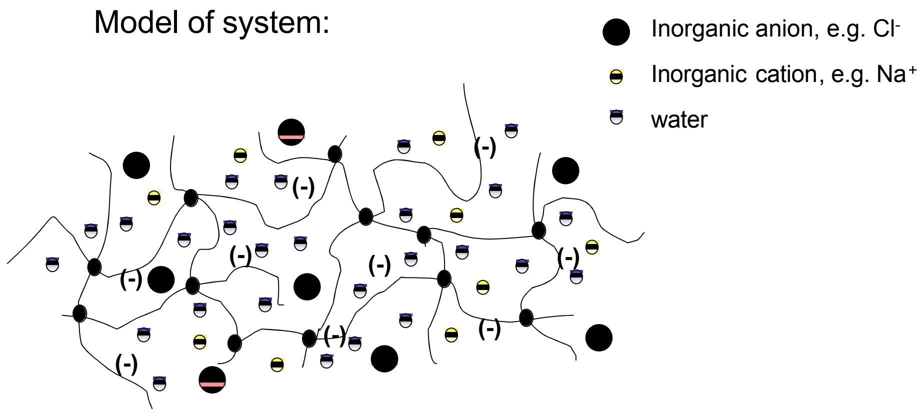
Brannon-Peppas theory of swelling in ionic hydrogels

- Original theory for elastic networks developed by Flory and Mehrer[1-3], refined for treatment of ionic hydrogels by Brannon-Peppas and Peppas[4,5]
- Other theoretical treatments[6]

Derivation of ionic hydrogel swelling

- Model structure of the system:



- System is composed of permanently cross-linked polymer chains, water, and salt
- We will derive the thermodynamic behavior of the ionic hydrogel using the model we previously developed for neutral hydrogels swelling in good solvent

- Model parameters:

a_+	activity of cations in gel
a_+^*	activity of cations in solution
a_-	activity of anions in gel
a_-^*	activity of anions in solution
c_+	concentration of cations in gel (moles/volume)
c_+^*	concentration of cations in solution (moles/volume)
c_-	concentration of anions in solution (moles/volume)
c_-^*	concentration of anions in solution (moles/volume)
c_s	concentration of electrolyte
c_2	concentration of ionizable repeat units in gel (moles/volume)
μ_1^*	chemical potential of water in solution
μ_1	chemical potential of water in the hydrogel
μ_1^0	chemical potential of pure water in standard state
M	Molecular weight of polymer chains before cross-linking
M_c	Molecular weight of cross-linked subchains
n_1	number of water molecules in swollen gel
χ	polymer-solvent interaction parameter

k_B	Boltzman constant
T	absolute temperature (Kelvin)
$v_{m,1}$	molar volume of solvent (water, volume/mole)
$v_{m,2}$	molar volume of polymer (volume/mole)
$v_{sp,1}$	specific volume of solvent (water, volume/mass)
$v_{sp,2}$	specific volume of polymer (volume/mass)
V_2	total volume of polymer
V_s	total volume of swollen hydrogel
V_r	total volume of relaxed hydrogel
ν	number of subchains in network
ν_e	number of 'effective' subchains in network
ν^+	stoichiometric coefficient for eletrolyte cation
ν^-	stoichiometric coefficient for eletrolyte anion
$\phi_{1,s}$	volume fraction of water in swollen gel
$\phi_{2,s}$	volume fraction of polymer in swollen gel
$\phi_{2,r}$	volume fraction of polymer in relaxed gel
x_1	mole fraction of water in swollen gel
x_1^*	mole fraction of water in solution

- Asterisks denote parameters in solution

- Free energy has 3 components: free energy of mixing, elastic free energy, and ionic free energy

Eqn 1 $$\Delta G_{total} = \Delta G_{mix} + \Delta G_{el} + \Delta G_{ion}$$

- At equilibrium, the chemical potential of water inside and outside the gel are equal:

Eqn 2 $\mu_1^* = \mu_1$

Eqn 3 $\mu_1^* - \mu_1^0 = \mu_1 - \mu_1^0$

- Solution contains ions so μ_1^* is not equal to μ_1^0

Eqn 4 $(\Delta\mu_1^*)_{TOTAL} = (\Delta\mu_1)_{TOTAL}$

Eqn 5 $(\Delta\mu_1^*)_{ion} = (\Delta\mu_1)_{mix} + (\Delta\mu_1)_{el} + (\Delta\mu_1)_{ion}$

- The equation we'll try to solve is a rearrangement of this:

Eqn 6 $(\Delta\mu_1^*)_{ion} - (\Delta\mu_1)_{ion} = (\Delta\mu_1)_{mix} + (\Delta\mu_1)_{el}$

- Contributions to the free energy:
  - Free energy of mixing:

Eqn 7 $\Delta G_{mix} = \Delta H_{mix} - T\Delta S_{mix}$

- We previously derived the contribution from mixing using the Flory-Rehner lattice model:

Eqn 8 $\Delta G_{mix} = k_BT[n_1\ln(1-\phi_{2,s}) + \chi n_1\phi_{2,s}]$

Eqn 9
$$(\Delta\mu_1)_{mix} = \left(\frac{\partial(\Delta G_{mix})}{\partial n_1}\right)_{T,P} = k_BT[\ln(1-\phi_{2,s}) + \phi_{2,s} + \chi\phi_{2,s}^2] = RT[\ln(1-\phi_{2,s}) + \phi_{2,s} + \chi\phi_{2,s}^2]$$

- Second expression puts us on a molar basis instead of per molecule

- Elastic free energy:

Eqn 10 $\Delta G_{el} = (3/2)k_BT\nu_e(\alpha^2 - 1 - \ln\alpha)$

Eqn 11
$$(\Delta\mu_1)_{el} = \left(\frac{\partial(\Delta G_{el})}{\partial n_1}\right)_{T,P} = \left(\frac{\partial(\Delta G_{el})}{\partial\alpha}\right)_{T,P}\left(\frac{\partial\alpha}{\partial n_1}\right)_{T,P} = RT\nu\left(1-\frac{2M_c}{M}\right)\frac{v_{m,1}}{V_r}\left[\left(\frac{\phi_{2,s}}{\phi_{2rs}}\right)^{1/3} - \frac{1}{2}\left(\frac{\phi_{2,s}}{\phi_{2rs}}\right)\right]$$
$$= RT\left(\frac{v_{m,1}}{v_{sp,2}M_c}\right)\left(1-\frac{2M_c}{M}\right)\phi_{2,r}\left[\left(\frac{\phi_{2,s}}{\phi_{2rs}}\right)^{1/3} - \frac{1}{2}\left(\frac{\phi_{2,s}}{\phi_{2rs}}\right)\right]$$

- Last equality uses:
  - $\nu = V_2/v_{sp,2}M_c$ (on handout)
  - $V_r = V_2/\phi_{2,r}$ (on handout)
  - Thus $\nu/V_r = \phi_{2,r}/v_{sp,2}M_c$

- Ionic free energy:
  - Term driving dilution of ions diffusing into gel to maintain charge neutrality
  - Chemical potential change in solution:

Eqn 12
$$(\Delta\mu_1)_{ion}^* = \mu_1^* - \mu_1^0 = RT\ln a_1^* \cong RT\ln x_1^* = RT\ln(1 - \sum_j^{all\_solutes} x_j^*)$$

- approximation in third equality is used for dilute solutions

Eqn 13 $$(\Delta\mu_1)^*_{ion} \cong -RT\sum_j^{all\_ions} x_j^* = -\frac{RT}{n}\sum_j^{all\_ions} n_j^* = -\frac{v_{m,1}RT}{v_{m,1}n}\sum_j^{all\_ions} n_j^* \cong -v_{m,1}RT\sum_j^{all\_ions} c_j^*$$

- The first approximation holds if Σx_j^* is small
- Fourth equality holds because we assume in the liquid lattice model that the molar volume of all species is the same, thus $v_{m,1}n = V$, the total volume of the system

- Chemical potential change in gel:

Eqn 14 $$(\Delta\mu_1)_{ion} = \mu_1 - \mu_1^0 = RT\ln a_1 \cong -v_{m,1}RT\sum_j^{all-ions} c_j$$

Eqn 15 $$(\Delta\mu_1)^*_{ion} - (\Delta\mu_1)_{ion} = v_{m,1}RT\sum_j^{all-ions}(c_j - c_j^*)$$

- The electrolyte dissolved in water provides mobile cations and anions in the solution and in the gel:
  - E.g. NaCl: $Na^+_{\nu+}Cl^-_{\nu+(s)} \rightarrow \nu^+Na^+_{(aq)} + \nu^-Cl^-_{(aq)}$
  - $\nu^+ = \nu^- = 1$ stoichiometric coefficients

Eqn 16 $$C^{z+}_{\nu_+}A^{z-}_{\nu_-} \rightarrow \nu_+C^{z+} + \nu_-A^{z-}$$

- e.g. $CaCl_2$: $\nu_+ = 1$, $\nu_- = 2$, $z_+ = 2$, $z_- = 1$

Eqn 17 $$\nu^+ + \nu^- = \hat{\nu}$$ **…for a 1:1 electrolyte**

Eqn 18 $$\nu^+ = \nu^- = \frac{\hat{\nu}}{2}$$ **…for a 1:1 electrolyte**

Eqn 19 $$c_+^* + c_-^* = (\nu^+ + \nu^-)c_s^* = \hat{\nu}c_s^*$$ **…total concentration of ions**

- We will derive equations for an anionic network
  - Assuming activities ~ concentrations
  - Inside gel:

Eqn 20 $c_+ = \nu_+c_s$

Eqn 21 $c_- = \nu_-c_s + ic_2/z_-$

- c_2 is the moles of ionizable repeat groups on gel chains per volume
- First term comes from electrolyte anions in gel, second term from ionized groups on the polymer chains
- The degree of ionization *i* can be related to the pH of the environment and the pKa of the network groups:

Eqn 22 $$K_a = \frac{[RCOO^-][H^+]}{[RCOOH]}$$

Eqn 23

$$i = \frac{[RCOO^-]}{[RCOOH] + [RCOO^-]} = \frac{\frac{[RCOO^-]}{[RCOOH]}}{1 + \frac{[RCOO^-]}{[RCOOH]}} = \frac{\frac{K_a}{[H^+]}}{1 + \frac{K_a}{[H^+]}} = \frac{K_a}{[H^+] + K_a} = \frac{K_a}{10^{-pH} + K_a} = \frac{10^{-pK_a}}{10^{-pH} + 10^{-pK_a}}$$

- Outside gel:

Eqn 24 $c_+^* = \nu_+ c_s^*$

Eqn 25 $c_-^* = \nu_- c_s^*$

- Our relationship for the ionic chemical potentials is now:

Eqn 26 $$(\Delta\mu_1)_{ion}^* - (\Delta\mu_1)_{ion} = v_{m,1}RT\sum_j^{all-ions}\left(c_j - c_j^*\right) = v_{m,1}RT\left(c_+ + c_- - c_+^* - c_-^*\right)$$

- Using Eqn 20, Eqn 21, Eqn 24, and Eqn 25, Eqn 26 becomes:

Eqn 27 $$(\Delta\mu_1)_{ion}^* - (\Delta\mu_1)_{ion} = v_{m,1}RT\left(\nu_+ c_s + \nu_- c_s + \frac{ic_2}{z_-} - \hat{\nu}c_s^*\right) = v_{m,1}RT\left(\hat{\nu}\, c_s + \frac{ic_2}{z_-} - \hat{\nu}c_s^*\right)$$

$$= v_{m,1}RT\left(\frac{ic_2}{z_-} - \hat{\nu}(c_s^* - c_s)\right)$$

- **How can we relate c_s and c_s^*?**
  - We can make simplifications for a 1:1 cation:anion electrolyte:
  - The chemical potentials of the mobile ions must also be equilibrated inside/outside the gel:

Eqn 28 $\mu_+ = \mu_+^*$

Eqn 29 $\mu_- = \mu_-^*$

- Add Eqn 29 to Eqn 28:

Eqn 30 $\mu_+ + \mu_- = \mu_+^* + \mu_-^*$

Eqn 31 $RT\ln a_+^{\nu_+} + RT\ln a_-^{\nu_-} = RT\ln a_+^{*\nu_+} + RT\ln a_-^{*\nu_-}$

- Therefore we can write:

Eqn 32 $a_+^{\nu+}a_-^{\nu-} = a_+^{*\nu+}a_-^{*\nu-}$

- Assuming dilute solutions where the activities are approximately equal to the concentrations:

Eqn 33 $$\left(\frac{c_+}{c_+^*}\right)^{\nu+} = \left(\frac{c_-^*}{c_-}\right)^{\nu-}$$

Eqn 34 $$\left(\frac{\nu_+ c_s}{\nu_+ c_s^*}\right)^{\nu+} = \left(\frac{\nu_- c_s^*}{\nu_- c_s + \frac{ic_2}{z_-}}\right)^{\nu-}$$

Eqn 35 $$\left(\frac{c_s}{c_s^*}\right)^{\nu+} = \left(\frac{c_s^*}{c_s + \frac{ic_2}{\nu_- z_-}}\right)^{\nu-}$$

Eqn 36

$$\frac{c_s^* - c_s}{c_s^*} = 1 - \left(\frac{c_s^*}{c_s + \frac{ic_2}{\nu_- z_-}} \right)^{\frac{\nu-}{\nu+}} = 1 - \frac{c_s^*}{c_s + \frac{ic_2}{\nu_- z_-}} = \frac{ic_2}{\hat{\nu} z_- c_s^*} - \left(\frac{1}{2 z_+ z_- \hat{\nu}^2} \right) \left(\frac{ic_2}{c_s^*} \right)^2$$

- Derivation of this equation in appendix
- Now Eqn 27 becomes:

Eqn 37

$$(\Delta\mu_1)_{ion}^* - (\Delta\mu_1)_{ion} = v_{m,1} RT \left(\frac{i^2 c_2^2}{2 z_+ z_- \hat{\nu} c_s^*} \right)$$

- But definition of ionic strength I is:

Eqn 38

$$I = \frac{1}{2} \sum_i^{all\_ions} z_i^2 c_i = \frac{z_+ z_- \hat{\nu} c_s^*}{2}$$ **…for a 1:1 electrolyte**

- Where z_i is the charge on ion *i*
- Therefore:

Eqn 39

$$(\Delta\mu_1)_{ion}^* - (\Delta\mu_1)_{ion} = v_{m,1} RT \left(\frac{i^2 c_2^2}{4I} \right) = v_{m,1} RT \left(\frac{i^2 \phi_{2,s}^2}{4 I v_{sp,2}^2 M_0^2} \right)$$

- (Using relation $c_2 = \frac{\phi_{2,s}}{v_{sp,2} M_0}$ =moles ionizable groups/volume)
- Eqn 39 can be re-cast in terms of the solution pH:

Eqn 40

$$(\Delta\mu_1)_{ion}^* - (\Delta\mu_1)_{ion} = \frac{v_{m,1} RT}{4I} \left(\frac{K_a}{10^{-pH} + K_a} \right)^2 \left(\frac{\phi_{2,s}}{z_- v_{sp,2} M_0} \right)^2 = v_{m,1} RT \left(\frac{K_a}{10^{-pH} + K_a} \right)^2 \left(\frac{\phi_{2,s}^2}{4 I v_{sp,2}^2 M_0^2} \right)$$

- Returning to the equilibrium criterion:

Eqn 41

$$v_{m,1} \left(\frac{10^{-pK_a}}{10^{-pH} + 10^{-pK_a}} \right)^2 \left(\frac{\phi_{2,s}^2}{4 I v_{sp,2}^2 M_0^2} \right) = \ln(1 - \phi_{2,s}) + \phi_{2,s} + \chi \phi_{2,s}^2 + \phi_{2,r} \left(\frac{v_{m,1}}{v_{sp,2} M_c} \right) \left(1 - \frac{2 M_c}{M} \right) \left[\left(\frac{\phi_{2,s}}{\phi_{2,r}} \right)^{1/3} - \frac{1}{2} \left(\frac{\phi_{2,s}}{\phi_{2,r}} \right) \right]$$

- Brannon-Peppas paper analyzes Polyacrylates/polymethacrylates:
  - In water pH 7.0 with I = 0.35
  - χ = 0.8
  - pK_a = 6.0
  - $v_{sp,2}$ = 0.8 cm^3/g
  - M = 75,000 g/mole
  - M_c = 12,000 g/mole
  - M_0 = 90 g/mole
  - $\phi_{2,r}$ = 0.5

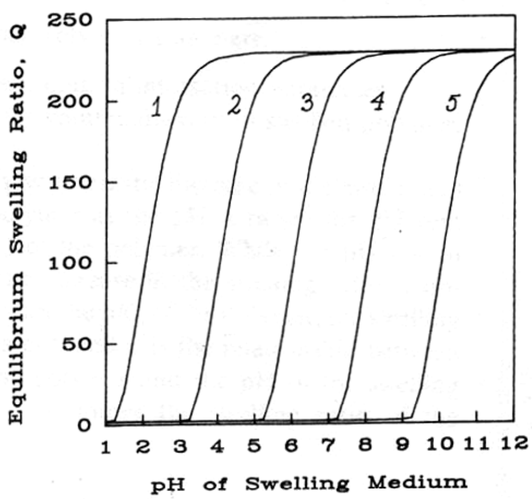


Fig. 3. Theoretical swelling predictions at comparable ionic strength conditions for an anionic network with: (1) $pK_a = 2.0$, (2) $pK_a = 4.0$, (3) $pK_a = 6.0$, (4) $pK_a = 8.0$, and (5) $pK_a = 10.0$.

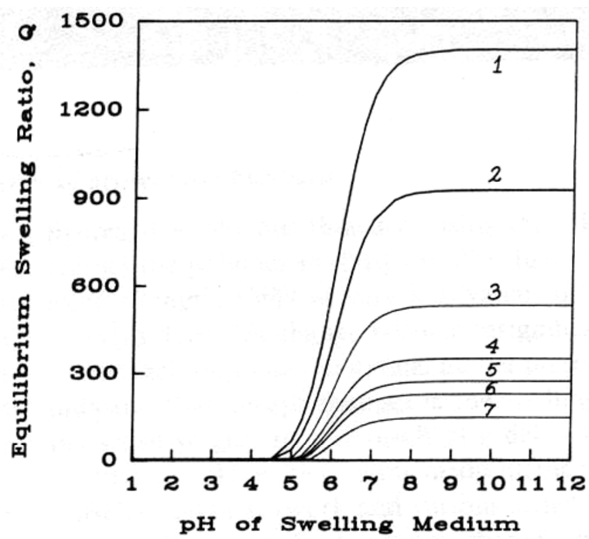


Fig. 4. Theoretical swelling predictions at comparable ionic strength conditions for an anionic network with: (1) $I = 0.05$, (2) $I = 0.1$, (3) $I = 0.25$, (4) $I = 0.5$, (5) $I = 0.75$, (6) $I = 1.0$, and (7) $I = 2.0$.

References

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