REDUCIBILITIES IN RECURSIVE FUNCTION THEORY

by

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(1963)

SUBMITTED IN PARTIAL FULFILLMENT

OF THE REQUIREMENTS FOR THE

DEGREE OF DOCTOR OF

PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF

TECHNOLOGY

June, 1966

Signature of Author ....
Department of Mathematics, May 13, 1966

Certified by ....
Thesis Supervisor

Accepted by ....
Chairman, Departmental Committee
on Graduate Students

ABSTRACT

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In this dissertation some reducibilities of recursive function theory are analyzed, with particular emphasis on the relationships between many-one reducibility and various kinds of truth-table reducibility.

In the first section, the theory of cylinders as developed by Rogers is given. Then the notion of "R-cylinder" is defined for any reducibility R, and the properties of R-cylinders are studied.

In the second section, the R-cylinders are characterized for many kinds of truth-table reducibilities. The characterizations are employed to prove that not every btt-degree has a maximum m-degree and several similar theorems. It is also shown that there are r.e., nonrecursive, noncreative sets A such that $A\times A \leq_m A$.

In the third section, it is pointed out that the reducibilities mentioned in the second section differ in general on the r.e. sets, but theorems are proved to show that they occasionally coincide under special hypotheses.

In the fourth section, the notion of "semirecursive set" is introduced and studied. It is shown that there are semirecursive sets in every tt-degree and hyperimmune semirecursive sets in every r.e. nonrecursive T-degree. It is proved that the p-degree of a semirecursive set consists of a single m-degree, where p-reducibility is as defined in section two. Priority constructions are used to prove that it is possible to have r.e. semirecursive sets A and B such that A join B is not semirecursive and r.e. sets A and B such that B is semirecursive, A is not semirecursive, and $A \leq_{btt} B$. Finally it is shown that immune semirecursive sets are hyperimmune, not hyperhyperimmune and in Σ_2 in the arithmetical hierarchy and that retraceable or effectively immune semi-recursive sets are co-r.e.

In the final section it is shown that the m-degrees of $A, A\times A, \ldots$ are all distinct for sets A such that A is simple but not hypersimple or $\bar{A}$ is immune, non-hyperimmune and retraced by a total function. From this it follows that every nonrecursive tt-degree has infinitely many m-degrees and every r.e. nonrecursive T-degree has infinitely many r.e m-degrees. It is also proved that every r.e. T-degree has an r.e. m-degree consisting of a single 1-degree. The theorems on r.e. nonrecursive T-degrees depend on a construction of Yates for simple but not hypersimple sets and use the propositional calculus as a tool. These theorems seem to be bound together by the fact that if A is simple but not hypersimple, then $\{x \mid D_x \subset A\}$ acts in many ways like a creative set. Finally, the notion of "inverse R-cylinder" is defined and shown to be relevant only for m-reducibility.

ACKNOWLEDGEMENTS

The author gratefully acknowledges the inspiration he has received from his thesis supervisor, Professor Hartley Rogers, Jr. through his course in recursive function theory, the many questions he has raised, and his helpful conversations on the subject of this thesis. He is also grateful to Professor Rogers for his many suggestions for improving the first draft of the thesis.

The author also wishes to thank Dr. P. R. Young for raising many interesting questions and for much helpful correspondence. Also, he is grateful to D. A. Martin, Prof. T. G. McLaughlin, and Dr. A. B. Manaster for solving some questions which are important for the thesis and for illuminating correspondence and conversation.

Finally, the author is grateful to his wife, Elizabeth A. Jockusch for a very thorough and helpful proof-reading of the manuscript.

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SECTION 0 PRELIMINARIES

The principal purpose of this dissertation is to study the relationships between various reducibilities, standard and otherwise, of recursive function theory. This study is carried out with the aid of the concept of "R-cylinder," the priority method of Friedberg, the notion of a "semirecursive set," and the propositional calculus. Among the theorems to be proved are the fact that not every btt-degree contains a maximum m-degree, the fact that every nonrecursive tt-degree contains infinitely many m-degrees, and the fact that every r.e. nonrecursive T-degree contains infinitely many r.e. m-degrees as well as an r.e. m-degree consisting of a single 1-degree.

It is assumed that the reader is familiar with elementary recursive function theory. Our notation, terminology, and point of view all follow closely those of Rogers [14] . In particular, proofs will be informal, and Church's Thesis will be freely used. Also, it is assumed that the reader is thoroughly familiar with the s-m-n theorem and the projection theorem (cf. Rogers [14]), as these theoremswill be freely, and often tacitly, applied.

We now give some of the notations to be used.

N is the set of all nonnegative integers.

<u>Functions</u>, denoted f,g,h, ..., are mapping from N to N.

<u>Partial functions</u>, denoted τ, ψ, are mappings from a subset of N into N.

<u>Sets</u>, denoted A,B,C, ..., are subsets of N.

$\bar{A}$ (the complement of A) is N - A.

Numbers (or integers) denoted u,v,w, ... are elements of N.

$A \leq_m B$ means ($\exists$recursive f) $(\forall x)\ [x \in A \Leftrightarrow f(x) \in B]$.

$A \leq_m B$ via f means f is recursive and $(\forall x)[x \in A \Leftrightarrow f(x) \in B]$.

$A \leq_1 B$ means ($\exists$ recursive 1-1 f) $(\forall x)\ [x \in A \Leftrightarrow f(x) \in B]$.

$A \leq_1 B$ via f means f is a 1-1 recursive function and

$$(\forall x)[x \in A \Leftrightarrow f(x) \in B].$$

If $x = 2^{x_1} + 2^{x_2} + \ldots + 2^{x_n}$, where the x_i are distinct, $D_x = \{x_1, x_2, \ldots, x_n\}$; $D_0 = \emptyset$

(Thus, given x, one may effectively write down a complete listing of the finite set D_x.)

The set A is <u>immune</u> if A is infinite but has no infinite r.e. subset.

The set A is <u>hyperimmune</u> if A is infinite and there is no recursive function f such that for all x and y, $(x \neq y) \Rightarrow D_{f(x)} \cap D_{f(y)} = \emptyset$ and $D_{f(x)} \cap A \neq \emptyset$

f <u>witnesses that</u> A <u>is not hyperimmune</u> if f is recursive and, for all x and y, $(x \neq y \Rightarrow D_{f(x)} \cap D_{f(y)} = \emptyset)$ and $D_{f(x)} \cap A \neq \emptyset$

A is <u>simple</u> if A is r.e. and $\bar{A}$ is immune.

A is <u>hypersimple</u> if A is r.e. and $\bar{A}$ is hyperimmune.

f <u>witnesses that</u> A <u>is not hypersimple</u> if f witnesses that A is not hyperimmune.

$A \times B$ is the set-theoretic cartesian product of A and B.

$\langle\langle A, B\rangle\rangle$ is the ordered pair formed from A and B.

For the following definitions, suppose that a 1-1 recursive function τ from $N \times N$ onto N (a <u>pairing</u> function) has been fixed.

$\langle x,y\rangle = \tau(x,y)$

$A \times B = \{\langle x,y\rangle \mid x \in A \text{ and } y \in B\}$ ("the cartesian product of A and B")

$A_1 \times A_2 \times \ldots \times A_n = (\cdots((A_1 \times A_2) \times A_3) \times \ldots \times A_n)$

$A \text{ join } B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$

φ_x is the x'th partial recursive function in a standard godel numbering.

W_x is the domain of φ_x

A set A is <u>productive</u> if there exists a partial recursive function ψ such that, for all x, $W_x \subset A \Rightarrow \psi(x)$ defined and $\psi(x) \in A - W_x$

A set B is <u>creative</u> if B is r.e. and $\bar{B}$ is productive.

A set A is <u>co</u>-blah just in case $\bar{A}$ is blah. (Example: A is cofinite means that $\bar{A}$ is finite.)

$|A|$ is the cardinality of the set A.

<u>Classes</u>, denoted $\mathcal{A}, \mathcal{B}, \mathcal{C}$, are collections of sets of integers.

$\mathcal{N}$ is class of all subsets of N.

$\lambda x[f(x)]$ is the function f.

$f^n(x) = \overbrace{ff\ldots f}^{n \text{ times}}(x)$; $f^0(x) = x$

SECTION 1. R-CYLINDERS

In this section, the notion of R-cylinder will be defined for any reducibility R. This concept will be used to prove that not every btt-degree has a maximum m-degree, in contrast to the situation for tt-reducibility. The special case of cylinders for m-reducibility, as developed by Rogers [14], will be developed before the general case to provide moti~~vi~~vation and tools for the general case.

DEFINITION 1.1 (Rogers) A set A is a cylinder if there exists a set B such that $A \equiv_1 B \times N$.

THEOREM 1.2 (Rogers) Let A be any set.

(i) $A \leq_1 A \times N$

(ii) $A \times N \leq_m A$

(iii) A is a cylinder $\Longleftrightarrow (C)\,[\,C \leq_m A \Rightarrow C \leq_1 A\,]$

$$\Longleftrightarrow A \times N \leq_1 A$$

Proof.

(i) $A \leq_1 A \times N$ via $\lambda x\,[\langle x, 0\rangle]$

(ii) $A \times N \leq_m A$ via $\lambda\langle x, y\rangle\,[x]$

(iii) First suppose that A is a cylinder, and let B be any set with $A \equiv_1 B \times N$. Assume that $C \leq_m A$. Then $C \leq_m A \leq_m B \times N \leq_m B$. Let $C \leq_m B$ via f. Then $C \leq_1 B \times N$ via $\lambda x\,[\langle f(x), x\rangle]$. It follows that $C \leq_1 A$, which was to be shown.

Now assume that for every C if $C \leq_m A$, the $C \leq_1 A$. By (ii), $A \times N \leq_m A$. Therefore, $A \times N \leq_1 A$, which was to be shown.

Finally assume that $A \times N \leq_1 A$. Since $A \leq_1 A \times N$ by (i), it follows that $A \equiv_1 A \times N$, so that A is a cylinder. qed.

COROLLARY 1.3 (Rogers) $A \leq_m B \iff A \times N \leq_1 B \times N$

Proof First suppose $A \leq_m B$. Then,

$$A \times N \leq_m A \leq_m B \leq_1 B \times N$$

So $A \times N \leq_m B \times N$. But $B \times N$ is a cylinder, so by (ii),

$$A \times N \leq_1 B \times N$$

Conversely, suppose $A \times N \leq_1 B \times N$. We have

$$A \leq_1 A \times N \leq_m B \times N \leq_m B$$

Therefore $A \leq_m B$. qed.

The Corollary shows that m-reducibility can be characterized in terms of 1-reducibility and in fact that there is a canonical homomorphism from the ordering of m-degrees to the ordering of 1-degrees which is given by mapping each m-degree to the maximum 1-degree in the m-degree.

The next theorem gives a useful characterization of cylinders due to Young [20]. A similar characterization can be found in Rogers [14].

THEOREM 1.4 (Young) A is a cylinder iff there exists a recursive function h such that, for all x, $W_{h(x)}$ is infinite, and

$$(x \in A \Rightarrow W_{h(x)} \subset A) \text{ and } (x \in \bar{A} \Rightarrow W_{h(x)} \subset \bar{A})$$

Proof First suppose that A is a cylinder, and $A \times N \leq_1 A$ via f. Let h be a recursive function such that, for all x,

$$W_{h(x)} = f(\{x\} \times N)$$

Then h has the desired properties.

Conversely, assume that h has the properties stated in the theorem. Let C be any set such that $C \leq_m A$. It will be shown that $C \leq_1 A$. Assume that $C \leq_m A$ via f, and define the recursive function g by induction as follows:

$$g(0) = f(0)$$

$$\vdots$$

$$g(m+1) = \text{the first number } y \text{ found in an effective listing of } W_{hf(m+1)} \text{ such that } y \notin \{g(0), g(1), \ldots, g(m)\}$$

Then $C \leq_1 A$ via g. qed.

Rogers [14] has developed an analogue to the above theory in which tt-reducibility takes the place of m-reducibility. This and several other examples will be considered in the framework now to be introduced.

<u>DEFINITION</u> <u>1.5</u> A reducibility is a transitive binary relation between sets of integers such that for all sets A and B

$$A \leq_1 B \Longrightarrow \langle\langle A, B \rangle\rangle \in R.$$

<u>NOTATIONS</u> <u>1.6</u>

(i) If R is a reducibility, $A \leq_R B$ shall mean $\langle\langle A,B\rangle\rangle \in R$.

(ii) The letters R, S, and T shall be understood to range over reducibilities.

The definition of R-cylinder is based on (iii) of theorem 1.2.

DEFINITION 1.7 An R-cylinder is a set A such that

$$(B)\ [\, B \leq_R A \Rightarrow B \leq_1 A \,]$$

Thus, every set is a 1-cylinder, and the m-cylinders are just the cylinders.

DEFINITION 1.8

(i) $A \equiv_R B$ means $A \leq_R B$ and $B \leq_R A$.

(ii) An R-degree is an equivalent class of the equivalence relation $\equiv_R$.

DEFINITION 1.9 A reducibility R is cylindrical if every R-degree contains an R-cylinder.

1-reducibility is trivially cylindrical, and m-reducibility is cylindrical by theorem 1.2. An example of a non-cylindrical reducibility would be the trivial reducibility in which all sets are interreducible. Indeed this reducibility has no cylinders. But later we shall see that some "natural" reducibilities are not cylindrical.

DEFINITION 1.10 If R is a cylindrical reducibility and A is a set, then A^R ("the R-cylindrification of A") denotes the 1-degree of any R-cylinder in the R-degree of A. (A^R must clearly be unique.)[1]

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[1] For particular cylindrical reducibilities R, A^R will often denote, by abuse of notation, a particular R-cylinder in the R-degree of A which can be found from A in a natural way. For example $A^1 = A$ and $A^m = A \times N$.

With this machinery, it is easy to prove an analogue to theorem 1.2.

<u>THEOREM</u> <u>1.11</u> Let R be a cylindrical reducibility.

(i) $A \leq_1 A^R$

(ii) $A^R \leq_R A$

(iii) A is an R-cylinder $\Leftrightarrow A^R \leq_1 A$

$A \equiv_1 B^R$ for some B

(iv) $A \leq_R B \Leftrightarrow A^R \leq_1 B^R$[2]

(v) $A \leq_R B \Leftrightarrow (C)\ [(C \text{ an R-cylinder and } B \leq_1 C) \Rightarrow A \leq_1 C]$

<u>Proof</u> Parts (i) - (iv) are either obvious from definitions or are proved just as in theorem 1.2 and corollary 1.3.

To prove part (v), first assume $A \leq_R B$. Let C be any R-cylinder, and assume that $B \leq_1 C$. We have

$$A \leq_R B \leq_1 C$$

so $A \leq_R C$. Since C is an R-cylinder, it follows that $A \leq_1 C$.

Conversely, assume that A is 1-1 reducible to every R-cylinder to which B is 1-1 reducible. Then, in particular, $A \leq_1 B^R$, so $A \leq_R B$. qed.

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[2] The statements of (i) - (iv) involve an obvious abuse of notation which is unimportant because of the assumption that $A \leq_1 B \Rightarrow A \leq_R B$.

<u>COROLLARY</u> <u>1.12</u> If two cylindrical reducibilities S and T have the same cylinders, i.e. if

$$(C)\,[\,C \text{ is an S-cylinder} \Longleftrightarrow C \text{ is a T-cylinder}\,]$$

Then S = T.

<u>Proof</u> Immediate from (v) qed.

Part (v) also suggests a way of obtaining a reducibility R from a given class of sets which are to be R-cylinders.

<u>DEFINITION</u> <u>1.13</u> Let $\mathcal{A}$ be a class of sets. Define a binary relation R($\mathcal{A}$) on sets on integers by

$$\langle\langle A, B\rangle\rangle \in R(\mathcal{A}) \Longleftrightarrow (C)\,[\,C \in \mathcal{A}\ \&\ B \leq_1 C \Rightarrow A \leq_1 C\,]$$

If R is a reducibility, let $\mathcal{C}(R)$ denote $\{C \mid C \text{ is an R-cylinder}\}$

If R and S are reducibilities R is <u>weaker</u> <u>than</u> S (S is <u>stronger</u> <u>than</u> R) if $S \subset R$, i.e. if

$$(\forall A)\,(\forall B)\,[\,A \leq_S B \Rightarrow A \leq_R B\,]$$

For example, if $\mathcal{A}$ is the class of all cylinders R($\mathcal{A}$) is m-reducibility. If $\mathcal{A}$ is empty, the R($\mathcal{A}$) is the reducibility in which any two sets are interreducible. If $\mathcal{A}$ is the collection of all finite sets, then it is easy to verify that

$$\langle\langle A, B\rangle\rangle \in R(\mathcal{A}) \Rightarrow |A| \leq |B|$$

<u>LEMMA</u> <u>1.14</u>

(i) For any $\mathcal{A}$, R($\mathcal{A}$) is a reducibility

(ii) $\mathcal{A} \subset \mathcal{B} \Rightarrow R(\mathcal{B}) \supset R(\mathcal{A})$ for any classes $\mathcal{A}, \mathcal{B}$.

(iii) $S \subset T \Rightarrow \mathcal{C}(S) \supset \mathcal{C}(T)$, for any reducibilities S,T.

(iv) $S \subset R\,\mathcal{C}(S)$ for any reducibility S.

(v) $\mathcal{a} \subset \mathcal{C}R(\mathcal{a})$ for any class $\mathcal{a}$.

(vi) $R(\mathcal{a}) = R\mathcal{C}R(\mathcal{a})$ for any class $\mathcal{a}$.

(vii) $\mathcal{C}(R) = \mathcal{C}R\mathcal{C}(R)$ for any reducibility R.

(viii) For any class $\mathcal{a}$, $R(\mathcal{a})$ is the unique weakest reducibility R such that every set in $\mathcal{a}$ is an R-cylinder.

<u>PROOF</u> Parts i-v are immediate from definitions.

Part vi is proved from parts i-v:

$\mathcal{a} \subset \mathcal{C}R(\mathcal{a})$ by (v)

$\therefore$ $R(\mathcal{a}) \supset R\mathcal{C}R(\mathcal{a})$ by (ii)

Also, $R(\mathcal{a}) \subset R\mathcal{C}R(\mathcal{a})$ by (iv)

$\therefore$ $R(\mathcal{a}) = R\mathcal{C}R(\mathcal{a})$

The proof of vii is dual to that of vi.

(viii) Every set in $\mathcal{a}$ is an $R(\mathcal{a})$-cylinder by (v).

Now assume that every set in $\mathcal{a}$ is an S-cylinder for some reducibility S.

Then $\mathcal{a} \subset \mathcal{C}(S)$

So $R(\mathcal{a}) \supset R\mathcal{C}(S) \supset S$. qed.

<u>DEFINITION 1.15</u> The <u>closure</u> of R (denoted $\bar{R}$) is $R\mathcal{C}(R)$. The <u>closure</u> of $\mathcal{a}$ (denoted $\bar{\mathcal{a}}$) is $\mathcal{C}R(\mathcal{a})$. R is <u>closed</u> if $R = \bar{R}$ and $\mathcal{a}$ is <u>closed</u> if $\mathcal{a} = \bar{\mathcal{a}}$.

Lemma 1.12 shows that the notion of closure defined above has some of the usual properties of closure. For instance, the closure of $\mathcal{a}$ is the smallest closed class containing $\mathcal{a}$. In particular $\bar{\bar{\mathcal{a}}} = \bar{\mathcal{a}}$. We also see from lemma 1.12 that R is closed iff $R = R(\mathcal{a})$ for some $\mathcal{a}$.

Therefore, every cylindrical reducibility is closed. On the other hand the trivial reducibility in which all sets are interreducible is closed but not cylindrical.

Lemma 1.12 shows that there is a natural 1-1 correspondence between closed reducibilities and closed classes that is in some ways analogous to the correspondence between the intermediate fields of a Galois extension and the closed subgroups of the Galois group which is studied in Galois theory. However, a basic difference between the two theories is that the closure operator defined here is not a Kuratowski closure operator, i.e. we do not get a topology on $\mathcal{N}$ from the above definition of "closed class" The problem is that, although arbitrary intersections of closed classes are closed, it is not always true that finite unions of closed ~~sets~~ classes are closed. To see this, let A be any set and define

$$\mathcal{A}_A = \{ B \mid B \equiv_1 A \}.$$

It is claimed that $\mathcal{A}_A$ is closed. Well,

$$C \leq_{R(\mathcal{A}_A)} D \Longleftrightarrow (D \leq_1 A \Longrightarrow C \leq_1 A)$$

Thus the reducibility $R(\mathcal{A}_A)$ has two degrees, one consisting of the sets which are 1-1 reducible to A and the other consisting of the sets not 1-1 reducible to A. Hence, the cylinders in the former degree must be 1-equivalent to A because they are of highest 1-degree in their $R(\mathcal{A}_A)$-degree, and the latter degree has no maximum 1-degree (and hence no $R(\mathcal{A}_A)$-cylinders) because it is uncountable. Thus every member of $\mathcal{C}R(\mathcal{A}_A)$ is in $\mathcal{A}_A$, and $\mathcal{A}_A$ is closed.

Now if A is a creative set and E is an infinite and coinfinite recursive set,

$$E \in \overline{a_A \cup a_A}$$

For let

$$D \leq_{R(a_A \cup a_{\bar{A}})} E$$

Then since $E \leq_1 A$ and $E \leq_1 \bar{A}$, we have by transitivity

$$D \leq_{R(a_A \cup a_{\bar{A}})} A \ \& \ D \leq_{R(a_A \cup a_{\bar{A}})} \bar{A}$$

But since A and $\bar{A}$ are $R(a_A \cup a_{\bar{A}})$-cylinders,

$$D \leq_1 A, \ D \leq_1 \bar{A},$$

whence D and $\bar{D}$ are r.e. so that D is recursive and $D \leq_1 E$. This shows that E is an $R(a_A \cup a_{\bar{A}})$-cylinder although $D \notin a_A \cup a_{\bar{A}}$, so that $a_A \cup a_{\bar{A}}$ is not closed.

The definition of R-cylinder we have chosen is in some respects arbitrary. For instance, we could have defined A to be an R-cylinder just in case the 1-degree of A was maximal among the 1-degrees occurring in the R-degree of A i.e. if $(B)\,[B \equiv_R A \ \& \ A \leq_1 B \Rightarrow B \leq_1 A]$, and this definition would also coincide with the definitions of Rogers for many-one and truth-table reducibilities, although it is superficially a much weaker requirement to put on A. We now show that these two definitions must yield the same R-cylinders for a certain important kind of reducibility.

<u>DEFINITION</u> <u>1.16</u> R is <u>regular</u> if every set of maximal 1-degree in its R-degree is an R-cylinder.

PROPOSITION 1.17 If for all A and B, the R-degree of A join B is the l.u.b. of the R-degrees of A and B (in the partial ordering of R-degrees induced by $\leq_R$)(i.e. if join is an l.u.b. for R), then R is regular.

Proof. Assume that join is an l.u.b. for R and let A have maximal 1-degree in its R-degree. Let $B \leq_R A$.
To show: $B \leq_1 A$. We have:

$$A \leq_1 B \text{ join } A$$

$$B \text{ join } A \leq_R A \text{ (since join is l.u.b. for R)}$$

So, B join $A \leq_1 A$ by maximality of the 1-degree of A.
Hence, $B \leq_1 A$.
Thus, A is an R-cylinder, which is the desired conclusion. qed.

The converse to the above proposition is false, for 1-reducibility is trivially regular, although join does not give a l.u.b. for 1-reducibility. On the other hand, any reducibilities other than $\leq_1$ which have been discussed in the literature do have join as a l.u.b. Also, a wide class of closed reducibilities as defined in definition 1.15 have join as an l.u.b.

PROPOSITION 1.18 If R is closed and weaker than $\leq_m$, then join is an l.u.b. for R.

Proof Since join gives a 1-upper bound, join is an upper bound operation for any reducibility R. Thus it suffices to show that

$$A \leq_R C,\ B \leq_R C \Rightarrow A \text{ join } B \leq_R C.$$

under the above hypotheses.

Let $R = R(\mathcal{a})$. Suppose $A \leq_R C$ and $B \leq_R C$. Let $D \in \mathcal{a}$ and $C \leq_1 D$. To show:

$$(A \text{ join } B) \leq_1 D.$$

Since D is an R-cylinder, and A and B are each R-reducible to D,

$$A \leq_1 D \text{ and } B \leq_1 D$$

Therefore, A join $B \leq_m D$ since join is a l.u.b. for m-reducibility. But since D is an R-cylinder and R is weaker than m-reducibility, D is a cylinder, so

$$A \text{ join } B \leq_1 D.$$

qed.

Proposition 1.17 and 1.18 show that every closed reducibility weaker than m-reducibility is regular. I do not know whether proposition 1.18 is true without the hypothesis that R is closed.

The collection of all classes (of sets) forms a lattice under class inclusion. We shall now show that the reducibilities also form a lattice in a natural way and investigate the connection between these lattices.

DEFINITION 1.19

$A \leq_{R \cap S} B$ means $A \leq_R B$ and $A \leq_S B$

$A \leq_{R \text{ join } S} B$ means that for some finite sequence of sets $C_1, C_2, \cdots, C_n$ $A = C_1$, $B = C_n$, and for each i, $1 \leq i \leq n-1$, $C_i \leq_R C_{i+1}$ or $C_i \leq_S C_{i+1}$.

PROPOSITION 1.20 $R \cap S$ and R join S are reducibilities. $R \cap S$ is the weakest reducibility stronger than both R and S and R join S is the strongest reducibility weaker than both R and S.

Proof Immediate

qed.

Of course, proposition 1.18 just says that the set of reducibilities forms a lattice with join as its l.u.b. operation and intersection as its g.l.b. operation when it is partially ordered under the relation "weaker than."

PROPOSITION 1.21

(i) $\mathcal{C}(R_1 \text{ join } R_2) = \mathcal{C}(R_1) \cap \mathcal{C}(R_2)$

(ii) $R(\mathcal{A} \cup \mathcal{B}) = R(\mathcal{A}) \cap R(\mathcal{B})$

(iii) If R_1 and R_2 are closed, $\mathcal{C}(R_1 \cap R_2) = \overline{\mathcal{C}(R_1) \cup \mathcal{C}(R_2)}$

(iv) If $\mathcal{A}$ and $\mathcal{B}$ are closed, $R(\mathcal{A} \cap \mathcal{B}) = \overline{R(\mathcal{A}) \text{ join } R(\mathcal{B})}$

Proof (i). Clearly, $\mathcal{C}(R_1 \text{ join } R_2) \subset \mathcal{C}(R_1) \cap \mathcal{C}(R_2)$, since R_1 join R_2 is weaker than R_1 and R_2. Now suppose that A is an R_1-cylinder and an R_2-cylinder, and let $B \leq_{R_1 \text{ join } R_2} A$. To show that A is an R_1 join R_2-cylinder, it must be shown that $B \leq_1 A$. By definition of R_1 join R_2, there is a finite sequence $C_1, C_2, \ldots, C_n$, such that $B = C_1$, $A = C_n$, and for each i, $1 \leq i \leq n-1$

$C_i \leq_{R_1} C_{i+1}$, or $C_i \leq_{R_2} C_{i+1}$. We will show by induction on i that $C_i \leq_1 A$. $C_1 \leq_1 A$ since $C_1 = A$. Now assume $C_i \leq_1 A$, where $1 \leq i \leq n-1$. Assume $C_{i+1} \leq_{R_1} C_i$. (The other case is the same.)

Then by transitivity $C_{i+1} \leq_{R_1} A$. Thus $C_{i+1} \leq_1 A$. Thus in particular, $C_n \leq_1 A$, i.e. $B \leq_1 A$.

The proof of (ii) is immediate and (iii) and (iv) follow from (ii) and (i) respectively.

SECTION 2. EXAMPLES AND APPLICATIONS OF R-CYLINDERS

In this section, we shall mostly be concerned with various reducibilities of the truth-table type. It will turn out that R-cylinders have convenient characterizations for such reducibilities. These reducibilities will be defined using propositional formulas rather than truth-table conditions.

DEFINITION 2.1 A propositional formula (or, simply, formula) is a statement built up in the usual way from statement letters $P_n (n \in N)$ and the propositional connectives $\vee, \wedge, \neg$ ("or", "and", and "not", respectively)[1]

We assume that we have fixed an effective coding from the set of formulas onto N. In fact, formulas will often be identified with their code numbers.

DEFINITION 2.2

(i) A propositional formula σ is true of a set A, just in case σ is true in the interpretation in which each P_n is true iff $n \in A$.

(Example: $P_5 \vee P_7$ is true of A iff $5 \in A$ or $7 \in A$)

(ii) The norm of a formula $\sigma (\|\sigma\|)$ is $|\{n \mid P_n \text{ occurs in } \sigma\}|$.

There are two natural ways to obtain a reducibility from a set of connectives. These are given in the following definition.

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[1]Most of the theorems to follow do not depend at all on this particular selection of connectives. However, the word "connective" as used here, will always mean one of the three connectives $\vee, \wedge$ and $\neg$.

DEFINITION 2.3 Let U be a set of connectives. Define two binary relations on sets of integers by:

$\langle\langle A,B\rangle\rangle \in \mathcal{V}(U) \Longleftrightarrow$ ($\exists$ recursive f)($\forall$x)[every connective in f(x) is in U and ($x \in A \Longleftrightarrow$ f(x) is true of B)]

$\langle\langle A,B\rangle\rangle \in \mathcal{B}(U)$ ($\exists$ recursive f) ($\exists$m) ($\forall$x) [every connective in f(x) is in U and $\|f(x)\| \leq m$ and ($x \in A \Longleftrightarrow$ f(x) is true of B)][2]

THEOREM 2.4 For any set U of connectives, $\mathcal{V}(U)$ and $\mathcal{B}(U)$ are reducibilities weaker than $\leq_m$. Also $\mathcal{V}(U)$ & $\mathcal{B}(U)$ have join as a l.u.b. operation.

Proof First suppose $A \leq_m B$. It will be shown that $\langle\langle A,B\rangle\rangle \in \mathcal{B}(U)$ Let $A \leq_m B$ via f. Then $x \in A \Longleftrightarrow P_{f(x)}$ is true of B.

Since $P_{f(x)}$ involves no connectives and has norm 1, we see that $\langle\langle A,B\rangle\rangle \in \mathcal{B}(U)$ and hence $\langle\langle A,B\rangle\rangle \in \mathcal{V}(U)$

Now suppose that $\langle\langle A,B\rangle\rangle \in \mathcal{V}(U)$ & $\langle\langle B,C\rangle\rangle \in \mathcal{V}(U)$. Let f and g be recursive functions such that

($x \in A \Longleftrightarrow$ f(x) is true of B) and f(x) uses only connectives from U

($x \in B \Longleftrightarrow$ g(x) is true of C) and g(x) uses only connectives from U.

Now let h(x) be the code number for the formula obtained by substituting for every statement letter P_n occurring in the formula f(x) the formula with code number g(n). It is immediate to verify that, for all x,

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[2]In this definition, the identification of formulas with their code numbers several times.

($x \in A \Leftrightarrow h(x)$ is true of C) and $h(x)$ uses only connectives from U.

Therefore $\langle\langle A, C\rangle\rangle \in \mathcal{T}(U)$ so $\mathcal{T}(U)$ is a transitive relation. The above proof also shows that $\mathcal{B}(U)$ is a transitive relation, so each relation is a reducibility weaker than $\leq_m$. It is immediate to show that join is a l.u.b. operation for $\mathcal{B}(U)$ and $\mathcal{T}(U)$. qed.

The above definitions yield ten reducibilities. All have been studied in the literature. The following table names the reducibilities.

Definition	Name	Abbreviation
$\mathcal{T}(\{\vee,\wedge,\neg\}) = \mathcal{T}(\{\vee,\neg\}) = \mathcal{T}(\{\wedge,\neg\})$	truth-table reducibility	tt
$\mathcal{B}(\{\vee,\wedge,\neg\}) = \mathcal{B}(\{\vee,\neg\}) = \mathcal{B}(\{\wedge,\neg\})$	bounded truth-table reducibility	btt
$\mathcal{T}(\{\vee,\wedge\})$	positive reducibility[3]	p
$\mathcal{B}(\{\vee,\wedge\})$	bounded positive reducibility	bp
$\mathcal{T}(\{\wedge\})$	conjunctive reducibility	c
$\mathcal{B}(\{\wedge\})$	bounded conjunctive reducibility	bc
$\mathcal{T}(\{\vee\})$	disjunctive reducibility	q
$\mathcal{B}(\{\vee\})$	bounded disjunctive reducibility	bq
$\mathcal{T}(\{\neg\}) = \mathcal{B}(\{\neg\})$	norm-1 reducibility	n
$\mathcal{T}(\emptyset) = \mathcal{B}(\emptyset)$	many-one reducibility	m

tt, btt, and m-reducibilities were introduced by Post in [13].

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[3] A positive formula is one which uses only the connectives $\vee$ and $\wedge$

q-reducibility and n-reducibility were introduced by Rogers [14]. The remaining reducibilities have been studied by Lachlan in [10] (with somewhat different terminology) and the writer.

We now characterize the R-cylinders for these reducibilities. Let A be a set. Each connective "operates" on A as follows:

$$A_\neg = \{m \mid \neg P_m \text{ is true of } A\} = \overline{A}$$

$$A_\wedge = \{\langle m,n\rangle \mid P_m \wedge P_m \text{ is true of } A\} = A \times A$$

$$A_\vee = \{\langle m,n\rangle \mid P_m \vee P_m \text{ is true of } A\} = \overline{\overline{A} \times \overline{A}}$$

<u>LEMMA</u> <u>2.5</u> For any connective ϵ,

$$A_\epsilon \le_{\mathcal{B}(\{\epsilon\})} A$$

<u>Proof</u> $A_\neg \le_{\mathcal{B}(\{\neg\})} A$ via $\lambda n[\neg P_n]$

$A_\wedge \le_{\mathcal{B}(\{\wedge\})} A$ via $\lambda\langle m,n\rangle[P_m \wedge P_n]$

$A_\vee \le_{\mathcal{B}(\{\vee\})} A$ via $\lambda\langle m,n\rangle[P_m \vee P_n]$[4]. qed.

<u>THEOREM</u> <u>2.6</u> Let U be a set of connectives, and let A be any set.

(i) $\mathcal{V}(U)$ is cylindrical. If $A \neq N$, the cylindrification of A is given by $A^{\mathcal{V}(U)} = \{x \mid$ every connective in x is in U and x is true of A$\} \times N$

(ii) The following statements are equivalent:

(a) A is a cylinder and $A_\epsilon \le_m A$ for each connective ϵ in U

(b) A is a $\mathcal{V}(U)$-cylinder

(c) A is a $\mathcal{B}(U)$-cylinder

(iii) $\overline{\mathcal{B}(U)} = \mathcal{V}(U)$

---

[4] The use of the word "via" is extended to truth-table reducibilities in the obvious way here.

(i) Suppose $A \neq N$.[5] It will be shown that $A^{\mathcal{V}(U)}$ is the $\mathcal{V}(U)$-cylindrification of A. $A \leq_1 A^{\mathcal{V}(U)}$ via $\lambda x[\langle P_x, 0\rangle]$. Let $a' \notin A$, and define h by

$$h(x) = \begin{cases} x \text{ if every connective in } x \text{ is in } U \\ a' \text{ otherwise} \end{cases}$$

Then $A^{\mathcal{V}(U)} \leq_{\mathcal{V}(U)} A$ via h, so $A^{\mathcal{V}(U)} \equiv_{\mathcal{V}(U)} A$.

Now it must be shown that $A^{\mathcal{V}(U)}$ is a $\mathcal{V}(U)$-cylinder.

Suppose $B \leq_{\mathcal{V}(U)} A^{\mathcal{V}(U)}$. Then, since $A^{\mathcal{V}(U)} \equiv_{\mathcal{V}(U)} A$, $B \leq_{\mathcal{V}(U)} A$.

Therefore there is a recursive function f such that

f(x) uses only connectives from U and

$x \in B \Longleftrightarrow f(x)$ is true of A

Then $B \leq_1 A^{\mathcal{V}(U)}$ via $\lambda x[\langle f(x), x\rangle]$. Therefore $A^{\mathcal{V}(U)}$ is a $\mathcal{V}(U)$-cylinder.

It may also be checked that N may be R-cylindrified as follows:

$$N^{p} = N^{c} = N^{q} = N^{m} = N$$

$$N^{tt} = N^{n} = \{2x \mid x \in N\}$$

(ii) (a)$\Rightarrow$(b). Assume that A is a cylinder and $A_{\varepsilon} \not\leq_m A$ for each connective ε in U. Assume also that $A \neq N$, since the case $A = N$, can be checked separately.[5]

---

[5] The exceptional case that A = N would not appear anywhere if we used, for each set U of connectives, an effective coding from the formulas having only connectives in U onto N.

To show that A is a $\mathcal{V}(U)$-cylinder, it is sufficient to show that $A^{\mathcal{V}(U)} \leq_1 A$, and therefore, since A is a cylinder, it is sufficient to show that

$$\{x \mid \text{every connective in } x \text{ is in } U \text{ and } x \text{ is true of } A\} \leq_m A.$$

If $\neg \in U$, let $A_\neg \leq_m A$ via f. If $\vee \in U$, let $A_\vee \leq_m A$ via g.
If $\wedge \in U$, let $A_\wedge \leq_m A$ via h.
Let $a' \notin A$.

A recursive function k will now be defined to give the desired reduction. If the formula x has some connective not in U, define $k(x) = a'$. $k(x)$ is defined for other arguments by induction on the number of connectives in the formula x. If x has no connectives, so that x is some P_n, define $k(x) = n$. Assume now that $k(y)$ has been defined for all formulas y having at most m connectives, all in U, and that $k(x)$ has $m + 1$ connectives, all in U. Then define

$$k(x) = \begin{cases} fk(y) \text{ if } x \text{ is } \neg y \\ g(\langle k(y),\ k(z)\rangle) \quad \text{if } x \text{ is } y \vee z \\ h(\langle k(y),\ k(z)\rangle) \quad \text{if } x \text{ is } y \wedge z \end{cases}$$

It is now immediate to verify by induction on the number of quantifiers in x that every connective in x is in U and x is true of A $\Longleftrightarrow$ $k(x) \in A$ so that k furnishes the desired reduction.

(b) $\Rightarrow$ (c). Trivial.

(c) $\Rightarrow$ (a). Assume that A is a $\mathcal{B}(U)$-cylinder. Then A is a cylinder since $\mathcal{B}(U)$ is weaker than m-reducibility. Also, by lemma 2.5, $A_\epsilon \leq_{\mathcal{B}(\{\epsilon\})} A$ for each ϵ in U. Thus, since A is a $\mathcal{B}(U)$-cylinder, $A_\epsilon \leq_1 A$ for each ϵ in U.

(iii) Since $\mathcal{V}(U)$ is cylindrical, $\mathcal{V}(U)$ is closed. By (ii)

$$\mathcal{C}(\mathcal{B}(U)) = \mathcal{C}(\mathcal{V}(U))$$

Therefore $\overline{\mathcal{B}(U)} = \mathcal{R}\mathcal{C}(\mathcal{B}(U)) = \mathcal{R}\mathcal{C}(\mathcal{V}(U)) = \overline{\mathcal{V}(U)} = \mathcal{V}(U)$

qed.

Since A_ϵ is "easily calculable" for each connective ϵ, the above theorem gives a convenient characterization of the R-cylinders for each reducibility R which has been considered. For instance:

C is a tt-cylinder $\Leftrightarrow$ C is a btt-cylinder $\Leftrightarrow C \times C \leq_m C$, $C \leq_m \bar{C}$ and C is a cylinder.

Thus, by theorem 1.11, it gives a method of defining each of these reducibilities from its cylinders, without referring to formulas or truth-tables. For instance,

$$A \leq_{tt} B \Leftrightarrow (C) \ [(C \times C \leq_m C \ \& \ C \leq_m \bar{C} \ \& \ C \times N \leq_1 C \ \& \ B \leq_1 C) \Rightarrow A \leq_1 C]$$

Finally, the theorem shows that the reducibility $\mathcal{V}(U)$ can be obtained in a "natural way" from the reducibility $\mathcal{B}(U)$.

<u>DEFINITION 2.7</u> A set A is <u>R - complete</u> if A is r.e. and $B \leq_R A$ for every r.e. set B.

<u>COROLLARY 2.8</u> Let U be any set of connectives.

(i) Each $\mathcal{B}(U)$-degree contains a maximal 1-degree iff $\mathcal{B}(U) = \mathcal{V}(U)$.

(ii) There are btt, bp, bc and bq degrees which have no maximal 1-degrees.

<u>Proof</u>

(i) Suppose that each $\mathcal{B}(U)$-degree contains a maximal 1-degree.

Then, since join is an l.u.b. for $\mathcal{B}(U)$, each set of maximal 1-degree in its $\mathcal{B}(U)$ degree is a $\mathcal{B}(U)$-cylinder by proposition 1.17, so each $\mathcal{B}(U)$-degree has a $\mathcal{B}(U)$-cylinder. Therefore $\mathcal{B}(U)$ is cylindrical. Since $\mathcal{V}(U)$ is also cylindrical, $\mathcal{B}(U)$ and $\mathcal{V}(U)$ are cylindrical reducibilities having the same cylinders and are thus identical.

Conversely, assume $\mathcal{B}(U) = \mathcal{V}(U)$. Then, since $\mathcal{V}(U)$ is cylindrical, each $\mathcal{B}(U)$ degree contains a maximal 1-degree.

(ii) Post [13] constructs a simple set S* which is c-complete and proves that no simple set is btt-complete. It follows immediately that,

$$tt \neq btt$$

$$p \neq bp$$

$$c \neq bc$$

Since $A \leq_q B$ iff $\bar{A} \leq_c \bar{B}$, it follows also that $bq \neq q$. qed.

The question of whether every btt-degree contains a maximal 1-degree is due to Rogers.

Although the definitions of various reducibilities R given in this section is convenient for the development of the theory of R-cylinders, a different kind of formulation, suggested by Rogers, is often more useful in applications. This will be given now.

<u>DEFINITION</u> <u>2.9</u> D'_x is D_x if $x \neq 0$; D'_0 is $\{0\}$.

<u>THEOREM</u> <u>2.10</u>

(i) $A \leq_c B \Leftrightarrow (\exists \text{ recursive } f)(\forall x)[x \in A \Leftrightarrow D'_{f(x)} \subset B]$[6]

[6]The non-empty sets D'_x are used to characterize these reducibilities roughly because every formula has at least one statement letter. However, the substitution of the D_x for the D'_x would at most change which sets are R-reducible to N and ϕ.

(ii) $A \leq_p B \Leftrightarrow (\exists \text{ rec. } f)(\forall x)[x \in A \Leftrightarrow (\exists y)[y \in D'_{f(x)} \& D'_y \subset B]]$

$\Leftrightarrow (\exists \text{ rec. } f)(\forall x)[x \in A \Leftrightarrow (\forall y)[y \in D'_{f(x)} \Rightarrow D'_y \cap B \neq \emptyset]]$

(iii) $A \leq_{bp} B \Leftrightarrow (\exists \text{ rec. } f)(\exists m)(\forall x)[|\bigcup_{y \in D'_{f(x)}} D'_y| \leq m \;\&$
$(x \in A \Leftrightarrow (\exists y)[y \in D'_{f(y)} \& D'_y \subset B]]$

(iv) (Rogers) $A \leq_{tt} B \Leftrightarrow (\exists \text{ rec, } f)(\forall x)[x \in A \Leftrightarrow$
$(\exists u)(\exists v)[\langle u,v\rangle \in D'_{f(x)} \& D'_u \subset B \& D'_v \subset \overline{B}]$

(v) $A \leq_{btt} B \Leftrightarrow (\exists \text{ rec. } f)(\exists m)(\forall x)[|\bigcup_{\langle u,v\rangle \in D'_{f(x)}} D'_u \cup D'_v| \leq m \;\&$
$(x \in A \Leftrightarrow (\exists u)(\exists v)[\langle u,v\rangle \in D'_{f(x)} \& D'_u \subset B \& D'_v \subset \overline{B}])]$

(vi) $A \leq_n B \Leftrightarrow (\exists \text{ rec. } f)(\forall x)[x \in A \Leftrightarrow f(x) \in B \text{ join } \overline{B}]$

Proof For all parts, observe that

$$P_{x_1} \wedge P_{x_2} \wedge \ldots \wedge P_{x_n} \text{ is true of } A \Leftrightarrow D'_x \subset A \text{ where } D'_x = \{x_1, x_2, \ldots, x_n\}$$

Similar statements can be made for formulas involving only disjunction or only negation.

(i) is immediate by the above remark.

(ii) will follow from the above remark if it can be shown that every positive formula (i.e. formula using only $\wedge$ and $\vee$) is equivalent both to a conjunction of disjunctions of statement letter and to a disjunction of conjunctions of statement letters. But both of these facts follow from an easy induction on the number of connectives in the positive formula.

(iii) follows by the same argument as (ii).

(iv) and (v) follow from the fact that every propositional formula is equivalent to a disjunction of formulas, each of which is a conjunction of stamtenet letter and negation of statement letters.

(vi) is immediate. qed.

Of course, similar characterizations could have been given for any reducibility that has been introduced above. The reducibilities in the theorem are those which have the most relevance for this paper.

COROLLARY 2.11 $A \le_p B$, B r.e. $\Rightarrow$ A r.e.

Proof Immediate from (ii) and the projection theorem. qed.

COROLLARY 2.12 (cf. definition 1.19).

(i) q join c = p

bq join bc = bp

n join c = n join q = n join p = tt

n join bc = n join bq = n join bp = btt

(ii) If V and W are sets of connectives, then[7]

$\mathcal{V}(V)$ join $\mathcal{V}(W) = \mathcal{V}(V \cup W)$

$\mathcal{B}(V)$ join $\mathcal{B}(W) = \mathcal{B}(V \cup W)$

Proof

(i) q join c $\subset$ p, trivially. Suppose $A \le_p B$. Then by the theorem,

$$A \le_c \{x \mid D'_x \cap B \ne \emptyset\} \le_q B$$

Therefore, $A \le_{q\,\mathrm{join}\,c} B$. Thus q join c = p. Similarly, bq join bc = bp

Now suppose $A \le_{tt} B$. Then, by the theorem,

$$A \le_p B \text{ join } \bar{B} \le_n B$$

---

[7]This statement, unlike the rest in this section, appears to depend on the choice of $\neg\, \vee$ and $\wedge$ as basic connectives. The writer has not explored the question of whether an analogue to (ii) holds for an arbitrary selection of connectives.

Therefore, $A \leq_{n\,join\,p} B$. Thus $tt \subset n$ join p.

Now it will be shown that $p \subset c$ join n. Suppose $A \leq_p B$.

Then

$$A \leq_c \{x \mid D_x' \cap B \neq \emptyset\} \leq_n \overline{\{x \mid D_x' \cap B \neq \emptyset\}} = \{x \mid D_x' \subset \overline{B}\} \leq_c \overline{B} \leq_n B$$

Therefore $A \leq_{c\,join\,n} B$, so $p \subset c$ join n.

Thus $tt \subset n$ join $p \subset n$ join c join n = n join c. Since n join $c \subset tt$ trivially, it follows that n join c = n join p = tt. Similarly, n join q = tt. The statements for btt follow by the same argument.

(ii). (i) shows that all non-trivial instances of (ii) are true.

qed.

It will be shown later that it is not necessarily true that

$$\mathcal{V}(V) \cap \mathcal{V}(W) = \mathcal{V}(V \cap W)$$

$$\text{or } \mathcal{B}(V) \cap \mathcal{B}(W) = \mathcal{B}(V \cap W)$$

and, in particular, that $m \neq bq \cap bc$.

The following four theorems concern c-reducibility.

<u>THEOREM</u> <u>2.13</u> (Fischer) If S* is the simple tt-complete set constructed by Post, then $S^* \times S^* \not\leq_m S^*$. Thus m-reducibility and btt-reducibility differ on the r.e. nonrecursive sets.

<u>Proof</u> It follows immediately from Post's construction of S* that S* is c-complete, so that if A is any r.e. set, $A \leq_c S^*$. Now assume that $S^* \times S^* \leq_m S^*$. Then by the characterization of c-cylinders, it follows that $A \leq_m S^*$, for any r.e. set A, so that S* is m-complete and thus creative. This contradicts the fact that S* is simple, so it follows that $S^* \times S^* \not\leq_m S^*$. Since $S^* \times S^* \leq_{btt} S^*$, it follows that m-reducibility and btt-reducibility differ on the r.e.

nonrecursive sets. qed.

Remark. The above theorem shows that if A is any c-complete noncreative set, then $A \times A \not\leq_m A$. Another generalization of this theorem, to be proved in section 5, is that if A is any set which is simple but not hypersimple, then $A \times A \not\leq_m A$.

THEOREM 2.14 Every c-degree contains a set A such that $A \times A \leq_m A$. Thus there are r.e. sets A which are neither recursive nor creative such that $A \times A \leq_m A$.

Proof Every c-degree contains a set A such that $A \times A \leq_m A$ since c-reducibility is cylindrical. In particular, the c-degree of a hypersimple set B contains a set A such that $A \times A \leq_m A$. A is r.e., but A is not recursive since B is not recursive, and A is not creative since B is not tt-complete and thus not c-complete.

Remark The above example answers a question of P.R. Young in [19].

A set of formulas in a first-order language is called a theory if every formula deducible from formulas in the set is itself in the set. A set of integers is called a theory if it is the image of a theory under some 1-1 effective coding from a first-order language onto the integers. As Rogers has pointed out in [14], if B is a theory, then $B \times B \leq_m B$; for if σ_1 and σ_2 are formulas of the language of the theory, then σ_1 and σ_2 are both in the theory iff the conjunction $\sigma_1 \wedge \sigma_2$ is in the theory. Also, it is easy to see from Young's characterization of cylinders that every theory is a cylinder. Hence every theory is a c-cylinder.

Feferman has shown that if A is any non-empty set, then there is a theory B such that $A \leq_1 B$ and $B \leq_c A$. Hence every non-empty c-cylinder is isomorphic to a theory, and, therefore, every non-empty c-cylinder is a theory. Since no theory can be empty, it follows that the theories are precisely the non-empty c-cylinders. The result can be stated in more standard terminology as follows: A is m-equivalent to some theory iff A is non-empty and $A \times A \leq_m A$.

<u>THEOREM</u> <u>2.15</u> A is a c-cylinder iff $A \times A \leq_1 A$ and $|A| \neq 1$.

<u>Proof</u> If A is a c-cylinder, then since $A \times A \leq_m A$ and A is a cylinder, $A \times A \leq_1 A$. Also, since A is a cylinder, $|A| \neq 1$.

Now assume that $A \times A \leq_1 A$ via f, and $|A| \neq 1$. If A is empty, A is certainly a c-cylinder. So assume $A \neq \emptyset$, and let m and n be distinct members of A. An infinite r.e. subset of A is defined as follows:

$$S_0 = \{m, n\}$$

$$\vdots$$

$$S_{n+1} = S_n \times S_n$$

$$B = \bigcup_{n=0}^{\infty} S_n$$

It is clear by induction that each S_n is a subset of A, so that B is a subset of A. Also, since f is 1-1, $|S_{n+1}| = |S_n|^2$. Thus, since $|S_0| = 2$, B is infinite. Now let g be a recursive function such that

$$W_{g(x)} = f(\{x\} \times B)$$

$W_{g(x)}$ is infinite since B is infinite and f is 1-1. Also $(x \in A \Rightarrow W_{g(x)} \subset A)$ & $(x \in \bar{A} \Rightarrow W_{g(x)} \subset \bar{A})$.

Thus $W_{g(x)}$ witnesses that A is a cylinder by Young's characterization. Therefore A is a c-cylinder, since $A \times A \leq_m A$ qed.

The above characterization could be applied to yield alternative descriptions of R-cylinders for reducibilities other than c-reducibility (such as tt-reducibility).

We now consider R-cylinders for a few reducibilities other than the truth-table reducibilities discussed before.

The writer knows of no good characterization of T-cylinders. Rogers [14] has shown that if K is a creative set, then no T-cylinders lie above K in the T-ordering. Hence the closure of T-reducibility has a maximum degree containing K, and T-reducibility certainly is not closed. On the other hand Martin (unpublished) has shown that if $B \leq_T A$, and there is no hyperimmune set in the T-degree of A, then $B \leq_{tt} A$. Thus if A is a tt-cylinder and the T-degree of A has no hyperimmune sets, A is a T-cylinder. Martin has also shown that nonrecursive T-degrees exist which contains no hyperimmune sets, so that nonrecursive T-cylinders do exist. It may be that the T-cylinders are just the tt-cylinders which have hyperimmune-free T-degree.[8]

<u>DEFINITION</u> <u>2.16</u> $A \leq_j B$ iff $(\exists$ p.r.$\psi)(\forall x)[x \in A \Longleftrightarrow \psi(x)$ cgt. & $\psi(x) \in B]$.

This reducibility is introduced to facilitate the discussion of i-reducibility which will follow. If one defines

$$A^j = \{x \mid \varphi_x(0) \text{ cgt. } \& \ \varphi_x(0) \in A\}$$

Then A^j is a j-cylindrification of A, and A is a j-cylinder iff $A \equiv_1 A^j$.

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[8](Added in proof) It is easy to verify that the T-cylinders are just the tt-cylinders of hyperimmune-free T-degree. The proof also shows that every T-degree either consists of a single tt-degree or contains infinitely many tt-degrees.

For,

$A^{j} \leq_{j} A$ via $\lambda x[\varphi_x(0)]$

and $A \leq_{1} A^{j}$ via f, where f is a 1-1 recursive function such that, for all x and y, $\varphi_{f(x)}(y) = x$.

Now, if $B \leq_j A$ via Ψ, and f is a 1-1 recursive function such that

$$\varphi_{f(x)}(y) = \begin{cases} \Psi(x) & \text{if } \Psi(x) \text{ cgt.} \\ \text{dgt.} & \text{otherwise} \end{cases}$$

Then $x \in B \Leftrightarrow \Psi(x)$ cgt. & $\Psi(x) \in A \Leftrightarrow \varphi_{f(x)}(0)$ cgt. & $\varphi_{f(x)}(0) \in A \Leftrightarrow f(x) \in A^{j}$

so $B \leq_1 A^{j}$ via f.

Thus, $B \leq_j A^j \Rightarrow B \leq_j A \Rightarrow B \leq_1 A^j$, so that A^j is a j-cylinder in the j-degree of A.

<u>DEFINITION 2.17</u> $A \leq_i B$ (A is isolically reducible to B) if

$$(\exists \text{ p.r. } 1\text{-}1\ \Psi)(\forall x)[x \in A \Leftrightarrow (\Psi(x) \text{ cgt. } \& \ \Psi(x) \in B)]$$

Using Young's characterization of cylinders, it is easy to show that

$A \leq_j B$, B a cylinder $\Rightarrow A \leq_i B$

Hence if A is a cylinder, A^j is an i-cylinder in the i-degree of A. However, if A is immune, then A^j is not in its i-degree, for every set i-reducible to an immune set is immune. I do not know whether any (or all) immune sets are i-equivalent to i-cylinders. In fact, the only i-cylinders I know of are the j-cylinders and the finite sets.

SECTION 3. REDUCIBILITIES ON THE R.E. SETS

It was mentioned in section 2 that many of the reducibilities defined there differed on the r.e. sets. Actually, it may be shown that the ten reducibilities defined in the table on page 22 all differ on the r.e. nonrecursive sets, except that n and m coincide on these sets. Lachlan has a general theorem to this effect in [10]. The theorem is proved by a priority argument, and, of course, each instance of the theorem may be proved by a straightforward Friedberg type priority argument. These arguments will not be presented.

In this section it will be shown that n and m reducibility coincide on the nonrecursive r.e. sets and that btt and bp reducibility coincide for certain special r.e. sets. Then it will be shown by a priority argument that there are r.e. sets A, B such that

$A \leq_{bc} B$, $A \leq_{bq} B$, and $A \not\leq_m B$.

Thus, m-reducibility is not the intersection of bq and bc.

PROPOSITION 3.1 Suppose that A and B are r.e. and $N \neq B \neq \emptyset$. Then $A \leq_n B \Rightarrow A \leq_m B$.

Proof Assume A and B are as above with f a recursive function such that $A \leq_m B$ join $\overline{B}$ via f. Let $b \in B$ and $b' \in \overline{B}$. To compute g(x), first compute f(x). If f(x) is even, let $g(x) = \frac{f(x)}{2}$. If f(x) is odd, then $x \in A \Longleftrightarrow \frac{f(x)-1}{2} \notin B$, so look for x in A and look for $\frac{f(x)-1}{2}$ in B by effectively listing these sets. If x is found in A, set $g(x) = b$. If $\frac{f(x)-1}{2}$ is found in B, set $g(x) = b'$. Then g is a recursive function, and $A \leq_m B$ via g.

DEFINITION 3.2 (Friedberg) A set A is maximal if A is r.e. and coinfinite and for every r.e. set C, either $C \cap \overline{A}$ or $\overline{C} \cap \overline{A}$ is finite.

THEOREM 3.3 If A is a maximal set and $A \leq_{btt} B$, where B is r.e., then $A \leq_{bp} B$.

Proof Assume the hypotheses of the theorem. Since $A \leq_{btt} B$, there is a recursive function f and a number m such that for all x

$$x \in A \Leftrightarrow (\exists u)(\exists v)[\langle u, v\rangle \in D_{f(x)} \ \&\ D_u \subset B \ \&\ D_v \subset \overline{B}]$$
$$\&\ \Big| \bigcup_{\langle u,v\rangle \in D_{f(x)}} D_u \cup D_v \Big| \leq m$$

The function f exists by part (iv) of theorem 2.9 and the observation that the sets D_x may be used in place of the D_x' in that part of the theorem.

Define

$$N(f) = \sup_x \big| \{ \langle u,v\rangle \mid \langle u,v\rangle \in D_{f(x)} \ \&\ D_v \neq \phi\} \big|$$

N(f) ("the negativity of f") measures, in a sense, the extent to which f fails to be a positive truth table reduction. In particular, if N(f) is zero, f immediately yields a bounded positive reduction, since in that case all the D_v's such that $\langle u,v\rangle$ occur in any $D_{f(x)}$ are empty. Thus the theorem holds if f has negativity zero. Now assume that the theorem holds for all f's which btt-reduce A to B as above and have negativity n. Let f' have negativity n + 1, and assume $A \leq_{btt} B$ via f' as above. Define a partial recursive function Υ by

$$\Upsilon(x) = \begin{cases} \text{the least } \langle u,v\rangle \text{ such that } \langle u,v\rangle \in D_{f(x)} \text{ and } v \neq 0, \text{ if such exists} \\ \text{divergent, otherwise} \end{cases}$$

(The element $\langle u,v\rangle$ will be "removed" from $D_{f(x)}$ to yield a reduction of lower negativity.)

Define the "projection functions" π_1 and π_2 to be $\lambda\langle x, y\rangle[x]$ and $\lambda\langle x, y\rangle[y]$ respectively.

Let $C = \{x \mid \Psi(x) \text{ convergent } \& \ D_{\pi_2\Psi(x)} \cap B \neq \phi\}$

Since C is r.e., either $C \cap \overline{A}$ is finite or $\overline{C} \cap \overline{A}$ is finite.

Case 1 $C \cap \overline{A}$ is finite. Define a recursive function f by

$$D_{f(x)} = \begin{cases} D_{f'(x)} & \text{if } \Psi(x) \text{ dgt. (i.e. if } D_{f'(x)} \text{ is "positive")} \\ (D_{f'(x)} - \{\Psi(x)\}) \cup \{\langle \pi_1\Psi(x), 0\rangle\} & \text{if } \Psi(x) \text{ cgt. } \& \ x \notin C \cap \overline{A} \\ E & \text{if } x \in C \cap \overline{A} \end{cases}$$

where E is a fixed positive "truth-table condition" which is false of B such as

$$\{\langle 2^{b'}, 0\rangle\}$$

where $b' \notin B$.

f is recursive since $C \cap \overline{A}$ is finite and the domain of Ψ is recursive. From the definition of C, $A \leq_{btt} B$ via f. Also, f has negativity at most n. Hence, by the induction assumption, $A \leq_{bp} B$.

Case 2. $\overline{C} \cap \overline{A}$ is finite. A recursive function g will be defined by the following instructions. Given x, see if $x \in \overline{C} \cap \overline{A}$. If so, give output 0. If not, then $x \in C \cup A$. Simultaneously list C and A until x appears in one or the other. Give output 1 if x first appears in C and output 2 if x first appears in A. Now define f:

$$D_{f(x)} = \begin{cases} E & \text{if } g(x) = 0 \\ D_{f'(x)} - \{\Psi(x)\} & \text{if } g(x) = 1 \\ F & \text{if } g(x) = 2 \end{cases}$$

where E is a fixed positive tt-condition false of B as in case 1 and F is a fixed positive tt-condition true of B. f gives a reduction of A to B, by the definition of C.

$$N(f) \leq n, \text{ so } A \leq_{bp} B.$$

Thus the theorem has been proved for all reductions of finite negativity, so it certainly holds for all bounded truth table reductions. qed.

It will be shown in section 4 that the following analogue to theorem 3.3. <u>fails</u>:

$$A \text{ maximal, } B \text{ r.e. } A \leq_{tt} B \Rightarrow A \leq_{p} B$$

It can also be shown that it is not true that

$$A \text{ hypersimple, } B \text{ r.e., } A \leq_{btt} B \Rightarrow A \leq_{bp} B.$$

<u>THEOREM</u> <u>3.4</u> There are r.e. sets A,B such that

$$A \leq_{bc} B, \ A \leq_{bq} B, \text{ and } A \not\leq_{m} B.$$

<u>Proof</u> The sets A and B will be such that, for all x

$$x \in A \Leftrightarrow (4x \in B \text{ and } 4x + 1 \in B)$$

and $$x \in A \Leftrightarrow (4x + 2 \in B \text{ or } 4x + 3 \in B)$$

Hence $A \leq_{bc} B$ and $A \leq_{bq} B$.

A straightforward priority construction of the Friedberg type will be used to ensure that $A \not\leq_{m} B$. We imagine that we have two infinite vertical lists of N which will be called the A-list and the B-list. We also have symbols "+" and "-" which can be associated with members of the A-list and the B-list as the construction proceeds. Finally, we have a movable marker $\boxed{i}$ for each $i \in N$ which can be associated with numbers in the A-list and which can be moved to larger numbers in the A-list as the construction proceeds.

The construction will be given inductively by stages, and A and B will be defined by

$$A = \{x \mid x \text{ receives a "+" in the A-list at some stage}\}$$

$$B = \{x \mid x \text{ receives a "+" in the B-list at some stage}\}$$

The purpose of the movable marker $\boxed{j}$ is to prevent φ_j from yielding an m-reduction of A to B. In particular, if $\boxed{j}$ is associated with a number a_j in the A-list, the construction will try to ensure that

$$a_j \in A \Longleftrightarrow \varphi_j(a_j) \notin B$$

Call an integer in the A-list <u>free</u> if neither x nor any larger number has any mark or marker associated with it in the A-list.and neither 4x nor any larger number has any mark associated with it in the B-list.

The construction is as follows:

<u>Stage</u> <u>n</u> $(n \geq 0)$

Associate $\boxed{n}$ with the least free integer in the A-list. Let $a_0, a_1, \ldots, a_n$ be the present positions of the markers $\boxed{0}$, $\boxed{1}$, ..., $\boxed{n}$. Let j be the smallest number i such that

a_i has neither a "+" nor a "-" in the A-list and

$\varphi_i(a_i)$ is convergent in n or fewer steps

(If no such i exists, go to stage n + 1.)

Let $c = \varphi_j(a_j)$ We will try to arrange that

$$a_j \in A \iff c \notin B$$

<u>Case 1.</u> c has a "+" in the B-list.

Put a "-" by a_j in the A-list and a "-" by each member of $\{4x, 4x+1, 4x+2, 4x+3\}$ in the B-list.

<u>Case 2.</u> c does not have a " + " in the B-List and $c \notin \{4x, 4x+1\}$

Put a "-" by c in the B-list and a "+" by a_j in the A-list. Also put a "+" by each member of $\{4x, \ldots, 4x+3\} - \{c\}$ in the B-list.

<u>Case 3.</u> $c = 4x$ or $c = 4x + 1$

Put a "+" by c in the B-list and a "-" by a_j in the A-list. Also put a " - " by each member of $\{4x, \ldots, 4x+3\} - \{c\}$ in the B-list.

In any case, if $j < n$, move the markers $\boxed{k}$ such that $j < k \leq n$ down to free integers in the A-list.

Note that the above stage was designed so that φ_j cannot give an m-reduction between A and B if the " - "symbols introduced at that stage are not disturbed by some later stage.

A and B are r.e. because, given n, it is possible to determine effectively what numbers are put into A and B at stage n.

$A \leq_{bc} B$ and $A \leq_{bq} B$ because the reductions given at the beginning of the proof hold true for the partial listings of A and B obtained at the conclusion of each stage.

Observe that a marker $\boxed{k}$ is caused to move only when a marker $\boxed{j}$ such that $j < k$ (i.e. a marker of higher priority) is <u>attacked</u> (i.e. plays the role of $\boxed{j}$ in the construction.) Also, a marker is attacked at most once at a given location. Hence, by a simple inductive argument, each marker moves only finitely often and thus achieves a final resting place.

Let j be any given number. Let a_j be the final resting place of $\boxed{j}$. It is easy to see that if $\varphi_j(a_j)$ is convergent, then $\boxed{j}$ must be attacked at some stage after $\boxed{j}$ achieves its final resting place. Assume that $\varphi_j(a_j)$ is convergent, and let n be a stage such that $\boxed{j}$ is associated with a_j at stage n and is attacked at that stage. Then none of the "$-$" symbols introduced at that stage can ever be changed to "$+$" signs at a later stage lest $\boxed{j}$ be caused to move. Thus, by the remark just after the construction, φ_j cannot yield an m-reduction of A to B.

Therefore $A \not\leq_m B$. qed.

Arguments similar to the above can be used to show the distinctness of the various reducibilities of section 2 on the r.e. sets.

SECTION 4 SEMIRECURSIVE SETS

In this section, the notion of recursiveness will be generalized to that of semirecursiveness, Some existence theorems for semirecursive sets will be proved, the properties of semirecursive sets will be studied, and the information thus obtained will be applied to the study of reducibilities.

DEFINITION 4.1 A set A is semirecursive if there exists a recursive function f of two variables such that, for all x and y,

(i) $f(x,y) = x$ or $f(x,y) = y$, and

(ii) $(x \in A \text{ or } y \in A) \Rightarrow f(x,y) \in A$.

Such a recursive function is called a selector function for A.

We now recall some standard definitions.

DEFINITION 4.2 (Dekker, Myhill, Tennenbaum)

(i) Let A be a set and let $a_0, a_1 \ldots$ be the members of A in increasing order. A is said to be retraceable if there is a partial recursive function ψ such that

$$\psi(a_o) = a_o$$

and $\psi(a_{i+1}) = a_i$ for each $i \geq 0$.

In such a case, ψ is called a partial retracing function for A.

(ii) Let A be a set, A is said to be regressive if there is an enumeration $a_o, a_1, \ldots$ of A and a partial recursive function ψ such that

$$\psi(a_o) = a_o$$

and $\psi(a_{i+1}) = a_i$ for each $i \geq 0$.

In such a case, Ψ is called a partial regressing function for A.

It is easy to show that a retraceable set is recursive in every infinite subset and that a regressive set is recursively enumerable in every infinite subset. (See Dekker and Myhill [3]). Hence every retraceable set is recursive or immune and every regressive set is r.e. or immune.

THEOREM 4.3 If A is r.e. and coregressive, then A is semirecursive

Proof Suppose that A is r.e. and $\bar{A}$ is regressive, with Ψ a partial regressing function for $\bar{A}$.

We will define a selector function f: Given x and y, simultaneously enumerate A and $\{\Psi^n(x) \mid n \geq 0\}$ and $\{\Psi^n(y) \mid n \geq 0\}$. Stop the procedure the first time any one of the following occurs:

(i) x is found in A

(ii) y is found in A

(iii) x is found in $\{\Psi^n(y) \mid n \geq 0\}$

(iv) y is found in $\{\Psi^n(x) \mid n \geq 0\}$

If event (i) or (iv) stops the procedure, set f(x,y) = x.

If event (ii) or (iii) stops the procedure, set f(x,y) = y.

f is partial recursive. Also f is total, for if for some x and y the procedure never stops, then $x \in \bar{A}$ and $y \in \bar{A}$. Then it is clear from the definition of regressiveness that event (iii) or (iv) must occur. Thus f is recursive.

Now suppose $f(x,y) \notin A$. It must be shown that $x \notin A$ and $y \notin A$. Since $f(x,y) \notin A$, f(x,y) was comupted via event (iii) or event (iv). Suppose, without loss of generality, that it was computed with event (iii). Thus f(x,y) = y, so $y \notin A$. Hence by the definition of regressiveness

$$\{\psi^n(y) \mid n \geq 0\} \subset \bar{A}$$

But, by (iii) $x \in \{\psi^n(y) \mid n \geq 0\}$, so $x \notin A$ and $y \notin A$. qed.

<u>DEFINITION</u> <u>4.4</u> An R-degree is <u>r.e.</u> [<u>recursive</u>] if it contains an r.e. [recursive] set.

<u>COROLLARY</u> <u>4.5</u> Every r.e. nonrecursive Turing degree contains a semirecursive hypersimple set.

<u>Proof</u> The hypersimple set constructed by Dekker in each r.e. nonrecursive T-degree has been shown by Dekker and Myhill to be coretraceable and thus coregressive. qed.

We now prove a more extensive existence theorem for semirecursive sets. The present writer introduced the notion of semirecursive set and the following construction was first used by McLaughlin and Martin to prove the existence of a continuum of semirecursive sets.

<u>THEOREM</u> <u>4.6</u> For any set A, there is a set B such that B is semirecursive, $B \leq_p A$, and $A \leq_{tt} B$.

<u>Proof</u> Let A be given. To avoid trivial cases, assume that A is infinite and coinfinite. Define a real number r by

$$r = \sum_{n \in A} 2^{-n}$$

For each integer x, define a rational number r_x by

$$r_x = \sum_{n \in D'_x} 2^{-n} \qquad \text{(cf. definition 2.8)}$$

Define $B = \{x \mid r_x \leq r\}$.

Now B is semirecursive with the following selector function:

$$f(x,y) = \begin{cases} x & \text{if } r_x \leq r_y \\ y & \text{if } r_y > r_x \end{cases}$$

To see that $B \leq_p A$, first define a recursive function h by

$h(x)$ = the largest member of D_x'

To see that $B \leq_p A$, it will be sufficient to show that

$x \in B \Longleftrightarrow$ for some y such that $D_y' \subset \{0,1, \ldots, h(x)\}$ & $r_y \geq r_x$

$D_y' \subset A$

Suppose $x \in B$. Let $D_y' = A \cap \{0,1,\ldots,h(x)\}$. Then, since A is coinfinite, it follows from the fact that $r_x \leq r$ and an elementary property of binary expansions, that $r_y \geq r_x$, so the desired y exists.

Now suppose that such a y exists. Since $D_y' \subset A$, $r_y \leq r$. Since $r_x \leq r_y$, it follows that $r_x \leq r$ and so $x \in B$.

To show that $A \leq_{tt} B$, we will show, by induction on n, how to define g(n) such that

$n \in A \Longleftrightarrow$ (the formula) g(n) is true of B

The induction will be uniform in n, so g will be recursive and it will follow that $A \leq_{tt} B$.

$0 \in A \Leftrightarrow r \geq 1 \Leftrightarrow 1 \in B$ (since $r_1 = \sum_{n \in D_1' = \{0\}} 2^{-n} = 1$)

So let g (0) be a code number for the formula "P_1."

Now assume that g(0), g(1), ..., g(n-1) have all been defined. Let D_{x_1}, D_{x_2}, ..., D_{x_k} ($k = 2^n$) be a list of all subsets of $\{0,1, \ldots, n-1\}$. To make the procedure definite, assume $x_1 < x_2 < \ldots < x_k$ Now,

$n \in A \Longleftrightarrow A \cap \{0,1,\ldots,n-1\} = D_{x_1}$ & $y_1 \in B$ (where $D_{y_1} = D_{x_1} \cup \{n\}$)

or

.

.

.

or

$A \cap \{0,1, \ldots, n-1\} = D_{x_k}$ & $y_k \in B$ (where $D_{y_k} = D_{x_k} \cup \{n\}$)

The above statement follows from the same reasoning about binary expansions that was used to show $B \leq_p A$. Now by the induction assumption, for each i the statement "$A \cap \{0,1, \ldots, n-1\} = D_{x_i}$" can be uniformly translated into an equivalent statement about B. Now let g(n) be the code number for the formula obtained from the right hand side of the above equivalence when these equivalent formulas are substituted in. This completes the induction. qed.

<u>COROLLARY</u> <u>4.7</u> (i) (McLaughlin, Martin) There exist $2^{\aleph_0}$ semirecursive sets.

(ii) Every r.e. tt-degree contains an r.e. semirecursive set.

<u>THEOREM</u> <u>4.8</u> The following statements are equivalent.

(i) A is semirecursive.

(ii) $A \times \bar{A}$ and $\bar{A} \times A$ are recursively separable.

(iii) $(\exists \text{ rec. } h)(\forall x)[D_x \cap A \neq \emptyset \Rightarrow h(x) \in D_x \cap A]$.

(iv) (McLaughlin, Appel) [unpublished] A is the lower half of of a cut in a recursive linear ordering of N (i.e. there is a recursive ordering linear $\leq_0$ of N such that $y \in A,\ x \leq_0 y \Rightarrow x \in A$.)

<u>Proof</u> (i)$\Leftrightarrow$(ii) and (iii)$\Rightarrow$(i) are trivial. Thus it will be sufficient to show that (i)$\Rightarrow$(iv)$\Rightarrow$(iii).

Assume (i), and let A be semirecursive with a selector function f. A function g mapping the integers 1-1 into the rationals will be defined such that there are recursive functions f_1, f_2, and f_3 with $g(x) = (-1)^{f_1(x)} \dfrac{f_2(x)}{f_3(x)}$. Then the desired recursive ordering will be defined by

$$x \leq_0 y \Longleftrightarrow g(x) \leq g(y)$$

$g(n)$ is defined by induction on n.

Define $g(0) = 0$

Now assume that $g(0), g(1), \ldots, g(n)$ are all defined. Let $x_0, x_1, \ldots, x_n$ be the integers from 0 to n arranged in such a way that $g(x_0) < g(x_1) < \ldots < g(x_n)$.

Case 1. $f(n + 1, x_0) = n + 1$ Then define $g(n + 1) = g(x_0) - 1$.

Case 2. $f(n + 1, x_n) = x_n$ Then define $g(n + 1) = g(x_n) + 1$.

Case 3. Neither case 1 nor case 2 applies. Then let j be the largest number i such that $f(n, x_i) = n$. Then define $g(n + 1) = \frac{g(x_j) + g(x_{j+1})}{2}$ Note that j exists and is less than n because neither case 1 nor case 2 applies.

This completes the definition of g. The recursive ordering $\leq_0$ is defined as above. It is straightforward to verify by induction on $\max \{x, y\}$ that

$$y \in A,\ x \leq_0 y \Longrightarrow x \in A$$

Thus (iv) is proved.

Now assume (iv), and let $\leq_0$ be a recursive linear ordering of N such that A is the lower half of a cut in $\leq_0$. Let $h(x)$ be the least member under $\leq_0$ of D_x if D_x is non-empty and 0 if D_x is empty. Then h is a recursive function, and

$$D_x \cap A \neq 0 \Rightarrow h(x) \in D_x \cap A$$

Thus (iii) is proved. It should be noted that (iii) can also be proved directly from (i) without difficulty by defining $h(x)$ inductively on the cardinality of D_x. qed.

The following theorem gives some simple properties of semi-recursive sets.

THEOREM 4.9 Let A be semirecursive and let B be any set. Then,

(i) $A \equiv_m A \times A$, $A \equiv_m \overline{\bar{A} \times \bar{A}}$

(ii) $B \leq_p A \Rightarrow B \leq_m A$

(iii) $B \leq_p A \Rightarrow B$ semirecursive

(iv) The positive degree of A consists of a single m-degree.

(v) A immune $\Rightarrow$ A hyperimmune

(vi) $A \leq_p \bar{A} \Rightarrow A$ recursive

Proof Let f be a selector function for A for remainder of proof.

(i) $\overline{A \times A} \leq_m A$ via f, or more precisely, via $\lambda \langle x,y \rangle [f(x,y)]$. Thus $\overline{\bar{A} \times \bar{A}} \leq_m A$. The complement of A is also semirecursive, so $A \times A \equiv_m A$.

(ii) By (i) and theorem 2.6 AxN is a p-cylinder. Thus (ii) follows.

(iii) Assume $B \leq_p A$. Then $B \leq_m A$ by (ii). Let $B \leq_m A$ via g. Define h:

$$h(x,y) = \begin{cases} x & \text{if} \quad f(g(x),g(y)) = g(x) \\ y & \text{if} \quad f(g(x),g(y)) \neq g(x) \end{cases}$$

Then h is a selector function for B, so B is semirecursive.

(iv) Assume $B \equiv_p A$. To show: $B \equiv_m A$. By (ii) $B \leq_m A$. By (iii), B is semirecursive. Hence by (ii) (applied with B and A interchanged), $A \leq_m B$.

(v) Assume that A is infinite and not hyperimmune. To show: A is not immune. Let k be a recursive function such that $D_{k(x)}$ witnesses the non-hyperimmunity of A i.e., for all x and y

$$D_{k(x)} \cap A \neq \emptyset \text{ and } (x \neq y) \Rightarrow (D_{k(x)} \cap D_{k(y)} = \emptyset)$$

Let h be a recursive function such that

$$D_x \cap A \neq \emptyset \Rightarrow h(x) \in A.$$

Then since for each x, $hk(x) \in D_{k(x)} \cap A$, the function hk is a 1-1 recursive function with range a subset of A, so A is not immune.

(vi) Suppose $A \leq_p \bar{A}$. Then $A \leq_m \bar{A}$ by (ii). Let $A \leq_m \bar{A}$ via g. Then

$$x \in A \Leftrightarrow f(x,g(x)) = x$$

Hence A is recursive. qed.

Many facts about reducibilities can now be deduced immediately from the preceding theorem and the constructions at the beginning of this section.

<u>COROLLARIES</u> <u>4.10</u>

(i) Each tt-degree contains a p-degree consisting of a single m-degree.

(ii) Each r.e. tt-degree contains an r.e. p-degree consisting of a single m-degree.

(iii) No p-complete set is semirecursive.

(iv) There exists a set which is tt-complete but not p-complete.

(v) Not every nonrecursive r.e. tt-degree contains a simple semirecursive set.

(vi) (Dekker) Each simple set having a regressive complement is hypersimple.

(vii) Each tt-degree contains incomparable p-degrees.

(viii) There exist hypersimple sets A such that AxA is a cylinder.

Proof

(i) follows from theorem 4.6 and (iv) of theorem 4.9

(ii) follows from Corollary 4.7 and (iv) of theorem 4.9

(iii) Assume A is p-complete, and let B be any set which is simple but not hypersimple. B is not semirecursive by (v) of 4.9. Thus, since B is not semirecursive and $B \leq_p A$, A is not semirecursive by (ii) of 4.9.

(iv) By (iii), the r.e. semirecursive set in the complete tt-degree is not p-complete.

(v) Any simple set in the complete tt-degree would be hypersimple, violating the theorem of Post that no hypersimple set is tt-complete.

(vi) A simple set with a regressive complement is semirecursive by theorem 4.3 and hence hypersimple by (v) of 4.9

(vii) The recursive tt-degree contains $\emptyset$ and N, which are p-incomparable. Any non-recursive tt-degree contains a semirecursive set which is p-incomparable with its complement by (vi) of 4.9.

(viii) Young [19] has shown that if A is simple, then AxA is a cylinder iff $AxA \leq_m A$. Thus if A is any hypersimple semirecursive set, AxA is a cylinder. qed.

Further results of this kind can be obtained from a theorem due to Yates. This theorem will be of fundamental importance in section 5.

THEOREM 4.11 (Yates) Each r.e. nonrecursive T-degree contains a simple set which is not hypersimple.

Proof. See Yates [17] . qed.

COROLLARY 4.12

(i) Each r.e. nonrecursive T-degree contains an r.e. set which is not semirecursive.

(ii) Each r.e. T-degree contains at least two p-degrees.

Proof (i) follows from the theorem and (v) of 4.9

(ii) follows from (i) above, from (iii) of 4.9 and corollary 4.5.

It will be shown that hyperhyperimmune sets are not semirecursive. The proof is a slight strengthening of an argument due to Martin.

THEOREM 4.13 (Martin) Every infinite semirecursive set has an infinite co-r.e. retraceable subset.

Proof Let A be infinite and semirecursive. We may assume that A is immune, since otherwise A has an infinite r.e. subset and hence an infinite recursive subset and the result is immediate. Suppose that A is the lower half of a cut in a recursive linear ordering $\leq_0$ of N. Define

$$B = \{x \mid (\forall y)[x \leq y \Rightarrow x \leq_0 y]\}$$

It is claimed that B is the desired infinite co-r.e. retraceable subset of A. Clearly B is co-r.e. To show that $B \subset A$, assume $x \in B$. Let y be any member of A which is greater than x. Then $x \leq_0 y$, since $x \in B$. Therefore $x \in A$ by the definition of "cut." Hence $B \subset A$. Let b_0 be the least member of B. (B will later be shown non-empty). Then the recursive function f will be a retracing function for B:

$$f(x) = \begin{cases} b_0 & \text{if } x \leq b_0 \\ \text{the largest number } z \text{ such that } z < x \text{ and} & \\ \quad (\forall u)[z \leq u \leq x \Rightarrow z \leq_0 u], & \text{otherwise} \end{cases}$$

f is total since if $x > b_0$ a number z with the required property, i.e. b_0, will exist, and hence a largest such z will exist. Now suppose $x \in B$ and $z \prec x$. Then

$$z \in B \Longleftrightarrow (\forall u) \left[z \preceq u \leq x \Longrightarrow z \preceq_0 u \right]$$

The implication to the right is immediate from the definition of B and the implication to the left follows from the fact that $x \in B$ and $\preceq_0$ is transitive.

Thus f maps the least member of B to itself and every other member of B to the next smaller member and is therefore a retracing function for B.

It remains to show that B is infinite. Assume not. Let j be a member of which A is larger than every member of B. Now the following recursive function g will enumerate an infinite r.e. subset of A. This will contradict the assumption that A was immune.

g is defined inductively:

$$g(0) = j.$$

$$\vdots$$

$g(n + 1) =$ the smallest number y such that

$$y > g(n) \text{ and } y \preceq_0 g(n)$$

Range g is a subset of A since $j \in A$ and g is a decreasing function with respect to the ordering $\preceq_0$. To show that g is total, assume the opposite and let n + 1 be the least argument for which g is not defined. Then for all y,

$$g(n) \leq y \Longrightarrow g(n) \preceq_0 y \qquad \text{(since } \preceq_0 \text{ is a total ordering)}$$

This says that $g(n) \in B$. But

$$g(n) \geq j$$

so the assumption that j was larger than every member of B is contradicted.

Range g is infinite and r.e. since g is a 1-1 recursive function.

qed.

Now that it has been shown that every infinite semirecursive set has an infinite retraceable subset, it is natural to inquire whether every infinite retraceable set has an infinite semirecursive subset. The following corollary shows that this is far from being the case.

<u>COROLLARY</u> <u>4.14</u> If a retraceable set A has an infinite semirecursive subset, then A is recursive in K, where K is any creative set.

<u>Proof</u> Suppose A is retraceable and has an infinite semirecursive subset B. By the theorem, B has an infinite co-r.e. subset C. Since C is an infinite subset of A and A is retraceable, A is recursive in C. Thus A is recursive in K. qed.

<u>COROLLARY</u> <u>4.15</u> Each infinite co-r.e. regressive set has an infinite co-r.e. retraceable subset.

<u>Proof</u> Each such set is semirecursive by theorem 4.3. qed.

The principal corollary of theorem 4.13 will be that no hyperhypersimple set is semirecursive. To be able to make a stronger statement, we state a definition and theorem.

<u>DEFINITION</u> <u>4.16</u>

(i) (Young, Martin) A set A is <u>finitely</u> <u>strongly</u> <u>hyperimmune</u> (FSHI) if A is infinite and there is no recursive function f such that and for all x and y,

$$[(x \neq y) \Rightarrow W_{f(x)} \cap W_{f(y)} = \phi] \ \& \ W_{f(x)} \text{ finite} \ \& \ W_{f(x)} \cap A \neq \phi \ \& \ \bigcup_x W_{f(x)} = N$$

(ii) (Yates) A function f is <u>basic</u> if f is finite-one i.e. if the set $f^{-1}(x)$ is finite for each x.

<u>THEOREM</u> <u>4.17</u> (Martin) A set A is FSHI iff it is infinite and has no infinite subset retraced by a basic recursive function.

<u>Proof</u> Just the "only if" part of the theorem will be needed, and only this part will be proved. Suppose that the basic recursive function f retraces an infinite subset of A. We may assume (cf. theorem 5.17) that $f(x) \leq x$ for each x. Then for every x there exists an n such that $f^{n+1}(x) = f^{n}(x)$. Hence if we define

$$W_{f(n)} = \{x \mid n \text{ is the least number } m \text{ such that } f^{m+1}(x) = f^{m}(x)\}$$

the sets $W_{f(n)}$ witness that A is not FSHI qed.

<u>COROLLARY</u> <u>4.18</u> No FSHI set is semirecursive.

<u>Proof</u> The retracing function f defined in the proof of theorem 4.13 is a basic function so that the semirecursive set A cannot be FSHI. It is not necessary to use the proof of 4.13, however, since it is easy to see that each co-r.e. retraceable set is retraced by some basic recursive function. qed.

Since hyperhyperimmune sets are trivially FSHI, it of course follows that no hyperhyperimmune set is semirecursive. Thus no hyperhypersimple set is m-reducible to a coregressive hypersimple set. However, Appel and McLaughlin have proved a stronger result by different methods.

<u>THEOREM</u> <u>4.19</u> (Appel and McLaughlin) Let A be a hyperhypersimple set and let B be hypersimple and coregressive. Then A and B are m-incomparable.

<u>Proof</u> See Appel and McLaughlin [1] .

<u>COROLLARY 4.20</u> If A is hyperhypersimple, then $\overline{\bar{A}x\bar{A}} \not\leq_m A$.

<u>Proof</u> Let A be a given hyperhypersimple set and let B be obtained by the Dekker construction for A. Thus B is hypersimple, coretraceable, and $B \leq_q A$. Now assume $\overline{\bar{A}x\bar{A}} \leq_m A$. Then $B \leq_m A$ by theorem 2.6.. But this contradicts theorem 4.19. qed.

Note that corollary 4.20 implies, independently of corollary 4.18, that no hyperhypersimple set is semirecursive.

Corollary 4.18 also implies that not every r.e. btt-degree contains an r.e. semirecursive set. In particular, no maximal set is btt-reducible to any r.e. semirecursive set, since otherwise the maximal set would also be bp-reducible to the semirecursive set by theorem 3.3, and hence would itself be semirecursive, contradicting corollary 4.18. Corollary 4.18 also shows that there are r.e. sets A and B such that $A \leq_{tt} B$ but A cannot be tt-reduced to B via any f with "finite negativity" in the sense of the proof of theorem 3.3. To see this, let A be a maximal set and B be any r.e. semirecursive set in the tt-degree of A The desired fact now follows from the proof theorem 3.3, since if A were tt-reducible to B via some f with finite negativity, A would be p-reducible to B and thus semirecursive.

Let A be any set. It is immediate from (vi) of theorem 4.9 that A join $\bar{A}$ semirecursive $\Rightarrow$ A recursive.

Now letting A be semirecursive but not recursive, it is clear that the join of two semirecursive sets need not be semirecursive and that some non-semirecursive set can be n-reduced (and hence btt-reduced) to a semirecursive set. The following theorems investigate whether such phenomena still occur when all sets involved are required to be r.e.

<u>THEOREM</u> <u>4.21</u> There are r.e. and coretraceable (and therefore semirecursive) sets A,B such that A join B is not semirecursive.

<u>Proof</u> The construction will define two 1-1 recursive functions f and g. Then if we define

$$A = \{x \mid (\exists y)\,[\, y > x \;\&\; f(y) < f(x)]\,\}$$
$$B = \{x \mid (\exists y)\,[\, y > x \;\&\; g(y) < g(x)]\,\}$$

A and B will be the desired sets.

The construction uses a single list and a set of movable markers. The movable markers will be associated with even integers in the list. An integer 2z is said to be <u>free</u> in the list at a given stage if there are no markers below it and f and g are undefined for all arguments $y \geq z$. A symbol $*$ will be placed beside a number in the list when the marker associated with it has been "attacked".

<u>Stage</u> <u>n</u> $(n \geq 0)$

Let 2k be the least free non-zero integer. Associate the marker $\boxed{n}$ with 2k.

Define $f(x) = 2x$ for each $x \leq k + 1$ such that $f(x)$ has not previously been defined.

Define $g(x) = 2x$ for each $x \leq k + 1$ such that $g(x)$ has not previously been defined.

Let $2a_0, 2a_1, \ldots, 2a_n$ be the present positions of $\boxed{0}, \boxed{1}, \ldots, \boxed{n}$. Calculate n steps in each of $\varphi_i\,(2a_i, 2a_i + 1)$, $0 \leq i \leq n$. Let j be the least number such that $\varphi_j\,(2a_j, 2a_j + 1)$ is found to be convergent in n steps and $2a_j$ does not have a $*$. (If no such j exists, go to stage n + 1.) Put a * by $2a_j$. If $\varphi_j(2a_j, 2a_j + 1) \notin \{2a_j, 2a_j + 1\}$, go to stage n + 1. Otherwise there are two cases.

<u>Case 1</u> $\varphi_j(2a_j, 2a_j + 1) = 2a_j$

To ensure that φ_j is not a selector function for A join B, we want to put a_j into $\bar{A} \cap B$. When the construction is complete, it will be clear that

$$f(a_j) = g(a_j) = 2a_j.$$

Hence we define

$$f(k + 2) = 2k + 4$$

(Recall that 2k was the least free non-zero integer.)

$$g(k + 2) = g(a_j) - 1 = 2a_j - 1$$

<u>Case 2</u> $\varphi_j(2a_j, 2a_j + 1) = 2a_j + 1$ [1]

In analogy to case 1, define

$$f(k + 2) = f(a_j) - 1 = 2a_j - 1$$

$$g(k + 2) = 2k + 4$$

Note that in case 1, each number $2a_k$, $j \leq k \leq n$ is thrown into B and that in case 2 each of those numbers is thrown into A. Hence in either case, if $j < n$, move each marker $\boxed{k}$, $j < k \leq n$ down to free (even) integers.

In either case 1 or 2, if the marker $\boxed{j}$ is not caused to move by a later stage, φ_j cannot be a selector function for A join B. But, by an inductive argument, each marker moves only finitely often. Thus if φ_j is a total function, the marker $\boxed{j}$ must be attacked at some stage after it achieves its final resting place, and hence φ_j cannot be a selector function for A join B. qed.

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[1] For this proof, assume that φ_j is the j th partial recursive function of two variables.

One immediate corollary to the above theorem is the fact that the join of immune retraceable co-r.e. sets need not be regressive. However, as Dekker has pointed out, this fact can easily be deduced from well-known theorems in the literature.

PROPOSITION 4.20 Let A and B be r.e. sets with $A \leq_n B$. If B is semirecursive, then A is semirecursive.

Proof If $B = \emptyset$ or $B = N$, the proposition is trivial. Otherwise by proposition 3.1, $A \leq_m B$, so A is semirecursive, if B is. qed.

Since every r.e. tt-degree contains an r.e. semirecursive set, the tt-analogue to the above proposition fails. The following theorem shows that even the btt-analogue to the proposition fails.

THEOREM 4.23 There are r.e. sets A and B with $A \leq_{btt} B$ and B semirecursive and A not semirecursive.

Proof The proof combines a priority argument with a Dekker construction in a manner similar to the proof of theorem 4.21.

The construction yields a 1-1 recursive function f. The set B is defined from f by

$$B = \{ x \mid (\exists y) [y > x \;\&\; f(y) < f(x)] \}$$

Then, as Dekker and Myhill have pointed out, the set B is r.e. and coretraceable.

The set A is defined from B by

$$2z \in A \iff (3z + 1 \notin B \;\&\; 3z + 2 \in B) \text{ or }$$
$$(3z \in B \;\&\; 3z + 1 \in B \;\&\; 3z + 2 \in B)$$
$$2z + 1 \in A \iff 3z + 1 \in B \;\&\; 3z + 2 \in B$$

Thus $A \leq_{btt} B$.

The function f will be defined with a priority argument in such a way to ensure that A is not semirecursive. The construction uses a single list of the integers (the A-list) and a movable marker $\boxed{i}$ for each integer i. The markers are associated with even numbers in the list and may be moved to larger numbers as the construction proceeds. The purpose of the i'th marker is to ensure that φ_i[2] is not a selector function for A. Also, a * will be associated with certain members of the A-list as the construction proceeds. An even number 2k in the B-list is called <u>free</u> at a given stage neither 2k nor any larger number has any marker beside it and f is undefined for all arguments y such that $y \geq 3k$.

The construction proceeds in stages.

<u>Stage n</u>

Let 2k be the least free integer. Associate the marker $\boxed{n}$ with 2k. Now for every number y such that $y \leq 3k + 2$ and f(y) has not previously been defined, set

$$f(y) = 2y$$

Let $2a_0, 2a_1, \ldots, 2a_n$ be the present position of $\boxed{0}, \boxed{1}, \ldots, \boxed{n}$. Let j be the smallest number i such that $2a_i$ does not have a * and $\varphi_i\,(2a_i, 2a_i + 1)$ is convergent in n or fewer steps. (If no such i exists, go to stage n + 1) Place a * by $2a_j$, and say that $\boxed{j}$ is <u>attacked</u>.

If $\varphi_j(2a_j, 2a_j + 1) \notin \{2a_j, 2a_j + 1\}$ then φ_j is not a selector function for any set. In this case, proceed to stage n + 1. Otherwise there are two cases.

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[2] For this proof, assume that φ_i is the i'th partial recursive function of two variables.

<u>Case 1</u> $\varphi_j(2a_j, 2a_j + 1) = 2a_j + 1$

To ensure that φ_j is not a selector function for A, it will be sufficient to put $2a_j$ into A and $2a_j + 1$ into $\overline{A}$. It follows from the definition of A from B that this will be accomplished if $3a_j$ and $3a_j+1$ are put into $\overline{B}$ and $3a_j+2$ is put into B. Now when the construction is complete it will be evident that, since $2a_j$ has a movable marker associated with it, $f(y) = 2y$ for $y \in \{3a_j, 3a_j+1, 3a_j+2\}$
Thus we define

$$f(3k+3) = f(3a_j+2) - 1 = 6a_j+3$$ (Recall: 2k was least free integer)

This definition puts each y, $3a_j+2 \leq y \leq 3k+2$ into B and hence may interfere with earlier stages. Thus, if $j < n$, move all markers $\boxed{k}$, $j < k \leq n$ to free integers in the A-list.

<u>Case 2</u> $\varphi_j(2a_j, 2a_j+1) = 2a_j$

To ensure that φ_j is not a selector function for A, it will be sufficient to put $2a_j$ into $\overline{A}$ and $2a_j+1$ into A, and this will be accomplished if $3a_j$ is put into $\overline{B}$ and $3a_j+1$ and $3a_j+2$ are put into B. Since $f(y) = 2y$ for $y \in \{3a_j, 3a_j+1, 3a_j+2\}$, define

$$f(3k+3) = f(3a_j+1) - 1 = 6a_j+1$$

This definition puts each y, $3a_j+1 \leq y \leq 3k+2$ into B and hence may interfere with earlier stages. Thus, if $j < n$, move all markers $\boxed{k}$, $j < k \leq n$ to free integers in the A-list.

In either case 1 or 2, if the marker $\boxed{j}$ is not caused to move by a later stage, φ_j cannot be a selector function for A. But, by an inductive argument, each marker moves only finitely often. Thus for any number j, if φ_j is a total function, the marker $\boxed{j}$ must be attacked at some stage after it achieves its final resting place, and hence cannot be a selector function for A. Therefore A is not semirecursive.

It remains only to verify that A is r.e. For convenience, the definition of A is repeated below:

$$2z \in A \Longleftrightarrow (3z + 1 \notin B \;\&\; 3z + 2 \in B) \text{ or }$$

$$(3z \in B \;\&\; 3z + 1 \in B \;\&\; 3z + 2 \in B)$$

$$2z + 1 \in A \Longleftrightarrow 3z + 1 \in B \;\&\; 3z + 2 \in B$$

From the above definition and the fact that B is r.e. it follows that

$$\{2z + 1 \mid 2z + 1 \in A\}$$

is r.e. Now is claimed that

$2z \in A \Longleftrightarrow$ there exists a stage n and a marker j such that j is associated with 2z at stage n and j is attacked at stage n and case 1 applies
or
$3z \in B$ and $3z + 1 \in B$ and $3z + 2 \in B$

If the above claim can be proved, it will follow that $\{2z \mid 2z \in A\}$ is r.e. and hence that A is r.e.

To prove the claim, first assume that $2z \in A$. Then, by the definition of A, either $3z + 1 \notin B$ and $3z + 2 \in B$ or $\{3z, 3z + 1, 3z + 2\} \subset B$ If the second condition holds, then the right-hand side of the claim trivially holds, so there is nothing to prove. Assume that $3z + 1 \notin B$ and $3z + 2 \in B$. Then it follows from the construction that

$$f(3z + 1) = 2(3z + 1) = 6z + 2$$

$$f(3z + 2) = 2(6z + 2) = 6z + 4$$

so it follows from the definition of B that, for some $y > 3z + 2$

$$f(y) = 6z + 3$$

Hence there is a stage n and a marker $\boxed{j}$ such that $\boxed{j}$ is associated with 2z at stage n and $\boxed{j}$ is attacked at stage n and case 1 applies.

Conversely, if $\{3z, 3z + 1, 3z + 2\} \subset B$, it follows from the definition of A that $2z \in A$. Now assume that n and $\boxed{j}$ exist as in the above paragraph. Then there are two cases:

(i) The marker $\boxed{j}$ is not caused to move at some stage later than n. Then $3z + 1 \notin B$ and $3z + 2 \in B$, so $2z \in A$.

(ii) The marker $\boxed{j}$ is caused to move at some stage after n. Then $\{3z, 3z + 1, 3z + 2\} \subset B$, so $2z \in A$.

This proves the claim and concludes the proof of the theorem. qed.

We have seen that immune semirecursive sets are hyperimmune but not FSHI. We will now show that such sets are in Σ_2 in the arithmetical hierarchy and that they can be shown with additional assumptions to be co-r.e. Some of the theorems will apply to immune sets A such that $\overline{\overline{A} x \overline{A}} \leq_m A$ rather than just immune semirecursive A.

<u>THEOREM</u> <u>4.24</u>

(i) If A is immune and $\overline{\overline{A} x \overline{A}} \leq_m A$, then $A \in \Sigma_2$.

(ii) If A is hyperimmune (or even if no sequence of sets of bounded cardinality witnesses A not hyperimmune), and, for any n, $\overbrace{\overline{A} x \overline{A} x \ldots x \overline{A}}^{n+1} \leq_m \overbrace{\overline{A} x \overline{A} x \ldots x \overline{A}}^{n}$, then $A \in \Sigma_2$.

<u>Proof</u>

(i) Suppose that A is immune and $\overline{\overline{A} x \overline{A}} \leq_m A$ via g. Then

$$(x \in A \text{ or } y \in A) \Longleftrightarrow g(\langle x,y \rangle) \in A$$

<u>Claim</u> $x \in A \Longleftrightarrow \{ g(\langle x,y \rangle) \mid y \in N\}$ is finite

If the claim is established, it will follow immediately that $A \in \Sigma_2$ for

$$x \in A \Longleftrightarrow (\exists u)\ (\forall y)\ [\ g(\langle x,y \rangle) \leq u]$$

To prove the claim, first assume $x \in A$. Then $\{g(\langle x,y\rangle) \mid y \in N\}$ is an r.e. subset of A and therefore finite.

Now suppose that there were a number x such that $x \in \bar{A}$ and $\{g(\langle x,y\rangle) \mid y \in N\}$ is finite. Then, for all z,

$$z \in A \Leftrightarrow g(\langle x,z\rangle) \in A \cap \{g(\langle x,y\rangle) \mid y \in N\}$$

Thus A is recursive, contrary to assumption.

This proves the claim and therefore part (i)

(ii) Suppose that A and n are such that

$$\neg(\exists \text{ rec. } f)(\exists m)(\forall x)(\exists y)[(x \neq y \Rightarrow D_{f(x)} \cap D_{f(y)} = \emptyset) \;\&\; D_{f(x)} \cap A \neq \emptyset \;\&\; |D_{f(x)}| \leq m]$$

$$\text{and } \overbrace{\bar{A}\times\ldots\times\bar{A}}^{n+1} \not\leq_m \overbrace{\bar{A}\times\ldots\times\bar{A}}^{n}$$

It must be shown that $A \in \Sigma_2$. This will be proved by induction on n. If n = 1, the result follows immediately from (i).

Now assume that the theorem is true for n = k. To prove the theorem for n = k + 1, assume that

$$\overbrace{\bar{A}\times\bar{A}\times\ldots\bar{A}}^{k+2} \leq_m \overbrace{\bar{A}\times\bar{A}\times\ldots\times\bar{A}}^{k+1}$$

and let g be a recursive function such that

$$(x_1 \in A \text{ or } x_2 \in A \text{ or} \ldots \text{ or } x_{k+2} \in A) \Leftrightarrow$$
$$D_{g(x_1,\ldots,x_{k+2})} \cap A \neq \emptyset \;\&\; |D_{g(x_1,\ldots,x_{k+2})}| \leq k+1$$

It will be shown that the equivalence

$$x \in A \Leftrightarrow (\exists D)[D \text{ is a finite set } \&$$
$$(\forall x_2)\ldots(\forall x_{k+2})[D_{g(x,x_2,\ldots x_{k+2})} \cap D \neq \emptyset]]$$

can be false only if $A \in \Sigma_2$. Since the above equivalence implies that $A \in \Sigma_2$, it will follow that $A \in \Sigma_2$.

First assume $x \in A$. Then each of the sets $D_{g(x,x_2,\ldots,x_n)}$ intersects A. Since these sets have cardinality bounded by k + 1, there must exist a finite set which intersects all of them, since otherwise a disjoint subcollection of the sets could be constructed to get a sequence of sets of bounded cardinality witnessing A not hyperimmune. (c.f. the proof of lemma 5.15)

Thus, if the above equivalence is false, there must be a number $x \in \bar{A}$ and a finite set D such that every set of the form $D_{g(x,x_2,\ldots x_n)}$ intersects D. Let a be a fixed member of A, and define a recursive function h of k + 1 variables by:

$$D_{h(x_2,\ldots,x_k)} = \begin{cases} D_{g(x,x_2,\ldots,x_k)} - D & \text{if } D \cap D_{g(x,x_2,\ldots x_k)} \cap A = \phi \\ \{a\} & \text{if } D \cap D_{g(x,x_2,\ldots,x_k)} \cap A \neq \phi \end{cases}$$

Then h shows that $\overbrace{\bar{A}x\bar{A}x\ldots x\bar{A}}^{k+1} \leq_m \overbrace{\bar{A}x\bar{A}x\ldots x\bar{A}}^{k}$, so $A \in \Sigma_2$ by the induction assumption. qed.

<u>COROLLARY 4.25</u>

(i) There exist $\aleph_0$ immune semirecursive sets.

(ii) If A is regressive and if there is an n such that $\overbrace{\bar{A}x\bar{A}x\ldots x\bar{A}}^{n+1} \leq_m \overbrace{\bar{A}x\bar{A}x\ldots x\bar{A}}^{n}$, then $A \in \Sigma_2$.

<u>Proof</u>

(i) By theorem 4.3. there exist at least $\aleph_0$ immune semirecursive sets, and by the present theorem there exist at most $\aleph_0$ such sets.

(ii) Suppose that A is regressive. If A is r.e., there is nothing to prove. Otherwise A is immune, and Appel and McLaughlin [1] have proved that no immune regressive set is witnessed non-hyperimmune by a sequence of sets of bounded cardinality. Therefore, the theorem applies. qed.

<u>DEFINITION 4.26</u> Let A be immune.

(i) (Smullyan) A is said to be <u>effectively immune</u> if there is a recursive function f such that, for all x,

$$W_x \subset A \Longrightarrow |W_x| \leq f(x)$$

(ii) (McLaughlin) A is said to be <u>strongly effectively immune</u> if there is a recursive function g such that, for all x,

$$W_x \subset A \Longrightarrow W_x \subset \{0, 1, \ldots, g(x)\}$$

<u>THEOREM 4.27</u>

(i) If A is effectively immune and $\overline{\overline{A}\times\overline{A}} \leq_m \overline{A}$, then $\overline{A}$ is r.e.

(ii) If A is strongly effectively immune and semirecursive, then $\overline{A}$ is r.e. and A is regressive.

<u>Proof</u>

(i) Suppose A is effectively immune and that $\overline{\overline{A}\times\overline{A}} \leq_m A$ via g. By the argument of the preceding theorem,

$$x \in \overline{A} \Longleftrightarrow \{g(\langle x,y\rangle) \mid y \in N\} \text{ is infinite}$$

Let f be a recursive function such that

$$W_x \subset A \Longrightarrow |W_x| \leq f(x)$$

Let h be a recursive function such that

$$W_{h(x)} = \{g(\langle x,y\rangle) \mid y \in N\}$$

Then $x \in \overline{A} \Longleftrightarrow$ there are more than $hf(x)$ numbers of the form $g(\langle x,y\rangle)$

Therefore, $\overline{A}$ is r.e.

(ii) Suppose that A is strongly effectively immune and semirecursive. Then $\overline{A}$ is r.e. by (i). Assume that A is the lower half of a cut in a recursive linear ordering $\leq_o$ of N. Say that y is an <u>o-predecessor</u> of x if $y \leq_o x$. Any x in A has only finitely many o-predecessors, because the set of its o-predecessors is an r.e. subset of A. Thus the restriction of $\leq_o$ to $A \times A$ is an ordering in which every element has only finitely many predecessors and hence is order-isomorphic to N with the usual ordering. Thus there is an enumeration $a_0, a_1, \ldots$ of A such that

$$a_0 <_o a_1 <_o a_2 <_o \cdots.$$

Observe that if x is given, it is possible to compute effectively an r.e. index for the set of o-predecessors of x. Thus, since A is strongly effectively simple, there is a recursive function g such that if x is in A, every o-predecessor of x is less than or equal to g(x).

Now the following recursive function Ψ will regress the enumeration $a_0, a_1, \ldots$ of A:

$$\Psi(x) = \begin{cases} a_0 & \text{if } x = a_0 \\ \text{the largest } y \text{ (with respect to the ordering } \leq_o) \\ \text{such that } y \neq x,\ y \leq_o x,\ \&\ y \leq g(x) \text{ if } x \neq a_0 \\ \text{and such a } y \text{ exists.} \\ 0 & \text{otherwise} \end{cases}$$

Therefore, A is regressive. qed.

<u>THEOREM</u> <u>4.28</u> Suppose that A is retraceable and that $\overline{\overline{A} \times \overline{A}} \leq_m A$. Then $\overline{A}$ is r.e.

<u>Proof</u> Suppose that Ψ is a partial recursive retracing function for A and that g is a recursive function such that for all x and y

$$x \in A \text{ or } y \in A \Longleftrightarrow g(x, y) \in A$$

Suppose also that A is nonrecursive, since otherwise the result is immediate.

Let a_0 be the least member of A. Define B by

$$B = \{x \mid (\exists n)(\exists y)[\, g(x,y) > x \;\&\; \psi^n g(x,y) = a_o \;\& \; x \notin \{\psi g(x,y), \psi^2 g(x,y), \ldots, \psi^n g(x,y)\}]\}$$

B is r.e. by the projection theorem. It is claimed that $B = \overline{A}$. To show that $B \subset \overline{A}$, assume that some number x were in $B \cap A$. Let n and y be such that

$$g(x,y) > x \;\&\; \psi^n g(x,y) = a_o \;\&\; x \notin \{\psi g(x,y), \psi^2 g(x,y), \ldots, \psi^n g(x,y)\}$$

Since $x \in A$, $g(x,y) \in A$. Since $\psi^n g(x,y) = a_o$, every member of A which is less than $g(x,y)$ is in $\{\psi g(x,y), \ldots, \psi^n g(x,y)\}$. In particular, $x \in \{\psi g(x,y), \ldots \psi^n g(x,y)\}$, contradicting the assumption on n and y.

To show that $\overline{A} \subset B$, assume that $x \in \overline{A}$. Then the set C is infinite, where

$$C = \{g(x,y) \mid y \in A\} \subset A$$

For if C were finite, the obvious equivalence

$$y \in A \Leftrightarrow g(x,y) \in C$$

would show that A is recursive.

Since C is infinite, there is a number $y \in A$, such that $g(x,y) > x$. Since $y \in A$, $g(x,y) \in A$, so there is a number n with $\psi^n g(x,y) = a_o$. Also, since $g(x,y) \in A$ and $x \in \overline{A}$,

$$x \notin \{\psi g(x,y), \psi^2 g(x,y), \ldots, \psi^n g(x,y)\}$$

Thus this n and this y show that $x \in B$.

Therefore $\overline{A} = B$, so $\overline{A}$ is r.e. qed.

The above theorem becomes false when the hypothesis that A is retraceable is weakened to the hypothesis that A is regressive, since, for example, if A is creative, then A is a regressive set such that $\overline{\bar{A} \times \bar{A}} \leq_m A$ and $\bar{A}$ is not r.e. However, it may be shown by a slight modification of the above proof that every regressive set A such that $\overline{\bar{A} \times \bar{A}} \leq_m A$ is the difference of r.e. sets. It is not known whether every such set is either r.e. or co-r.e.

The theorem makes it easy to give some necessary and sufficient conditions for a retraceable set to be semirecursive.

<u>COROLLARY</u> <u>4.29</u> Let A be retraceable. Then the following conditions are equivalent:

(i) A is semirecursive

(ii) $\overline{\bar{A} \times \bar{A}} \leq_m A$

(iii) $\bar{A}$ is r.e.

<u>Proof</u> Let A be retraceable.

(i) $\Longrightarrow$ (ii) by part (i) of theorem 4.9

(ii) $\Longrightarrow$ (iii) by the above theorem

(iii) $\Longrightarrow$ (i) by theorem 4.3 qed.

<u>COROLLARY</u> <u>4.30</u> If A is retraceable, immune, and non-hyperimmune, then $\overline{\bar{A} \times \bar{A}} \not\leq_m A$.

<u>Proof</u> If A is a retraceable set and $\overline{\bar{A} \times \bar{A}} \leq_m A$, then by the theorem $\bar{A}$ is r.e. Thus, since A is retraceable, if A is immune, then A is hyperimmune. qed.

It is not known whether the conclusion of corollary 4.30 can be strengthened to read that the m-degrees of $\bar{A}$, $\bar{A} \times \bar{A}$, $\bar{A} \times \bar{A} \times \bar{A}$, ... are all distinct. However, in section 5 it will be shown under the assumpton that A is immune, non-hyperimmune and retraced by a <u>total</u> recursive function that these m-degrees are all distinct.

The preceding two theorems characterize semirecursive sets which are strongly effectively immune or retraceable fairly adequately. However, they do not go far towards classifying all immune semirecursive sets. In particular, the following elementary questions remain unanswered:

(i) Is every immune semirecursive set regressive?

(ii) Is every immune semirecursive set co-r.e.?

(iii) Are there semirecursive sets which are both immune and co-immune?

The existence of semirecursive sets which are both immune and co-immune would be of particular interest, since by some of the theorems in this section, such sets would have several interesting properties.

SECTION 5. RELATIONSHIPS BETWEEN REDUCIBILITIES

In the previous section it was proved that every tt-degree contains a p-degree consisting of a single m-degree and contains incomparable p-degrees. In this section, more results of this kind will be proved, but by quite different methods. Certain types of immune but not hyper-immune sets will be studied, with propositional logic used as a tool in this study. Also it will be shown that each r.e. nonrecursive T-degree contains r.e. sets which have many of the properties of creative sets.

THEOREM 5.1 Let A be a simple set which is not hypersimple. Let B be any set. Then $\{x \mid D_x \subset A\} \leq_m B \Rightarrow \overline{B}$ not immune

Proof

Propositional logic will be used to abbreviate the proof. First we introduce some conventions. Recall what it means for a propositional formula σ to be true of a set A:

σ is true of A iff σ is true when each statement letter P_n is interpreted as true when $n \in A$ and false when $n \notin A$.

Hence, when we know that a formula σ will be interpreted in a set A, we may use the symbol " $n \in A$" in place of the statement letter P_n. Abbreviating further, we may use the symbol "$D_x \subset A$" in place of the statement

$$x_1 \in A \wedge x_2 \in A \wedge \dots \wedge x_k \in A \quad \text{where } D_x = \{x_1, x_2, \dots, x_k\}$$

Similar abbreviations will be freely used.

Finally statements referring to two sets may be thought of as propositional formulas to be interpreted in the join of the two sets, e.g

$5 \in A \wedge 6 \notin B$ abbreviates the statement

$10 \in A$ join B & $13 \notin A$ join B.

We will use two facts from elementary logic: the set of logical consequences of a recursively enumerable set of formulas is itself recursively enumerable (when formulas are coded effectively to integers) and when every formula in a some set of formulas is true in some fixed interpretation, then every formula deducible from that set is also true in the interpretation. The latter result is called the "soundness theorem."

Now assume the theorem false, so that A is simple and not hypersimple and B is coimmune, and

$$\{x \mid D_x \subset A\} \leq_m B$$

Let $\{D_{f(x)}\}$ witness the non-hyperimmunity of A. Let $\{x \mid D_x \subset A\} \leq_m B$ via g.

Consider the following set of axioms T:

1_x	$D_{f(x)} \cap \overline{A} \neq \emptyset$	all x
2_x	$x \in A$	all $x \in A$
3_x	$D \cap \overline{A} \neq \emptyset \Longleftrightarrow g(x) \in \overline{B}$	all x

If σ is a propositional formula, let $\vdash_T \sigma$ mean that σ is provable from the above statements by the rules of propositional calculus when they are viewed as propositional formulas.

Now consider the set C, where

$$C = \{x \mid \vdash_T x \in \overline{B}\}$$

Since the axioms are all true and form a recursively enumerable set of statements, C is an r.e. subset of $\overline{B}$, and thus finite. But by the axioms 3_x

$$\vdash_T D_x \cap \overline{A} \neq \emptyset \Longleftrightarrow \vdash_T g(x) \in \overline{B} \Longleftrightarrow g(x) \in C$$

Thus the set D is recursive, where

$$D = \{x \mid \vdash_T D_x \cap \overline{A} \neq \emptyset\}$$

For convenience in giving later proofs, the rest of this proof will be given in a lemma.

<u>LEMMA 5.2</u> Suppose that A is simple and not hypersimple, and that $\{D_{f(x)}\}$ witnesses that A is not hypersimple. Suppose that $\mathcal{C}$ is a class of <u>non-empty</u> finite sets such that

$$\text{(i)}\quad D_{f(x)} \in \mathcal{C} \qquad \text{all } x$$

$$\text{(ii)}\quad D_x \in \mathcal{C},\ y \in A \Rightarrow (D_x - \{y\}) \in \mathcal{C}$$

Then $\{x \mid D_x \in \mathcal{C}\}$ is not recursive.

First we note that the theorem now follows at once from the lemma. For if we let $\mathcal{C} = \{D_x \mid \vdash_T D_x \cap \overline{A} = \phi\}$, then each member of $\mathcal{C}$ is non-empty because each member of $\mathcal{C}$ intersects $\overline{A}$, by the soundness theorem. $\mathcal{C}$ satisfies conditions (i) and (ii) because of the axioms 1_x and 2_x. Finally, it has already been remarked that $D = \{x \mid D_x \in \mathcal{C}\}$ is recursive, which contradicts the lemma.

<u>Proof of Lemma</u>

Suppose the lemma is false. Let $\mathcal{C}$ be a class of finite sets satisfying the hypotheses of the lemma such that $\{x \mid D_x \in \mathcal{C}\}$ is recursive. Let

$$M = \{x \mid D_x \in \mathcal{C}\ \&\ (\forall y)[D_y \subsetneq D_x \Rightarrow D_y \notin \mathcal{C}]\}$$

and let $\quad G = \bigcup_{x \in M} D_x$

It is claimed that G is an infinite r.e. subset of $\overline{A}$. If this claim is proved, the simplicity of A will be contradicted.

Since $\{x \mid D_x \in \mathcal{C}\}$ is recursive, M is recursive, and thus G is r.e. Call a set D_x <u>minimal</u> if $x \in M$. Now any member of $\mathcal{C}$ which intersects A has a proper subset in $\mathcal{C}$, by condition (ii) on $\mathcal{C}$. Hence every minimal set is a subset of $\overline{A}$, and G is a subset of $\overline{A}$. Finally note that every member D_x of $\mathcal{C}$ has a minimal subset, e.g. any subset D_y of D_x which has minimal cardinality among the subsets of D_x which are members of $\mathcal{C}$. Thus each $D_{f(x)}$ has a minimal subset and therefore each $D_{f(x)}$ intersects G, since all members of $\mathcal{C}$ are non-empty. Hence G is infinite. qed.

COROLLARY 5.3 If A is simple but not hypersimple, then $A \times A \not\leq_m A$.

Proof Assume that A is simple and not hypersimple and that $A \times A \leq_m A$ Then $A \times N$ is a c-cylinder and, since $\{x \mid D_x \subset A\} \leq_c A$, $\{x \mid D_x \subset A\} \leq_m A$. This contradicts the theorem. qed.

Since there are hypersimple semirecursive sets, the above corollary, and hence the theorem, fails when the hypothesis "A is not hypersimple" is dropped.

Theorem 5.1 implies in particular that no creative set can be m-reduced to a coimmune set, i.e. no productive set is immune. In this section, several other facts about creative sets will be generalized to sets of the form $\{x \mid D_x \subset A\}$ for simple, non-hypersimple A. The proofs will thus give an alternate method for proving some of the standard facts about creative sets. More importantly, since Yates has shown the existence in every r.e. nonrecursive Turing degree of a simple set which is not hypersimple, the theorems to come will show that r.e. sets which share many of the properties of creative sets exist in every r.e. nonrecursive T-degree. This in turn will make it possible to show that each nonrecursive r.e. Turing degree shares some of the standard properties of the complete degree.

DEFINITION 5.4 B is a strong cylinder if B is a cylinder and

$$(\forall C)\,[\,B \leq_m C \implies B \leq_1 C\,]$$

PROPOSITION 5.5. If B is a strong cylinder, the m-degree of B consists of a single 1-degree.

Proof Trivial. qed.

It is well known that creative sets are strong cylinders. We are heading towards a generalization of this fact. We first "localize" the notion of cylinder in a way motivated by Young's characterization of cylinders (theorem 1.4.)

DEFINITION 5.6 For sets D and E, D is a E-cylinder if there is a recursive function h such that, for all x

$$(x \in D \implies W_{h(x)} \subset D)\ \&\ (x \in \overline{D} \implies W_{h(x)} \subset \overline{D})\ \&$$
$$(x \in E \implies W_{h(x)} \text{ infinite})$$

LEMMA 5.7 $F \leq_m D$ via g, D a (range g)-cylinder $\implies F \leq_1 D$.

Proof The proof is a straightforward relativization of the proof of theorem 1.4). Assume $F \leq_m D$ via g and let h be a recursive function showing that D is a (range g)-cylinder. Define a recursive 1-1 function k by induction:

$$k(0) = g(0)$$
$$\vdots$$
$$k(n+1) = \text{the first number found in } W_{hg(n+1)} \text{ which is not}$$
$$\text{a member of } \{0,1,\ldots,k(n)\}$$

Then $F \leq_1 D$ via h. qed.

THEOREM 5.8 Let A be simple but not hypersimple and let C be any non-empty set. Then $\{x \mid D_x \subset A\} \times C$ is a strong cylinder.

<u>Proof</u> Let a function i, (the index function) mapping finite sets to integers, be defined by

$$i(D_x) = x$$

To see that $\{x \mid D_x \subset A\} \times C$ is a cylinder, let

$$W_{p(\langle u,v\rangle)} = \{\langle i(D_u \cup D_x), v\rangle \mid D_x \subset A\}$$

Then, $W_{p(\langle u,v\rangle)}$ shows that $\{x \mid D_x \subset A\} \times C$ is a cylinder, by Young's characterization of cylinders.

Now let $B = \{x \mid D_x \subset A\}$, and suppose $B \times C \leq_m D$ via g. It must be shown that $B \times C \leq_1 D$, so by the lemma it suffices to show that D is a (range g)-cylinder. It follows from theorem 5.1 that D has an infinite r.e. subset, since

$$\{x \mid D_x \subset A\} \leq_m \{x \mid D_x \subset A\} \times C \leq_m D$$

(It is at his point that the fact that C is non-empty is used.)

Let G be an infinite r.e. subset of D.

Let $\{D_{f(x)}\}$ witness that A is not hypersimple.

We now write everything we know in the form of axioms :

$1_{\langle u,v\rangle}$	$\langle u,v\rangle \in B \times C \iff g(\langle u,v\rangle) \in D$	all $\langle u,v\rangle$
2_x	$x \notin D$	all $x \in G$
3_x	$x \in A$	all $x \in A$
4_x	$D_{f(x)} \cap \overline{A} \neq \emptyset$	all x

The set of axioms given above is recursively enumerable and each axiom is true when given the obvious interpretation. Hence there is a recursive function h such that

$$W_{h(x)} = \{y \mid \vdash_T y \in D \leftrightarrow x \in D\}$$

and $\quad x \in D \Rightarrow W_{h(x)} \subset D \;\&\; x \notin D \Rightarrow W_{h(x)} \subset \overline{D}$, where as before

$\vdash_T \sigma$ means σ is provable from the above axiom.

Thus to show that D is a (range g)-cylinder it suffices to show that

$$x \in \text{range } g \Rightarrow W_{h(x)} \text{ infinite}$$

Assume the above is false. Let x_0 be a member of range g such that $W_{h(x_0)}$ is finite. Let $x_0 = g(\langle u_o, v_o\rangle)$. Define

$$\mathcal{C} = \{D_u \mid \nvdash_T [(D_u \cup D_{u_o}) \subset A \ \& \ v_o \in C] \leftrightarrow [D_{u_o} \subset A \ \& \ v_o \in C]\}$$

($\nvdash_T \sigma$ means that σ cannot be proved from T)

We will show that $\mathcal{C}$ gives a counterexample to lemma 5.2 and thus obtain a contradiction. First note that every member of $\mathcal{C}$ is non-empty, for trivially,

$$\vdash_T [(\phi \cup D_{u_o}) \subset A \ \& \ v_o \in C] \leftrightarrow [D_{u_o} \subset A \ \& \ v_o \in C]$$

Also $\{u \mid D_u \in \mathcal{C}\}$ is recursive, since

$$D_u \in \mathcal{C} \Leftrightarrow \vdash_T [(D_u \cup D_{u_o}) \subset A \ \& \ v_o \in C] \leftrightarrow [D_{u_o} \subset A \ \& \ v_o \in C]$$

$$\Leftrightarrow \vdash_T g(\langle i(D_u \cup D_{u_o}), v_o\rangle) \in D \leftrightarrow \underbrace{g(\langle u_o, v_o\rangle)}_{= x_o} \in D$$

$$\Leftrightarrow g(\langle i(D_u \cup D_{u_o}), v_o\rangle) \in W_{h(x_o)}$$

Since $W_{h(x_0)}$ was finite, the last line gives an effective test to see whether $D_u \in \mathcal{C}$.

Now it must be shown that each $D_{f(x)} \in \mathcal{C}$. Assume the contrary: let x be such that $D_{f(x)} \notin \mathcal{C}$. Then

$$\vdash_T [D_{f(x)} \cup D_{u_o} \subset A \ \& \ v_o \in C] \leftrightarrow [D_{u_o} \subset A \ \& \ v_o \in C]$$

But by axiom 1_x: $\vdash_T D_{f(x)} \cap \bar{A} \neq \phi$

$$\vdash_T \neg(D_{u_o} \subset A \ \& \ v_o \in C)$$

$$\vdash_T x_o \notin D$$

All members of the set G are also provably not in D. Thus

$$G \subset W_{g(x_o)}$$

This contradicts the assumption that $W_{g(x_0)}$ was finite.

Finally it must be shown that $D_u \in \mathcal{C},\ y \in A \Rightarrow (D_u - \{y\}) \in \mathcal{C}$. This will be proved in the form: $(D_u - \{y\}) \notin \mathcal{C},\ y \in A \Rightarrow D_u \notin \mathcal{C}$

Suppose $(D_u - \{y\}) \notin \mathcal{C}$ and $y \in A$. Thus

$$\vdash_T [(D_u - \{y\}) \cup D_{u_0} \subset A \ \&\ v_0 \in C] \leftrightarrow [D_{u_0} \subset A \ \&\ v_0 \in C]$$

But then, since $y \in A$, by axiom 3_y:

$$\vdash_T [(D_u - \{y\}) \cup D_{u_0} \subset A \ \&\ v_0 \in C] \leftrightarrow [D_u \cup D_{u_0} \subset A \ \&\ v_0 \in C]$$

Combining these two equivalences it follows that

$$\vdash_T [D_u \cup D_{u_0} \subset A \ \&\ v_0 \in C] \longleftrightarrow [D_{u_0} \subset A \ \&\ v_0 \in C]$$

Therefore $D_u \notin \mathcal{C}$, which was to be shown.

Thus it has been shown that the class $\mathcal{C}$ provides a counterexample to lemma 5.2, and the theorem is proved. qed.

The first corollary is a special case of the theorem which generalizes the fact that creative sets are strong cylinders.

<u>COROLLARY</u> <u>5.9</u> If A is simple but not hypersimple, then $\{x \mid D_x \subset A\}$ is a strong cylinder.

<u>Proof</u> Take C = N in the theorem. qed.

Of course, the theorem also shows that the cartesian product of a creative set with any non-empty set is a strong cylinder. This fact does not seem to the writer to be obvious from the classical proof with the recursion theorem that creative sets are strong cylinders, although the present proof would not have been simplified (except notationally) by considering $\{x \mid D_x \subset A\}$ rather than $\{x \mid D_x \subset A\} \times C$. However, this is the only example known where the present method yields new information about creative sets.

COROLLARY 5.10 Every r.e. T-degree contains an r.e. m-degree consisting of a single 1-degree.

Proof The recursive T-degree certainly contains such an m-degree. By the theorem of Yates mentioned in the previous section (theorem 4.11) each nonrecursive T-degree contains a simple but not hypersimple set A and thus contains a strong cylinder, e.g. $\{x \mid D_x \subset A\}$. By proposition 5.5, the m-degree of this strong cylinder consists of a single 1-degree.

qed.

Corollary 5.10 answers a question raised by P.R. Young, who showed in [20] that every nonrecursive m-degree either consists of a single 1-degree or contains a linearly ordered collection of 1-degrees with the order type of the rationals and inquired whether there were nonrecursive, noncreative r.e. m-degree consisting of a single 1-degree.

We now prove a weakened analogue of the theorem that btt-complete sets are not simple.

THEOREM 5.11 Let A be simple but not hypersimple, and suppose that

$$\{x \mid D_x \subset A\} \leq_{bp} C$$

Then C is not simple.

Proof Suppose the theorem is false. Let A be simple, and let $\{D_{f(x)}\}$ witness that A is not hypersimple. Suppose that C is a simple set such that there exists a recursive function g and a number m such that, for all x,

$$D_x \subset A \Longleftrightarrow (\exists u)[u \in D_{g(x)} \;\&\; D_u \subset \bar{C}] \;\&\; \Big|\bigcup_{u \in D_{g(x)}} D_u\Big| \leq m$$

Now consider the following axioms T :

1_x $\quad D_x \cap \bar{A} \neq \phi \Longleftrightarrow (\forall u)\,[u \in D_{g(x)} \Longrightarrow D_u \cap \bar{C} \neq \phi]$ $\qquad$ all x

2_x $\quad x \in A$ $\qquad$ all $x \in A$

$3_x \quad x \in C$ all $x \in C$

$4_x \quad D_{f(x)} \cap A \neq \emptyset$ all x

Let $\vdash_T \sigma$ mean that the formula σ is provable from the above axioms.

Let $\mathcal{C} = \{D_x \mid \vdash_T D_x \cap \bar{A} \neq \emptyset\}$

We will apply lemma 5.2. $\mathcal{C}$ is clearly a class of non-empty sets which satisfies conditions (i) and (ii) in lemma 5.2. Thus, to get a contradiction, it is sufficient to show that $\{x \mid D_x \in \mathcal{C}\}$ is recursive. Observe that by axiom 1_x,

$$\vdash_T D_x \cap \bar{A} \neq \emptyset \Leftrightarrow \vdash_T D_{u_1} \cap \bar{C} \neq \emptyset \ \& \vdash_T D_{u_2} \cap \bar{C} \neq \emptyset \ \& \ldots \& \vdash_T D_{u_n} \cap \bar{C} \neq \emptyset$$

where $D_{g(x)} = \{u_1, u_2, \ldots, u_n\}$. By the assumption on the boundedness of the reduction, $|D_u| \leq m$, for u in any $D_{g(x)}$. Thus to see that $\{x \mid \vdash_T D_x \cap \bar{A} \neq \emptyset\}$ is recursive, it is sufficient to show that S_n is recursive for all n where

$$S_n = \{u \mid |D_u| \leq n \ \& \vdash_T D_u \cap \bar{A} \neq \emptyset\}$$

This is proved by induction on n using techniques similar to those in the proof of lemma 5.2. S_0 is trivially recursive. Now assume that S_n is recursive. Let

$$M_{n+1} = \{u \mid |D_u| \leq n+1 \ \& \vdash_T D_u \cap \bar{C} \neq \emptyset \ \& \ (\forall v)[D_v \subsetneq D_u \Rightarrow \not\vdash_T D_v \cap \bar{C} \neq \emptyset]\}$$

$$= \{u \mid u \in S_{n+1} \ \& \ (\forall v)[D_v \subsetneq D_u \Rightarrow v \notin S_n]\}$$

Since the set of axioms given above is r.e. S_{n+1} is r.e. By the induction assumption S_n is recursive. Hence M_{n+1} is r.e. Let

$$G_{n+1} = \bigcup_{u \in M_{n+1}} D_u$$

Since M_{n+1} is r.e., G_{n+1} is r.e. Also, G_{n+1} is a subset of $\bar{C}$, since, by axioms 3, $x \in M_{n+1} \Rightarrow D_x \subset \bar{C}$. Thus M_{n+1} is finite and hence G_{n+1} is finite. But now, the equivalence

$$u \in S_{n+1} \Longleftrightarrow u \in G_{n+1} \text{ or, for some } v \text{ such that } D_v \subset D_u,\ v \in S_n$$

shows that S_{n+1} is recursive, completing the induction. qed.

COROLLARY 5.12 If A is simple but not hypersimple, then the m-degrees of A, AxA, AxAxA, ... are all distinct.

Proof Suppose the corollary is false. Let A be a simple but not hypersimple set and n a number such that $A^{n+1} \leq_m A^n$, where for all $k > 0$,

$$A^k = \overbrace{A x \ldots x A}^{k \text{ factors}}$$

Let $A_k = \{x \mid |D_x| \leq k \ \&\ D_x \subset A\}$. It is easy to check that for all $k > 0$,

$$A^k \equiv_m A_k$$

Hence $A_{n+1} \leq_m A_n$, say via g, so that

$$D_x \subset A \ \&\ |D_x| \leq n+1 \Longleftrightarrow D_{g(x)} \subset A \ \&\ |D_x| \leq n$$

A recursive function h will now be defined so that $\{x \mid D_x \subset A\} \leq_m A_n$ via h.

If $|D_x| \leq n$, define $h(x) = x$.

If $|D_x| > n$, let y be the smallest number so that $D_y \subset D_x$ and $|D_y| = n+1$. Let $D_{x'} = (D_x - D_y) \cup D_{g(y)}$. Observe that

$$|D_{x'}| < |D_x| \text{ and } (D_{x'} \subset A \Leftrightarrow D_x \subset A)$$

If for any set D_x all the sets $D_{x'}, D_{x''}, \ldots$ had cardinality greater than n, then the numbers $|D_{x'}|, |D_{x''}| \ldots$ would form a strictly decreasing chain of integers, which is impossible.

Thus if $|D_x| > n$, define $D_{h(x)} = D_{x\overbrace{'' \cdots '}^{m}}$, where m is the smallest number such that $|D_{x\overbrace{'' \cdots '}^{m}}| \leq n$.

Clearly, $\{x \mid D_x \subset A\} \leq_m A_n$ via h. But also, $A_n \leq_{bp} A$, so $\{x \mid D_x \subset A\} \leq_{bp} A$, which contradicts the theorem, with A = C. qed.

COROLLARY 5.13 Every r.e. nonrecursive T-degree contains infinitely many r.e. m-degrees.

Proof By the theorem of Yates, each such T-degree contains a simple but not hypersimple set A, and hence the m-degrees of

A, AxA . . .

from the desired infinite collection qed.

In view of the well known theorem of Post that no creative set can be btt-reduced to a simple set, it is natural to inquire whether sets of the form $\{x \mid D_x \subset A\}$ for simple, non-hypersimple A can be btt-reduced to simple sets. The writer has been unable to answer this question, although the methods of the previous theorem do show some promise. More precisely, if the conclusion of lemma 5.2 could be strengthened to:

"Then $\{x \mid D_x \not\subset C\}$ is not recursively separable from $\{x \mid D_x \subset A\}$" it would follow by an elaboration of the methods of theorem 5.11 that no set of the form $\{x \mid D_x \subset A\}$ for simple but not hypersimple A could be btt-reduced to a simple set.

It is also natural to ask whether sets of the form $\{x \mid D_x \subset A\}$ for simple but not hypersimple A can be tt-reduced to hypersimple sets. Since $A \equiv_p \{x \mid D_x \subset A\}$, this is equivalent to asking whether simple but not hypersimple sets can be tt-reduced to hypersimple sets. Here again the answer is not known, but we can prove a positive analogue to the classical theorems.

THEOREM 5.14 No simple, non-hypersimple set can be p-reduced to any hypersimple set.

Proof The first part of the proof consists of a lemma which shows that it suffices to prove the theorem for c-reducibility.

LEMMA 5.15 C hyperimmune $\Rightarrow \{x \mid D_x \subset C\}$ hyperimmune

Proof of Lemma Suppose that $\{x \mid D_x \subset C\}$ is infinite and not hyperimmune. Let $D_{h(x)}$ witness that $\{x \mid D_x \subset C\}$ is not hyperimmune. Assume $0 \notin \bigcup_x D_{h(x)}$ to avoid difficulties with the empty set. Now define a recursive function k by

$$D_{k(x)} = \{u \mid u \text{ is the largest member of some set } D_y \text{ with } y \in D_{h(x)}\}$$

Now each $D_{k(x)}$ intersects C, because each $D_{k(x)}$ contains some member of a subset of C. However, the sets $D_{k(x)}$ need not be disjoint. On the other hand, they can be made disjoint using a simple techniquue due to Post: define a recursive function l by

$$D_{l(0)} = D_{k(0)}$$

$$D_{l(n+1)} = D_{k(y)} \text{ where } y \text{ is the smallest number such that } D_{k(y)} \text{ is disjoint from } \bigcup_{i=0}^{n} D_{l(i)}$$

To show that C is not hypersimple, it is sufficient to show that l is total, i.e. that the number y referred to in the definition of l always exists. Suppose that this y fails to exist for some n + 1. Then every $D_{k(y)}$ intersects the finite set $\bigcup_{i=0}^{n} D_{l(i)}$. Thus some number u is in infinitely many $D_{k(y)}$. Thus, by the definition of $D_{k(y)}$, u is the largest member of infinitely many sets, which is impossible. Thus l is total. qed.

Proof of theorem Suppose the theorem is false, and let $A \leq_p B$, where B is hypersimple and A is simple but not hypersimple. Let

$$D = \{ x \mid D_x \cap B \neq \emptyset \}$$

Since D is r.e. and

$$\overline{D} = \{ x \mid D_x \subset \overline{B} \}$$

D is hypersimple by lemma 5.15. Also $A \leq_c D$, by theorem 2.9. Let f be a recursive function such that, for all x,

$$x \in A \Leftrightarrow D_{f(x)} \subset D$$

Let $D_{g(x)}$ witness that A is not hypersimple. Define a recursive function p by

$$D_{p(x)} = \bigcup_{y \in D_{g(x)}} D_{f(y)}$$

Each $D_{p(x)}$ intersects $\overline{D}$. Define the r.e. set E by

$$E = \{ y \mid (\exists x) \, [(D_y \cup D) \supset D_{p(x)}] \}$$

Since each $D_{p(x)}$ intersects $\overline{B}$, each member of E is the canonical index of a set intersecting $\overline{B}$. Let k be a recursive function with range E. Now, just as in the proof of lemma 5.15, the $D_{k(x)}$ can be replaced by a subsequence of disjoint sets to witness B not hypersimple unless there is some finite set, say F, which intersects every $D_{k(x)}$. If F is such a set, then $F \cap \overline{B}$ intersects every $D_{p(x)}$. Thus some number $u \in \overline{B}$, would be in $D_{p(x)}$ for infinitely many x and

$$\{ x \mid u \in D_{f(x)} \}$$

would be an infinite r.e. subset of $\overline{A}$, contradicting the assumption that A is simple. Thus the $D_{k(x)}$ can be disjointified to witness B not hypersimple. qed.

A counterpart to the above theorem for m-reducibility was first proved by Young. Martin has proved the btt-analogue of the above theorem.

One corollary of the above theorem is that every r.e. Turing degree contains at least two p-degrees. However, this fact has already been pointed out as corollary 3.10.

We now turn to non-r.e. sets. The goal of the present section is to prove that every nonrecursive tt-degree contains infinitely many m-degrees. For the present methods, the analogue of the simple but not hypersimple sets will be the immune but not hyperimmune sets which are retraced by total functions.

<u>PROPOSITION 5.16</u> Every nonrecursive tt-degree contains a set which is immune but not hyperimmune and retraced by a (total) recursive function.

<u>Proof</u>

The <u>binary tree</u> is the collection of all finite sequences of 0's and 1's.

Let σ be a 1-1 effective coding of the binary tree onto the integers such that, for any sequences a and b in the binary tree,

$$a \text{ longer than } b \Longrightarrow \sigma(a) > \sigma(b)$$

Let B be any given nonrecursive set. With B associate first the infinite sequence $S = c_B(0), c_B(1), \ldots,$ where c_B is the characteristic function of B. Now associate with B the set A, where A is defined by

$$A = \{\sigma(a) \mid a \text{ is a finite initial subsequence of } S\}$$

It is claimed that A is the desired immune not hyperimmune set retraced by a recursive function such that $A \equiv_{tt} B$.

First we show that $B \leq_{tt} A$. We have $B \leq_{q} A$, since, for all n

$n \in B \Longleftrightarrow$ some sequence of length n ending in a 1 is in A and

given n, one can effectively compute the cannonical index for the set of all code numbers for sequences of length n which end in a 1. Also, $A \leq_{tt} B$, since for all n

$n \in A \iff$ the sequence with code number n is a finite initial subsequence of

$$\langle c_B(0),\ c_B(1),\ \ldots \rangle$$

and the right hand side of the above equivalence can be written as a tt-condition on B uniformly in n.

Now we show that A is retraceable. Define a function f which maps finite sequences to finite sequences by

$$f(a) = \begin{cases} a, \text{if } a \text{ is the empty sequence} \\ \text{the sequence obtained from } a \text{ by deleting the last} \\ \qquad \text{term of } a, \text{ otherwise} \end{cases}$$

Now let f' be the corresponding function mapping N to N;

$$f' = \sigma f \sigma^{-1}$$

By the condition that longer sequences have larger code numbers, f' is a retracing function for A. By the effectiveness of the coding, f' is recursive.

Since A is retraceable and nonrecursive, A is immune. The sets $\{D_{g(n)}\}$ witnesses that A is not hyperimmune, where

$$D_{g(n)} = \{x \mid x \text{ is the code number for a sequence of length } n\}$$

qed.

THEOREM 5.17 Suppose that A is retraced by a total recursive function. Then,

(i) $A \times A \leq_m A$

(ii) If A is immune but not hyperimmune, then the m-degrees of

$$\bar{A},\ \bar{A}\times\bar{A},\ \bar{A}\times\bar{A}\times\bar{A}, \ldots$$

are all distinct.

(iii) If A is immune but not hyperimmune, then A and $\overline{A}$ are m-incomparable.

(iv) If A is not hyperimmune, $A \equiv_p \overline{A}$.

<u>Proof</u> Suppose for the proof of all parts, that A is retraced by the recursive function f'. Then the recursive function f also retraces A, where

$$f(x) = \begin{cases} f'(x) & \text{if } f'(x) \leq x \\ x & \text{if } f'(x) > x \end{cases}$$

f will be used throughout the proof because it has the useful property that $f(x) \leq x$ for all x.

Some terminology will be introduced now for this proof only. Let "x retraces to y" mean

$$(\exists n) \left[n \geq 0 \ \& \ f^n(x) = y \right]$$

Since $f(x) \leq x$ for all x and there are no infinite descending chains of integers, $\{\langle x,y\rangle \mid x \text{ retraces to } y\}$ is a recursive set.

Let "x and y are comparable" mean that x retraces to y or y retraces to x, and let "x is incomparable with y" mean that x and y are not comparable.

(i) To show that $A \times A \leq_m A$, let a' be a fixed member of $\overline{A}$, and define the recursive function g' by

$$g'(\langle x,y\rangle) = \begin{cases} x & \text{if } x \text{ retraces to } y \\ y & \text{if } y \text{ retraces to } x \\ a' & \text{if } x \text{ and } y \text{ are incomparable} \end{cases}$$

It is claimed that $A \times A \leq_m A$ via g'. If both x and y are in A, then x and y are comparable, so $g'(\langle x,y\rangle) = x$ or $g'(\langle x,y\rangle) = y$.

Therefore $g'(\langle x,y \rangle) \in A$. Conversely, if $g'(\langle x,y \rangle) \in A$, then $g'(\langle x,y \rangle) \neq a'$, so $g'(\langle x,y \rangle)$ retraces to both x and y. Therefore, x and y are in A. This proves part (i).

(ii) To prove part (ii), assume for <u>reductio</u> <u>ad</u> <u>absurdum</u> that A is immune and not hyperimmune and

$$\overbrace{\bar{A}x\bar{A}x\ldots x\bar{A}}^{k+1 \text{ factors}} \leq_m \overbrace{\bar{A}x\bar{A}x\ldots x\bar{A}}^{k \text{ factors}}$$

where k is fixed. Since it is easy to check that for all $j > 0$

$$\overbrace{\bar{A}x\bar{A}x\ldots x\bar{A}}^{j \text{ factors}} \equiv_m \overline{\{y \mid |D_y| \leq j \;\&\; D_y \cap A \neq \emptyset\}}$$

it follows that

$$\{y \mid D_y \leq k+1 \;\&\; D_y \cap A \neq \emptyset\} \leq_m \{y \mid |D_y| \leq k \;\&\; D_y \cap A \neq \emptyset\}$$

Let g be a recursive function such that for all y with $|D_y| \leq k+1$,

$$|D_{g(y)}| \leq k \text{ and } (D_y \cap A \neq \emptyset \Longleftrightarrow D_{g(y)} \cap A \neq \emptyset)$$

The properties of f and g are now used to obtain an r.e. subset B of $\bar{A}$. Let $B = B_1 \cup B_2$, where

$$B_1 = \{x \mid (\exists y)[\,|D_y| \leq k+1 \;\&\; x \in D_y \;\&\; x \text{ is incomparable with every member of } D_{g(y)}]\}$$

$$B_2 = \{x \mid (\exists y)[\,|D_y| \leq k+1 \;\&\; x \text{ is incomparable with every member of } D_y \;\&\; x \text{ retraces to some member of } D_{g(y)}]\}$$

B_1 and B_2 are r.e., so B is r.e. To see that $B_1 \subset \bar{A}$, suppose that some number x were in $B_1 \cap A$. Let y be such that $|D_y| \leq k+1$ and $x \in D_y$ and x is incomparable with every member of $D_{g(y)}$. Since $x \in D_y \cap A$, $D_y \cap A \neq \emptyset$; hence $D_{g(y)} \cap A \neq \emptyset$. But any element of A is comparable with x, so $D_{g(y)}$ contains a number comparable with x, contrary to assumption. Thus $B_1 \subset \bar{A}$.

To see that $B_2 \subset \bar{A}$, suppose that some number x were in $B_2 \cap A$.

Let y be such that $|D_y| \leq k+1$ and x is incomparable with every member of D_y and x retraces to some member of $D_{g(y)}$. Since $x \in A$, and x is incomparable with every member of D_y, $D_y \cap A = \emptyset$. Hence $D_{g(y)} \cap A = \emptyset$. But x retraces to some member of $D_{g(y)}$, so x retraces to a nonmember of A, which is impossible. Thus $B_2 \subset \overline{A}$, so $B \subset \overline{A}$.

It will be shown that B is "large" in a sense to be made precise with the use of the recursive function n defined below:

$$n(x) = \text{the least number } m \text{ such that } f^{m+1}(x) = f^m(x)$$

Observe that $n(x)$ ("the norm of x") is always defined because $f(x) \leq x$ for all x and there are no infinite descending chains of integers.

(Note: This proof can be visualized in terms of the "retracing tree." (cf. Rogers [14]). For example, $n(x)$ is the "level" of x in the retracing tree.)

Now it is claimed that for every j there are at most $2k$ numbers which have norm j and are not in B. (Recall that k was fixed earlier.) To facilitate the proof of the claim, a partial ordering $\leq_\ell$ of N will be defined such that any two numbers x and y are comparable with respect to $\leq_\ell$ if and only if $n(x) = n(y)$. $x \leq_\ell y$ is defined inductively on $n(x) = n(y)$:

$x \leq_\ell y$ means $x \leq y$ if $n(x) = n(y) = 0$

$\vdots$

$x \leq_\ell y$ means $nf(x) \leq_\ell nf(y)$ or $(f(x) = f(y)$ and $x \leq y)$ if $n(x) = n(y) > 0$, and $u \leq_\ell v$ has already been defined for all u,v with $n(u) = n(v) < n(x)$.

The above definition makes sense because, if $n(x) > 0$, then $nf(x) < n(x)$.

It is easy to verify that $\leq_\ell$ is a partial ordering under which any two numbers of equal norm are comparable and that for any x,y, and z

$$x \leq_{\ell} y \implies f^z(x) \leq_{\ell} f^z(y)$$

(Intuitively, $x \leq_{\ell} y$ means that x and y have the same level in the retracing tree, and x is to the left of y in the tree, if the tree is coded so that code numbers increase as one moves from left to right at a given level.)

Now suppose the claim made above is false, i.e assume that there is some number j such that 2k + 1 nonmembers of B have norm j. Let $x_1, \ldots, x_{2k+1}$ be 2k + 1 nonmembers of B of norm j. Assume that the x_i are indexed so that

$$x_1 <_{\ell} x_2 <_{\ell} \cdots <_{\ell} x_{2k+1}$$

Let $D_y = \{x_1, x_3, x_5, \ldots, x_{2k+1}\}$. Since no member of D_y is in B_1, every member of D_y is comparable with some member of $D_{g(y)}$. But since $|D_y| \leq k + 1$, $|D_{g(y)}| \leq k$. Thus there is some member w of $D_{g(y)}$ which is comparable with two distinct members, say x_{2m+1} and x_{2n+1} of D_y. Thus, since x_{2m+1} and x_{2n+1} have the same level, there is a z such that

$$f^z(x_{2m+1}) = f^z(x_{2n+1}) = w$$

Assume $m < n$. It will be shown that $x_{2m+2} \in B_2$. This will give the desired contradiction and prove the claim. Note that

$$x_{2m+1} <_{\ell} x_{2m+2} <_{\ell} x_{2n+1}$$

Thus $\underset{\overset{\shortparallel}{w}}{f^z(x_{2m+1})} \leq_{\ell} f^z(x_{2m+2}) \leq_{\ell} \underset{\overset{\shortparallel}{w}}{f^z(x_{2n+1})}$

So $f^z(x_{2m+2}) = w$.

Now consider the set $D_y (= \{x_1, x_3, \ldots, x_{2k+1}\})$. Every member of D_y is incomparable with x_{2m+2}, since $x_{2m+2} \notin D_y$, and every member of D_y has the same norm as x_{2m+2}. Also x_{2m+2}

retraces to some member, w, of $D_{g(y)}$. Thus $x_{2m+2} \in B_2$, which was to be shown.

The fact that A is not hyperimmune will now be used. Let $a_0, a_1, \ldots$ be the members of A in increasing order. Rice has shown that since A is infinite and not hyperimmune there is a recursive function h which majorizes A, i.e. which is such that for all n,

$$h(n) > a_n$$

We now define a sequence of disjoint sets $D_{p(x)}$ all intersecting A. To find $D_{p(x)}$, list B until at most 2j numbers which have norm x and are less than h(x) have not appeared in the list of B. By the previous argument, this state of affairs must be reached for every x. Then let $D_{p(x)}$ be the set whose members are these at most 2j numbers which have norm x and are less than h(x) and have not yet appeared in the listing of B. Now $D_{p(x)} \cap A \neq \emptyset$ because $a_x \in D_{p(x)} \cap A$. Now the proof could be conculded at this point by quoting a lemma of Appel and McLaughlin [1] which states that no regressive immune set is witnessed nonhyperimmune by a collection of sets $\{D_{p(x)}\}$ of bounded cardinality, since we have, for all x, $|D_{p(x)}| \leq 2j$. However, we prove below just the special case of the Appel-McLaughlin lemma needed for the proof.

<u>LEMMA 5.18</u> If A is an immune set retraced by a total function and $\{D_{k(x)}\}$ witnesses that A is not hyperimmune, then there is no constant c such that, for all x

$$|D_{p(x)}| \leq c$$

<u>Proof of lemma</u> Suppose there were such a constant c. Consider

$$C = \{ y \mid (\exists x) [\text{Every member of } D_{p(x)} \text{ retraces to } y]\}$$

(We continue to use the terminology of the proof of the theorem)

Since each $D_{p(x)}$ intersects A, C is an r.e. subset of A and thus finite.

Let y_o be a member of A which is not in C. Define $D_{p'(x)}$ by

$$D_{p'(x)} = D_{p(x)} - \{ z \mid z \in D_{p(x)} \text{ \& } z \text{ does not retrace to } y_o \}$$

It is claimed that all but finitely many $D_{p'(x)}$ intersect A. This is so because if $D_{p'(x)}$ fails to intersect A, then $D_{p(x)}$ must have contained a member of A which did not retrace to y_o, i.e. $D_{p(x)}$ must have contained a number to which y_o retraced, and there are only finitely many such numbers. Thus by eliminating these finitely many sets we obtain the sets $D_{p''(x)}$ which witness A nonhypersimple and which are bounded in cardinality by c-1, since each $D_{p(x)}$ contained a number which did not retrace to y_o. Iterating this procedure c times, we obtain a sequence of empty sets witnessing the nonhypersimplicity of A, which is absurd. qed.

Since the $D_{p(x)}$ defined in the proof of the theorem are bounded in cardinality, we have contradiction, and part (ii) is proved.

(iii) To prove part (iii), assume that A is immune and not hyperimmune. Assume also that A and $\overline{A}$ are m-comparable, so that $A \equiv_m \overline{A}$. Then by part (i),

$\overline{A} x \overline{A} \equiv_m A x A \equiv_m A$

This contradicts part (ii).

To prove part (iv), assume that A is non-hyperimmune. Let the functions n and h be as in the proof of part (ii).

Then, for any x

$$x \in \overline{A} \Leftrightarrow (\exists y) [y \neq x \ \& \ n(y) = n(x) \ \& \ y \leq hn(x) \ \& \ y \in A]$$

Thus $\overline{A} \leq_p A$, so $A \leq_p \overline{A}$ and $A \equiv_p \overline{A}$. qed.

<u>COROLLARY 5.19</u>

(i) Every nonrecursive tt-degree contains infinitely many m-degrees.

(ii) Every nonrecursive tt-degree contains a p-degree with incomparable m-degrees.

<u>Proof</u>

(i) By proposition 5.16 each nonrecursive tt-degree contains an immune but not hyperimmune set A which is retraced by a recursive function. Thus it contains the m-degrees of

$\bar{A}, \bar{A} \times \bar{A}, \ldots$

and by theorem 5.18 this is an infinite collection of m-degrees.

(ii) By proposition 5.16 each nonrecursive tt-degree contains an immune but not hyperimmune set A which is retraced by a total function. By theorem 5.17 the positive degree of A contains the m-degrees of A and $\bar{A}$ which are incomparable. qed.

We now show that every r.e. nonrecursive tt-degree contains a strong cylinder.

<u>THEOREM 5.20</u> If A is an r.e. nonrecursive set, A^{tt} is a strong cylinder, where A^{tt} is defined as $\{x \mid x \text{ is true of } A\}$. (cf. theorem 2.6)

<u>Proof</u> For any A, A^{tt} is a cylinder. Let A be r.e. and nonrecursive and suppose $A \leq_m B$ via g. It must be shown that $A \leq_1 B$, so by lemma 5.7 it is sufficient to show that B is a (range g)-cylinder. Let h be a recursive function such that for all y,

$$W_{h(y)} = \{f(x \wedge (n \in A)) \mid f(x) = y \ \& \ n \in A\} \cup \{f(x \vee (\neg n \in A)) \mid f(x) = y \ \& \ n \in A\}$$

(Recall that formulas are conventionally identified with their code numbers so that, for example, $x \wedge (n \in A)$ refers to the code number for the formula obtained by conjoining the formula with code number x with the formula "$n \in A$".)

To show that B is a (range g)-cylinder it is sufficient to show that

$$y \in B \Rightarrow W_{h(y)} \subset B, \qquad y \in \overline{B} \Rightarrow W_{h(y)} \subset \overline{B}$$
$$\& \quad y \in \text{range } g \Rightarrow W_{h(y)} \text{ infinite}$$

Note that for any x,y, and n with $n \in A$ and $f(x) = y$

$$\begin{array}{ccccc} y \in B \Leftrightarrow x \in A^{tt} \Leftrightarrow & (x \wedge n \in A) \in A^{tt} & \Leftrightarrow & (x \vee \neg n \in A) \in A^{tt} \\ & \Updownarrow & & \Updownarrow \\ & g(x \wedge n \in A) \in B & & g(x \vee \neg n \in A) \in B \end{array}$$

Thus $(y \in B \Rightarrow W_{h(y)} \subset B)$ & $(y \in \overline{B} \Rightarrow W_{h(y)} \subset \overline{B})$.

Now suppose that $W_{h(y_0)}$ were finite for some number y_0 in range g. Let $y_0 = g(x_0)$. We will get a contradiction by showing that A is recursive.

First suppose that $y_0 \in B$. Then, for all n,

$$n \in A \Leftrightarrow g(x_0 \wedge n \in A) \in W_{h(y_0)}$$

The arrow to the right above is immediate from the definition of $W_{h(y_0)}$. To prove the arrow to the left, assume that $g(x_0 \wedge n \in A) \in W_{h(y_0)}$. Since $y_0 \in B$, $W_{h(y_0)} \subset B$, so $g(x_0 \wedge n \in A) \in B$. Thus $(x_0 \wedge n \in A) \in A^{tt}$, so $(n \in A) \in A^{tt}$. Therefore, $n \in A$.

But the equivalence proved above shows that A is recursive, since $W_{h(y_0)}$ is finite.

Now suppose that $y_0 \notin B$. Then, for all n,

$$n \in A \Leftrightarrow g(x_0 \vee \neg n \in A) \in W_{h(y_0)}$$

The arrow to the right above is immediate from the definition of

$W_{h(y_0)}$. To prove the arrow to the left, assume that $g(x_0 \vee \neg n \in A) \in W_{h(y_0)}$. Since $y_0 \in \overline{B}$, $W_{h(y_0)} \subset \overline{B}$, so $g(x_0 \vee \neg n \in A) \in \overline{B}$. Thus $(x_0 \vee \neg n \in A) \notin A^{tt}$, so $(n \notin A) \notin A^{tt}$. Therefore $n \in A$.

Again, we have that A is recursive.

Thus we see that $W_{h(y)}$ is infinite for $y \in$ range g, and the proof is complete. qed.

<u>COROLLARY 5.21</u> Every r.e. tt-degree contains an m-degree consisting of a single 1-degree.

<u>Proof</u> The recursive tt-degree obviously contains an m-degree consisting of a single 1-degree and each nonrecursive r.e. tt-degree contains a strong cylinder, which, by proposition 5.5, belongs to an m-degree consisting of a single 1-degree. qed.

It should be noted that corollary 5.21 is not a generalization of corollary 5.10, which states that every r.e. Turing degree contains an <u>r.e.</u> m-degree consisting of a single 1-degree. The common generalization of these two corollaries, i.e. the statement that every r.e. tt-degree contains an r.e. m-degree consisting of a single 1-degree, would follow immediately from corollary 5.9 if it could be shown that every r.e. nonrecursive tt-degree contains a simple set which is not hypersimple. Likewise, it would follow that every r.e. nonrecursive tt-degree contains infinitely many m-degrees. However, it is probably not the case that every r.e. nonrecursive tt-degree contains a simple set which is not hypersimple.

In theorem 5.21 the hypothesis that A is not recursive obviously cannot be dropped. The writer does not know whether the theorem remains true when the requirement that A be r.e. is dropped.

We now study an inversion of the notion of R-cylinder. As we shall see, the notion seems to be of interest only for m-reducibility.

DEFINITION 5.22 Let R be a reducibility. A is an inverse R-cylinder if

$$(\forall B)[A \leq_R B \Rightarrow A \leq_1 B]$$

A is an inverse cylinder if A is an inverse m-cylinder.

Since strong cylinders are precisely the sets which are both cylinders and inverse cylinders, we have already shown that a variety of inverse cylinders exist. We now show, however, that practically no reducibilities R strictly weaker than m-reducibility have any inverse R-cylinders.

THEOREM 5.23 No reducibility R weaker than bq (or bc) reducibility has any inverse R-cylinders.

Proof Suppose it can be shown that there are no inverse bq-cylinders. Then it follows trivially that no reducibility R weaker than bq has any inverse R-cylinders. Also, since

$$A \leq_{bq} B \Leftrightarrow \bar{A} \leq_{bc} \bar{B} \quad \& \quad A \leq_1 B \Leftrightarrow \bar{A} \leq_1 \bar{B}$$

it follows that there are no inverse bc-cylinders and hence no inverse R-cylinders for any reducibility weaker than bc-reducibility.

Thus it is sufficient to show that there are no inverse bq-cylinders. Suppose that some set A is an inverse bq-cylinder.

Case 1 A is finite. Then let B be any coimmune set. We have

$$A \leq_m B.$$

Thus, $A \leq_{bq} B$.

Thus, since A is an inverse bq-cylinder, $A \leq_1 B$. Hence $\bar{A} \leq_1 \bar{B}$.

Thus a cofinite set is 1-1 reducible to an immune set, which is impossible.

Case 2 A is infinite. Let B be a set which is both immune and coimmune. (Post has shown the existence of such sets.) Let C be given by

$$C = (A \text{ join } A) \cap (B \text{ join } \bar{B})$$

Note that the set C is a subset of the immune set B join $\bar{B}$ and that C is infinite, since A is infinite. Thus C is immune. Note also that

$$x \in A \Longleftrightarrow (2x \in C) \text{ or } (2x + 1) \in C$$

so that $A \leq_{bq} C$, so $A \leq_1 C$. Thus since A is infinite, A is immune. Let a be any member of A. We have $A \leq_m A - \{a\}$, so $A \leq_{bq} A - \{a\}$, and therefore $A \leq_1 A - \{a\}$. But this last statement contradicts a well-known theorem of Dekker and Myhill, since A is immune. qed.

We now study inverse (m)-cylinders. Note that it follows from the proof of the previous theorem that no inverse cylinder is m-reducible to any immune set. Hence not all m-degrees contain inverse cylinders and, in particular, there are cylinders which are not inverse cylinders. However, it is not known whether there are inverse cylinders which are not cylinders. It is also not known whether the join of two inverse cylinders is an inverse cylinder, although, of course, the join of two cylinders is a cylinder. Below will be given two theoremswhich will make it easy to explore the connection between these questions.

THEOREM 5.24 For any set A, the following statements are equivalent.

(i) A is a cylinder

(ii) A join A is a cylinder

(iii) A join $A \leq_1 A$

Proof (i) $\Rightarrow$ (ii) since the join of two cylinders is always a cylinder. (i) $\Rightarrow$ (iii) since it is always true that A join $A \leq_m A$.

To prove (ii) $\Rightarrow$ (i), assume that A join A is a cylinder and that g is a recursive function such that, for all x

$$W_{g(x)} \text{ is infinite } \& \ (x \in A \text{ join } A \Rightarrow W_{g(x)} \subset A \text{ join } A)$$
$$\& \ (x \in \overline{A \text{ join } A} \Rightarrow W_{g(x)} \subset \overline{A \text{ join } A})$$

g exists by Young's characterization of cylinders. Now let h be a recursive function such that, for all x,

$$W_{h(x)} = \{x \mid 2x \in W_{g(x)} \text{ or } (2x+1) \in W_{g(x)}\}$$

Then h witnesses that A is a cylinder by Young's characterization.

Now suppose that A join $A \leq_1 A$. It must be shown that A is a cylinder. Let a number x be given. Define a sequence $\{S_i\}$ of finite sets inductively:

$$S_0 = \{x\}$$
$$\vdots$$
$$S_{n+1} = \{f(2x) \mid x \in S_n\} \cup \{f(2x+1) \mid x \in S_n\}$$

Since f is 1-1, $|S^n| = 2^n$. Thus $\bigcup_{n=0}^{\infty} S^n$ is infinite. Also, if $y \in S^n$, $y \in A$ iff $x \in A$. Thus if we let h be a recursive function such that, for each x, $W_{h(x)}$ is $\bigcup_{n=0}^{\infty} S^n$, h witnesses that A is a cylinder.

qed.

The above theorem allows one to prove a special case of Young's result that every nonrecursive m-degree either consists of a single 1-degree or contains a collection of 1-degrees which is linearly ordered under $\leq_1$ with the order type of rationals.

COROLLARY 5.25 Every m-degree either consists of a single 1-degree or contains an infinite collection of 1-degrees with the order type of the integers.

Proof If an m-degree does not consist of a single 1-degree it contains a non-cylinder A. Define the sequence A^i inductively:

$$A^0 = A$$
$$A^{n+1} = A^n \text{ join } A^n$$

Then from the equivalence (i)$\Longleftrightarrow$(ii) in the theorem, each A^n is a non-cylinder, so by the equivalence (i)$\Longleftrightarrow$(iii) we have

$$A^n <_1 A^{n+1}.$$

Since for all n, $A^n \equiv_m A$, the proof is complete. qed.

THEOREM 5.26 If either A or B is a cylinder, then the 1-degree of A join B is the least upper bound to the 1-degrees of A and B in the 1-ordering.

Proof For any A and B, the 1-degree of A join B is an upper bound to the 1-degrees of A and B. Now assume that A is a cylinder. To show that the 1-degree of A join B is the l.u.b. to the 1-degrees of A and B we must show that for any set C with $A \leq_1 C$ and $B \leq_1 C$, it is the case that

$$A \text{ join } B \leq_1 C$$

Assume that $A \leq_1 C$ via f and $B \leq_1 C$ via g. Let h be a recursive function such that, for all x,

$$W_{h(x)} \text{ infinite } \;\&\; (x \in A \Rightarrow W_{h(x)} \subset A) \;\&\; (x \in \bar{A} \Rightarrow W_{h(x)} \subset \bar{A})$$

We now define a 1-1 recursive function k by induction so that A join B $\leq_1$ C via k:

$$k(0) = f(0)$$

$\vdots$

$k(2n) = y$ where y is the first number found in an effective listing of $f(W_{h(n)})$ such that $y \notin \{k(0), k(1), \ldots k(2n-1)\}$. $(n > 0)$

$k(2n+1) = g(n)$ if $g(n) \notin \{k(0), k(1), \ldots, k(2n)\}$. Otherwise use the instructions below to compute $k(2n+1)$.

If $g(n) \in \{k(0), k(1), \ldots k(2n)\}$, list the sets $f(W_{h(x)})$ for $x \leq n$ until $g(n)$ is found in one of these sets, say $f(W_{h(z)})$. Then let $k(2n+1) = y$, where y is the first number found in an effective listing of $f(W_{h(z)})$ such that $y \notin \{k(0), k(1), \ldots, k(2n)\}$.

Clearly, if k is total, A join $B \leq_1 C$ via k.

k is clearly defined for even arguments and for odd arguments $2n+1$ such that $g(n) \notin \{k(0), k(1), \ldots, k(2n)\}$. So it is sufficient to show that $k(2n+1)$ is defined when $g(n) \in \{k(0), k(1), \ldots, k(2n)\}$. It follows from the definition of k, that for any $m \leq 2n$, either $k(m) = g(u)$ where $u < n$, or $k(m) \in \bigcup_{x=0}^{n} f(W_{h(x)})$. Since $g(n) \in \{k(0), k(1), \ldots, k(2n)\}$, then $g(n) = g(u)$ where $u < n$, or $g(n) \in \bigcup_{x=0}^{n} f(W_{h(x)})$. Since g is 1-1, it follows that $g(n) \in \bigcup_{x=0}^{n} f(W_{h(x)})$ so it is apparent from the definition of k that $k(2n+1)$ is defined. qed.

In contrast to the above theorem, it may be shown that A join B is never a least upper bound to A and B in the 1-ordering when A and B are immune. In fact, Young [18] has shown that if A and B are simple sets incomparable under $\leq_1$, then A and B have no l.u.b. in the 1-ordering.

COROLLARY 5.27

(i) If A is a strong cylinder and B is an inverse cylinder, then A join B in an inverse cylinder.

(ii) If A and B are strong cylinders, then A join B is a strong cylinder.

Proof

(i) Suppose that A is a strong cylinder and B is an inverse cylinder and

$$A \text{ join } B \leq_m C.$$

It must be shown that A join B $\leq_1$ C. Since A and B are m-reducible to A join B, A and B are each m-reducible to C. Since A and B are inverse cylinders, it follows that A and B are each 1-reducible to C. Since A is a cylinder, the theorem implies that A join B is 1-reducible to C.

(ii) Suppose that A and B are strong cylinders. Then by part (i), A join B is an inverse cylinder. Also, since A and B are cylinders, A join B is a cylinder, so A join B is a strong cylinder. qed.

The question of whether every inverse cylinder is a cylinder or, equivalently, whether every inverse cylinder is a strong cylinder, has been left open. The following corollary gives an alternative formulation of the question.

COROLLARY 5.28 The following two propositions are equivalent.

(i) Every inverse cylinder is a cylinder.

(ii) The join of any two inverse cylinders is an inverse cylinder.

Proof

Assume (i) and let A and B be inverse cylinders. Thus A and B are strong cylinders. Then by the preceding corollary, A join B is a strong cylinder and thus an inverse cylinder.

Assume (ii), and let A be an inverse cylinder. By (ii), A join A is an inverse cylinder. Thus, since A join A $\leq_m$ A, A join A $\leq_1$ A. Now it follows from theorem 5.25 that A is a cylinder. qed.

It is easy to show that if A is a cylinder and B is any set, then AxB is a cylinder. The corresponding statement for inverse cylinders is false, for if A is an inverse cylinder and B is empty, then AxB is empty and hence not an inverse cylinder. However, the writer does not know whether AxB is an inverse cylinder when A is an inverse cylinder and B is non-empty. The inverse cylinders exhibited in theorem 5.8, i.e sets of the form $\{x \mid D_x \subset A\} \times C$ for simple but not hypersimple A and non-empty C, have the property that their cartesian product with any non-empty set is still an inverse cylinder. However, it does not seem clear that the inverse cylinders exhibited in theorem 5.20, i.e. sets of the form A^{tt} for r.e. but not recursive A, share this property, much less whether all inverse cylinders share this property.

BIBLIOGRAPHY

[1] K. I. Appel and T. G. McLaughlin, "On properties of Regressive Sets," Transactions of the American Mathematical Society, vol. 115, 1965, pp. 83-93.

[2] J.C.E. Dekker, "A Theorem on Hypersimple Set," Proceedings of the American Mathematical Society, vol. 5, 1954, pp. 791-796.

[3] J.C.E. Dekker and J. Myhill, "Retraceable Sets," Canadian Journal of Mathematics, vol. 10, 1958, pp. 357-373

[4] J.C.E. Dekker, "Infinite Series of Isols," Proceedings of the Symposia in Pure Mathematics, vol. V: Recursive Function Theory, American Mathematical Society, Providence, Rhode Island, 1962, pp. 77-96.

[5] P. C. Fischer, "A Note on Bounded-truth-table Reducibility," Proceedings of the American Mathematical Society, vol. 14, 1963 pp. 875-877.

[6] S. Feferman, "Degrees of Unsolvability Associated with Classes of Formalized Theories," Journal of Symbolic Logic, vol. 22, 1957 pp. 161-175.

[7] R. Friedberg, "Two Recursively Enumerable Sets of Incomparable Degrees of Unsolvability," Proceedings of the National Academy of Sciences, USA, vol. 43, 1958, pp.236-238

[8] R. M. Friedberg, "Three Theorems on Recursive Enumeration. I. Decomposition II. Maximal Set III. Enumeration with Duplication," Journal of Symbolic Logic, vol. 23, 1958, pp.309-316.

[9] R. M. Friedberg and H. Rogers, Jr., "Reducibility and Completeness for Sets of Integers," Zeitschrift fur Mathematische Logik und Grundlagen der Mathematik, vol. 5, 1959, pp. 117-125.

[10] A. H. Lachlan, "Some Notions of Reducibility and Productiveness," Zeitschrift fur Mathematische Logik und Grundlagen der Mathematik, vol. 11, 1965, pp. 17-44

[11] T. G. McLaughlin, "On a Class of Complete Simple Sets," Canadian Mathematical Bulletin, vol. 8, 1965, pp. 33-37.

[12] J. Myhill, "Recursive Digraphs, Splinters, and Cylinders," Mathematische Annalen, vol. 138, 1959. pp. 211-218.

[13] E. L. Post, "Recursively Enumerable Sets of Integers and their Decision Problems," Bulletin of the American Mathematical Society, vol. 50, 1944, pp. 284-316.

[14] H. Rogers, Jr., Theory of Recursive Functions and Effective Computability, McGraw-Hill, 1966.

[15] R. M. Smullyan, Theory of Formal Systems, Annals of Mathematics Studies, No. 47, Princeton, New Jersey, 1961.

[16] C.E.M. Yates, "Recursively Enumerable Sets and Retracing Functions," Zeitschrift fur Mathematische Logik und Grundlagen der Mathematik, vol. 8, 1962, pp. 331-345.

[17] C.E.M. Yates, "Three Theorems of the Degrees of Recursively Enumerable Sets," Duke Mathematical Journal, vol. 32, 1965, pp.461.

[18] P. R. Young, On the Structure of Recursively Enumerable Sets, MIT Doctoral Dissertation, 1963.

[19] P. R. Young, "A Note on Pseudo-creative Sets and Cylinders," Pacific Journal of Mathematics, vol. 14, 1964, pp. 749-753.

[20] P. R. Young, "Linear Orderings under 1-1 Reducibility," to appear in Journal of Symbolic Logic.

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"Cylinders and Positive Reducibility," vol. 12 (1965), p. 720

"Semirecursive Sets," vol. 12 (1965), p. 816

"Relationships between Reducibilities," vol. 13 (1966), p. 383

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