easily to hypermatroids. If $\mathcal{H} = (E, r)$ is a hypermatroid, then the function $r^*(X) = \#X + r(E - X) - r(E)$, defined for $X \subseteq E$, is a rank function on 2^E. Hence it defines a hypermatroid $\mathcal{H}^* = (E, r^*)$ which we call the *dual* of $\mathcal{H}$. The next proposition is immediate from the definitions and generalizes [16, Prop. 6.2.4]. Note also that $\mathcal{H}^{**} = \mathcal{H}$, i.e. the duality operation is an involution on the set of hypermatroids with edge set E.

Proposition 9.1.4 *For all hypermatroids $\mathcal{H}$,*

$$T_{\mathcal{H}^*}(x, y) = T_{\mathcal{H}}(y, x). \quad \square$$

The direct sum of two hypermatroids $\mathcal{H}_1 = (E_1, r_1)$ and $\mathcal{H}_2 = (E_2, r_2)$, where E_1 and E_2 are disjoint, is the hypermatroid on the edge set $E = E_1 \cup E_2$ with rank function defined by $r(X \cup Y) = r_1(X) + r_2(Y)$ for all $X \subseteq E_1$ and $Y \subseteq E_2$. We denoted it by $\mathcal{H}_1 \oplus \mathcal{H}_2$. As in the special case of matroids, the following proposition holds.

Proposition 9.1.5 *For hypermatroids $\mathcal{H}_1$ and $\mathcal{H}_2$ with disjoint edge sets,*

$$T_{\mathcal{H}_1 \oplus \mathcal{H}_2}(x, y) = T_{\mathcal{H}_1}(x, y)\, T_{\mathcal{H}_2}(x, y). \quad \square$$

Independence. A subset X of E, the ground set of $\mathcal{H} = (E, r)$, is *independent* if $r(X) = \#X$, otherwise $r(X) < \#X$ and X is *dependent*. Also, X is *spanning* if it has full rank, i.e. if $r(X) = r(E)$. A subset B of E is called a *basis* if it is both independent and spanning. As Figure 9.1 shows, not all hypermatroids have a basis. The independent subsets of E form a simplicial complex, that is an order ideal in the Boolean algebra 2^E, which we will refer to as the *independence complex*, or simply the *complex* of the hypermatroid $\mathcal{H}$ and denote it by $IN(\mathcal{H})$. Similarly, the spanning subsets form a dual order ideal and the bases an antichain.

Note that the independent sets in $\mathcal{H}$ are the complements of the spanning sets in $\mathcal{H}^*$ and that the bases of $\mathcal{H}$ are the complements of the bases of $\mathcal{H}^*$, for any $\mathcal{H}$. We say that $\mathcal{H}$ is a *hypergeometry* if all subsets of E having at most two elements are independent.

To illustrate the concepts introduced so far, consider the following two examples:

Example 9.1.6 Take our edge set to be the vertex set V of a finite graph G and for $X \subseteq V$, let $r(X)$ be the minimal number of stable subsets of V whose union is X. In other words, $r(X)$ is the chromatic number of the induced subgraph G_X. The corresponding hypermatroid contains no loops and its independent subsets are exactly the cliques of G. A basis is a clique whose size is the chromatic number of G. Thus, not all hypermatroids of this form have a basis.

Example 9.1.7 Now suppose that $\mathcal{G} = (E, \mathcal{F})$ is a greedoid (for basic definitions about greedoids we refer the reader to [10]). We construct a hypermatroid $\mathcal{H}(\mathcal{G})$ as follows: For $X \subseteq E$, let the rank of X in $\mathcal{H}(\mathcal{G})$ be the basis rank $\beta(X)$ in $\mathcal{G}$, defined as

$$\beta(X) = \max\{\#(X \cap F) : F \in \mathcal{F}\}.$$

It is easy to see that β has the unit increase property. The concepts of a loop, isthmus (coloop for greedoids), basis and spanning set are identical for $\mathcal{G}$ and the corresponding hypermatroid. The collection of independent subsets $IN(\mathcal{H}(\mathcal{G}))$ coincides with the order ideal in 2^E generated by the collection $\mathcal{F}$. It is an immediate consequence of [10, Thm. 8.6.2] that the Tutte polynomial of $\mathcal{H}(\mathcal{G})$

$$T(x,y) = T_{\mathcal{H}(\mathcal{G})}(x,y) = \sum_{X \subseteq E} (x-1)^{\beta(E)-\beta(X)}(y-1)^{\#X-\beta(X)}$$

has the property $T(1,y) = \lambda_{\mathcal{G}}(y)$, where $\lambda_{\mathcal{G}}$ is the Greedoid polynomial [10, §8.6]. Unfortunately, the contraction and deletion operations for greedoids and hypermatroids are not compatible with the correspondence described above and hence the two theories of Tutte invariants do not seem to be related.

9.2 Lattices and hypergraphs

As we remarked early in this chapter, the concept of a hypermatroid is too broad to be of independent interest. In this section we introduce the standard hypermatroids by assigning hypermatroids to lattices, or equivalently, to hypergraphs. This will give us a convenient way to restrict the class of hypermatroids and define the Tutte polynomial for any hypergraph H. We need to consider objects slightly more general than lattices to take care of the cases in which H is not an antichain.

Generalized lattices. Let L be a finite lattice with unique minimal element $\hat{0}$. Suppose also that E is a finite set and $f : E \longrightarrow L$ is a map. For $X \subseteq E$, we will denote by $\vee X$ the join of the elements of L in the image $f(X)$ and adopt the convention $\vee\emptyset = \hat{0}$. Inequalities of the type $\vee X \leq \vee Y$ will always refer to the partial order in L, sometimes denoted as $\leq_L$. We call the triple $\mathcal{L} = (L, E, f)$ a *generalized lattice.* We will often assume that the elements of $f(E)$ join-generate the lattice L, meaning that any element of L can be written as $\vee X$ for some $X \subseteq E$. We will call such a generalized lattice *atomic.* This assumption will usually be inessential, but it is reasonable since most of our constructions will depend only on the join-sublattice of L generated by $f(E)$. For instance, E would typically be a subset of L, like the set of atoms and f the inclusion map.

Definition 9.2.1 *Let $\mathcal{L} = (L, E, f)$ be a generalized lattice, as described above. For $X \subseteq E$ let $r(X)$ be the minimal k such that $\vee Y \geq \vee X$ for some $Y \subseteq E$ with $\#Y = k$. We call r the standard rank function of $\mathcal{L}$ and the corresponding hypermatroid the standard hypermatroid defined by $\mathcal{L}$. We also call a rank function on 2^E compatible (with $\mathcal{L}$) if the rank of $X \subseteq E$ depends only on $\vee X$.*

The standard rank function defined above, which is clearly compatible, is a rank function in the sense of Definition 9.1.1. Indeed, the first condition is trivial. To check the second, note that $r(X \cup e) \geq r(X)$ since $\vee(X \cup e) \geq \vee X$. Also, if $\vee Y \geq \vee X$ then

$\vee(Y \cup e) \geq \vee(X \cup e)$ and this implies $r(X \cup e) \leq r(X) + 1$. Note that the subset Y in Definition 9.2.1 has standard rank $r(Y) = \#Y$.

Our standard hypermatroids include all matroids. Indeed, if L is the geometric lattice associated to a matroid $M = (E, r)$ in the sense of [27] and f is the map sending an element of E to the corresponding flat in L, then $\mathcal{L} = (L, E, f)$ is atomic and its standard rank function agrees with the rank function r of M.

Hypergraphs. Now let $H = (E, V)$ be a hypergraph on a finite vertex set V, that is a finite collection E of nonempty subsets of V, called the edges of H. Let Π_V be the lattice of (set) partitions of V. The *lattice of contractions* or *bond lattice* L_H of H is the join-sublattice of Π_V generated by all partitions with a unique nonsingleton block $e \in H$, corresponding to an edge e of H with at least two elements. Note that L_H includes the empty join $\hat{0}$, the partition of V all of whose blocks are singletons. Thus, L_H is the set of all "connected" partitions of V, partially ordered by refinement.

We can also describe L_H as the intersection lattice of a certain subspace arrangement associated to H. Indeed, suppose we lebel the vertices of H so that $V = \{1, 2, \ldots, n\}$. If $\mathcal{A}_H$ is the arrangement of subspaces of $\mathbb{R}^n$ of the form

$$x_{i_1} = x_{i_2} = \cdots = x_{i_k},$$

where $\{i_1, i_2, \ldots, i_k\} \in E$, then L_H is isomorphic to the lattice of intersections of the subspaces in $\mathcal{A}_H$, ordered by reverse inclusion.

If $f_H : E \longrightarrow L_H$ associates to an edge e the partition of V having e as its only (possibly) nontrivial block, then we will refer to the standard rank function of the generalized lattice (L_H, E, f_H) as the standard rank function of H. We define the Tutte polynomial of the hypergraph $H = (E, V)$ to be the Tutte polynomial of the corresponding standard hypermatroid $\mathcal{H}$ on E and denote it by $T_H(x, y)$. Note that an edge of H is a singleton if and only if it is a loop of $\mathcal{H}$. Also, $\mathcal{H}$ is a hypergeometry if and only if H an antichain.

Note that, using (9.6), we get

$$n^{d-r(E)}\, t^{r(E)}\, T_H(1 + \frac{n}{t}, 1 + t) = \sum_{X \subseteq E} n^{d-r(X)}\, t^{\#X},$$

where r is our standard rank function and $d = \#V$, as opposed to Stanley's proposed

$$n^{c(E)}\, t^{r(E)}\, T_H(1 + \frac{n}{t}, 1 + t) = \sum_{X \subseteq E} n^{c(X)}\, t^{\#X},$$

where $c(X)$ stands for the number of connected components of the spanning subhypergraph of H with edge set E ([61, §3.3]). Of course, our Tutte polynomial reduces to the classical one when H is a graph.

Lattices. In the same way, we can define the standard hypermatroid and Tutte polynomial for any atomic lattice L by associating to it the generalized lattice (L, E, f), where E is the set of atoms of L and f the identity map. Note that, if H is an antichain containing no singletons, then (L_H, E, f_H) is the generalized lattice associated to the lattice of contractions L_H. From now and on, any concept we define for hypermatroids will be defined for atomic lattices and hypergraphs by passing to the corresponding standard hypermatroid.

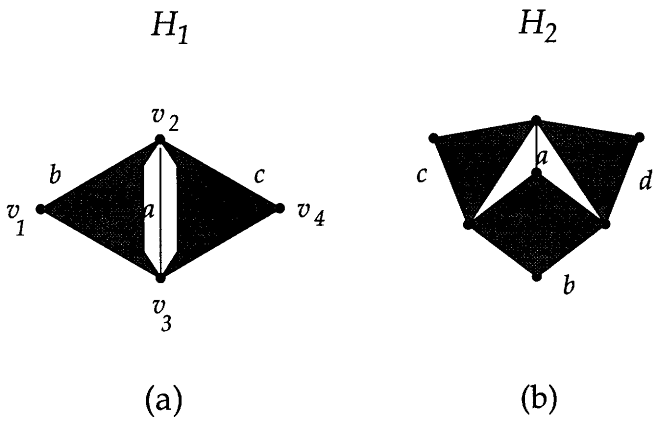

Figure 9.2: Two hypergraphs

Two examples. In general, as expected, the standard rank function of a hypergraph is not locally semimodular (see the note following Definition 9.1.1). For the hypergraph H_1 in Figure 9.2 (a), we have $r_1(a) = r_1(a \cup b) = r_1(a \cup c) = 1$, but $r_1(a \cup b \cup c) = 2$. Similarly for H_2 we have $r_2(a \cup b) = r_2(a \cup b \cup c) = r_2(a \cup b \cup d) = 2$, but $r_2(a \cup b \cup c \cup d) = 3$. The Tutte polynomials, computed from (9.6), are $T_{H_1}(x, y) = x^2 + 2xy - x - y$ and $T_{H_2}(x, y) = x^3 + x^2 + 2xy - x - y$. Unfortunately the class of standard hypermatroids is not closed under contraction. Indeed, H_1/a consists of two loops whose union has rank 1 and hence cannot be standard.

The hypergraph H_1 has the unique basis $\{b, c\}$ and the rest of its independent sets are $\emptyset$, $\{a\}$, $\{b\}$ and $\{c\}$. The hypergraph H_2 defines a hypergeometry with bases $\{a, c, d\}$ and $\{b, c, d\}$. Every basis is a maximal face of the complex $IN(\mathcal{H})$ but, as our examples show, not conversely. Thus $IN(\mathcal{H})$ need not be pure, as is the case for matroids [4, §3]. In general a hypermatroid need not have any bases, as we noted in §9.1. However, a hypermatroid defined by a standard generalized lattice $\mathcal{L} = (L, E, f)$ has always at least one basis. Indeed, if B is of minimal cardinality satisfying $\vee B = \vee E$, then B is independent and, by definition, of full rank.

Note that even if we consider only standard hypergraphs, we don't have any hope of achieving a reasonable exchange axiom for bases, as it happens with matroids. If, for instance, H is the hypergraph obtained from H_1 by adding the edge $d =$

$\{v_1, v_2, v_4\}$, then the corresponding standard hypermatroid has the disjoint sets $\{a, d\}$, $\{b, c\}$ as bases.

We call a hypergraph H *intersecting* if $A \cap B \neq \emptyset$ for all $A, B \in H$. In this case, all connected partitions of V have at most one nontrivial block and hence the standard rank function r on 2^E is easier to describe: For $X \subseteq E$, $r(X)$ is the smallest k for which $\cup X \subseteq \cup Y$ for some $Y \subseteq E$ with $\#Y = k$. As in Part II, we denote by $\cup X$ the union of the elements of X. In fact it is easy to see that the standard rank function of any generalized lattice is of this type.

Proposition 9.2.2 *Let $\mathcal{H}$ be the standard hypermatroid defined by the generalized lattice $\mathcal{L} = (L, E, f)$. Then there exists an intersecting hypergraph H whose standard hypermatroid is isomorphic to $\mathcal{H}$.*

Proof: Let $\hat{L}$ be the lattice obtained by adjoining another $\hat{0}$ element to L, so that $f(e) > \hat{0}$ in $\hat{L}$ for all $e \in E$. For $x \in L$, let $V_x = \{y \in L : y \geq x\}$ be the principal dual order ideal of L corresponding to x. Note that for all $x, y \in L$ we have $V_{x \vee y} = V_x \cap V_y$, and hence $\hat{L} - V_{x \vee y} = (\hat{L} - V_x) \cup (\hat{L} - V_y)$. Let H be the collection of sets $\hat{L} - V_{f(e)}$ for $e \in E$, all of which contain $\hat{0}$, with the standard rank function described above. Then the bijection mapping an element $e \in E$ to $\hat{L} - V_{f(e)}$ induces a rank preserving bijection of Boolean algebras, as required. $\square$

9.3 Tutte Invariants and the characteristic polynomial

In this section we discuss the basic Tutte invariants for hypermatroids. Let $b(\mathcal{H})$, $i(\mathcal{H})$ and $s(\mathcal{H})$ denote the number of bases, independent sets and spanning sets of the hypermatroid $\mathcal{H}$ respectively. The following proposition is a direct generalization of [16, Prop. 6.2.11]. It is an immediate consequence of the defining equation (9.4) for $S_{\mathcal{H}}(x, y)$ and (9.5).

Proposition 9.3.1 *The numbers $b(\mathcal{H})$, $i(\mathcal{H})$, $s(\mathcal{H})$ and $2^{\#E}$ are $T - G$ invariants. More specifically, we have*

(i) $b(\mathcal{H}) = T_{\mathcal{H}}(1,1) = S_{\mathcal{H}}(0,0)$;

(ii) $i(\mathcal{H}) = T_{\mathcal{H}}(2,1) = S_{\mathcal{H}}(1,0)$;

(iii) $s(\mathcal{H}) = T_{\mathcal{H}}(1,2) = S_{\mathcal{H}}(0,1)$;

(iv) $2^{\#E} = T_{\mathcal{H}}(2,2) = S_{\mathcal{H}}(1,1)$.

□

Let f be an isomorphism invariant defined on a class of hypermatroids closed under deletions and contractions. As in [16], we call f a T-G *group invariant*, or simply a group invariant, if it satisfies the first relation in (9.3). Examples of group invariants are the coefficients b_{ij} of $T_{\mathcal{H}}(x,y)$. Now let $i_k(\mathcal{H})$ denote the number of independent sets of $\mathcal{H}$ with k elements. We also write r for the rank $r(\mathcal{H})$ of $\mathcal{H}$. In the case of matroids, [16, Prop. 6.2.9] gives i_{r-k} explicitly in terms of the coefficients b_{ij}. The proof below is slightly simpler and extends to our more general setup.

Proposition 9.3.2 *Let k be a nonnegative integer. The quantity $i_{r-k}(\mathcal{H})$ is a group invariant and if $T_{\mathcal{H}}(x,y) = \sum_i \sum_j b_{ij}x^iy^j$, then*

$$i_{r-k}(\mathcal{H}) = \sum_i \sum_j b_{ij}\binom{i}{k}.$$

Proof: By (9.4), $i_{r-k}(\mathcal{H})$ is the coefficient of x^k in $S_{\mathcal{H}}(x,y)$ and hence a group invariant. Now, writing $S_{\mathcal{H}}(x,y) = \sum_i \sum_j b_{ij}(x+1)^i(y+1)^j$ and calculating the coefficient of x^k in the right hand side, we get the desired expression. □

We now discuss the linear equalities holding among the coefficients of the Tutte polynomial of a hypermatroid. Brylawski ([15, Prop. 6.3] [16, Thm. 6.2.13], see also [13]) has characterized all such equalities holding among the coefficients of the Tutte polynomial of a matroid. We observe here that the same equalities hold more generally for hypermatroids and hence the characterization carries over. We give a new simple proof of condition (vi) below, which avoids the use of induction.

Theorem 9.3.3 *The following identities form a basis for the affine linear equalities that hold among the coefficients b_{ij} of the Tutte polynomial*

$$T_{\mathcal{H}}(x,y) = \sum_{i\geq 0}\sum_{j\geq 0} b_{ij}x^iy^j,$$

where $\mathcal{H}$ is a hypergeometry of rank r with m edges, none of which is an isthmus.

(i) $b_{ij} = 0$ *for all* $i > r$ *and* $j \geq 0$;

(ii) $b_{r0} = 1$; $b_{rj} = 0$ *for all* $j > 0$;

(iii) $b_{r-1,0} = m - r$; $b_{r-1,j} = 0$ *for all* $j > 0$;

(iv) $b_{ij} = 0$ *for all* i *and* j *such that* $1 \leq i \leq r-2$ *and* $j \geq m-r$;

(v) $b_{0,m-r} = 1$; $b_{0j} = 0$ *for all* $j > m-r$;

(vi) $\sum_{s=0}^{k}\sum_{t=0}^{k-s}(-1)^t\binom{k-s}{t}b_{st}=0$ *for all* k *such that* $0\le k\le m-3$.

Moreover, (i) *holds for all hypermatroids* $\mathcal{H}(E)$, (ii) *holds for all loop-free* $\mathcal{H}(E)$, (iii) *holds for all hypergeometries* $\mathcal{H}(E)$, (iv) *and* (v) *hold for all* $\mathcal{H}(E)$ *with no isthmuses and* (vi) *holds for all* $\mathcal{H}(E)$ *with* $\#E>k$. *For* $k=0,1$ (vi) *gives* $b_{00}=0$ *if* $\#E\ge 1$ *and* $b_{10}=b_{01}$ *if* $\#E\ge 2$ *respectively.*

Proof: The conditions (i)-(v) are immediate from the definitions. We verify (vi). Note that

$$(1+x)^k\,T_{\mathcal{H}}\left(\frac{x}{1+x},-x\right)=(1+x)^k\sum_{s,t\ge 0}b_{st}x^s(1+x)^{-s}(-x)^t=$$

$$=\sum_{s,t\ge 0}b_{st}(-1)^t(1+x)^{k-s}x^{s+t}$$

and hence the coefficient of x^k in this expression is

$$\sum_{s,t\ge 0}b_{st}(-1)^t\binom{k-s}{k-s-t}=\sum_{\substack{s,t\ge 0\\ s+t\le k}}(-1)^t\binom{k-s}{t}b_{st}.$$

On the other hand,

$$(1+x)^k\,T_{\mathcal{H}}\left(\frac{x}{1+x},-x\right)=(1+x)^k\sum_{X\subseteq E}\left(\frac{-1}{1+x}\right)^{r(E)-r(X)}(-x-1)^{\#X-r(X)}=$$

$$=(-1)^{r(E)}(1+x)^{k-r(E)}\sum_{X\subseteq E}(-1)^{\#X}(1+x)^{\#X}=(-1)^{r(E)}(1+x)^{k-r(E)}(-x)^{\#E}.$$

The coefficient of x^k in the last expression is 0 if $\#E>k$, as desired. □

The characteristic polynomial. Suppose now that $\mathcal{H}$ is a hypermatroid defined by an atomic generalized lattice $\mathcal{L}=(L,E,f)$ and some rank function r. We define the characteristic polynomial of $\mathcal{H}$, on which we will focus in the next few sections, and relate it to the Tutte polynomial. We refer the reader to [73] for a brief exposition of the characteristic polynomial of a matroid and to [60] for background on Möbius functions. The definition of the Möbius function appears in §4.1 in this thesis.

Our main results about the characteristic polynomial in this and the next section continue to hold under the milder assumption that $f(E)$ contains the atoms of L. However, we find it convenient here to assume that $\mathcal{L}$ is atomic. This will not be restrictive for our purposes since the generalized lattices in which we are primarily interested in, the ones defined by hypergraphs, are atomic. Unfortunately, the characteristic polynomial $p_H(\lambda)$ we associate here to a hypergraph H which is not a graph, by passing to its associated hypermatroid, is unrelated to the characteristic polynomial $\chi(\mathcal{A}_H, \lambda)$ of the subspace arrangement $\mathcal{A}_H$ corresponding to H.

We first introduce some more of the language of matroid theory for atomic generalized lattices. To each element y of the lattice L we associate the subset $F_y = \{e \in E : f(e) \leq y\}$ of E and call it the *flat* corresponding to y. In what follows we do not distinguish y from its associated flat since this hardly creates any ambiguity. Indeed, the map $y \longrightarrow F_y$ is injective and $y \leq z$ in L if and only if $F_y \subseteq F_z$. Clearly, $F \subseteq E$ is a flat if and only if $F = \{e \in E : f(e) \leq \vee F\}$. We denote by $\mathcal{F}_{\mathcal{L}}$ the collection of flats of $\mathcal{L}$. Conversely, for any $F \subseteq E$ we denote by $\bar{F}$ the flat associated to $\vee F$ and call it the *closure* of F.

As in [73, §1], for $F \subseteq E$ and $G \subseteq E$ a flat, we define

$$\mu_{\mathcal{L}}(F,G) = \begin{cases} \mu_L(F,G), & \text{if } F \text{ is a flat;} \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial, as in [73, §2], is defined by the formula

$$p_{\mathcal{H}}(\lambda) = p_{\mathcal{L}}(\lambda) = \sum_{G \in \mathcal{F}_{\mathcal{L}}} \mu_{\mathcal{L}}(\emptyset, G)\ \lambda^{r(E)-r(G)}. \tag{9.8}$$

Note that $p_{\mathcal{H}}(\lambda) = 0$ if $f(e) = \hat{0}$ for some $e \in E$, since in this case, $\emptyset$ is not a flat. Such an e will necessarily be a loop if r is a compatible rank function, in the sense of Definition 9.2.1.

The characteristic polynomials of the standard hypermatroids defined in Figure 2 are $p_{H_1}(\lambda) = \lambda^2 - \lambda$ and $p_{H_2}(\lambda) = \lambda^3 - 4\lambda^2 + 4\lambda - 1$.

Proposition 9.3.4 *Let $F, G \subseteq E$ with $F \subseteq G$ and $G \in \mathcal{F}_{\mathcal{L}}$. Then*

$$\mu_{\mathcal{L}}(F,G) = \sum_{\substack{F \subseteq X \subseteq G \\ \vee X = G}} (-1)^{\#(X-F)}. \tag{9.9}$$

Proof: If F is not a flat, then there is an $e \in E - F$ such that $\vee(F \cup e) = \vee F$. Since G is a flat, $e \in G$. The involution $X \longrightarrow X \bigtriangleup e$ on the set of $X \subseteq E$ described under the sum in (9.9) shows that the sum is 0, as desired. Here $\bigtriangleup$ denotes symmetric difference. Suppose now that F is a flat and let $\nu(F,G)$ be the right hand side of (9.9). It suffices to check the defining relations (14) in [60, §3.7] of the Möbius function for the function ν. Clearly $\nu(F,F) = 1$, and for $F \neq G$,

$$\sum_{\substack{F \subseteq S \subseteq G \\ S \in \mathcal{F}_{\mathcal{L}}}} \nu(F,S) = \sum_{\substack{F \subseteq S \subseteq G \\ S \in \mathcal{F}_{\mathcal{L}}}} \sum_{\substack{F \subseteq X \subseteq S \\ \vee X = S}} (-1)^{\#(X-F)} =$$

$$= \sum_{F \subseteq X \subseteq G} (-1)^{\#(X-F)} = (1-1)^{\#(G-F)} = 0,$$

as desired. $\square$

Assume now that r is a compatible rank function, so that $r(X) = r(G)$, where G is the closure $\bar{X}$ of X. Plugging in (9.8) the expression for $\mu_{\mathcal{L}}(\emptyset, G)$ from (9.9), we get

$$p_{\mathcal{H}}(\lambda) = \sum_{G \in L} \sum_{\vee X = G} (-1)^{\# X} \lambda^{r(E)-r(G)} = \sum_{X \subseteq E} (-1)^{\# X} \lambda^{r(E)-r(X)} =$$

$$= (-1)^{r(E)} S_{\mathcal{H}}(-\lambda, -1) = (-1)^{r(E)} T_{\mathcal{H}}(1-\lambda, 0).$$

Thus we have deduced the following extension of [16, (6.20)].

Proposition 9.3.5 *Let $\mathcal{H}$ be a hypermatroid defined by a compatible rank function on an atomic generalized lattice. Then the characteristic polynomial of $\mathcal{H}$ is given by the formula*

$$p_{\mathcal{H}}(\lambda) = (-1)^{r(E)}\, T_{\mathcal{H}}(1-\lambda, 0). \tag{9.10}$$

$\square$

Chapter 10

NBC sets and the Möbius function

We say that the coefficients of the polynomial $p(t) = \sum_{i=0}^{m} a_i t^{m-i}$ (weakly) *alternate in sign* if $(-1)^i a_i \geq 0$. Note that the coefficients of p_{H_1} and p_{H_2}, computed in the previous section, alternate in sign. In this chapter we will see that this is true for a class of hypergraphs which strictly includes all graphs. We achieve this by generalizing Rota's NBC theorem [48] to a class of hypermatroids. Our treatment was inspired by the one given in [4] for the matroid case. A similar generalization was obtained independently by Sagan [49] and another one is included in [9, §3]. When restricted to standard lattices, our result turns out to be a special case of that of Sagan, as we comment later on. Sagan's proof is a more combinatorial version of ours, which follows the one given in [4, Prop. 7.4.5] and [9, Thm. 3.11]. Basic definitions and facts about simplicial complexes can be found in [4]. For information about broken circuit complexes of matroids we refer the reader to [4] and [14].

10.1 The NBC Theorem

Let $\mathcal{H}$ be the hypermatroid defined by an atomic generalized lattice $\mathcal{L} = (L, E, f)$ and some compatible rank function r and fix a total order ω of its edge set E. We denote by $\mathcal{H}(E, \omega)$ the hypermatroid $\mathcal{H}$ together with the order ω and we call it an *ordered hypermatroid.* The terms "minimal," "smallest" etc will refer to the total order ω, unless stated otherwise.

The broken circuit complex. A *circuit* of $\mathcal{H}(E, \omega)$ is a minimal (with respect to inclusion) dependent subset of E. A *broken circuit* is a circuit with its minimal element (with respect to ω) removed. Let $BC_\omega(\mathcal{H})$ be the collection of subsets of E that contain no broken circuit, also called *NBC sets.* It is easy to check that $BC_\omega(\mathcal{H})$ is a simplicial complex, called the *broken circuit complex* of $\mathcal{H}(E, \omega)$. It is a cone with apex the smallest element of E. Also note that $BC_\omega(\mathcal{H}) \subseteq IN(\mathcal{H})$, since faces of $BC_\omega(\mathcal{H})$ do not contain any circuits and thus are independent. For $G \in L$, let

$$NBC(G) = \{X \in BC_\omega(\mathcal{H}) : \vee X = G\}.$$

The following theorem is Rota's NBC result, as phrased by Björner in [4, Prop. 7.4.5]. It implies that the coefficients of the characteristic polynomial of any matroid alternate in sign; in fact that the Möbius function of a geometric lattice alternates in sign [48, §7, Thm. 4] [60, Prop. 3.10.1]. Recall from §9.2 that a matroid gives rise naturally to an atomic generalized lattice.

Theorem 10.1.1 *Let $\mathcal{L} = (L, E, f)$ come from a loop-free ordered matroid $M(E, \omega)$ with rank function r. Then, for any $G \in L$,*

$$\mu_L(\hat{0}, G) = (-1)^{r(G)} \# NBC(G). \quad \square$$

Our generalization can be stated as follows. We denote by $\leq_L$ the partial order of L to avoid confusion with the ordering ω of E.

Theorem 10.1.2 *Let $\mathcal{H}(E, \omega)$ be any ordered hypermatroid defined by an atomic generalized lattice $\mathcal{L} = (L, E, f)$ and a compatible rank function r. Suppose the total order ω is such that, for any circuit C of $\mathcal{H}$ with minimal edge e with respect to ω, we have*

$$e \leq_L \vee(C - e). \tag{10.1}$$

Then, for any $G \in L$,

$$\mu_{\mathcal{L}}(\emptyset, G) = (-1)^{r(G)} \# NBC(G). \tag{10.2}$$

Proof: The result is trivial if $\emptyset$ is not a flat. Indeed, in this case $\mathcal{H}$ has a loop and $\emptyset$ is a broken circuit, so there are no NBC sets. We now assume that $\mathcal{H}$ is loop-free and show that the right hand side of (10.2) satisfies the same recursion as $\mu_L(\hat{0}, G)$. It is clear now that $\# NBC(\emptyset) = 1$.

Let G be a nonempty flat and let $\mathcal{H}'(G, \omega')$ be the ordered hypermatroid obtained by restricting $\mathcal{H}$ to G and by restricting the ordering ω on G. We will show that a subset of G is a broken circuit of $\mathcal{H}$ if and only if it is a broken circuit of $\mathcal{H}'$. Indeed, suppose C is a circuit of $\mathcal{H}$ with minimal edge e and that $(C - e) \subseteq G$. By our assumption (10.1), we have $e \leq_L \vee(C - e) \leq_L G$, and hence $e \in G$. By the definition of ω', $C - e$ is a broken circuit in $\mathcal{H}'$. The other implication is obvious. Hence

$$BC'_\omega(\mathcal{H}') = \bigcup_{F \leq_L G} NBC(F),$$

where F runs through all flats contained in G and the union is disjoint. Since subsets of E which contain no broken circuit are independent, we have

$$\sum_{\hat{0} \leq_L F \leq_L G} (-1)^{r(F)} \# NBC(F) = -\chi(BC_{\omega'}(\mathcal{H}')),$$

where χ denotes Euler characteristic. But the Euler characteristic of any cone is 0 and this completes the proof. $\square$

Corollary 10.1.3 *Let $\mathcal{H}$ and ω be as in Theorem 10.1.2 and satisfy the same hypotheses. Then the coefficients of $p_{\mathcal{H}}$ alternate in sign. Their absolute values, i.e. the coefficients of $(-1)^{r(E)}p_{\mathcal{H}}(-\lambda) = T_{\mathcal{H}}(1+\lambda, 0)$ are the face numbers of the broken circuit complex $BC_\omega(\mathcal{H})$:*

$$(-1)^{r(E)}p_{\mathcal{H}}(-\lambda) = \sum_{i=0}^{r} f_i \lambda^{r(E)-i}, \tag{10.3}$$

where f_i denotes the number of faces of $BC_\omega(\mathcal{H})$ with cardinality i. $\square$

Consider once again the hypergraphs H_1 and H_2 of Figure 9.2. The circuits for H_1 are $\{a,b\}$ and $\{a,c\}$ and the ordering $a < b < c$ satisfies the hypotheses (10.1) of Theorem 10.1.2. Similarly, the circuits for H_2 are $\{a,b,c\}$ and $\{a,b,d\}$ and the ordering $a < b < c < d$ again satisfies (10.1). The broken circuits are $\{b,c\}$ and $\{b,d\}$. The corresponding broken circuit complex has faces $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a,b\}$, $\{a,c\}$, $\{a,d\}$, $\{c,d\}$, $\{a,c,d\}$ and hence it has face numbers $1,4,4,1$, as predicted by Corollary 10.1.3 and the computation $p_{H_2}(\lambda) = \lambda^3 - 4\lambda^2 + 4\lambda - 1$.

The argument in the proof of the next corollary is due to Brylawski ([14, Prop. 3.5]).

Corollary 10.1.4 *Let $\mathcal{H}$ and ω be as in Theorem 10.1.2 and satisfy the same hypotheses. Then, the number of ways to color the edges in E with the colors $0,1,\ldots,n$ so that no broken circuit is colored entirely with 0's is $n^{\#E-r(E)}\ T_{\mathcal{H}}(1+n,0)$.*

Proof: Using Corollary 10.1.3, the quantity

$$n^{\#E-r(E)}\ T_{\mathcal{H}}(1+n,0) = \sum_{i=0}^{r} f_i n^{\#E-i}$$

counts the number of ways to choose a face of $BC_\omega(\mathcal{H})$, color its vertices with 0 and color the rest of the elements of E with the colors $1,\ldots,n$. Here f_i has the same meaning as in Corollary 10.1.3. The assertion follows from the definition of the broken circuit complex. $\square$

We remark here that, given $\mathcal{H}$ as in the beginning of this section, it is quite simple to decide whether there exists an ordering of its edge set E which satisfies (10.1). Indeed, imagine that an element e of a circuit C is circled if (10.1) holds. Thus, we have a finite family $\mathcal{C}$ of sets with some of their elements circled and we want to check whether there exists a total order ω of the union E of the sets in $\mathcal{C}$, such that the minimal element in each set of $\mathcal{C}$ is circled. If such an order exists, then there should be an element e_1 of E which is circled in all sets of $\mathcal{C}$ in which it appears, since the minimal element with respect to ω would have this property. Now erase all sets in $\mathcal{C}$ which contain e_1. The new family $\mathcal{C}_1$ should also have the same property, that is an element e_2 of E which is circled in all sets of $\mathcal{C}_1$ in which it appears. In this way, we should be able to reach the empty collection starting with $\mathcal{C}$. Conversely, this fact guarantees that an ordering ω with the desirable property exists.

10.2 An application to hypergraphs

In this section we describe a class of hypergraphs for which an ordering of the edges satisfying the hypothesis of Theorem 10.1.2 exists.

We say that a generalized lattice $\mathcal{L} = (L, E, f)$ with standard rank function r is *normal*, if for all $X \subseteq E$, there exists a $Y \subseteq X$ with $\vee Y = \vee X$ and $\#Y = r(X)$. In other words, we demand that for each X, the subset Y in Definition 9.2.1 can be chosen to be a subset of X. A hypergraph is said to be normal if the corresponding generalized lattice (L_H, E, f_H) is so, and similarly for an atomic lattice L. The hypergraph of Figure 10.1 fails to be normal since the subset $X = \{a, b, c\}$ has rank 2, but no proper subset defines the same partition in the bond lattice as X. The next proposition gives a necessary condition for a standard generalized lattice to satisfy the hypothesis of Theorem 10.1.2.

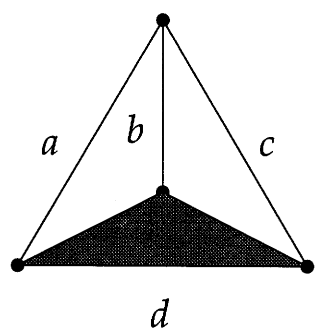

Figure 10.1: A hypergraph which is not normal

Proposition 10.2.5 *Let $\mathcal{H}(E,\omega)$ be a standard ordered hypermatroid defined by an atomic generalized lattice $\mathcal{L} = (L, E, f)$. If $\mathcal{H}$ satisfies condition (10.1) for all circuits C, then it is normal.*

Proof: Let X be a subset of E. We have to show that there exists an independent subset Y of X with $\vee Y = \vee X$. We do this by induction on $\#X$. If X is independent there is nothing to prove. Otherwise X is dependent ($\#X \geq 1$) and hence contains a circuit C. If e is the minimal element of C with respect to ω, then (10.1) implies that $\vee(C - e) = \vee C$ and hence that $\vee(X - e) = \vee X$. Induction applies to $X - e$ and completes the proof. $\square$

In view of the previous proposition, it is easy to see that the following is true: if $\mathcal{L}$ is standard, corresponding to the lattice L and together with ω satisfies the hypotheses of Theorem 10.1.2, then our notion of independence coincides with the one introduced by Sagan in [49]. That is, $X \subseteq E$ is independent if and only if $\vee Y <_L \vee X$ for all proper subsets Y of X. Hence Theorem 10.1.2 becomes a special case of [49, Thm. 1.2].

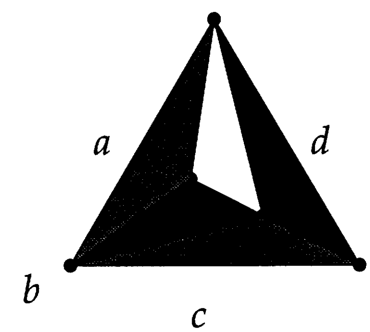

Figure 10.2: A hypergraph which is normal but does not satisfy (10.1)

The converse to Proposition 10.2.5 is not true, as Figure 10.2 shows. The hypergraph shown there has rank 2 and the circuits are the four subsets of $E = \{a, b, c, d\}$ with 3 elements. It is easily seen to be normal. If there existed a total order ω of E satisfying (10.1) for all circuits C, then the minimal element e of E should satisfy the relations $e \leq_L \vee\{f, g\}$ for all three 2- element subsets $\{f, g\}$ of $E - e$. This is impossible though, since the relations $a \leq_L \vee\{c, d\}$, $b \leq_L \vee\{c, d\}$, $c \leq_L \vee\{a, b\}$ and $d \leq_L \vee\{a, b\}$ are not valid in the lattice of contractions. The next result gives the class of hypergraphs promised in the beginning of this section.

Proposition 10.2.6 *Let $H = (E, V)$ be a normal hypergraph containing no loops. Assume that $r(X) = \#X$ for all $X \subseteq E$ which consist only of large edges, i.e. edges e with $\#e \geq 3$. Let ω be any total order of E in which every edge with two elements precedes any large edge. Then ω satisfies the hypothesis of Theorem 10.1.2. In particular, the coefficients of p_H alternate in sign.*

Proof: Let C be a circuit with minimal edge e. We have to show that (10.1) holds, where $L = L_H$.

If $X \subseteq E$, we say that a sequence of vertices $v_0, v_1, \ldots, v_k$ of H is an X-path joining v_0 with v_k if for all $1 \leq i \leq k$, $\{v_{i-1}, v_i\}$ is contained in some edge in X. Since H is normal and C has rank $|C| - 1$, there exists an $f \in C$, say $f = \{u_0, \ldots, u_m\}$, such that $f \leq_L \vee(C - f)$. By the definition of the lattice of contractions L of H, this means that for each $1 \leq i \leq m$, there exists a $(C - f)$-path joining u_{i-1} with u_i. Choose these paths to be of minimal length. We can assume without loss of generality that $m = 1$. Indeed, if not, by the independence assumption on the large edges, one of the paths has the form $u_{i-1}, \ldots, u, v, \ldots, u_i$ where uv (short for $\{u, v\}$) is a 2-edge in $C - f$. By minimality of the path, $v, \ldots, u_i, u_{i-1}, \ldots, u$ is a $(C - uv)$-path joining v with u and hence $uv \leq_L \vee(C - uv)$.

We now assume that $m = 1$, that is $f = \{u_0, u_1\}$. As already noted above, there exists a sequence of vertices $u_0 = w_0, w_1, \ldots, w_l = u_1$ with the w_i's all distinct, such that for each $1 \leq i \leq l$, $\{w_{i-1}, w_i\}$ is contained in e_i, where $e_i \in (C - f)$. Since C is a minimal dependent set, the e_i's are the only edges of $C - f$. Hence, either $e = f$ or $e = \{w_{i-1}, w_i\}$ for some $1 \leq i \leq l$ and we easily conclude as before that $e \leq_L \vee(C - e)$. $\square$

A typical situation in which the collection $\mathcal{E}$ of large edges of H satisfies the condition mentioned in Theorem 10.2.5 is when for each $f \in \mathcal{E}$, we have

$$f - \bigcup_{e \in \mathcal{E} - f} e \neq \emptyset.$$

Chapter 11

The Orlik-Solomon Algebra

The Orlik-Solomon algebra $\mathcal{A}(L)$ is an anticommutative L-graded algebra defined for each geometric lattice L. It was introduced by Orlik and Solomon in [40] and shown to be isomorphic to the cohomology ring of the complement in $\mathbb{C}^d$ to a finite union of central hyperplanes with intersection lattice L.

The basis theorem for $\mathcal{A}(L)$ [4, Thm. 7.10.2(ii)] (see also [9, Thm. 5.2]) asserts that a linear basis for $\mathcal{A}(L)$ is induced by the elements of the broken circuit complex corresponding to L. By construction, $\mathcal{A}(L)$ depends only on the collection $\mathcal{C}$ of circuits of L. In this section we show that the basis theorem remains valid if we replace the condition that $\mathcal{C}$ is the collection of circuits of a matroid with the slightly weaker condition described in Definition 11.1.1. The collection of circuits of a standard hypermatroid satisfying the hypothesis of Theorem 10.1.2 also satisfies this new condition. Thus we are able to define the Orlik-Solomon algebra for more general lattices than the geometric ones, in a way that preserves the basis theorem.

For more information about the Orlik-Solomon algebra of a geometric lattice L we refer the reader to [4, §11] and the references cited there. For a homological interpretation of $\mathcal{A}(L)$ related to the Whitney homology of L see also [4, §10].

11.1 Definition of $\mathcal{A}(L)$ and the basis theorem

Let E be a finite ground set and let ω be a total order of E. To make our notation below seem more reasonable, we will assume that $E = \{1, 2, \ldots, m\}$ where $1 < 2 < \cdots < m$ is the order ω. By abuse of notation, we denote by

$$\Lambda E = \bigoplus_{p=0}^{m} \Lambda^p E \tag{11.1}$$

the exterior algebra over $\mathbb{Z}$ of the free abelian group with basis $\{e_1, e_2, \ldots, e_m\}$. As usual, we write $e_A = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}$ whenever $A = \{i_1, i_2, \ldots, i_p\}$ with $1 \leq i_1 < i_2 < \cdots < i_p \leq m$, written shortly as $A = \{i_1, i_2, \ldots, i_p\}_<$. We denote i_1 by $\min A$, always refering to the total order ω.

For each $C = \{x_1, x_2, \ldots, x_k\} \subseteq E$ with $x_1 < x_2 < \cdots < x_k$, let

$$\partial(e_C) = \sum_{i=1}^{k} (-1)^i e_{C-x_i}.$$

Let $\mathcal{C}$ be a collection of nonempty subsets of E and let $I_\mathcal{C}$ be the ideal of ΛE generated by the elements $\partial(e_C)$ for all $C \in \mathcal{C}$. Thus $I_\mathcal{C}$ is linearly generated by the elements $e_S \wedge \partial(e_C)$, where $S \subseteq E$ and $C \in \mathcal{C}$. We define the *Orlik-Solomon algebra* of $\mathcal{C}$ as the quotient

$$\mathcal{A}(\mathcal{C}) = \Lambda E / I_\mathcal{C}.$$

Clearly,

$$\mathcal{A}(\mathcal{C}) = \bigoplus_{p=0}^{m} \mathcal{A}_p(\mathcal{C}), \tag{11.2}$$

where $\mathcal{A}_p(\mathcal{C}) = \Lambda^p E / (\Lambda^p E) \cap I_\mathcal{C}$, since each $\partial(e_C)$ is homogeneous with respect to the decomposition (11.1). If $\mathcal{C}$ is the collection of circuits of a geometric lattice L with atom set E, then this algebra is the classical Orlik-Solomon algebra of L.

The map ∂ extends, by linearity, to a map $\partial : \Lambda E \longrightarrow \Lambda E$ (where $\partial(e_\emptyset) = 0$ by convention). The identities

$$\partial^2 v = 0, \quad \partial(u \wedge v) = \partial u \wedge v + (-1)^p u \wedge \partial v \tag{11.3}$$

for $u \in \Lambda^p E$ and $v \in \Lambda E$ imply that ∂ preserves $I_\mathcal{C}$, and hence induces a map

$$\partial : \mathcal{A}(\mathcal{C}) \longrightarrow \mathcal{A}(\mathcal{C}).$$

One of the cryptomorphic ways of defining a matroid is to give axioms that its collection of circuits $\mathcal{C}$ should satisfy [39, Prop. 2.2.4]. The elimination axiom requires that for all $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq C_2$ and any $b \in E$, there exists a $C \in \mathcal{C}$ such that $C \subseteq C_1 \cup C_2 - \{b\}$. The condition described next is a weaker version of the elimination axiom. Here and in what follows, we use the notation $\bar{C} = C - \min C$ for all $C \subseteq E$.

Definition 11.1.1 *We say that $\mathcal{C} \subseteq 2^E$ satisfies the weak elimination axiom with respect to the total order ω of E, if for any $C_1, C_2 \in \mathcal{C}$ such that $\min C_1 \neq \min C_2$ and any $b \in \bar{C}_1 \cap \bar{C}_2$, there exists a $C \in \mathcal{C}$ such that*

$$C \subseteq C_1 \cup C_2 - \{b\}.$$

We define $BC_\omega(\mathcal{C})$ to be the collection of subsets of E that do not contain a "broken circuit" $\bar{C}$, $C \in \mathcal{C}$. As in [4], we denote by $\bar{e}_A$ the coset of e_A in $\mathcal{A}$. We now state the main result of this section.

Theorem 11.1.2 *Suppose that $\mathcal{C}$ satisfies the weak elimination axiom with respect to the total order ω on E. Then $\{\bar{e}_A : A \in BC_\omega(\mathcal{C})\}$ is a linear basis for $\mathcal{A}(\mathcal{C})$. In particular, $\mathcal{A}(\mathcal{C})$ is free as an abelian group.*

The following corollary describes a situation in which the weak elimination axiom is satisfied trivially.

Corollary 11.1.3 *Suppose that the sets in the collection $\mathcal{C}$ have the same minimal element in the order ω. Then $\{\bar{e}_A : A \in BC_\omega(\mathcal{C})\}$ is a linear basis for $\mathcal{A}(\mathcal{C})$. In particular, $\mathcal{A}(\mathcal{C})$ is free as an abelian group.*

The circuits $\{a, b, c\}$ and $\{a, b, d\}$ of the hypergraph H_2 of Figure 9.2 (b) satisfy trivially the weak elimination axiom with respect to the ordering $a < b < c < d$, but not its stronger matroid theoretic analogue. This is not an isolated example, as the next lemma shows.

Lemma 11.1.4 *The collection of circuits of a standard ordered hypermatroid that satisfies the hypothesis of Theorem 10.1.2 also satisfies the weak elimination axiom.*

Proof: To check this, suppose that $\mathcal{L}$ and r are as in Theorem 10.1.2. Suppose C_1, C_2 are circuits, $i_1 = \min C_1$, $i_2 = \min C_2$ and that $i_1 \neq i_2$, say $i_1 < i_2$. Let $b \in B = \bar{C}_1 \cap \bar{C}_2$. Condition (10.1) gives $i_1 \leq_L \vee\bar{C}_1$, $i_2 \leq_L \vee\bar{C}_2$. It follows that

$$\vee(C_1 \cup C_2 - \{b\}) \leq_L \vee(\bar{C}_1 \cup \bar{C}_2) = \vee(\bar{C}_1 \cup \bar{C}_2 - \{i_2\}).$$

Note that $\bar{C}_1 \cup \bar{C}_2 - \{i_2\}$ has one element less than $C_1 \cup C_2 - \{b\}$. Thus, by definition of the standard rank function, $C_1 \cup C_2 - \{b\}$ is dependent and hence contains some circuit C. $\square$

Clearly, we can associate an Orlik-Solomon algebra to any generalized lattice. We consider only lattices at this point for reasons of simplicity. The restriction that L is atomic is also not essential, but we include it here since the algebra $\mathcal{A}(L)$, defined for any lattice L, will depend only on the lattice join-generated by the atoms of L. So let L be a finite atomic lattice with set of atoms E. Let r be the standard rank function on 2^E associated to L and let $\mathcal{C}$ be the collection of circuits of the corresponding standard hypermatroid. Then we denote $\mathcal{A}(\mathcal{C})$ by $\mathcal{A}(L)$ and call it the Orlik-Solomon algebra of L. Note that $\mathcal{A}(L)$ in general is not L-graded, as it happens in the classical case [4, (7.52)], since the join of the elements of $C - x_i$, where C is a circuit, might as well depend on i when L is not geometric.

The following is an immediate corollary of Theorem 11.1.2 and Lemma 11.1.4. Again, ω is a total order of E and $BC_\omega(L)$ is the corresponding broken circuit complex.

Corollary 11.1.5 *Suppose that the total order ω on E satisfies the hypothesis of Theorem 10.1.2. Then $\{\bar{e}_A : A \in BC_\omega(L)\}$ is a linear basis for $\mathcal{A}(L)$. In particular, $\mathcal{A}(L)$ is free as an abelian group.*

We remark that the corresponding assertion fails to be true in Sagan's more general setup [49].

11.2 Proof of the basis theorem

One half of Theorem 11.1.2 is part of the following lemma. Its proof is the same as the one given in [4, Lemma 7.10.1], but we include it here as a warm up for the more involved proof of the other half of Theorem 11.1.2. Note that there is no assumption on $\mathcal{C}$ in this lemma. By abuse of language, we refer to the elements C of $\mathcal{C}$ as circuits and to the sets $\bar{C}$ as broken circuits.

Lemma 11.2.1

(i) $\bar{e}_A = 0$ *if A is dependent, i.e. if it contains a circuit.*

(ii) $\{\bar{e}_A : A \in BC_\omega(\mathcal{C})\}$ *linearly generates $\mathcal{A}(\mathcal{C})$.*

Proof: *(i)* A contains a circuit C. If t is any element of C, then $e_A = \pm e_t \wedge \partial(e_C) \wedge e_{A-C} \in I_\mathcal{C}$.

(ii) Suppose $u \in \Lambda E$ is written in the form $u = \sum_{A \subseteq E} a_A e_A$, where $a_A \in \mathbb{Z}$. Suppose that $a_A \neq 0$ for some A which contains a broken circuit, say $C - x_1$, where $C = \{x_1, x_2, \ldots, x_k\}_<$. Then the relation $\partial(e_C) \wedge e_{A-C} \in I_\mathcal{C}$ yields

$$\bar{e}_A = \sum_{i=2}^{k} (-1)^i \, \bar{e}_{(A \cup x_1) - x_i},$$

if x_1 is not in A and 0 otherwise. Thus we can express $\bar{e}_A$ in terms of elements $\bar{e}_B$ with B lexicographically smaller than A. After repeating this process a finite number of steps, we get an expression for u involving only monomials $\bar{e}_B$ with B containing no broken circuit. □

Before we go on with the proof of Theorem 11.1.2, we need to introduce a total order τ on $\Lambda^p E$, for each p. We first recall the definition of the antilexicographic order $<_{AL}$ on the set $\binom{E}{p}$ of p-element subsets of E, which is a total order on $\binom{E}{p}$. For $A = \{a_1, a_2, \ldots, a_p\}_<$ and $B = \{b_1, b_2, \ldots, b_p\}_<$, we have $A <_{AL} B$ if and only if $a_i < b_i$, where i is the largest index j for which $a_j \neq b_j$. We will also use below the following total order of the integers, denoted by $\trianglelefteq$, in which 0 is the minimal element:

$$0 \lhd -1 \lhd 1 \lhd -2 \lhd 2 \lhd \cdots .$$

Now, if $u, v \in \Lambda^p E$, $u = \sum_{A \subseteq E} a_A e_A$, $v = \sum_{A \subseteq E} b_A e_A$, we say that u is less than v in the order τ, written $u <_\tau v$, if $a_A \lhd b_A$ where A is the largest subset B, in the antilexicographic order of $\binom{E}{p}$, for which $a_B \neq b_B$. Note that any nonempty subset of $\Lambda^p E$ has a minimal element with respect to τ. The proof given below has a flavor of Kimmo Eriksson's theory of strongly convergent games [26].

Proof of Theorem 11.1.2: By Lemma 11.2.1 (ii) and the direct sum decomposition (11.2), it remains to show that $\{\bar{e}_A : A \in BC_\omega(\mathcal{C}), |A| = p\}$ is linearly independent for each $0 \leq p \leq m$. So from now and on we fix such a p.

Consider a loop-free graph $\mathcal{D}$ on the infinite vertex set $\Lambda^p E$. Two distinct vertices u, v are joined by an edge if and only if $u - v = \pm e_S \wedge \partial(e_C)$ for some circuit C and some $S \subseteq E$. A vertex of $\mathcal{D}$ will be called an *NBC vertex* if it belongs to the $\mathbb{Z}$-span of $\{e_A : A \in BC_\omega(\mathcal{C}), |A| = p\}$. We also consider an orientation o of $\mathcal{D}$ in which any edge between u and v is oriented "from u towards v" if and only if $v <_\tau u$. By a *path* in $\mathcal{D}$ we will mean an alternating sequence of vertices and edges $P = (u_0, \varepsilon_1, u_1, \ldots, \varepsilon_l, u_l)$, where the edge ε_i joins u_{i-1} and u_i for $1 \leq i \leq l$. We say that P has initial vertex u_0 and terminal vertex u_l. We will use the terms *forward path*, respectively *backward path*, when every edge ε_i is oriented by o from u_{i-1} towards u_i, respectively from u_i towards u_{i-1}. We will denote by ${}^tP$ the path $(u_l, \varepsilon_l, u_{l-1}, \ldots, \varepsilon_1, u_0)$ and call it the *reverse* of P, so that the reverse of a forward path is backward and vice versa.

Note that u and v are equal in $\mathcal{A}(\mathcal{C})$ if and only if there is a path in $\mathcal{D}$ joining u and v. Thus, the connected components of $\mathcal{D}$ correspond to the elements of $\mathcal{A}(\mathcal{C})$. Another fact that will be of importance to us is that for any vertex u of $\mathcal{D}$ there is a forward path in $\mathcal{D}$ with initial vertex u and terminal vertex an NBC vertex. This follows immediately from the proof of Lemma 11.2.1 (ii). Lastly, we remark that any NBC vertex u has outdegree 0 with respect to o. Indeed, if $\pm e_S \wedge \partial(e_C)$ is nonzero, then it contains a monomial $\pm e_G$, where G contains the broken circuit $\bar{C}$, and other monomials e_F (if $S \cap C = \emptyset$) with $F <_{AL} G$. Since the coefficient of e_G in u is zero, by adding $\pm e_S \wedge \partial(e_C)$ to u we can only go higher in the order τ.

To prove our theorem, it suffices to show that no two distinct NBC vertices u and v are connected in $\mathcal{D}$. So we assume that such vertices exist and obtain a contradiction.

Let $P = (u = u_0, \varepsilon_1, u_1, \ldots, \varepsilon_l, u_l = v)$ be a path joining u and v, where ε_i is an edge joining u_{i-1} with u_i.

Claim 1 *For some i with $0 < i < l$ there exist two forward paths in $\mathcal{D}$ which have a common initial vertex u_i and whose terminal vertices are two distinct NBC vertices of $\mathcal{D}$.*

Proof of Claim 1: Since $u_l = v$ is an NBC vertex, ε_l is oriented from u_{l-1} towards u_l. Let t be the smallest index for which $P_{tl} = (u_t, \varepsilon_{t+1}, \ldots, \varepsilon_l, u_l)$ is a forward path and let s be the smallest index for which $s \leq t$ and $P_{st} = (u_s, \varepsilon_{s+1}, \ldots, \varepsilon_t, u_t)$ is a backward path. Let Q be a forward path with initial vertex u_s and terminal vertex an NBC vertex w. If $w \neq v$, then the two paths P_{tl} and ${}^tP_{st}$ followed by Q give the desired pair of forward paths, starting at u_t. Note that, if $s = 0$, we can choose Q to be of zero length and $w = u_0 = u \neq v$, so the same argument applies. Assume now that $s > 0$ and $w = v$. Then $w \neq u$. Thus, we can replace the parts P_{st} and P_{tl} of P with Q to get a new path P' joining two distinct NBC vertices, namely u and v. The value of s, defined as above, is smaller for P' than for P and hence induction completes the proof of Claim 1. $\square$

Now let $\tilde{u}$ be the minimal vertex of $\mathcal{D}$ in the order τ for which there exist forward paths P_1 and P_2 with initial vertex $\tilde{u}$ and terminal vertices two distinct NBC vertices v_1 and v_2. Suppose that the first edges γ_1 and γ_2 of P_1, P_2 join $\tilde{u}$ to w_1 and w_2 respectively.

Claim 2 *There exist paths Q_1, Q_2 with initial vertices w_1 and w_2 respectively and common terminal vertex, such that all the vertices visited by Q_1 and Q_2 precede $\tilde{u}$ in the order τ.*

Before proving Claim 2, we show how it implies our theorem. Let $\tilde{w}$ be the common endpoint of Q_1 and Q_2 and let Q be a forward path with initial vertex $\tilde{w}$ and terminal vertex the NBC vertex $\tilde{v}$. Since $v_1 \neq v_2$, $\tilde{v}$ is different from one of the two, say $\tilde{v} \neq v_1$. Let $\tilde{P}_1$ be the path obtained from P_1 by deleting its first vetrex and edge. Thus $\tilde{P}_1$ has initial vertex w_1. The path ${}^t\tilde{P}_1$ followed by Q_1 and Q gives a path R in $\mathcal{D}$ which joins the distinct NBC vertices v_1 and $\tilde{v}$ and whose vertices precede $\tilde{u}$ in the order τ. Then, Claim 1 guarantees that one of the vertices of R violates the minimality condition imposed on $\tilde{u}$, and thus gives the desired contradiction.

Proof of Claim 2: Let $w_i = \tilde{u} + x_i$, where $x_i = \pm e_{S_i} \wedge \partial(e_{C_i})$ for $i = 1, 2$. By assumption, $w_i <_\tau \tilde{u}$. As we have commented earlier, any nonzero element of $\Lambda^p E$ of the form $\pm e_S \wedge \partial(e_C)$ contains a monomial e_G, where G contains the broken circuit $\bar{C} = C - \min C$, and other monomials e_F (if $S \cap C = \emptyset$) with $F <_{AL} G$. We can always assume that $G = S \cup \bar{C}$. Indeed, this is clear if $S \cap C = \emptyset$. If not, then

$S \cap C = \{c\}$ and $\pm e_S \wedge \partial(e_C) = \pm e_{S-c} \wedge e_C = \pm e_{S'} \wedge \partial(e_C)$, where

$$S' = \begin{cases} S, & \text{if } c = \min C; \\ (S - c) \cup \min C, & \text{otherwise} \end{cases}$$

and hence $G = S' \cup \bar{C}$.

Let e_{G_1} and e_{G_2} be the monomials corresponding to x_1 and x_2 respectively. Thus, the coefficient of e_{G_i} is smaller, with respect to $\lhd$, in w_i than it is in $\tilde{u}$, for $i = 1, 2$. We distinguish two cases.

Case 1: $G_1 \neq G_2$, say $G_1 <_{AL} G_2$. Then it is easy to see that $\tilde{u} + x_1 = w_1 >_\tau w_1 + x_2$. If also $w_2 >_\tau w_1 + x_2$, then we can choose Q_1, Q_2 to be forward paths with vertices $w_1, w_1 + x_2$ and $w_2, w_1 + x_2$ in order, respectively. If $w_2 <_\tau w_1 + x_2$ we can choose Q_1 to be the forward path with vertices $w_1, w_1 + x_2, w_2$ in order and Q_2 a path of zero length.

Case 2: $G_1 = G_2 = G$. Let $i = \min C_1$, $j = \min C_2$. We use again the notation $\bar{C}_1 = C_1 - i$, $\bar{C}_2 = C_2 - j$, so that $G = S_1 \cup \bar{C}_1 = S_2 \cup \bar{C}_2$. We write

$$G = T \cup A_1 \cup B \cup A_2 \quad \text{(disjoint union)},$$

where $\bar{C}_1 = A_1 \cup B$, $\bar{C}_2 = B \cup A_2$. Thus $B = \bar{C}_1 \cap \bar{C}_2$, $S_1 = T \cup A_2$, $S_2 = T \cup A_1$. Let $\# A_1 = k$. We distinguish two subcases.

Case 2 (i): $i \neq j$, say $i < j$. Since $C_1 = i \cup A_1 \cup B$, $C_2 = j \cup B \cup A_2$, we can write

$$x_1 = \sigma_1 \; e_T \wedge \partial(e_i \wedge e_{A_1} \wedge e_B) \wedge e_{A_2}$$

and

$$x_2 = \sigma_2 \; e_T \wedge e_{A_1} \wedge \partial(e_j \wedge e_B \wedge e_{A_2}),$$

where $\sigma_1, \sigma_2 = \pm 1$. Since both edges γ_1, γ_2 are forward edges with common initial vertex $\tilde{u}$, the coefficients of e_G in x_1 and x_2 should be equal, so $\sigma_1 = \sigma_2$ and we can assume for convenience that they are both 1.

Using formulas (11.3) and a straightforward computation, we find that

$$\tilde{u} + x_1 + y_1 + z_1 = \tilde{u} + x_2 + y_2 = \tilde{w}, \tag{11.4}$$

where

$$\begin{aligned} y_1 &= e_T \wedge e_i \wedge \partial e_{A_1} \wedge \partial(e_j \wedge e_B \wedge e_{A_2}), \\ z_1 &= (-1)^{k+1} e_T \wedge \partial(e_i \wedge e_j \wedge e_{A_1} \wedge \partial(e_B) \wedge e_{A_2}), \\ y_2 &= e_T \wedge \partial(e_i \wedge e_{A_1} \wedge e_B) \wedge e_j \wedge \partial(e_{A_2}). \end{aligned}$$

Since $i = \min C_1$ and $j = \min C_2$, y_1 and y_2 can be written as sums of elements of the form $\pm e_S \wedge \partial(e_C)$ with C a circuit, such that all monomials e_F that appear in these sums satisfy $F <_{AL} G$. Thus, if we show that the same is true for z_1, we get two paths Q_1 and Q_2 with initial vertices $w_1 = \tilde{u} + x_1$ and $w_2 = \tilde{u} + x_2$ and terminal vertex $\tilde{w}$, whose vertices precede $\tilde{u}$ in the order τ, as required in Claim 2.

We consider z_1. It is 0 if B is empty, otherwise it is an integer linear combination of

$$z_{1q} = e_T \wedge \partial(e_i \wedge e_j \wedge e_{A_1} \wedge e_{B_q} \wedge e_{A_2}),$$

where B_q is B with its q^{th} element removed. Since $i < j$, weak elimination implies that $D_q = i \cup j \cup A_1 \cup B_q \cup A_2$ is dependent. If we write $D_q = F \cup C$ where C is a circuit and $t \in C$, then

$$\partial(e_{D_q}) = \partial(e_F) \wedge e_C \ \pm\ e_F \wedge \partial(e_C) = \pm\partial(e_F) \wedge e_t \wedge \partial(e_C) \ \pm\ e_F \wedge \partial(e_C)$$

is a linear combination of elements of the form $\pm e_S \wedge \partial(e_C)$. Hence, so is each z_{1q}. Moreover, all monomials $e_{\hat{D}_q}$ appearing in the last sum, where $\hat{D}_q$ stands for D_q with an arbitrary element removed, still satisfy $T \cup \hat{D}_q <_{AL} G$, as desired.

Case 2 (ii) $i = j$. The same proof works. It is much easier now to check (11.4) since $z_1 = 0$ and y_1, y_2 expanded have a much simpler form. □

This completes the proof of the theorem. □

11.3 Epilogue

As we have seen in the last three chapters, our generalized Tutte polynomial preserves many important properties of the classical one. It fails, though, to preserve other properties and it seems unfair not to mention them here.

The chromatic polynomial $\chi_\Gamma(\lambda)$ of a graph Γ is, except for a power of λ, the characteristic polynomial of the corresponding matroid [16, Prop. 6.3.1] and hence a specialization of the Tutte polynomial of Γ. We don't see how to generalize this result to hypergraphs. The paper [22], which recently came to our attention, introduces a different approach to this problem. Furthermore, results related to orientations of graphs, like Stanley's theorem about the number of acyclic orientations of a graph [16, Prop. 6.3.17] [58, Cor. 1.3] do not seem to generalize to our setup for hypergraphs. Also, the hypergraph of Figure 9.2 (b) seems to be a counterexample to a possible extension of Stanley's factorization theorem for supersolvable lattices [57, Thm. 4.1] (see also [56, Thm. 2]). Such an extension was obtained by Blass and Sagan [12, §6] by using a suitable notion of rank.

It might be an interesting problem to characterize the standard rank functions on a finite Boolean algebra, or at least to study the complexity of the problem to decide when a given rank function is standard. Similar problems in the theory of hyperplane arrangements have been proven to be NP-hard (see [5, §4.1]).

It could also be quite interesting to find any homological properties of our generalized Orlik-Solomon algebra. We hope that this algebra can be related to the theory of subspace arrangements [5] (see also Part II in this thesis) for some very special arrangements, but we have no indication, other than Corollary 11.1.5, that such a connection might exist. We remark here that a simpler proof of Theorem 11.1.2 would be desirable.

Blass and Sagan [12] have given a generalization of the NBC theorem that improves by far the one that appears in [49]. This theorem is valid for any lattice L and was obtained by replacing the total order on the atom set of L by an arbitrary partial order.

Lastly, we mention a problem in asymptotic combinatorics suggested by Definition 2.1. Let $R(n)$ be the number of rank functions on the Boolean algebra B_n on a set with n elements. It is quite simple to show that $R(n) \geq 2^{2^{n-1}}$ by assigning rank i to each set having cardinality $2i$, where $i \leq n/2$. The first few values of R are shown in the following table:

n	1	2	3	4	5
$R(n)$	2	6	38	990	395094

The only other information we have for the asymptotics of $R(n)$ is the following: If $R(n) = 2^{c_n 2^{n-1}}$ then the sequence $\{c_n\}$ is decreasing for $n \geq 3$. Thus $R(n) \leq 2^{c_5 2^{n-1}}$ for $n \geq 5$, where $c_5 = 1.161\ldots$ All these facts seem to support the following conjecture.

Conjecture 11.3.1

$$R(n) = 2^{2^{n-1}(1+o(1))}.$$

In other words, $\{c_n\}$ *converges to* 1.

The number of rank functions on B_n taking only the values 0 and 1 is one less than the number of order ideals of B_n, a quantity that has been studied a lot in the past (see for example [35]).

"No eternal reward will forgive us now for wasting the dawn."

James Douglas Morrison

Bibliography

[1] R. M. ADIN AND C. A. ATHANASIADIS, On Hanlon's eigenvalue conjecture, *J. Combin. Theory Ser. A* **73** (1996), 360–367.

[2] V. ANANTHARAM AND P. TSOUCAS, A proof of the Markov chain tree theorem, *Stat. Probab. Letters* **8** (1989), 189–192.

[3] M. BARNABEI, G. NICOLETTI AND L. PEZZOLI, The symmetric exchange property for poset matroids, *Advances in Math.* **102** (1993), 230–239.

[4] A. BJÖRNER, The homology and shellability of matroids and geometric lattices, Chapter 7 in *Matroid Applications,* N. White ed., Cambridge University Press, Cambridge, 1991, pp. 226–283.

[5] A. BJÖRNER, Subspace arrangements, in *Proc. of the first European Congress of Mathematics, Paris 1992,* A. Joseph et al., Progress in Math. **119**, Birkhäuser 1994, pp. 321–370.

[6] A. BJÖRNER AND T. EKEDAHL, Subspace arrangements over finite fields: cohomological and enumerative aspects, in preparation.

[7] A. BJÖRNER AND L. LOVAŚZ, Linear decision trees, subspace arrangements and Möbius functions, *J. Amer. Math. Soc.* **7** (1994), 677–706.

[8] A. BJÖRNER AND M. WACHS, Shellable nonpure complexes and posets, I, *Trans. Amer. Math. Soc.*, to appear.

[9] A. BJÖRNER AND G. M. ZIEGLER, Broken circuit complexes: Factorizations and generalizations, *J. Combin. Theory Ser. B* **51** (1991), 96–126.

[10] A. BJÖRNER AND G. M. ZIEGLER, Introduction to Greedoids, Chapter 8 in *Matroid Applications,* N. White ed., Cambridge University Press, Cambridge, 1991, pp. 284–357.

[11] A. BLASS AND B. E. SAGAN, Characteristic and Ehrhart polynomials, *Trans. Amer. Math. Soc.*, submitted.

[12] A. Blass and B. E. Sagan, Möbius functions of lattices, preprint dated December 1, 1995.

[13] T. Brylawski, A decomposition for combinatorial geometries, *Trans. Amer. Math. Soc.* **171** (1972), 235–282.

[14] T. Brylawski, The broken-circuit complex, *Trans. Amer. Math. Soc.* **234** (1977), 417–433.

[15] T. Brylawski, The Tutte polynomial I: General Theory, in *Matroid Theory and its Applications (Proc. C. I. M. E., Varenna, 1980),* Naples 1982, 125–276.

[16] T. Brylawski and J. Oxley, The Tutte polynomial and its applications, Chapter 6 in *Matroid Applications, N. White ed.,* Cambridge University Press, Cambridge, 1991, pp. 123–225.

[17] S. Chaiken and D. J. Kleitman, Matrix tree theorems, *J. Combin. Theory Ser. A* **24** (1978), 377–381.

[18] T. Chow, Spectra and complexity of periodic strips, unpublished manuscript.

[19] H. H. Crapo, The Tutte polynomial, *Aequationes Math.* **3** (1969), 211–229.

[20] H. Crapo and G.-C. Rota, *On the Foundations of Combinatorial Theory: Combinatorial Geometries*, preliminary edition, M.I.T. press, Cambridge, MA, 1970.

[21] D. Cvetković, M. Doob and H. Sachs, *Spectra of graphs*, Academic Press, New York, 1979.

[22] K. Dohmen, A Broken-Circuits-Theorem for hypergraphs, *Arch. Math.* **64** (1995), 159–162.

[23] A. Edelman, private communication.

[24] P. H. Edelman and V. Reiner, Free arrangements and rhombic tilings, *Discr. Comp. Geometry,* to appear.

[25] N. Elkies, G. Kuperberg, M. Larsen and J. Propp, Alternating sign matrices and domino tilings, *J. Alg. Combin.* **1** (1992), 111–132.

[26] K. Eriksson, Strong convergence and the polygon property of 1-player games, in *Formal Power Series and Algebraic Combinatorics,* Proceedings of the Fifth Conference (Florence, June 21-25, 1993), A. Barlotti, M. Delest and R. Pinzani eds., University of Florence, pp. 187–200.

[27] U. Faigle, Lattices, Chapter 3 in *Theory of Matroids,* N. White ed., Cambridge University Press, Cambridge, 1986, pp. 45–61.

[28] I. M. Gelfand, M. I. Graev and A. Postnikov, Hypergeometric functions associated with positive roots, in *Arnold-Gelfand Math Seminars 1993-95*, to appear.

[29] C. D. Godsil, *Algebraic combinatorics*, Chapman & Hall, New York, 1993.

[30] M. D. Haiman, Conjectures on the quotient ring by diagonal invariants, *J. Alg. Combin.* **3** (1994), 17–76.

[31] P. Hanlon, Some remarkable combinatorial matrices, *J. Combin. Theory Ser. A* **59** (1992), 218–239.

[32] P. Headley, On reduced words in affine Weyl groups, in *Formal Power Series and Algebraic Combinatorics, May 23-27, 1994*, DIMACS, pp. 225–242.

[33] P. Headley, *Reduced expressions in infinite Coxeter groups*, Ph.D. thesis, University of Michigan, 1994.

[34] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics **29**, Cambridge University Press, Cambridge, England, 1990.

[35] D. Kleitman and G. Markowsky, On Dedekind's problem: The number of isotone boolean functions. II, *Trans. Amer. Math. Soc.* **213** (1975), 373–390.

[36] B. Kostant, Lie algebra cohomology and the generalized Borel-Weil theorem, *Ann. of Math.* **74** (1961), 329–387.

[37] P. McMullen, On zonotopes, *Trans. Amer. Math. Soc.* **159** (1971), 91–109.

[38] R. Merris, Laplacian matrices of graphs: A survey, *Linear Algebra Appl.* **197**, **198** (1994), 143–176.

[39] G. Nicoletti and N. White, Axiom systems, Chapter 2 in *Theory of Matroids,* N. White ed., Cambridge University Press, Cambridge, 1986, pp. 29–44.

[40] P. Orlik and L. Solomon, Combinatorics and topology of complements of hyperplanes, *Invent. Math.* **56** (1980), 167–189.

[41] P. Orlik and H. Terao, *Arrangements of Hyperplanes*, Grundlehren 300, Springer-Verlag, New York, NY, 1992.

[42] W. V. Parker, The matrices AB and BA, *Amer. Math. Monthly* **60** (1953), p. 316.

[43] L. Pezzoli, On D-complementation, *Advances in Math.* **51** (1984), 226–239.

[44] A. Postnikov, Intransitive trees, preprint dated November 1, 1994.

[45] A. Postnikov and R. P. Stanley, Deformed Coxeter hyperplane arrangements, in preparation.

[46] J. Propp, private communication.

[47] J. Propp, Lattice structure for orientations of graphs, unpublished manuscript dated October 1, 1993.

[48] G.-C. Rota, On the foundations of combinatorial theory I. Theory of Möbius functions, *Z. Wahrscheinlichkeitstheorie* **2** (1964), 340–368.

[49] B. E. Sagan, A generalization of Rota's NBC theorem, *Advances in Math.*, to appear.

[50] B. E. Sagan, Why the characteristic polynomial factors, preprint dated September 24, 1995.

[51] J. Schmid, A remark on characteristic polynomials, *Amer. Math. Monthly* **77** (1970), 998–999.

[52] W. M. Scott, On characteristic roots of matrix products, *Amer. Math. Monthly* **48** (1941), 201–203.

[53] J.-Y. Shi, *The Kazhdan-Lusztig cells in certain affine Weyl groups*, Lecture Notes in Mathematics, no. 1179, Springer-Verlag, Berlin/Heidelberg/New York, 1986.

[54] J.-Y. Shi, Sign types corresponding to an affine Weyl group, *J. London Math. Soc.* **35** (1987), 56–74.

[55] R. P. Stanley, *Ordered structures and partitions*, Ph.D. thesis, Harvard Univ., Cambridge, MA, 1971.

[56] R. P. Stanley, Modular elements of geometric lattices, *Algebra Universalis* **1** (1971), 214–217.

[57] R. P. Stanley, Supersolvable lattices, *Algebra Universalis* **2** (1972), 197–217.

[58] R. P. Stanley, Acyclic orientations of graphs, *Discrete Math.* **5** (1973), 171–178.

[59] R. P. Stanley, A zonotope associated with graphical degree sequences, in *Applied Geometry and Discrete Combinatorics*, DIMACS Series in Discrete Mathematics, vol. 4, 1991, pp. 555–570.

[60] R. P. Stanley, *Enumerative Combinatorics*, vol. 1, Wadsworth & Brooks/Cole, Belmont, CA, 1992.

[61] R. P. STANLEY, Graph colorings and related symmetric functions: Ideas and applications, *Discrete Math.*, to appear.

[62] R. P. STANLEY, Hyperplane arrangements, interval orders and trees, *Proc. Nat. Acad. Sci. U.S.A.*, to appear.

[63] R. P. STANLEY, *Enumerative Combinatorics*, vol. 2, in preparation.

[64] H. TERAO, Arrangement of hyperplanes and their freeness I, II, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **27** (1980), 293–320.

[65] H. TERAO, Generalized exponents of a free arrangement of hyperplanes and the Shepherd-Todd-Brieskorn formula, *Invent. Math.* **63** (1981), 159–179.

[66] H. S. THURSTON, On the characteristic equations of products of square matrices, *Amer. Math. Monthly* **38** (1931), 322–324.

[67] W. T. TUTTE, A contribution to the theory of chromatic polynomials, *Canad. J. of Math.* **6** (1953), 80–91.

[68] H. WHITNEY, A logical expansion in mathematics, *Bull. Amer. Math. Soc.* **38** (1932), 572–579.

[69] T. ZASLAVSKY, Facing up to arrangements: face-count formulas for partitions of space by hyperplanes, *Mem. Amer. Math. Soc.* vol. 1, no. 154, (1975).

[70] T. ZASLAVSKY, The geometry of root systems and signed graphs, *Amer. Math. Monthly* **88**, (1981), 88–105.

[71] T. ZASLAVSKY, Signed graph coloring, *Discrete Math.* **39**, (1982), 215–228.

[72] T. ZASLAVSKY, Chromatic invariants of signed graphs, *Discrete Math.* **42**, (1982), 287–312.

[73] T. ZASLAVSKY, The Möbius function and the characteristic polynomial, Chapter 7 in *Combinatorial Geometries,* N. White ed., Cambridge University Press, Cambridge, 1987, pp. 114–138.

[74] T. ZASLAVSKY, Orientation of signed graphs, *European J. Combin.* **12**, (1991), 361–375.