B and D Analogues of Stable Schubert Polynomials and Related Insertion Algorithms

by

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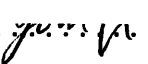
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Abstract

Our main objective is to study the analogues of the stable Schubert polynomials G_w, H_w for the classical Weyl groups B_n and D_n respectively.

Using the nilCoxeter algebra for B_n, Fomin and Kirillov have shown that G_w is a symmetric function and can be written as an integer combination of Schur P-functions. Using the Kraśkiewicz insertion, we can show that they are actually non-negative integer combinations of the Schur P-functions.

Next, looking into other properties of the Kraśkiewicz insertion, we are able to give nice descriptions for some G_w. Relations between the Kraśkiewicz insertion and Haiman's promotion sequence and shifted mixed insertion are found.

A variation of the Kraśkiewicz insertion is found which gives similar results for the D_n analogues of the stable Schubert polynomials.

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Title: Professor

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Contents

Chapter 1

Introduction

In [3], Edelman and Greene gave an insertion algorithm that presents a bijection between the reduced words of the symmetric group, S_n and the set $\{(P,Q) : P$ is a tableau word and Q is a Young tableau$\}$. An analogue for the hyperoctahedral group B_n was developed by Kraśkiewicz in [9]. In this thesis, we will explore some of its properties and applications. One of the applications of this is in the enumeration of the number of reduced word of a signed permutation. This question first appeared in a paper by Stanley[14]. For each permutation of S_n, he constructed a symmetric function that provided an answer after expressing it in terms of Schur functions. The Edelman-Greene insertion can be used to show that the coefficients are nonnegative. It turns out that this function is also a stable Schubert polynomial.

In [9], the Kraśkiewicz insertion is used to give a formula for the number of reduced words of signed permutations. We intend to use the Kraśkiewicz insertion to show that the B_n analogue of the stable Schubert polynomials can be expressed as a nonnegative sum of Schur P-functions. This will be the main goal of Chapter 2. We will also try to develop analogues for D_n, the subgroup of signed permutations with even number of signs in Chapters 3 and 4. In particular, we will answer the question on the number of reduced words of an element in D_n and also show that the D_n stable Schubert polynomials can be expressed as nonnegative sums of Schur P-functions.

In this chapter, we will introduce the terminology and definitions that we will need. Section 1.1 will be a short description of B_n and reduced words. Section

1.2 will cover some basic terminology in the theory of symmetric functions. The Kraśkiewicz insertion is introduced in Section 1.3. We will expand on the paper by Kraśkiewicz. This will spill over into Section 1.4. The notation and terminology there has been adapted to our purposes.

1.1 The Hyperoctahedral Group

We will spend some time on giving definitions and notations here. First, some basic facts about B_n. The main reference used is [8]. The hyperoctahedral group is the finite Coxeter group corresponding to the root system, B_n. Abusing notation, we will denote the group by B_n. Consider the n-dimensional vector space over the real numbers with standard basis $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_n\}$. The *simple roots* are vectors denoted by α_i's satisfying certain properties. We will not state these properties but instead state explicitly the simple roots. Here, we will depart from the usual notation. We set

$$\alpha_0 = \epsilon_1$$
$$\alpha_1 = \epsilon_2 - \epsilon_1$$
$$\alpha_2 = \epsilon_3 - \epsilon_2$$
$$\cdots$$
$$\alpha_{n-1} = \epsilon_n - \epsilon_{n-1}$$

The corresponding *simple reflections* are denoted by s_i's. Each s_i is the reflection of the vector space in the hyperplane perpendicular to α_i. They generate the hyperoctahedral group, B_n. So, every element $w \in B_n$ can be expressed as a product of s_i's. Any such expression of shortest length is called a *reduced word*. We will call this unique shortest length the *length* of w and denote it by $l(w)$. The collection of reduced words of w is denoted by $R(w)$. When we write a reduced word, we will often only write the subscripts of the simple reflections. So, $s_3s_2s_1s_0s_3s_2s_3$ is written simply as 3210323. The reduced words of $R(w)$ for a fixed $w \in B_n$ are related by the

Coxeter relations for B_n. They are:

$$
\begin{aligned}
0101 &\sim 1010 \\
a\,b &\sim b\,a & b > a+1 \\
a\,a+1\,a &\sim a\,a+1\,a & a \neq 0
\end{aligned}
$$

An element w of B_n is uniquely determined by its action on $\epsilon_i, 1 \leq i \leq n$. So, we can write w as a signed permutation, $w_1 w_2 \cdots w_n$, where

$$
w_i = \begin{cases} j \text{ if } w(\epsilon_i) = \epsilon_j \\ \bar{j} \text{ if } w(\epsilon_i) = -\epsilon_j \end{cases}
$$

Then, using the one-line notation, the simple reflections are

$$
\begin{aligned}
s_0 &= \bar{1}2\cdots n \\
s_i &= 12\cdots i-1\,i+1\,i\,i+2\cdots n \text{ for } 1 \leq i < n
\end{aligned}
$$

When we multiply s_i's, we do it from the right. The reason is that it facilitates us in computing an element of B_n from a given word(not necessary reduced). Under this convention, a simple reflection s_i acts on 1-line notation by switching the numbers in positions i and $i+1$ if $i \neq 0$ and changing the sign of the first number if $i = 0$.

Example: To find the element w with the reduced word 21032, we write up the table as follows:

	1234
2	1324
1	3124
0	$\bar{3}$124
3	$\bar{3}$142
2	$\bar{3}$412

Then $w = \bar{3}412$.

1.2 Tableaux and Symmetric Functions

In this section, we will give a short description of all the necessary definitions and results that we will need later. Almost all the results are stated without proof. The proofs can be found in standard texts like [10], [13] and [16]. The reader who is familiar with the subject can skip this section and refer to it later on to clarify the notation.

Consider a *Young diagram of shape* $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$, $\lambda_i \geq \lambda_{i+1}$ for $1 \leq i < l$. We represent it as rows of boxes, the first row of length λ_1 on top, the second row of length λ_2 below it and so on, justified to the left. For example, if $\lambda = (5,3,3)$, the Young diagram is:
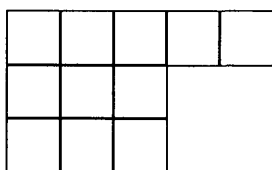
The number of boxes used is denoted by $|\lambda| = \sum_{i=1}^{l} \lambda_i$ and the number of rows is denoted by $l(\lambda)$. Each box is identified by its coordinates (i,j) where i is the row index and j is the column index.

Similarly, we can define a *shifted Young diagram* to be an arrangement of boxes of shape $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$, $\lambda_i > \lambda_{i+1}$ for $1 \leq i < l$. All the rows must have different lengths. When we represent it as rows of boxes, each subsequent row is indented 1 box to the right. For example, if $\lambda = (5,3,2)$, the Young diagram is:
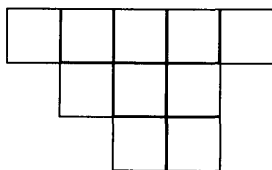
Each box is identified by its coordinates in the same manner as the unshifted Young diagram. So, the shifted Young diagram always lies in the octant $\{(i,j) : 1 \leq i \leq j\}$. To distinguish the 2 types of Young diagrams, we would sometimes called the first

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kind *unshifted* Young diagrams. A *tableau T* is a Young diagram with boxes filled in with numbers. It is called a *shifted tableau* if we use a shifted Young diagram to begin with. We collect a few definitions and notations.

Definition 1.1 *Let T be a Young diagram, shifted or unshifted.*

1. *The set of numbers used, counting multiplicity, is called the* content *of the tableau and we denote it by* $\mathrm{con}(T)$.

2. *The shape of T is denoted by* $\mathrm{sh}(T)$.

3. *The number of rows in T is denoted by* $l(T)$.

4. *The ith row of T is denoted by* T_i.

5. *The* reading word *of T is* $\pi_T = T_l T_{l-1} \cdots T_2 T_1$ *where each row is treated as a sequence of numbers.*

Example:

5	4	1	2
	4	2	3

is a shifted tableau of shape $(4, 3)$, size 7, content $(5, 4, 4, 3, 2, 2, 1)$ and reading word 4235412.

Definition 1.2

1. *A tableau T is called a* standard Young tableau *if*

    (a) $\mathrm{con}(T) = \{1, 2, \cdots, |\mathrm{sh}(T)|\}$

    (b) the numbers in each row is strictly increasing

    (c) the numbers in each column is strictly increasing

2. *A tableau T is called* semistandard *if*

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(a) the numbers in each row is weakly increasing

(b) the numbers in each column is strictly increasing

Note that there is no condition on $\mathrm{con}(T)$ for a semistandard tableau.

There are some standard tableaux which we will come across often. For convenience, we name them below.

Definition 1.3 *Let T be a standard Young tableau of shape λ.*

1. *The transpose of T denoted by T^{t} is the standard Young tableau that is obtained by reflecting T in the diagonal $i = j$.*

2. *T is called* row-wise *if*

$$
\begin{aligned}
T_1 &= 12\cdots\lambda_1 \\
T_2 &= \lambda_1 + 1\ \lambda_1 + 2\cdots\lambda_2 \\
T_3 &= \lambda_2 + 1\ \lambda_2 + 2\cdots\lambda_3 \\
&\cdots
\end{aligned}
$$

3. *T is called* column-wise *if T^{t} is row-wise.*

Consider the ring of formal power series with indeterminates $x_1, x_2, \cdots$ over the field of rational numbers, $\mathbf{Q}[[\boldsymbol{x}]]$. Given a semistandard tableau T, we can associate with it a monomial in $\mathbf{Q}[[\boldsymbol{x}]]$ as follows. Let $\mathrm{con}(T) = \{i_1, i_2, \cdots i_m\}$. Define

$$
\boldsymbol{x}^T = x_{i_1} x_{i_2} \cdots x_{i_m}
$$

Let us denote the subalgebra of $\mathbf{Q}[[\boldsymbol{x}]]$ consisting of symmetric functions as Λ. This is the ring that we are interested in.

Theorem 1.4 *Let*

$$
e_k(\boldsymbol{x}) = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_k}
$$

$$h_k(\boldsymbol{x}) \;=\; \sum_{i_1 \le i_2 \le \cdots \le i_k} x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_k}$$

$$p_k(\boldsymbol{x}) \;=\; x_1^k + x_2^k + x_3^k + \cdots$$

Then, $\{e_k : k = 1, 2, \cdots\}, \{h_k : k = 1, 2, \cdots\}$ and $\{p_k : k = 1, 2, \cdots\}$ are algebra bases of Λ.

Definition 1.5 *Let λ be of unshifted shape. A* Schur function *of shape λ is*

$$s_\lambda(\boldsymbol{x}) = \sum_T \boldsymbol{x}^T$$

where the sum is over all semistandard Young tableau of shape λ.

The Schur functions are symmetric functions. This is not clear from this definition. More importantly,

Theorem 1.6 *The Schur functions, s_λ where λ range over all unshifted shapes form a vector space basis of Λ.*

There is a similar definition for shifted shapes. However, we have to deal with filling the shape with 2 different types of numbers, barred and unbarred, and we assign a linear order on them as follows:

$$\bar{1} < 1 < \bar{2} < 2 < \bar{3} < \cdots$$

Definition 1.7 *A shifted tableau T filled with barred and unbarred numbers is called* P-semistandard *if*

1. *all the numbers are increasing along each row and each column*

2. *the unbarred numbers are weakly increasing in each row but strictly increasing in each column*

3. *the barred numbers are strictly increasing in each row but weakly increasing in each column*

4. the numbers in the first box of each row is unbarred

Definition 1.8 *Let T be a shifted Young tableau be filled with barred and unbarred numbers such that*

1. *all the numbers are increasing along each row and each column*

2. *the unbarred numbers are weakly increasing in each row but strictly increasing in each column*

3. *the barred numbers are strictly increasing in each row but weakly increasing in each column*

Then T is called a Q-semistandard Young tableau.

Note that, unlike the P-semistandard Young tableau, there is no restriction on the number in the first box of each row.

Let the $\text{con}(T) = \{f_1, f_2, \cdots, f_m\}$. We associate T with the monomial

$$\boldsymbol{x}^T = x_{|f_1|} x_{|f_2|} \cdots x_{|f_m|}$$

where $|f_i|$ is just the number without the bar.

Definition 1.9 *Let λ be of shifted shape.*

1. *Let*

$$P_\lambda(\boldsymbol{x}) = \sum_T \boldsymbol{x}^T$$

where the sum is over all P-semistandard Young tableau of shape λ. P_λ is called a Schur P-function.

2. *Let*

$$Q_\lambda(\boldsymbol{x}) = \sum_T \boldsymbol{x}^T$$

where the sum is over all Q-semistandard Young tableau of shape λ. Q_λ is called a Schur Q-function.

It is obvious that

$$Q_\lambda = 2^{l(\lambda)} P_\lambda$$

where $l(\lambda)$ denote the number of rows in λ. Just like the Schur functions, they are symmetric. However, they do not generate the whole algebra of symmetric functions.

Theorem 1.10 *Let $\tilde{\Lambda}$ be the subalgebra of Λ generated by $\{p_k : k$ odd $\}$. Then*

$$\tilde{\Lambda} = \{f(\boldsymbol{x}) \in \Lambda : f(-x_2, x_2, x_3, x_4, \cdots) = f(x_3, x_4, \cdots)\}$$

The P_λ's where λ range over all shifted shapes form a vector space basis for $\tilde{\Lambda}$. Similarly, the Q_λ's form a vector space basis for $\tilde{\Lambda}$.

Definition 1.11 *Take 2 Young diagrams, S and T of shape λ and μ respectively. Suppose $T \subset S$. If we remove T from S, the resulting configuration of boxes is called a* skew Young diagram *of shape λ/μ. If we filled in the skew Young diagram with numbers such that each row and column is increasing, then the resulting tableau is called a skew Young tableau.*

A skew Young diagram is said to be *connected* if we can always get from 1 box to another by moving left, right, up or down without leaving the diagram.

There are some special skew shapes that we will mentioned here.

Definition 1.12

1. *A* horizontal strip *is a skew Young diagram where each column can have at most 1 box.*

2. *A* vertical strip *is a skew Young diagram where each row can have at most 1 box.*

3. *A* rim hook *is a skew Young diagram that does not contain a 2×2 configuration of boxes. Equivalently, each top-left to bottom-right diagonal has at most 1 box.*

Let T be a semistandard Young tableau of unshifted shape λ. If we look at all the boxes in T that have the same entry, they form a horizontal strip since T is strictly increasing in its columns.

If T is a Q-semistandard Young tableau of shifted shape μ, then all the boxes in T that have the same barred entry form a vertical strip and those with the same unbarred entry form a horizontal strip. This is because the barred numbers have to be strictly increasing in each row and the unbarred numbers are strictly increasing in each column. If we instead look at all the boxes filled with the entries $l, \bar{l}$ for a fixed l, they form a rim hook. This is because there is no way of filling a configuration of 2×2 boxes with l or $\bar{l}$ such that it is Q-semistandard.

1.3 Kraśkiewicz Insertion

In this section, we will present Kraśkiewicz insertion. All the results here can be found in [9]. The presentation is different from that in [9]. Firstly, we have used s_0 as the special reflection instead of s_n. So, the numbers that are used in a reduced word for B_n will range from 0 to $n-1$. Secondly, our unimodal sequence will be a sequence of numbers that is initially strictly decreasing, then strictly increasing; that is

$$\boldsymbol{a} = a_1 > a_2 > \cdots > a_k < a_{k+1} < \cdots < a_l$$

The decreasing part of $\boldsymbol{a}$ is defined to include the minimum, that is

$$\boldsymbol{a}{\downarrow} = a_1 > a_2 > \cdots > a_k$$

The increasing part forms the remainder. We denote it by

$$\boldsymbol{a}{\uparrow} = a_{k+1} < a_{k+2} < \cdots < a_l$$

For example, 21056 is a unimodal sequence with decreasing part 210 and increasing part 56. 2489 is unimodal with decreasing part 2 and increasing part 489. 7521 is

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unimodal with decreasing part 7521 and no increasing part. We note that a unimodal sequence always has a decreasing part.

Other than these differences, this section and the next are basically an expansion of the paper [9]. The reader who is familiar with results in [9] can skip the rest of this chapter but keeping in mind the differences mentioned above.

In what follows, P_i denotes the ith row of the tableau P. Very often, we will abuse notation and use P_i to denote the sequence of numbers in that row.

Definition 1.13 *Let P be a shifted tableau with l rows such that*

1. $\pi_P = P_l P_{l-1} \cdots P_2 P_1$ is a reduced word of w

2. P_i is a unimodal subsequence of maximum length in $P_l P_{l-1} \cdots P_{i+1} P_i$

Then, P is called a standard decomposition tableau *of w and we denote the set of such tableaux by* SDT(w).

This is essentially the same as the definition of a tableau word in [9]. An example of a standard decomposition tableau is

5	4	1	2
	4	2	3

Note that if P is a standard decomposition tableau, we can create new standard decomposition tableau by removing the first row of P. Another method is by deleting the first entry of the last row.

Let $w \in B_n$ and $\boldsymbol{a} = a_1 a_2 \cdots a_m \in R(w)$. The insertion algorithm will give a map

$$a_1 a_2 \cdots a_m \overset{K}{\rightarrow} (P, Q)$$

We will drop the K from the arrow if the insertion is clear from context. P is called the *insertion tableau* and Q is called the *recording tableau*. It will be shown later that the insertion tableau is always a standard decomposition tableau.

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We will have to first construct a sequence of pairs of tableaux

$$(\emptyset, \emptyset) = (P^{(0)}, Q^{(0)}), (P^{(1)}, Q^{(1)}), \cdots, (P^{(m)}, Q^{(m)}) = (P, Q)$$

$\text{sh}(P^{(i)}) = \text{sh}(Q^{(i)})$ for $i = 0, 1, \cdots, m$. Each tableau $P^{(i)}$ is obtained by inserting a_i into $P^{(i-1)}$. We denote this as

$$P^{(i-1)} \leftarrow a_i = P^{(i)}$$

Insertion Algorithm:

Input: a_i and $(P^{(i-1)}, Q^{(i-1)})$. Output: $(P^{(i)}, Q^{(i)})$.

Step 1: Let $a = a_i$ and $R = $ 1st row of $P^{(i-1)}$.

Step 2: Insert a into R as follows:

- Case 0: $R = \emptyset$. If the empty row is the kth row, we write a indented $k - 1$ boxes away from the left margin. This new tableau is $P^{(i)}$. To get $Q^{(i)}$, we add i to $Q^{(i-1)}$ so that $P^{(i)}$ and $Q^{(i)}$ have the same shape. Stop.

- Case 1: Ra is unimodal. Append a to R and let $P^{(i)}$ be this new tableau. To get $Q^{(i)}$, we add i to $Q^{(i-1)}$ so that $P^{(i)}$ and $Q^{(i)}$ have the same shape. Stop.

- Case 2: Ra is not unimodal. Some numbers in the increasing part of R is greater than a. Let b be the smallest number in $R\uparrow$ bigger than or equal to a.

  - Case 2.0: $a = 0$ and R contains 101 as a subsequence. We leave R unchanged and go to Step 2 with $a = 0$ and R equal to the next row.

  - Case 2.1.1: $b \neq a$. We put a in b's position and let $c = b$.

  - Case 2.1.2: $b = a$. We leave the increasing part, $R\uparrow$ unchanged and let $c = a + 1$.

  We insert c into the decreasing part, $R\downarrow$. Let d be the biggest number in $R\downarrow$ which is smaller than or equal to c. This number always exists because the minimum of a unimodal sequence is in its decreasing part.

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– Case 2.1.3: $d \neq c$. We put c in d's place and let $a' = d$.

– Case 2.1.4: $d = c$. We leave $R{\downarrow}$ unchanged and let $a' = c - 1$.

Step 3: Repeat Step 2 with $a = a'$ and R equal to the next row. $\qquad\qquad\square$

For convenience, we will use the notation,

$$R \overset{in}{\leftarrow} a = a' \overset{out}{\leftarrow} R'$$

to mean that inserting a into row R turns R into R' and gives a' as the number to be inserted into the next row. Right now, it is not clear to us that P is a standard decomposition tableau and Q is a shifted standard Young tableau. But, let us try an example.

Example: Let $a = 3121034310 \in R(\bar{2}\bar{4}531)$.

$$P^{(1)} = \boxed{3} \qquad\qquad Q^{(1)} = \boxed{1}$$

$$P^{(2)} = \boxed{3\ 1} \qquad\qquad Q^{(2)} = \boxed{1\ 2}$$

$$P^{(3)} = \boxed{3\ 1\ 2} \qquad\qquad Q^{(3)} = \boxed{1\ 2\ 3}$$

In the above, we had only to use the Case 0 and Case 1 of the insertion algorithm. In the next step, we will have to use the other cases. We use the symbol "|" to separate the unimodal sequence into decreasing and increasing parts when necessary.

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$$P^{(3)} \leftarrow 1 \;=\; \boxed{3}\;\boxed{1}\;\Big|\;\boxed{2} \quad \leftarrow 1$$

$$\overset{\overset{2}{\leftarrow}}{=} \quad \boxed{3}\;\boxed{1}\;\Big|\;\boxed{1}$$

$$=\; \begin{array}{ccc} \boxed{3} & \boxed{2} & \boxed{1} \end{array} \quad \underset{\leftarrow 1}{}$$

$$=\; \begin{array}{ccc} \boxed{3} & \boxed{2} & \boxed{1} \\ \boxed{1} & & \end{array} \quad =\; P^{(4)}$$

and

$$Q^{(4)} \;=\; \begin{array}{ccc} \boxed{1} & \boxed{2} & \boxed{3} \\ & \boxed{4} & \end{array}$$

$$P^{(5)} \;=\; \begin{array}{cccc} \boxed{3} & \boxed{2} & \boxed{1} & \boxed{0} \\ & \boxed{1} & & \end{array} \qquad Q^{(5)} \;=\; \begin{array}{cccc} \boxed{1} & \boxed{2} & \boxed{3} & \boxed{5} \\ & \boxed{4} & & \end{array}$$

$$P^{(6)} \;=\; \begin{array}{ccccc} \boxed{3} & \boxed{2} & \boxed{1} & \boxed{0} & \boxed{3} \\ & \boxed{1} & & & \end{array} \qquad Q^{(6)} \;=\; \begin{array}{ccccc} \boxed{1} & \boxed{2} & \boxed{3} & \boxed{5} & \boxed{6} \\ & \boxed{4} & & & \end{array}$$

$$P^{(7)} \;=\; \begin{array}{cccccc} \boxed{3} & \boxed{2} & \boxed{1} & \boxed{0} & \boxed{3} & \boxed{4} \\ & \boxed{1} & & & & \end{array}$$

$$Q^{(7)} \;=\; \begin{array}{cccccc} \boxed{1} & \boxed{2} & \boxed{3} & \boxed{5} & \boxed{6} & \boxed{7} \\ & \boxed{4} & & & & \end{array}$$

$$P^{(7)} \leftarrow 3 \;=\; \begin{array}{|c|c|c|c|c|c|} \hline 3 & 2 & 1 & 0 & 3 & 4 \\ \hline \end{array} \;\leftarrow 3$$

$$\begin{array}{c} 1 \end{array}$$

(with the $\leftarrow 3$ arrow at the right, and the box 1 below the second cell)

$$= \quad \overset{\overset{4}{\longleftarrow}}{\begin{array}{|c|c|c|c|c|c|} \hline 3 & 2 & 1 & 0 & 3 & 4 \\ \hline \end{array}}$$

$$\begin{array}{c} 1 \end{array}$$

$$= \quad \begin{array}{|c|c|c|c|c|c|} \hline 4 & 2 & 1 & 0 & 3 & 4 \\ \hline \end{array}$$

$$\begin{array}{c} 1 \end{array} \quad \leftarrow 3$$

$$= \quad \begin{array}{|c|c|c|c|c|c|} \hline 4 & 2 & 1 & 0 & 3 & 4 \\ \hline \end{array}$$

$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline \end{array} \qquad = \; P^{(8)}$$

and

$$Q^{(8)} \quad = \quad \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 6 & 7 \\ \hline \end{array}$$

$$\begin{array}{|c|c|} \hline 4 & 8 \\ \hline \end{array}$$

$$P^{(8)} \leftarrow 1 \quad = \quad \boxed{4}\,\boxed{2}\,\boxed{1}\,\boxed{0}\,\big|\,\boxed{3}\,\boxed{4} \qquad \leftarrow 1$$
$$\boxed{1}\,\boxed{3}$$

$$= \qquad \overset{\overset{3}{\leftarrow}}{\boxed{4}\,\boxed{2}\,\boxed{1}\,\boxed{0}\,\big|\,\boxed{1}\,\boxed{4}}$$
$$\boxed{1}\,\boxed{3}$$

$$= \qquad \boxed{4}\,\boxed{3}\,\boxed{1}\,\boxed{0}\,\boxed{1}\,\boxed{4}$$
$$\boxed{1}\,\boxed{3} \qquad \leftarrow 2$$

$$= \qquad \boxed{4}\,\boxed{3}\,\boxed{1}\,\boxed{0}\,\boxed{1}\,\boxed{4}$$
$$\boxed{1}\,\boxed{2}$$
$$\overset{3}{\leftarrow}$$

$$= \qquad \boxed{4}\,\boxed{3}\,\boxed{1}\,\boxed{0}\,\boxed{1}\,\boxed{4} \qquad = \; P^{(9)}$$
$$\boxed{3}\,\boxed{2}$$
$$\boxed{1}$$

and

$$Q^{(9)} \qquad = \qquad \boxed{1}\,\boxed{2}\,\boxed{3}\,\boxed{5}\,\boxed{6}\,\boxed{7}$$
$$\boxed{4}\,\boxed{8}$$
$$\boxed{9}$$

$$P^{(9)} \leftarrow 0 \quad = \quad \begin{array}{|c|c|c|c|c|c|} \hline 4 & 3 & 1 & 0 & 1 & 4 \\ \hline \end{array} \leftarrow 0$$

$$\begin{array}{cc} & \begin{array}{|c|c|} \hline 3 & 2 \\ \hline \end{array} \\ & \begin{array}{|c|} \hline 1 \\ \hline \end{array} \end{array}$$

$$= \quad \begin{array}{|c|c|c|c|c|c|} \hline 4 & 3 & 1 & 0 & 1 & 4 \\ \hline \end{array}$$

$$\begin{array}{|c|c|} \hline 3 & 2 \\ \hline \end{array} \leftarrow 0$$

$$\begin{array}{|c|} \hline 1 \\ \hline \end{array}$$

$$= \quad \begin{array}{|c|c|c|c|c|c|} \hline 4 & 3 & 1 & 0 & 1 & 4 \\ \hline \end{array} \quad = \quad P^{(10)}$$

$$\begin{array}{|c|c|c|} \hline 3 & 2 & 0 \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline 1 \\ \hline \end{array}$$

and

$$Q^{(10)} \quad = \quad \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 6 & 7 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|} \hline 4 & 8 & 10 \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline 9 \\ \hline \end{array}$$

1.4   B-Coxeter-Knuth Relations

In the Edelman-Greene insertion, there is a Coxeter-Knuth relation that relates any reduced word of S_n with the reading word of the insertion tableau, P. In the Kraśkiewicz insertion, there is an analogue of this relation on the reduced words of $R(w)$ which we call the B-Coxeter-Knuth relations. They play an important role in the insertions. These relations are also given in [9]. We have translated them into our notation. The reverse of a word, $\boldsymbol{a} = a_1 a_2 \cdots a_m$ is defined to be

$$\boldsymbol{a}^{\mathrm{r}} = a_m a_{m-1} \cdots a_2 a_1$$

In what follows, $a < b < c < d$.

Definition 1.14 (Elementary B-Coxeter-Knuth Relations)

$$0101 \sim 1010 \tag{1}$$

$$a\ b\ b+1\ b \sim a\ b+1\ b\ b+1 \tag{2}$$

$$b\ a\ b+1\ b \sim b\ b+1\ a\ b \tag{3}$$

$$a\ a+1\ b\ a \sim a\ a+1\ a\ b \qquad a+1 < b \tag{4}$$

$$a+1\ a\ b\ a+1 \sim a+1\ b\ a\ a+1 \qquad a+1 < b \tag{5}$$

$$abdc \sim adbc \tag{6}$$

$$acdb \sim acbd \tag{7}$$

$$adcb \sim dacb \tag{8}$$

$$badc \sim bdac \tag{9}$$

and the reverse of these.

We mean to say that if $a \sim b$ is in the list above, then $a^r \sim b^r$ is also an elementary B-Coxeter-Knuth relation.

Definition 1.15 *Let $a, b \in R(w)$ for some $w \in B_n$. If $a = cxd$ and $b = cyd$ where both x and y have length 4 and (x, y) appeared in the list above, we say a is elementary B-Coxeter-Knuth related to b.*

Let $e, f \in R(w)$. If there exists a sequence of reduced words $e = a_1, a_2, \cdots, a_k = f \in R(w)$ such that each pair (a_i, a_{i+1}) is elementary B-Coxeter-Knuth related, we say e and f are B-Coxeter-Knuth related. We denote this as $e \sim f$.

These relations are refinements of the Coxeter relations for B_n. That is to say, if $a \sim b$, then they are also related by the Coxeter relations for B_n. So, if a is reduced, then so is b. This set of relations also appeared in [6, Table 3]. There, they were obtained by considering promotion sequences.

Lemma 1.16 ([9, Lemma 4.7]) *Let R be a unimodal reduced word of B_n and a, $0 \le a < n$ such that Ra is also reduced. If*

$$R \overset{in}{\leftarrow} a = a' \overset{out}{\leftarrow} R'$$

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then

$$Ra \sim a'R'.$$

Proof: We will imitate all the different cases of insertion by the B-Coxeter-Knuth relations. We have indicated the elementary relation used in each step where possible. This is tedious and the reader may skip it.

Case 1: Ra is unimodal. $a' = \emptyset$ and $a'R' = R' = Ra$.

Case 2: Ra is not unimodal. Let

$$R = r_1 > r_2 > \cdots > r_k < r_{k+1} < \cdots < r_m$$

Let r_b be the smallest number in $R\uparrow$ bigger than or equal to a. We will imitate all the insertion with B-Coxeter-Knuth relations. In the following, we have underlined the number that is moved. Suppose $a < r_{m-2} < r_{m-1} < r_m$. Then we use ($8^r$) to get

$$
\begin{aligned}
Ra &= r_1 r_2 \cdots r_{m-3} r_{m-2} r_{m-1} r_m \underline{a} \\
&\sim r_1 \cdots r_{m-3} r_{m-2} r_{m-1} \underline{a} r_m
\end{aligned}
$$

We keep applying (8^r) until we reach the index b where r_b is the smallest number bigger or equal to a.

$$
\begin{aligned}
Ra &\sim r_1 \cdots r_b \cdots r_{m-3} r_{m-2} \underline{a} r_{m-1} r_m \\
&\quad \cdots \\
&\sim r_1 \cdots r_b r_{b+1} r_{b+2} \underline{a} r_{b+3} \cdots r_m \\
&\sim r_1 \cdots r_b r_{b+1} \underline{a} r_{b+2} r_{b+3} \cdots r_m
\end{aligned}
$$

Next, we will have to consider the different cases.

Case 2.0: $a = 0$ and R contains 101 as a subsequence. So, our $r_{k-1} = 1$, $r_k = 0$ and $r_b = 1$.

$$
\begin{aligned}
Ra &\sim r_1 \cdots r_{k-2} 101 r_{k+2} \underline{0} r_{k+3} \cdots r_m \\
&\sim r_1 \cdots r_{k-2} 101 \underline{0} r_{k+2} r_{k+3} \cdots r_m \quad \text{by (4)} \\
&\sim r_1 \cdots r_{k-2} \underline{0} 101 r_{k+2} r_{k+3} \cdots r_m \quad \text{by (1)} \\
&\sim r_1 \cdots \underline{0} r_{k-2} 101 r_{k+2} r_{k+3} \cdots r_m \quad \text{by (4^r)}
\end{aligned}
$$

Case 2.1.1: $r_b \neq a$

$$
\begin{aligned}
Ra \quad \sim \quad & r_1 \cdots r_{b-2} r_{b-1} r_b r_{b+1} \underline{a} r_{b+2} \cdots r_m \\
\sim \quad & r_1 \cdots r_{b-2} r_{b-1} r_b \underline{a} r_{b+1} \cdots r_m && \text{by (7)} \\
\sim \quad & r_1 \cdots r_{b-2} \underline{r_b} r_{b-1} a r_{b+1} \cdots r_m && \text{by (6)} \\
& \cdots \\
\sim \quad & r_1 \cdots r_k r_{k+1} \underline{r_b} r_{k+2} \cdots r_{b-1} a r_{b+1} \cdots r_m && \text{by (6)}
\end{aligned}
$$

Case 2.1.2: $r_b = a$. Since the word is still reduced, r_{b+1} must be $a + 1$.

$$
\begin{aligned}
Ra \quad \sim \quad & r_1 \cdots r_{b-2} r_{b-1} a \; a + 1 \; \underline{a} r_{b+2} \cdots r_m \\
\sim \quad & r_1 \cdots r_{b-2} r_{b-1} \underline{a + 1} \; a \; a + 1 r_{b+2} \cdots r_m && \text{by (2)} \\
& \cdots \\
\sim \quad & r_1 \cdots r_k r_{k+1} \underline{a + 1} r_{k+2} \cdots a \; a + 1 r_{b+1} \cdots r_m && \text{by (6)}
\end{aligned}
$$

Next we have to move the number towards the left through the minimum. the number of cases is many but manageable. It can be verified indeed

$$
Ra \sim a' R'
$$

We omit the details. $\qquad \square$

Theorem 1.17 ([9, Lemma 4.7]) *Let* $a = a_1 a_2 \cdots a_m \in R(w)$ *,* $w \in B_n$ *and*

$$
a \xrightarrow{K} (P, Q)
$$

Then, $a \sim \pi_P$.

Proof: Let $P^{(i)}, 1 \leq i \leq m$ be the insertion tableau that is constructed using $a_1 a_2 \cdots a_i$. We claim that $\pi_{P^{(i-1)}} a_i \sim \pi_{P^{(i)}}$. Let R_j be the jth row of $P^{(i-1)}$ and

$d_0 = a_i$. Then the insertion gives

$$R_1 \stackrel{in}{\leftarrow} d_0 = d_1 \stackrel{out}{\leftarrow} R_1'$$

$$R_2 \stackrel{in}{\leftarrow} d_1 = d_2 \stackrel{out}{\leftarrow} R_2'$$

$$\cdots$$

$$R_l \stackrel{in}{\leftarrow} d_{l-1} = d_l \stackrel{out}{\leftarrow} R_l'$$

By Lemma 1.17, we have

$$\pi_{P(i-1)} a_i = R_l \cdots R_2 R_1 d_0$$

$$\sim R_l \cdots R_2 d_1 R_1'$$

$$\cdots$$

$$\sim d_l R_l' \cdots R_2' R_1'$$

$$= \pi_{P(i)}$$

Therefore, we have

$$a_1 a_2 \cdots a_m \sim \pi_{P(1)} a_2 a_3 \cdots a_m$$

$$\sim \pi_{P(2)} a_3 \cdots a_m$$

$$\cdots$$

$$\sim \pi_{P(m-1)} a_m$$

$$\sim \pi_{P(m)}$$

$$= \pi_P$$

$\square$

The next two results are not mentioned in [9].

Theorem 1.18 *Let $a, b \in R(w)$. $a \sim b$ iff they have the same insertion tableau.*

Proof: ($\Leftarrow$) Let a and b have the same insertion tableau, P. From Theorem 1.17, $a \sim \pi_P \sim b$.

($\Rightarrow$)The proof for this direction is long and tedious case by case analysis. We leave it till Appendix B. $\square$

Corollary 1.19 *Let P be a standard decomposition tableau. Then*

$$\pi_P \overset{K}{\rightarrow} (P, Q)$$

Proof: This follows from the previous theorem. □

In Edelman-Greene insertion, if $a \in S_n$ is mapped into (P, Q), the length of the longest increasing subsequence in a is equal to the the length of the 1st row of P. There is an analogous result in the Kraśkiewicz insertion. A unimodal subsequence of $a = a_1 a_2 \cdots a_m$ is a subsequence of the form,

$$a_{i_1} > a_{i_2} > \cdots > a_{i_k} < a_{i_{k+1}} < \cdots < a_{i_l}$$

where $i_1 < i_2 < \cdots < i_l$.

Lemma 1.20 ([9, Lemma 4.8]) *Let $a, b \in R(w)$ and $a \sim b$. If c is a unimodal subsequence of length k in a, then there is a unimodal subsequence d of length k in b.*

Proof: It suffices to show this when a is related to b by an elementary relation. Clearly, if the elementary relation does not affect the order of the elements in c, then we can take $d = c$ as a subsequence of b of length k . We list below the cases where the order of some of the elements in c are changed. It is very long and the reader can skip it. In the left column, we underline the elements of the elementary relation that are in c. d is obtained by taking all the elements of c that are not affected by the elementary relation and combining it with the numbers that are underlined on the right. The guidelines for choosing the elements on the righthand side is

1. if the subsequence on the lefthand side is decreasing, we try to change the first element to one which is smaller

2. if it is increasing, we replace the last element by a smaller one.

(2)	$a\ \underline{b+1}\ b\ \underline{b+1}$	$\underline{a}\ b\ \underline{b+1}\ b$
(3)	$b\ \underline{a}\ \underline{b+1}\ b$	$b\ \underline{b+1}\ \underline{a}\ b$
	$\underline{b}\ \underline{a}\ b+1\ b$	$\underline{b}\ b+1\ \underline{a}\ b$
	$b\ \underline{b+1}\ \underline{a}\ b$	$\underline{b}\ \underline{a}\ b+1\ b$
	$b\ \underline{b+1}\ \underline{a}\ b$	$\underline{b}\ \underline{a}\ b+1\ \underline{b}$
(4)	$a\ a+1\ \underline{b}\ \underline{a}$	$a\ \underline{a+1}\ \underline{a}\ b$
	$a\ \underline{a+1}\ \underline{a}\ b$	$a\ \underline{a+1}\ \underline{b}\ a$
(5)	$a+1\ \underline{a}\ \underline{b}\ a+1$	$a+1\ b\ \underline{a}\ \underline{a+1}$
	$\underline{a+1}\ a\ b\ a+1$	$\underline{a+1}\ b\ \underline{a}\ a+1$
	$a+1\ \underline{b}\ \underline{a}\ a+1$	$\underline{a+1}\ \underline{a}\ b\ a+1$
	$a+1\ \underline{b}\ \underline{a}\ a+1$	$\underline{a+1}\ \underline{a}\ b\ \underline{a+1}$
(6)	$a\ \underline{b}\ \underline{d}\ c$	$a\ d\ \underline{b}\ \underline{c}$
	$\underline{a}\ b\ \underline{d}\ c$	$\underline{a}\ d\ \underline{b}\ c$
	$a\ \underline{d}\ \underline{b}\ c$	$a\ b\ \underline{d}\ \underline{c}$
	$a\ \underline{d}\ \underline{b}\ \underline{c}$	$\underline{a}\ \underline{b}\ d\ \underline{c}$
(7)	$a\ c\ \underline{d}\ \underline{b}$	$a\ \underline{c}\ \underline{b}\ d$
	$a\ c\ \underline{b}\ \underline{d}$	$a\ \underline{c}\ \underline{d}\ b$
	$a\ \underline{c}\ \underline{b}\ d$	$\underline{a}\ c\ \underline{d}\ b$
	$\underline{a}\ c\ \underline{b}\ d$	$\underline{a}\ c\ \underline{d}\ b$
(8)	$\underline{a}\ \underline{d}\ c\ b$	$d\ \underline{a}\ \underline{c}\ b$
	$\underline{d}\ \underline{a}\ c\ b$	$a\ \underline{d}\ \underline{c}\ b$
	$\underline{d}\ \underline{a}\ c\ b$	$a\ \underline{d}\ \underline{c}\ b$
	$\underline{d}\ \underline{a}\ c\ \underline{b}$	$a\ \underline{d}\ \underline{c}\ b$
(9)	$b\ \underline{a}\ \underline{d}\ c$	$b\ d\ \underline{a}\ \underline{c}$
	$\underline{b}\ \underline{a}\ d\ c$	$\underline{b}\ d\ \underline{a}\ c$
	$b\ \underline{d}\ \underline{a}\ c$	$\underline{b}\ \underline{a}\ d\ c$
	$b\ \underline{d}\ \underline{a}\ c$	$\underline{b}\ \underline{a}\ d\ \underline{c}$

$\square$

Lemma 1.21 ([9, Section 4]) *Let $P \in \mathrm{SDT}(w)$ and a such that $\pi_P a$ is a reduced*

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word of length greater than $l(w)$. Let $v = ws_a$. If

$$P \leftarrow a = P'$$

then $P' \in \mathrm{SDT}(v)$.

Proof: We will prove this by induction on the number of rows of P. Note that the induction step is only used in the second case. By Theorem 1.17, $\pi_{P'} \in R(v)$. Let P_i and P'_i be the rows of P and P' respectively.

Case: $P_1 \overset{in}{\leftarrow} a = P_1 a$.

This gives

$$
\begin{aligned}
P'_1 &= P_1 a \\
P'_i &= P_i \quad \text{for } i > 1
\end{aligned}
$$

Then, $P_1 a$ is a unimodal sequence of longest length in $\pi_{P'}$ and each subsequent row, P'_i is a unimodal subsequence of longest length in $P'_l \cdots P'_i$. Therefore, $P' \in \mathrm{SDT}(v)$.

Case: $P_1 \overset{in}{\leftarrow} a = a' \overset{out}{\leftarrow} P'_1$.

Let R, R' be the tableaux obtained from deleting the 1st row of P and P' respectively. Clearly, R is a standard decomposition tableau and

$$R \leftarrow a' = R'$$

Since the number of rows in R is 1 less than that of P, we can apply the induction hypothesis on R. So R' is a standard decomposition tableau. Now, if we can show that P'_1 is a unimodal sequence of greatest length in $\pi_{P'}$, then P' is a standard decomposition tableau. By Lemma 1.20, the length of the longest unimodal subsequence in $\pi_{P'}$ is the same as the length of the longest unimodal subsequence in $\pi_P a$. It suffices to show that any unimodal subsequence $c = c_1 c_2 \cdots c_k$ of $\pi_P a$ has length less than $|P_1|$. If $c_k \neq a$, then c is in π_P and $|c| \leq |P_1|$. If $c_k = a$, note that the last number d in P_1 is greater than a. Let d be the last element in P_1. Then, $c_1 c_2 \cdots c_{k-1} d$ is a unimodal subsequence in π_P. Hence, $|c| \leq |P_1|$. $\qquad \square$

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Theorem 1.22 ([9, Section 4]) *Let $a \in R(w)$. If*

$$a \overset{K}{\to} (P, Q)$$

then $P \in \mathrm{SDT}(w)$ and Q is a standard shifted Young tableau.

Proof: The insertion algorithm gives

$$(\emptyset, \emptyset) = (P^{(0)}, Q^{(0)}), (P^{(1)}, Q^{(1)}), \cdots, (P^{(m)}, Q^{(m)}) = (P, Q)$$

A tableau of with only 1 box is clearly a standard decomposition tableau, that is $P^{(1)} \in \mathrm{SDT}(s_{a_1})$. By Lemma 1.21,

$$P^{(i)} \in \mathrm{SDT}(s_{a_1} s_{a_2} \cdots s_{a_i})$$

Also, we have $\mathrm{sh}(P^{(i)}) \supset \mathrm{sh}(P^{(i-1)})$ and $\mathrm{sh}(P^{(i)})$ is a shifted partition. So, Q is a standard shifted Young tableau. $\qquad\square$

This result is implicit in the definition of the insertion tableau.

Corollary 1.23 *Let $a \to (P, Q)$ and λ_1 be the length of the 1st row of P. Then the length of the longest unimodal subsequence in a is λ_1.*

Proof: This follows easily from Lemma 1.20 and that P is a standard decomposition tableau. $\qquad\square$

This property of λ_1 has analogues in the Edelman-Greene insertion and the Robinson-Schensted insertion(see [3] and [13]). As for λ_2, the length of the second row of the insertion tableau, one might think that $\lambda_1 + \lambda_2$ is the maximum of the sum of lengths of two disjoint unimodal sequence in the reduced word a. However, this is not the case.

Example: The reduced word 650871032343 of the signed permutation $\bar{2}473\bar{1}5968$ gets inserted into

8	7	3	0	2	3	4
	6	5	0	1	3	

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However, the reduced word cannot be divided into two disjoint unimodal subsequences.

Lemma 1.24 ([9, Equation 3.2]) *Let R be a unimodal reduced word and a be such that Ra is reduced but not unimodal. If*

$$R \overset{in}{\underset{\longleftarrow}{}} a = a' \overset{out}{\underset{\longleftarrow}{}} R'$$

then

$$R'^{\mathrm{r}} \overset{in}{\underset{\longleftarrow}{}} a' = a \overset{out}{\underset{\longleftarrow}{}} R^{\mathrm{r}}$$

Proof: We examine the different cases that could possibly arise. Again, let b be the smallest number in $R\!\uparrow$ which is bigger than or equal to a, c be the number that is to be inserted into $R\!\downarrow$ and d be the biggest number in $R\!\downarrow$ which is smaller than or equal to c. Let $R\!\downarrow$ end in e and $R\!\uparrow$ begin with f. In notation,

$$R \overset{in}{\underset{\longleftarrow}{}} a = \underbrace{\cdots d \cdots e}_{R\downarrow} \Big| \underbrace{\overset{c}{\underset{\longleftarrow}{}} f \cdots b \cdots}_{R\uparrow} \overset{in}{\underset{\longleftarrow}{}} a$$

Case: $a = 0$ and R contain the subsequence 101.

$$\underbrace{\cdots 1\ 0\ 1 \cdots}_{R} \overset{in}{\underset{\longleftarrow}{}} 0 = 0 \overset{out}{\underset{\longleftarrow}{}} \underbrace{\cdots 1\ 0\ 1 \cdots}_{R' = R}$$

$$\Rightarrow \underbrace{\cdots 1\ 0\ 1 \cdots}_{R^{\mathrm{r}}} \overset{in}{\underset{\longleftarrow}{}} 0 = 0 \overset{out}{\underset{\longleftarrow}{}} \underbrace{\cdots 1\ 0\ 1 \cdots}_{R^{\mathrm{r}}}$$

Case: $a = b, c = d$. The presence of $a + 1\ a$ in $R\!\downarrow$ and $a\ a + 1$ in $R\!\uparrow$ are forced by the word being reduced. Also, $e < a$.

$$\underbrace{\cdots a + 1\ a \cdots e}_{R\downarrow} \Big| \underbrace{\overset{a+1}{\underset{\longleftarrow}{}} \cdots a\ a + 1 \cdots}_{R\uparrow} \overset{in}{\underset{\longleftarrow}{}} a = a \overset{out}{\underset{\longleftarrow}{}} \underbrace{\cdots a + 1\ a \cdots e \cdots a\ a + 1 \cdots}_{R' = R}$$

Clearly,

$$R \overset{in}{\underset{\longleftarrow}{}} a = a \overset{out}{\underset{\longleftarrow}{}} R$$

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Case: $a = b, c \neq d$. There are 2 subcases to consider.

Subcase: $d = e$.

$$\underbrace{\cdots d}_{R\downarrow} \mid \underbrace{f \cdots a\ a+1 \cdots}_{R\uparrow} \overset{in}{\underset{a+1}{\overset{\longleftarrow}{\Leftarrow}}} a = d \overset{out}{\Leftarrow} \underbrace{\cdots a+1\ f \cdots a\ a+1 \cdots}_{R'}$$

$$\Rightarrow \underbrace{\cdots a+1\ a \cdots f}_{R'^r \downarrow} \mid \underbrace{a+1 \cdots}_{R'^r \uparrow} \overset{in}{\underset{a+1}{\overset{\longleftarrow}{\Leftarrow}}} d = a \overset{out}{\Leftarrow} \underbrace{\cdots a+1\ a \cdots f\ d \cdots}_{R}$$

Subcase: $d \neq e$.

$$\underbrace{\cdots d \cdots e}_{R\downarrow} \mid \underbrace{f \cdots a\ a+1 \cdots}_{R\uparrow} \overset{in}{\underset{a+1}{\overset{\longleftarrow}{\Leftarrow}}} a = d \overset{out}{\Leftarrow} \underbrace{\cdots a+1 \cdots e\ f \cdots a\ a+1 \cdots}_{R'}$$

$$\Rightarrow \underbrace{\cdots a+1\ a \cdots f\ e}_{R'^r \downarrow} \mid \underbrace{\cdots a+1 \cdots}_{R'^r \uparrow} \overset{in}{\underset{a+1}{\overset{\longleftarrow}{\Leftarrow}}} d = a \overset{out}{\Leftarrow} \underbrace{\cdots a+1\ a \cdots f\ e \cdots d \cdots}_{R^r}$$

Case: $a \neq b, c = d$. Since Ra is reduced, $R\downarrow$ must contain $b\ b-1$. We have to look into three subcases depending on the value of a.

Subcase: $a = b - 1$. Then $e < b - 1$ in order for Ra be reduced.

$$\underbrace{\cdots b\ b-1 \cdots e}_{R\downarrow} \mid \underbrace{\cdots b \cdots}_{R\uparrow} \overset{in}{\underset{b}{\overset{\longleftarrow}{\Leftarrow}}} a = b - 1 \overset{out}{\Leftarrow} \underbrace{\cdots b\ b-1 \cdots e \cdots a \cdots}_{R'}$$

$$\Rightarrow \underbrace{\cdots a \cdots e}_{R'^r \downarrow} \mid \underbrace{\cdots b-1\ b \cdots}_{R'^r \uparrow} \overset{in}{\underset{b}{\overset{\longleftarrow}{\Leftarrow}}} b - 1 = a \overset{out}{\Leftarrow} \underbrace{\cdots b \cdots e \cdots b-1\ b \cdots}_{R^r}$$

Subcase: $e < a < b - 1$. Same as above.

Subcase: $a < e \leq b - 1$. Then, $R\uparrow$ must start with b.

$$\underbrace{\cdots b\ b-1\cdots e}_{R\downarrow}\ \Big|\ \underbrace{b\cdots}_{R\uparrow}\overset{b}{\leftarrow}\ \overset{in}{\leftarrow}\ a=b-1\ \overset{out}{\leftarrow}\ \underbrace{\cdots b\ b-1\cdots e\ a\cdots}_{R'}$$

$$\Rightarrow\ \underbrace{\cdots a}_{R'^r\downarrow}\ \Big|\ \underbrace{e\cdots b-1\ b\cdots}_{R'^r\uparrow}\overset{b}{\leftarrow}\ \overset{in}{\leftarrow}\ b-1=a\ \overset{out}{\leftarrow}\ \underbrace{\cdots b\ e\ \cdots b-1\ b\cdots}_{R^r}$$

Case: $a\neq b, c\neq d$.

Subcase: $e<a,d$.

$$\underbrace{\cdots d\cdots e}_{R\downarrow}\ \Big|\ \underbrace{\cdots b\cdots}_{R\uparrow}\overset{b}{\leftarrow}\ \overset{in}{\leftarrow}\ a=d\ \overset{out}{\leftarrow}\ \underbrace{\cdots b\cdots e\cdots a\cdots}_{R'}$$

$$\Rightarrow\ \underbrace{\cdots a\cdots e}_{R'^r\downarrow}\ \Big|\ \underbrace{\cdots b\cdots}_{R'^r\uparrow}\overset{b}{\leftarrow}\ \overset{in}{\leftarrow}\ d=a\ \overset{out}{\leftarrow}\ \underbrace{\cdots b\cdots e\cdots d\cdots}_{R^r}$$

Subcase: $e=d<a$.

$$\underbrace{\cdots d}_{R\downarrow}\ \Big|\ \underbrace{f\cdots b\cdots}_{R\uparrow}\overset{b}{\leftarrow}\ \overset{in}{\leftarrow}\ a=d\ \overset{out}{\leftarrow}\ \underbrace{\cdots b\ f\cdots a\cdots}_{R'}$$

$$\Rightarrow\ \underbrace{\cdots a\cdots f}_{R'^r\downarrow}\ \Big|\ \underbrace{b\cdots}_{R'^r\uparrow}\overset{b}{\leftarrow}\ \overset{in}{\leftarrow}\ d=a\ \overset{out}{\leftarrow}\ \underbrace{\cdots b\cdots f\ d\cdots}_{R^r}$$

Subcase: $a<e\leq d$. This forces $f=b$.

$$\underbrace{\cdots d\cdots e}_{R\downarrow}\ \Big|\ \underbrace{b\cdots}_{R\uparrow}\overset{b}{\leftarrow}\ \overset{in}{\leftarrow}\ a=d\ \overset{out}{\leftarrow}\ \underbrace{\cdots b\cdots e\ a\cdots}_{R'}$$

$$\Rightarrow\ \underbrace{\cdots a}_{R'^r\downarrow}\ \Big|\ \underbrace{e\cdots b\cdots}_{R'^r\uparrow}\overset{b}{\leftarrow}\ \overset{in}{\leftarrow}\ d=a\ \overset{out}{\leftarrow}\ \underbrace{\cdots b\ e\cdots d\cdots}_{R^r}$$

$\square$

Lemma 1.25 *Given (P,Q), let Q' be the standard shifted Young tableau obtained by removing the largest entry from Q. There exists a unique $a, 0\leq a<n$ and a unique $P'\in\mathrm{SDT}(ws_a)$ such that*

$$P'\leftarrow a = P$$

and $\mathrm{sh}(P') = \mathrm{sh}(Q')$.

Proof: Let the largest entry be in the row j and b be the number in the corresponding box in P. Denote row i of P by P_i. We want to reverse the insertion procedure that has put b into row j. So, we insert b into P^r_{j-1} getting P'^r_{j-1} and a_{j-1}. We then insert a_{j-1} into P^r_{j-2} and so on. Note that $a_i P_{i-1}$ is reduced. Suppose a_i is empty for some i. That is to say, the reverse procedure ends at row i.

$$P^r_i \leftarrow a_{i+1} = P'^r_i$$

Consider the tableau R formed by rows $i, i+1, \cdots, j$ of P. It is a standard decomposition tableau. So, the length of the longest unimodal subsequence in R is $|P_i|$. But P'_i is a unimodal subsequence of length $|P_i| + 1$. This contradicts Lemma 1.20. Hence, a_i always exists and the insertion goes through to the 1st row of P. If we let $a = a_1$ and P' to be the tableau with $P'_i, 1 \le i < j$ as the first $j-1$ rows and $P_i, i \ge j$ as the succeeding rows, then $P' \in \mathrm{SDT}(ws_a)$.

To show uniqueness, we observe from Lemma 1.24, that each a_i is uniquely determined. Hence, the lemma holds. $\square$

Theorem 1.26 ([9, Theorem 5.2]) *The Kraśkiewicz insertion is a bijection between $R(w)$ and pairs of tableaux (P, Q) where $P \in \mathrm{SDT}(w)$ and Q is a standard shifted Young tableau.*

Proof: By Theorem 1.22, we know that $P \in \mathrm{SDT}(w)$ and Q is a standard shifted Young tableau. For the inverse map, given (P, Q), we will apply Lemma 1.25 repeatedly. Let a_i be the unique number obtained when the entry removed is i and $P^{(i)}$ be

the unique tableau obtained there.

$$
\begin{aligned}
\boldsymbol{a} &= a_1 a_2 \cdots a_m \\
&\sim \pi_{P^{(1)}} a_2 a_3 \cdots a_m \\
&\sim \pi_{P^{(2)}} a_3 \cdots a_m \\
&\quad \cdots \\
&\sim \pi_P \\
\Rightarrow \boldsymbol{a} &\in R(w)
\end{aligned}
$$

Then, it's easy to see that the $P^{(i)}$'s are exactly the tableaux that were obtained when we apply the Kraśkiewicz insertion on $\boldsymbol{a}$. Hence, the insertion is a bijection as required. $\qquad\square$

A first application of the Kraśkiewicz insertion gives us:

Corollary 1.27 ([9, Section 6]) *Let $w \in B_n$.*

$$
|R(w)| = \sum_{P \in \mathrm{SDT}(w)} g^{\mathrm{sh}(P)}
$$

where g^λ is the number of standard shifted Young tableau of shape λ.

In the next chapter, we will use this insertion to arrive at some properties of B_n stable Schubert polynomials.

Chapter 2

B_n Stable Schubert Polynomials

In this chapter, we will introduce B_n stable Schubert polynomials. The definitions are similar to those in [4] and [5]. Much work has been done on the stable Schubert polynomials for the symmetric group S_n and we will try to derive the various analogous results. In the Section 2.1, we will define the B_n stable Schubert polynomials using the nilCoxeter algebra for B_n. In Section 2.2, we will express the B_n stable Schubert polynomials in terms of Schur P-functions using the Kraśkiewicz insertion.

2.1   The NilCoxeter Algebra

The introduction of nilCoxeter algebra gave a new point of view in the study of Schubert polynomials. In [5], Fomin and Stanley used it to provide some simple proofs for alternate description of Schubert polynomials. It is also a starting point to generalize the Schubert polynomials into the other classical groups. See [4]. This use of the nilCoxeter algebra has also been independently studied by Stembridge[15].

Following the presentation in [4], we will define the nilCoxeter algebra for B_n. From there, we will give a generating function for the B_n stable Schubert polynomials.

Definition 2.1 *Let $\mathcal{B}_n$ be the nilCoxeter algebra for B_n. It is a non-commutative*

algebra generated by $u_0, u_1, \cdots, u_{n-1}$ with the relations:

$$
\begin{aligned}
u_i^2 &= 0 & i &\geq 0 \\
u_i u_j &= u_j u_i & |i - j| &> 1 \\
u_i u_{i+1} u_i &= u_{i+1} u_i u_{i+1} & i &> 0 \\
u_0 u_1 u_0 u_1 &= u_1 u_0 u_1 u_0
\end{aligned}
$$

The nilCoxeter algebra has a vector space basis of elements of B_n. This is clear since the last three relations listed above are exactly the Coxeter relations for B_n.

Next consider the polynomial ring $\mathcal{B}_n[x]$. We state the next result without proof.

Lemma 2.2 ([4, Proposition 4.2]) *Let*

$$
\begin{aligned}
B(x) &= (1 + x u_{n-1})(1 + x u_{n-2}) \cdots (1 + x u_1)(1 + x u_0) \\
&\quad (1 + x u_1) \cdots (1 + x u_{n-2})(1 + x u_{n-1})
\end{aligned}
$$

Then,

$$
B(x)B(y) = B(y)B(x)
$$

Definition 2.3 *Let $\mathcal{B}_n[\boldsymbol{x}]$ denote the polynomial ring with indeterminates $x_1, x_2, \cdots$. Consider the expansion of the formal power series*

$$
B(x_1)B(x_2)\cdots = \sum_{w \in B_n} G_w(\boldsymbol{x})w
$$

The $G_w(\boldsymbol{x})$ are called the B_n stable Schubert polynomials.

From Lemma 2.2, it is clear that they are symmetric functions. Moreover, if we replace x_1 by $-x_2$, we get

$$
\begin{aligned}
\sum_{w \in B_n} G_w(-x_2, x_2, x_3, x_4, \cdots)w &= B(-x_2)B(x_2)B(x_3)B(x_4)\cdots \\
&= B(x_3)B(x_4)\cdots \\
&= \sum_{w \in B_n} G_w(x_3, x_4, \cdots)w
\end{aligned}
$$

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Therefore,

$$G_w(-x_2, x_2, x_3, x_4, \cdots) = G_w(x_3, x_4, \cdots)$$

Using Theorem 1.10, this gives the next result.

Theorem 2.4 $G_w(x) \in \tilde{\Lambda}$

We would like to express the G_w's in terms of the basis $\{P_\lambda\}$ where λ is allowed to range over all shifted shapes. Before we go into that we give an alternate description following the presentation and notations in [2].

Definition 2.5 *Let* $a = a_1 a_2 \cdots a_m \in R(w)$. *We say that a sequence of positive integers* $i = (i_1, i_2, \cdots, i_m)$ *is an* a-*compatible sequence if*

1. $i_1 \leq i_2 \leq \cdots \leq i_m$

2. $i_j = i_{j+1} = \cdots = i_k$ *occurs only when* $a_j, a_{j+1}, \cdots, a_k$ *is a unimodal sequence*

Denote the set of a-*compatible sequence as* $K(a)$.

As in Chapter 1, we use $l(w)$ to denote the length of w.

Definition 2.6 *Let* $l_0(w)$ *denote the number of bars in the 1-line notation of* w. *Let* $l(i)$ *be the number of distinct integers in* i.

Note that the number of 0's in any reduced word of w is equal to $l_0(w)$.

Theorem 2.7 ([4, Equation 6.3], [1, Proposition 3.4])

$$G_w(x) = \sum_{a \in R(w)} \sum_{i \in K(a)} 2^{l(i) - l_0(w)} x_{i_1} x_{i_2} \cdots x_{i_m}$$

Proof: First, we look at all the monomials corresponding to a reduced word $a = a_1 a_2 \cdots a_m$ in the expansion of

$$B(x_1)B(x_2)B(x_3) \cdots$$
$$= (1 + x_1 u_{n-1}) \cdots (1 + x_1 u_1)(1 + x_1 u_0)(1 + x_1 u_1) \cdots (1 + x_1 u_{n-1})$$
$$(1 + x_2 u_{n-1}) \cdots (1 + x_2 u_1)(1 + x_2 u_0)(1 + x_2 u_1) \cdots (1 + x_2 u_{n-1})$$
$$(1 + x_3 u_{n-1}) \cdots (1 + x_3 u_1)(1 + x_3 u_0)(1 + x_3 u_1) \cdots (1 + x_3 u_{n-1})$$
$$\cdots \cdots$$

A typical monomial corresponding to $u_{a_1} u_{a_2} \cdots u_{a_m}$ is obtained as the product

$$x_{i_1} u_{a_1} x_{i_2} u_{a_2} \cdots x_{i_m} u_{a_m}$$

Clearly, $\boldsymbol{i} = (i_1, i_2, \cdots i_m)$ is a weakly increasing sequence. Also, $i_j = i_{j+1} = \cdots = i_k$ means that $a_j a_{j+1} \cdots a_k$ has to be a unimodal sequence. So, $\boldsymbol{i}$ is an $\boldsymbol{a}$-compatible sequence. Conversely, any $\boldsymbol{a}$-compatible sequence gives rise to such a monomial. We note that for each constant subsequence $i_j = i_{j+1} = \cdots = i_k$ of $\boldsymbol{i}$ with $i_{j-1} < i_j$ and $i_k < i_{k+1}$, there are at most 2 ways of getting this constant subsequence. If the corresponding unimodal subsequence $a_j a_{j+1} \cdots a_k$ contains 0, then there's only 1 way. If $a_j a_{j+1} \cdots a_k$ does not contain 0, then there are 2 ways corresponding to the 2 possible choices of u_{a_l} where a_l is the smallest element in $a_j a_{j+1} \cdots a_k$. Hence, we get

$$G_w(\boldsymbol{x}) = \sum_{\boldsymbol{a} \in R(w)} \sum_{\boldsymbol{i} \in K(\boldsymbol{a})} 2^{l(\boldsymbol{i}) - l_0(w)} x_{i_1} x_{i_2} \cdots x_{i_m}$$

$\square$

In the next section, we will slowly work towards showing that the G_w's are non-negative integer combinations of Schur P-functions, P_λ.

2.2 More Kraśkiewicz Insertion

We will begin this section with a study of the behaviour of the Kraśkiewicz insertion with respect to inserting 2 consecutive terms.

Lemma 2.8 *Let $P \in \mathrm{SDT}(w)$ and let a, a' with $a < a'$ be 2 numbers such that $\pi_P a a'$ is a reduced word. Suppose the insertion of a into P ends in box (i, j) and that of a' ends in box (i', j'). Then $i \geq i'$ and $j < j'$.*

Proof: Let's consider the insertion of a, a' into a unimodal word R where Raa' is a reduced word.

Case 1: Ra is unimodal. Then, Raa' is also unimodal and

$$R \overset{in}{\longleftarrow} aa' = Raa'$$

Case 2: Ra is not unimodal but $R'a'$ is. Then,

$$R \overset{in}{\longleftarrow} aa' = e \overset{out}{\longleftarrow} R'a'$$

In the next 2 cases, both $Ra, R'a'$ is are not unimodal.

Case 3: R contains the subsequence 101 and $a = 0$. $R'a'$ is not unimodal. Then,

$$\begin{aligned} R \overset{in}{\longleftarrow} 0a' &= 0 \overset{out}{\longleftarrow} R \overset{in}{\longleftarrow} a' \\ &= 0e' \overset{out}{\longleftarrow} R' \end{aligned}$$

Since, $0e'R'$ is reduced, we must have $e' > 0$. We note that $a' \neq 0$.

Case 4: We do not have both R containing 101 and $a = 0$ at the same time. Let b be the smallest number in R bigger than a and S be the subsequence of $R\uparrow$ consisting of all the numbers after b.

$$\begin{aligned} R \overset{in}{\longleftarrow} aa' &= R\downarrow \,|\, \underbrace{\cdots b\, S}_{R\uparrow} \overset{in}{\longleftarrow} aa' \\ &= R\downarrow \,|\, \overset{\overset{c}{\longleftarrow}}{\cdots a}\, S \overset{in}{\longleftarrow} a' \\ &= e \overset{out}{\longleftarrow} \cdots c \cdots \,|\, \cdots a\, S \overset{in}{\longleftarrow} a' \end{aligned}$$

Clearly, S is also a subsequence of $R'\uparrow$ since the insertion of a did not change this part of R. We remark that after insertion, it is possible for a to end up in $R'\downarrow$. However, this is not going to affect our result. If b' is the smallest number in R' bigger than or equal to a', then it must be in S. Hence, $c' > c$. Next, we want to compare e and e', the numbers that are bumped out after inserting c and c' into $R\uparrow$ and $R'\uparrow$

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respectively.

$$e \overset{out}{\longleftarrow} \underbrace{\cdots c \cdots}_{R' \downarrow} | \underbrace{\cdots S}_{R \uparrow} \overset{in}{\longleftarrow} a' \;=\; e \overset{out}{\longleftarrow} \cdots c \cdots \overset{\overset{\overset{c'}{\longleftarrow}}{}}{|} \cdots a' \cdots$$

$$=\; ee' \overset{out}{\longleftarrow} \cdots c' \cdots | \cdots a' \cdots$$

Let d' be the biggest number in $R' \downarrow$ which is smaller or equal to c'. If $d' \neq c'$, we would have $e' = d' > e$. If $d' = c'$, we would then have $e' = c' - 1 \geq c > e$. So, in both cases, we have $e < e'$.

In conclusion, as we look at the insertion of a and a' into P, each time the pair of numbers bumped out from each row still have the same relative order until we either reach Case 1 or Case 2. In Case 1, the insertion of a and a' end in the same row. So, we have $(i', j') = (i, j + 1)$. In Case 2, the insertion of a' ends first. This gives $i > i', j < j'$. $\qquad \square$

Now, suppose in the previous lemma, we have $a > a'$ instead and let us examine the possible outcomes of inserting aa' into R.

Case 1: Raa' is unimodal. This is only possible when Raa' is a decreasing sequence and we get

$$R \overset{in}{\longleftarrow} aa' = Raa'$$

Case 2: Ra is unimodal but Raa' is not. This can only occur if Ra is unimodal but not decreasing.

$$R \overset{in}{\longleftarrow} aa' \;=\; Ra \overset{in}{\longleftarrow} a'$$
$$=\; e' \overset{out}{\longleftarrow} R''$$

Case 3: Ra is not unimodal but $R'a'$ is. This case is possible only if R' is a decreasing sequence ending in a.

$$R \overset{in}{\longleftarrow} aa' \;=\; e \overset{out}{\longleftarrow} R' \overset{in}{\longleftarrow} a'$$
$$=\; e \overset{out}{\longleftarrow} R'a'$$

Case 4: Ra is not unimodal, $a' = 0$ and R' contains 101.

$$R \overset{in}{\leftarrow} aa' \;=\; e \overset{out}{\leftarrow} R' \overset{in}{\leftarrow} a'$$
$$=\; e0 \overset{out}{\leftarrow} R''$$

Since $e0R''$ is reduced, $e > 0$.

In the following cases, we will assume that we do not have both R' contains 101 and $a' = 0$ at the same time. Also, both Ra and $R'a'$ are not unimodal.

Case 5: a lands in $R'\uparrow$ after insertion. Using the same notations as before(see Case 4 of Lemma 2.8).

$$R \overset{in}{\leftarrow} aa' \;=\; e \overset{out}{\leftarrow} \underbrace{\cdots c \cdots}_{R'\downarrow} | \underbrace{\cdots a \cdots}_{R'\uparrow} \overset{in}{\leftarrow} a'$$
$$=\; ee' \overset{out}{\leftarrow} \underbrace{\cdots c' \cdots}_{R''\downarrow} | R''\uparrow$$

We have $b' \leq a < c$. If $a' = b'$, then $c' = a' + 1 \leq a < c$. If $a' \neq b'$, we then have $c' = b' \leq a < c$. So, we always end up with $c > c'$. Using the same arguments, it can be shown that $e > e'$.

Case 6: a lands in $R'\downarrow$ after insertion. This means that a is the smallest number in R' and e, the number bumped out by a, is the smallest number in R.

$$R \overset{in}{\leftarrow} aa' \;=\; e \overset{out}{\leftarrow} \underbrace{\cdots a}_{R'\downarrow} | \underbrace{b' \cdots}_{R'\uparrow} \overset{in}{\leftarrow} a'$$
$$=\; e \overset{out}{\leftarrow} \underbrace{\cdots a}_{R'\downarrow} | \overset{c'}{\overset{\leftarrow}{\underbrace{a' \cdots}}}$$
$$=\; ee' \overset{out}{\leftarrow} \underbrace{\cdots a'}_{R''\downarrow} | R''\uparrow$$

Since all the numbers in R' is bigger than or equal to a, we must have $e < a \leq e'$.

We compile the effect of all these different cases in the following lemma.

Lemma 2.9 *Let $P, a, a', (i, j), (i', j')$ be as in the previous lemma but with $a > a'$. If the insertion at each row is of the cases 2, 4 or 5, we have $i < i'$ and $j \geq j'$. If the*

insertion involves Case 1, 2 or 6 at some point, then $i \geq i'$ and $j < j'$.

For the next theorem, the definitions of rim hooks, vertical strips and horizontal strips can be found in Section 1.2.

Theorem 2.10 *Let $\boldsymbol{a} = a_1 a_2 \cdots a_m \in R(w)$ and*

$$\boldsymbol{a} \overset{K}{\to} (P, Q)$$

If $a_j a_{j+1} \cdots a_k$, a subsequence of $\boldsymbol{a}$ is unimodal, then in Q, the boxes with the entries $j, j+1, \cdots, k$ form a rim hook. Moreover, the way the entries appear in the rim hook is as follows:

1. *the entries $j, j+1, \cdots, l$ form a vertical strip where l is the entry in the leftmost and lowest box of the rim hook*

2. *these entries are increasing down the vertical strip*

3. *the entries $l+1, \cdots, k-1, k$ form a horizontal strip*

4. *these entries are increasing from left to right*

5. *the boxes in the vertical strip is always left of any box in the horizontal strip which is in the same row*

Proof: Let us consider the insertion of a unimodal sequence $\boldsymbol{b} = b_1 b_2 \cdots b_m$ into a unimodal word R and that $R b_1 b_2 \cdots b_m$ is reduced. Let k be the subscript of the smallest number in $\boldsymbol{b}$ and for all i, let $R^{(i)}$ be what becomes of R after inserting $b_1 b_2 \cdots b_i$.

Let us deal with $b_k b_{k+1} \cdots b_m$ first. Let $l, k \leq l \leq m$ be the smallest subscript such that $R^{(l-1)} b_l$ is unimodal. By Case 1 of Lemma 2.8, we get

$$R^{(k-1)} \overset{in}{\Leftarrow} b_k b_{k+1} \cdots b_m = b'_k b'_{k+1} \cdots b'_{l-1} \overset{out}{\Leftarrow} R^{(l-1)} b_l b_{l+1} \cdots b_m$$

where $b'_k < b'_{k+1} < \cdots < b'_{l-1}$ follows from Case 3 and Case 4 of Lemma 2.8.

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Now, we deal with $b_1 b_2 \cdots b_k$. Let $j, 1 \leq j < k$ be the smallest subscript such that the insertion of $b_j b_{j+1}$ is Case 1, 3 or 6 of Lemma 2.9. We note from Lemma 2.9, that each $b_i, j < i \leq l$ is smaller than all the numbers in $R^{(i-1)}$. Also, from Lemma 2.9, if the insertion of $b_h b_{h+1}$ for some $h, j \leq h < k$ is Case 3, then $b_{h+1} b_{h+2}$ has to be Case 1 since $R^{(h+1)}$ is a decreasing sequence. For the same reason, all subsequent insertion of pairs of numbers must be Case 1. Hence, the insertion for the pairs of consecutive terms in $b_j b_{j+1} \cdots b_k$ can be divided into these 3 sections with $b_j b_{j+1} \cdots b_h$ made up of Case 6 only, $b_h b_{h+1}$ as Case 3 and $b_{h+1} b_{h+2} \cdots b_k$) involving Case 1 only.

$$R^{(j-1)} \overset{in}{\leftarrow} b_j b_{j+1} \cdots b_k = b'_j b'_{j+1} \cdots b'_h \overset{out}{\leftarrow} R^{(h)} b_{h+1} b_{h+2} \cdots b_l$$

where $b'_j < b'_{j+1} < \cdots < b'_h$. Note that it is possible that some of these cases do not appear in the insertion of $b_j b_{j+1} \cdots b_k$ into R.

In the insertion of $b_1 b_2 \cdots b_j$, we only have Case 2, Case 4 or Case 5 of Lemma 2.9 occuring. For Case 2 of Lemma 2.9, we observe that it can only occur during the insertion of $b_1 b_2$. The reason is that if the insertion of $b_1 b_2$ into R is Case 1, 3 or 6, then from above, neither Case 2, Case 4 nor Case 5 of Lemma 2.9 can occur. If the insertion of $b_1 b_2$ is of Case 4 or Case 5, then the next time that the $R^{(i-1)} b_i$ is unimodal for some $i, i > 1$ is when b_i is smaller than all the numbers in $R^{(i-1)}$. But this would lead to Case 1 instead of Case 2.

Hence, in conclusion, for the insertion of $b_1 b_2 \cdots b_k$, we either have

$$R \overset{in}{\leftarrow} b_1 b_2 \cdots b_k = b'_2 b'_3 \ldots b'_h \overset{out}{\leftarrow} R^{(h)} b_{h+1} b_{h+2} \cdots b_k$$

with $b'_2 > b'_3 > \cdots > b'_j < b'_{j+1} < \cdots < b'_h$; or

$$R \overset{in}{\leftarrow} b_1 b_2 \cdots b_k = b'_1 b'_2 b'_3 \ldots b'_h \overset{out}{\leftarrow} R^{(h)} b_{h+1} b_{h+2} \cdots b_k$$

with $b'_2 > b'_3 > \cdots > b'_j < b'_{j+1} < \cdots < b'_h$.

Back to the original problem, let

$$a_1 a_2 \cdots a_{j-1} \overset{K}{\rightarrow} (P', Q')$$

and

$$a_1 a_2 \cdots a_{k+1} \overset{K}{\rightarrow} (P'', Q'')$$

We then look at the insertion of $a_j a_{j+1} \cdots a_k$ into P'. Clearly, the boxes containing the entries form a skew Young of shape $\text{sh}(Q'')/\text{sh}(Q')$.

From the above analysis, the insertion of $a_j a_{j+1} \cdots a_k$ into P' can be divided into 2 sections. Let l be such that the insertion of $a_j a_{j+1} \cdots a_l$ into P' only involves Cases 2, 4 and 5 of Lemma 2.9. Then, the entries $j, j+1, \cdots, l$ form a vertical strip in Q. The entries $l+1, l+2, \cdots, k$ form a horizontal strip in Q. If some row contains boxes from both the vertical strip and the horizontal strip, then the box from the vertical strip appears to the left of all the boxes from the horizontal strip in that row. Because of this, the boxes containing these entries have to form a rim hook. $\square$

Example: In the example in Chapter 1, the insertion of $\boldsymbol{a} = 3121034310$ gives the recording tableau

$$Q \quad = \quad \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & 6 & 7 \\\hline \end{array}$$

with the tableau:

1	2	3	5	6	7
4	8	10			
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$\boldsymbol{a}$ contains the unimodal subsequence 21034 and the entries $3, 4, 5, 6, 7$ appear in a rim hook in Q.

Now, if when we insert a reduced word $\boldsymbol{a}$ and find a rim hook in the recordiong tableau Q filled with consecutive numbers, then we would hope to see whether the corresponding subsequence in $\boldsymbol{a}$ is unimodal. As before, we would have to work out the details of the insertion.

Lemma 2.11 *Let $P \in \text{SDT}(w)$ and let a, a', a'' be three numbers such that the word $\pi_P a a' a''$ is reduced. Suppose $a < a'$ and $a' > a''$. Let $(i, j), (i', j')$ and (i'', j'') be*

the boxes that the respective insertions of a, a' and a'' end. Then $i \geq i', j < j'$ and $i' < i'', j' \geq j''$.

Proof: As in the proof of Lemma 2.8, we consider the insertion of a, a', a'' into a unimodal word R.

Case 1: Ra is unimodal. Then, Raa' is unimodal but $Raa'a''$ is not.

$$R \xleftarrow{in} aa'a'' = e'' \xleftarrow{out} Raa'$$

Case 2: Ra is not unimodal but $R'a'$ is. Note that $R'a'$ cannot be a decreasing sequence as R' contains a which is smaller than a'. Then $R'a'a''$ is not unimodal. This gives

$$R \xleftarrow{in} aa'a'' = ee'' \xleftarrow{out} Raa'$$

In the next 2 cases, both Ra and $R'a'$ are not unimodal.

Case 3: a is in $R'\uparrow$. This means that when we insert a' into R', a will remain in R''. a may end up in either $R''\uparrow$ or $R''\downarrow$ but a' will always be in $R''\uparrow$.

$$
\begin{aligned}
R \xleftarrow{in} aa'a'' &= e \xleftarrow{out} R'\downarrow \,|\, \underbrace{\cdots a \cdots}_{R'\uparrow} \xleftarrow{in} a'a'' \\
&= ee' \xleftarrow{out} R''\downarrow \,|\, \underbrace{\cdots a' \cdots}_{R''\uparrow} \xleftarrow{in} a'' \\
&= ee'e'' \xleftarrow{out} R'''
\end{aligned}
$$

The insertion of $a'a''$ will correspond to Case 4 or Case 5 of Lemma 2.9. This gives $e < e'$ and $e' > e''$.

Case 4: a appears in $R'\downarrow$.

$$R \xleftarrow{in} aa'a'' = e \xleftarrow{out} \underbrace{\cdots c \cdots a}_{R'\downarrow} \,|\, R'\uparrow \xleftarrow{in} a'a''$$

where c is the number that is bumped into $R\downarrow$ by a. When we next insert a' into R', we note that a will not be affected since $c' \geq c$. Hence, a will be the smallest number

in R'' and a' will end up in $R''\uparrow$.

$$e \overset{out}{\longleftarrow} \underbrace{\cdots c \cdots a}_{R'\downarrow} | R'\uparrow \overset{in}{\longleftarrow} a'a'' \quad = \quad ee' \overset{out}{\longleftarrow} \underbrace{\cdots a}_{R''\downarrow} | \underbrace{\cdots a' \cdots}_{R''\uparrow} \overset{in}{\longleftarrow} a''$$

$$= \quad ee'e'' \overset{out}{\longleftarrow} R'''$$

Then, as in the case above, the insertion of $a'a''$ will correspond to Case 4 or Case 5 of Lemma 2.9 and thus, $e < e'$ and $e' > e''$.

From here, it is not difficult to see that the insertion of a will end in a lower than or same row as that of a' and the insertion of a' will end in a strictly higher row than that of a''. This translates to $i \geq i', j < j'$ and $i' < i'', j' \geq j''$. $\qquad\square$

Theorem 2.12 *Let $a \to (P, Q)$. Let the boxes with entries $i, i+1, \cdots, k$ form a rim hook in Q. Suppose the entries increase down the vertical strip and then along the horizontal strip from left to right. Then, the corresponding subsequence $a_i a_{i+1} \cdots a_k$ is unimodal.*

Proof: Suppose $a_i a_{i+1} \cdots a_k$ is not unimodal. Then there exists $j, i < j < k$ such that $a_{j-1} < a_j$ and $a_j > a_{j+1}$. From the previous lemma, the entries $j - 1, j, j + 1$ cannot satisfy the hypothesis of theoerm. This is a contradiction and hence $a_i a_{i+1} \cdots a_k$ is unimodal. $\qquad\square$

Now, we are ready to prove the major theorem of this whole investigation.

Theorem 2.13 *For all $w \in B_n$,*

$$G_w(\boldsymbol{x}) = \sum_{R \in \text{SDT}(w)} 2^{l(R) - l_0(w)} P_{\text{sh}(R)}(\boldsymbol{x})$$

Proof: We will, instead, show that

$$2^{l_0(w)} G_w(\boldsymbol{x}) = \sum_{R \in \text{SDT}(w)} Q_{\text{sh}(R)}(\boldsymbol{x})$$

since $Q_\lambda = 2^{l(\lambda)} P_\lambda$. Fix $w \in B_n$ and let $m = l(w)$. To achieve this, we generalize the idea of $\boldsymbol{a}$-compatible sequence to include sequences with barred and unbarred

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numbers. Let $\boldsymbol{f} = (f_1, f_2, \cdots, f_m)$ be a sequence where $f_j \in \{\bar{1}, 1, \bar{2}, 2, \cdots\}$. This is called a *generalized* sequence. We give the barred and unbarred numbers the linear order,

$$\bar{1} < 1 < \bar{2} < 2 < \cdots$$

We say that $\boldsymbol{f}$ is $\boldsymbol{a}$-compatible if

1. $f_1 \leq f_2 \leq \cdots \leq f_m$

2. $f_j = f_{j+1} = \cdots = f_k = \bar{l}$ occurs only when $a_j > a_{j+1} > \cdots > a_k$

3. $f_j = f_{j+1} = \cdots = f_k = l$ occurs only when $a_j < a_{j+1} < \cdots < a_k$

Let $K'(\boldsymbol{a})$ be the set of all $\boldsymbol{a}$-compatible generalized sequences. In what follows, $\boldsymbol{i}$ will always be a sequence of unbarred numbers and $\boldsymbol{f}$ will denote a generalized sequence. We define

$$|f_j| = l \quad \text{if } f_j = \bar{l} \text{ or } f_j = l.$$

and

$$|\boldsymbol{f}| = (|f_1|, |f_2|, \cdots, |f_m|)$$

We will associate each generalized sequence $\boldsymbol{f}$, the monomial $x_{|f_1|} x_{|f_2|} \cdots x_{|f_m|}$.

Suppose $\boldsymbol{i}$ is a $\boldsymbol{a}$-compatible sequence, that is $\boldsymbol{i} \in K(\boldsymbol{a})$. We want to find all the $\boldsymbol{a}$-compatible generalized sequence $\boldsymbol{f}$ such that $|\boldsymbol{f}| = \boldsymbol{i}$. First, suppose $i_j = i_{j+1} = \cdots = i_k = l$ is a constant subsequence in $\boldsymbol{i}$ and $i_{j-1} < i_j$ and $i_k < i_{k+1}$. Then $a_j a_{j+1} \cdots a_k$ must be a unimodal sequence with, say a_h, as the smallest number in it. Any $\boldsymbol{a}$-compatible generalized sequence $\boldsymbol{f}$ such that $|\boldsymbol{f}| = \boldsymbol{i}$, has to have

$$f_j = f_{j+1} = \cdots = f_h = \bar{l} \quad \text{and} \quad f_{h+1} = f_{h+2} = \cdots = f_k = l$$

or

$$f_j = f_{j+1} = \cdots = f_{h-1} = \bar{l} \quad \text{and} \quad f_h = f_{h+1} = \cdots = f_k = l$$

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Hence, from Theorem 2.7, we get

$$
\begin{aligned}
2^{l_0(w)}G_w(\boldsymbol{x}) &= \sum_{\boldsymbol{a}\in R(w)}\sum_{\boldsymbol{i}\in K(\boldsymbol{a})} 2^{l(\boldsymbol{i})} x_{i_1}x_{i_2}\cdots x_{i_m} \\
&= \sum_{\boldsymbol{a}\in R(w)}\sum_{\boldsymbol{f}\in K'(\boldsymbol{a})} x_{|f_1|}x_{|f_2|}\cdots x_{|f_m|}
\end{aligned}
$$

Now, we will exhibit a bijection Φ from $\{(\boldsymbol{a},\boldsymbol{f}) : \boldsymbol{a}\in R(w), \boldsymbol{f}\in K'(\boldsymbol{a})\}$ to $\{(P,T) : P\in \mathrm{SDT}(w), T$ a Q-semistandard Young tableau such that $\mathrm{sh}(T)=\mathrm{sh}(P)\}$.

Step 1: Apply Kraśkiewicz insertion to $\boldsymbol{a}$. Let

$$
\boldsymbol{a} \xrightarrow{K} (P,Q)
$$

This P is the standard decomposition tableau we want.

Step 2: Take a Young diagram of the same shape as Q. We fill each box by $|f_j|$ when the corresponding box in Q has the entry j. Then, the new tableau is weakly increasing along the columns and rows because

$$
j < k \;\Rightarrow\; |f_j| \le |f_k|
$$

Step 3: For each subsequence $f_j f_{j+1}\cdots f_k$ such that $|f_j|=|f_{j+1}|=\cdots=|f_k|=l$ with $|f_{j-1}|<|f_j|$ and $|f_k|<|f_{k+1}|$, we know that $a_j a_{j+1}\cdots a_k$ is unimodal. Let a_h be the smallest number in it. Furthermore, from Theorem 2.10, the entries $j, j+1, \cdots k$ form a rim hook in Q. Let g be the entry of the box in the lowest and leftmost box in the rim hook. So, $j \le g \le k$.

1. if f_h is unbarred, we add a bar to all the new entries from f_j to f_{g-1}

2. if f_h is barred, we add a bar to all the new entries from f_j to f_g

This will give us a Young tableau with barred and unbarred numbers. This will be our T. Clearly, $\mathrm{sh}(T)=\mathrm{sh}(Q)=\mathrm{sh}(P)$. We will have to show that T is a Q-semistandard Young tableau.

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From the analysis in Theorem 2.10, we know that the boxes with entries $j, j +$ $1, \cdots, g$ in Q form a vertical strip and the boxes with entries $g, g + 1, \cdots k$ form a horizontal strip in Q. So, if we replace them by $f_j, f_{j+1}, \cdots, f_k$ and add bars to them according to the rule, we see that the vertical strip is filled with $\bar{l}$ while the horizontal strip is filled with l. The box with the entry g in Q can take either $\bar{l}$ or l. As remarked earlier, every box in the vertical strip is to the left of any box in the horizontal strip which is also in the same row. So, this shows that T is weakly increasing along columns and rows with respect to the linear order on the barred and unbarred numbers. Also, the unbarred numbers are strictly increasing along the columns and the barred numbers are strictly increasing in each row. Therefore, T is a Q-semistandard Young tableau as desired and we set

$$\Phi(\boldsymbol{a}, \boldsymbol{f}) = (P, T)$$

We have illustrated this procedure with an example at the end of the proof.

For the inverse map, given (P, T), we will first construct Q.

Step 1: Take a Young diagram of the same shape as T. We fill all the boxes with distinct numbers $1, 2, \cdots m$ as follows:

1. the entries in the Young diagram preserve the order of the entries in T

2. for all the boxes in T with the same barred number, these form a vertical strip and we fill the corresponding boxes in an increasing order from top to bottom

3. for all the boxes in T with the same unbarred number, these form a horizontal strip and we fill the corresponding boxes in an increasing order from left to right

This will be our Q. It is clearly a standard shifted Young tableau with the same shape as P.

Step 2: $\boldsymbol{a}$ is obtained from (P, Q) by the inverse Kraśkiewicz insertion. This is the reduced word that we want.

Step 3: To get $\boldsymbol{f}$, remove all the bars in T and let $\boldsymbol{i} = i_1 i_2 \cdots i_m$ be the content of this new tableau laid out in weakly increasing order. Fix a number l, we know that

in T, all the boxes with the entry $\bar{l}$ or l form a rim hook in T. The corresponding boxes in Q are filled with consecutive numbers, $j, j+1, \cdots, k$. Also, they satisfy the hypothesis in Theorem 2.12. Hence, $a_j a_{j+1} \cdots a_k$ is a unimodal sequence with smallest number, say a_h. Now, let (I, J) be the coordinates of the lowest and leftmost boxes in the rim hook which has entry $\bar{l}$ or l.

1. If (I, J) has the entry $\bar{l}$, then we add bars to $i_j, i_{j+1}, \cdots i_h$.

2. If (I, J) has the entry l, we add bars to $i_j, i_{j+1}, \cdots, i_{h-1}$.

This generalized sequence will be our $\boldsymbol{f}$.

By construction, it is $\boldsymbol{a}$-compatible. So, $\Phi^{-1}(P, T) = (\boldsymbol{a}, \boldsymbol{f})$ and indeed this is the inverse.

Hence, if we look at the associated monomials for generalized sequence and the Q-semistandard Young tableau, we see that they have to be equal. This gives

$$
\begin{aligned}
2^{l_0(w)} G_w(\boldsymbol{x}) &= \sum_{\boldsymbol{a} \in R(w)} \sum_{\boldsymbol{f} \in K'(\boldsymbol{a})} x_{|f_1|} x_{|f_2|} \cdots x_{|f_m|} \\
&= \sum_{P \in \text{SDT}(w)} \sum_{\text{sh}(T)=\text{sh}(P)} \boldsymbol{x}^T \\
&= \sum_{P \in \text{SDT}(w)} Q_{\text{sh}(P)}(\boldsymbol{x})
\end{aligned}
$$

Therefore,

$$
G_w(\boldsymbol{x}) = \sum_{R \in \text{SDT}(w)} 2^{l(R)-l_0(w)} P_{\text{sh}(R)}(\boldsymbol{x})
$$

$\square$

Note that $l(R) > l_0(w)$ since every row of R can have at most one 0. This gives the following corollary.

Corollary 2.14 *For all $w \in B_n$, $G_w(\boldsymbol{x})$ is a nonnegative integer linear combination of Schur P-functions.*

Example: Let $w = \bar{3}41\bar{2}$. Let $(\boldsymbol{a}, \boldsymbol{f}) = (10231023, 1\bar{2}2\bar{3}\bar{3}336)$. $\boldsymbol{f}$ is $\boldsymbol{a}$-compatible. We will now construct $(P, T) = \Phi(\boldsymbol{a}, \boldsymbol{f})$ as described in the proof. Applying the Kraśkiewicz on $\boldsymbol{a}$, we get

$$(P,Q) = \begin{pmatrix} \begin{array}{|c|c|c|c|c|} \hline 3 & 1 & 0 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 1 & 0 & 2 \\ \cline{2-4} \end{array} & \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 8 \\ \hline \multicolumn{1}{c|}{} & 5 & 6 & 7 \\ \cline{2-4} \end{array} \end{pmatrix}$$

We replace each entries i in Q by $|f_i|$ and adding bars following the procedures as described, we get

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & \bar{2} & 2 & \bar{3} & 6 \\ \hline \multicolumn{1}{c|}{} & 3 & 3 & 3 \\ \cline{2-4} \end{array}$$

The reader can verify that the $\Phi^{-1}(P,T)$ gives back $(\boldsymbol{a}, \boldsymbol{f})$. With some more work, it can be shown that $\mathrm{SDT}(\bar{3}41\bar{2})$ contains only

$$\begin{array}{|c|c|c|c|c|} \hline 3 & 1 & 0 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 2 & 0 \\ \cline{2-3} \multicolumn{2}{c|}{} & 1 \\ \cline{3-3} \end{array} \qquad \begin{array}{|c|c|c|c|c|} \hline 3 & 1 & 0 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 1 & 0 & 2 \\ \cline{2-4} \end{array}$$

Therefore,

$$G_w(\boldsymbol{x}) = 2P_{(5,2,1)}(\boldsymbol{x}) + P_{(5,3)}(\boldsymbol{x})$$

In [1, Section 3], an alternate definition of the B_n stable Schubert polynomial is given. It is denoted as $F_w(X)$. It turns out that their $F_w(X) = 2^{lo(w)}G_w(X)$.

Next, we would like to give a simple description of $G_{w_{B_n}}$ where $w_{B_n} = \bar{1}\bar{2}\cdots\bar{n}$ in 1-line notation. It is the element of longest length in B_n.

Theorem 2.15 ([9, Corollary 5.3]) *Let w_{B_n} be the longest element in B_n. Then* $\mathrm{SDT}(w_{B_n})$ *contains only one standard decomposition tableau of shape* $(2n-1, 2n-3, \cdots, 3, 1)$. *Hence,*

$$G_{w_{B_n}} = P_{(2n-1, 2n-3, \cdots, 3, 1)}$$

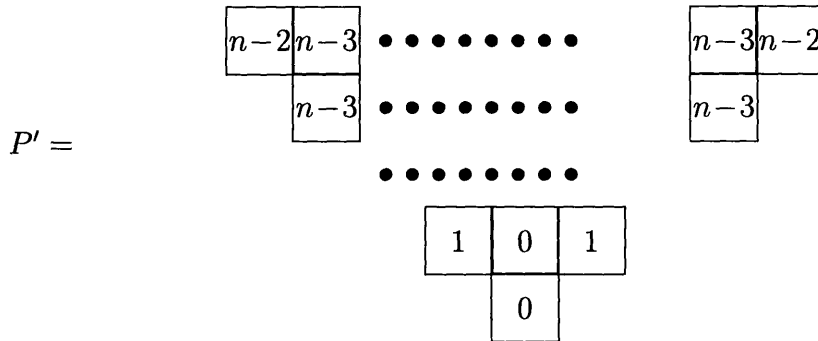Proof: We will prove by induction on n that $\mathrm{SDT}(w_{B_n})$ contains only

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$$\begin{array}{|c|c|}\hline n-1 & n-2 \\\hline & n-2 \\\hline\end{array}\ \bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\qquad \begin{array}{|c|c|}\hline n-2 & n-1 \\\hline n-2 & \\\hline\end{array}$$

$$\begin{array}{ccc} & \bullet\bullet\bullet\bullet\bullet\bullet\bullet & \\ \hline 1 & 0 & 1 \\ \hline & 0 & \\ \hline\end{array}$$
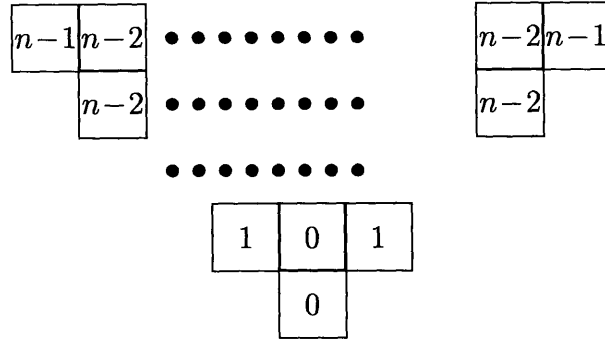
Case: $n = 1$. This is trivial since $w_{B_1} = \bar{1}$.

Case: $n > 1$. It is not difficult to show that in any reduced word of w_B, there exists a unimodal subsequence of length $2n - 1$. Using the method described in the example in Section 1 of Chapter 1, we just pick out the simple reflections that affect the number n when converting the identity permutation to w_B. So, the first row of P has to have length $2n - 1$. This means that it can only be

$$P_1 = n - 1\ n - 2\ \cdots\ 1\ 0\ 1\ \cdots\ n - 1$$

If P' is the tableau with the first row removed, $\pi_{P'}$ has to be a reduced word of $w_{B_{n-1}}$. By induction, we get

$$P' = \qquad \begin{array}{|c|c|}\hline n-2 & n-3 \\\hline & n-3 \\\hline\end{array}\ \bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\qquad \begin{array}{|c|c|}\hline n-3 & n-2 \\\hline n-3 & \\\hline\end{array}$$

Hence, P is as desired and the formula for G_{w_B} follows easily. $\qquad\square$

We would like to note that given any $w \in B_n$, a reduced word a of w can always be extended into a reduced word of w_B. This would mean that any standard decomposition tableau, P of w with shape λ must have $\lambda \subseteq (2n - 1, 2n - 3, \cdots, 3, 1)$.

The above theorem can be generalized.

Theorem 2.16 *Let $w \in B_n$ with $w_n = \bar{n}$ and $v \in B_n$ with $v_i = w_i, 1 \le i < n$ and $v_n = n$. Then there is a bijection Φ between $\mathrm{SDT}(v)$ and $\mathrm{SDT}(w)$. Moreover,*

$$G_w(x) = \sum_{R \in \mathrm{SDT}(v)} 2^{l(R) - l_0(v)} P_{(2n-1, \mathrm{sh}(R))}(x)$$

where $(2n - 1, \lambda) = (2n - 1, \lambda_1, \lambda_2, \cdots)$.

Proof: As above, we note that given any reduced word in $R(w)$, there exists a unimodal subsequence of length $2n - 1$. So, for all $P \in \mathrm{SDT}(w)$, the first row has to have length $2n - 1$ and this gives $P_1 = n - 1\ n - 2 \cdots 1\ 0\ 1 \cdots n - 1$. Let $\Psi(P)$ be the tableau P with its first row deleted. Clearly, $\Psi(P)$ is a standard decomposition tableau of $w s_{n-1} s_{n-2} \cdots s_1 s_0 s_1 \cdots s_{n-1} = v$. Therefore, $\Psi(P) \in \mathrm{SDT}(v)$.

For the inverse, take $R \in \mathrm{SDT}(v)$ and add $n - 1\ n - 2 \cdots 1\ 0\ 1 \cdots n - 1$ on top of R. Denote this new tableau by $\Phi(R)$. Clearly,

$$\pi_{\Phi(R)} = \pi_R 2n - 1\ 2n - 2 \cdots 1\ 0\ 1 \cdots 2n - 1$$

$$\in R(v s_{n-1} s_{n-2} \cdots s_1 s_0 s_1 \cdots s_{n-1}) = R(w)$$

Furthermore, since any unimodal subsequence of any reduced word in B_n cannot exceed $2n - 1$, the first row is clearly a unimodal sequence of maximal length in $\pi_{\Phi(R)}$. So $\Phi(R) \in \mathrm{SDT}(w)$. It is obvious that $\Psi = \Phi^{-1}$. Also, we note that

$$l(\Phi(R)) = l(R) + 1$$
$$l_0(w) = l_0(v) + 1$$

The formula follows easily from here. $\qquad \square$

Making use of the bijection, we were able to prove a conjecture of Stembridge([15]).

Corollary 2.17 *Let $w \in B_n$ be such that*

$$w_i = \begin{cases} \bar{i} & for\ i = i_1, i_2, \cdots i_k \\ i & otherwise \end{cases}$$

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where $i_1 < i_2 < \cdots < i_k$ Then,

$$G_w = P_{(2i_k-1,\cdots,2i_2-1,2i_1-1)}$$

Proof: Since $w_i = i$ for all $i > i_k$, we can treat w as an element in B_{i_k}. We then apply the previous theorem to get a bijection between SDT(w) and SDT(v). Next, we observe that v has the description

$$v_i = \begin{cases} \bar{i} & \text{for } i = i_1, i_2, \cdots i_{k-1} \\ i & \text{otherwise} \end{cases}$$

So, we can repeat the procedure and finally get down to a bijection between SDT(w) and SDT(1) where 1 is the identity element. This shows that SDT(w) contains only 1 tableau with k rows and each row is of the form

$$i_j \; i_j - 1 \; \cdots 1 \; 0 \; 1 \; \cdots i_j - 1 \; i_j$$

Hence,

$$G_w(\boldsymbol{x}) = P_{(2i_k-1,\cdots,2i_2-1,2i_1-1)}(\boldsymbol{x})$$

$\square$

Chapter 3

Further Properties of the Kraśkiewicz Insertion

In the previous section, we have used the Kraśkiewicz insertion to express the B_n stable Schubert polynomials, G_w in terms of P_λ. We would like to make use of further properties of the Kraśkiewicz insertion to provide more information on G_w. A number of these results were conjectured by Stembridge. Some were obtained independently by Stembridge, Billey and Haiman, Fomin and Kirillov. See [1], [4]. Section 3.1 covers some properties of the insertion tableau. In Section 3.2, we look at the Kraśkiewicz insertion applied on permutations of S_n. Section 3.3 associates the Edelman-Greene insertion with the Kraśkiewicz insertion in a surprising manner. Section 3.4 looks into properties of the recording tableau. Relations with the short promotion sequence of [6] are established in Section 3.5. Finally, in Section 3.6, we relate the Kraśkiewicz insertion with another insertion algorithm called the shifted mixed insertion. The shifted mixed insertion first appeared in [7].

3.1 Decreasing Parts

Let $a \to (P, Q)$ and let $\text{sh}(P) = (\lambda_1, \lambda_2, \cdots, \lambda_l)$. From Chapter 1, we have seen that λ_1 is the length of the longest unimodal subsequence. In general, it is difficult to give similar properties for $\lambda_i, i \neq 1$. However, we are able to say something about the

decreasing parts of each row of P.

Theorem 3.1 *Let P be a standard decomposition tableau and let $P\Downarrow$ be the tableau that is obtained when we delete the increasing parts of each row of P. Then, $P\Downarrow$ is a shifted tableau which is strictly decreasing in each row and in each top-left to bottom-right diagonal.*

Remarks: If we left justify $P\Downarrow$, it becomes an unshifted tableau of shape μ such that $\mu_1 > \mu_2 > \cdots$ and it is strictly decreasing in rows and columns.

Proof: Obviously, the numbers in each row of $P\Downarrow$ is strictly decreasing. For the other properties, it suffices to show for 2 consecutive rows in P that the decreasing parts have the properties mentioned above. So, let

$$
\begin{aligned}
R &= \underbrace{a_1 a_2 \cdots a_g}_{R\downarrow} \mid \underbrace{a_{g+1} \cdots a_h}_{R\uparrow} \\
S &= \underbrace{b_1 b_2 \cdots b_k}_{S\downarrow} \mid \underbrace{b_{k+1} \cdots b_l}_{S\uparrow}
\end{aligned}
$$

be 2 consecutive rows in P with R on top of S.

We examine the Kraśkiewicz insertion of the reduced word SR. It is clear that $g \geq k$ since the length of the decreasing part of the first row can never decrease during the insertion. We will now show by induction on $|S| = l$ that $g > k$ and $a_j > b_j$ for all $j \leq k$.

Case: $|S| = 1$. This means that $S = b_1$ and $k = 1$. We note that $b_1 a_1 a_2$ is not a unimodal sequence. Hence, $a_1 > b_1$ and $a_1 > a_2$. This shows that $g > k = 1$.

Case: $|S| > 1$. Let $j \leq k$. Let P' be the standard decomposition tableau that is obtained when we apply the Kraśkiewicz insertion on $b_1 b_2 \cdots b_k b_{k+1} \cdots b_l a_1 \cdots a_j$. Then

$$
P' = \begin{array}{|c|c|c|c|c|c|c|}
\hline
c_1 & c_2 & \bullet\,\bullet & c_{j-1} & b_j & \bullet\,\bullet & c_{l+1} \\
\hline
\end{array}
$$

with second row
$$
\begin{array}{|c|c|c|c|}
\hline
b_1 & b_2 & \bullet\,\bullet & b_{j-1} \\
\hline
\end{array}
$$

P' is a standard decomposition tableau and by induction hypothesis, $c_{j-1} > b_{j-1} > b_j$. When we next insert a_{j+1} into P', b_j gets bumped into the second row and it has

to be Case 2.1.3 of the insertion algorithm since $c_{j-1} > b_j + 1$. This means that the new entry in the jth box in the first row is strictly bigger than b_j. This would imply $a_j > b_j$.

Suppose $g = k$, when we insert a_{k+1}, b_k gets bumped into the second row. Again, by the same argument as above with j replaced by k, this bumping process involves Case 2.1.3 of the insertion algorithm. But b_k is the smallest element in the first row. This causes the length of the decreasing part of the first row to increase. Hence, $g > k$ and the entries in $P \Downarrow$ have to be strictly decreasing in top-left to bottom-right diagonals. □

Remarks: Using the above method, it can be shown that $a_1 > a_j$ for all $j \leq l + 1$. This means that the entry in box $(i, 1)$ is strictly bigger than entries in boxes (j, k) where $j > i$ or when $j = i, k \leq \lambda_{i+1} + 1$.

Theorem 3.2 *Let $\boldsymbol{a} \to (P, Q)$ and let* $\mathrm{sh}(P \Downarrow) = (\mu_1, \mu_2, \cdots, \mu_l)$. *Then, the longest strictly decreasing subsequence in $\boldsymbol{a}$ has length μ_1.*

Proof: First, we will verify that any longest strictly decreasing subsequence in π_P has length μ_1. Clearly, the decreasing part of the first row achieves this length. Next, we will show that this length is preserved by the B-Coxeter-Knuth relations.

For the first part, we look at how a strictly decreasing subsequence $\boldsymbol{d}$ is distributed across the rows of P. It can only pick up at most 1 element in the increasing part of each row. By Theorem 3.1, $P \Downarrow$ is strictly decreasing in rows and top-left to bottom-right diagonals. This means that any decreasing subsequence consisting only of entries in $P \Downarrow$ cannot have length more than μ_1.

Suppose $\boldsymbol{d}$ contains an entry in the increasing part of some row. We examine this situation for two consecutive rows SR as in the proof of Theorem 3.1. So, let b_j be in $\boldsymbol{d}$ where $j > k$. Let b_i be the smallest number with $i \leq k$ such that $b_i > b_j$. We claim that $a_{i+1} \geq b_j$. If b_i does not exist, we note that $a_1 > b_j$.

Let P' be the insertion tableau of $b_1 \cdots b_l a_1 \cdots a_j$.

$$P' = \begin{array}{|c|c|c c|c|c|c c|c|c c|c|}\hline c_1 & c_2 & \bullet\ \bullet & c_i & c_{i+1} & \bullet\ \bullet & b_j & \bullet\ \bullet & c_{l+1} \\\hline & b_1 & b_2 & \bullet\ \bullet & b_i & \bullet\ \bullet\ \bullet\ \bullet & b_{j-1} \\\hline\end{array}$$

In P', b_j is in the decreasing part of the first row and must lie to the right of c_i since $c_i > b_i > b_j$. So, it has to be in some box $(1, f)$ where $i < f \leq g$. After inserting $a_{j+1} \cdots a_h$ into P, we eventually have a_f in box $(1, f)$. This shows that $a_f \geq b_j$. Thus, $a_{i+1} \geq b_j$. This shows that if $\boldsymbol{d}$ picks up a number in the increasing part of P_j, it can then take 1 less number from $P \Downarrow$. Hence, $|\boldsymbol{d}|$ is less than μ_1.

For the second part, we can verify that the length of the longest strictly decreasing subsequence is preserved by the B-Coxeter-Knuth relations explicitly. This is just like the proof of Theorem 1.20 in Chapter 1. However, this is tedious. Another method is to note that the elementary B-Coxeter-Knuth relations are refinements of the Coxeter-Knuth relations except for the special relation $0101 \sim 1010$. In [3], we know that the Coxeter-Knuth relations preserve the length of the greatest decreasing subsequence. Our special relation $0101 \sim 1010$ also preserves this length. Hence, the B-Coxeter-Knuth relations preserve this length too. We are done. $\square$

Using what we have achieved so far, we will give an example of G_w that has an easy description in terms of the Schur P-functions.

Theorem 3.3

$$G_{\bar{n}\cdots\bar{2}\bar{1}}(\boldsymbol{x}) = P_{(n, n-1, \cdots, 2, 1)}(\boldsymbol{x})$$

Proof: Let $w = \bar{n} \cdots \bar{2}\bar{1}$. If $P \in \mathrm{SDT}(w)$, then n rows of P each contains a 0. Therefore, $|P \Downarrow| \geq n(n-1)/2$. But, $l(w) = n(n-1)/2$. This forces $P = P \Downarrow$ and $l(P) = n$. There is only one such P. Thus, $\mathrm{SDT}(w)$ contains only

$$
\begin{array}{|c|c|c c c c|c|}
\hline
n-1 & n-2 & \bullet & \bullet & \bullet & \bullet & 0 \\
\hline
 & n-2 & \bullet & \bullet & \bullet & \bullet & 0 \\
\hline
\end{array}
$$

$$\bullet\ \bullet\ \bullet\ \bullet$$

$$
\begin{array}{|c|c|}
\hline
1 & 0 \\
\hline
 & 0 \\
\hline
\end{array}
$$
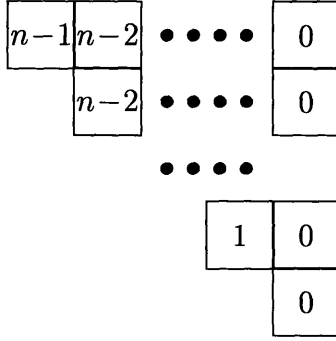
This gives

$$G_w(x) = P_{(n,n-1,\cdots,2,1)}(x)$$

$$\square$$

Theorem 3.4 *Let* $w = \bar{n}w_2w_3\cdots w_n$ *and* $v = w_2w_3\cdots w_n n$ *be 2 elements of* B_n. *Also, let* $\mathrm{SDT}_n(w) = \{P \in \mathrm{SDT}(w) : |P_1| = n\}$. *Then, there exists an injection* Φ *from* $\mathrm{SDT}_n(w)$ *into* $\mathrm{SDT}(v)$. *Furthermore, if* $\forall a \in R(v)$, *the length of the longest unimodal subsequence in* a *is strictly less than* n, *then* Φ *is a bijection.*

Proof: Let $a \in R(w)$. The longest strictly decreasing subsequence in a has to be $n-1\ n-2\cdots 1\ 0$. This corresponds to a sequence of simple reflections that affect the number n. So, if $P \in \mathrm{SDT}_n(w)$, by Theorem 3.2,

$$P_1 = n-1\ n-2\cdots 1\ 0$$

Define $\Phi(P)$ to be the tableau obtained by removing the first row of P. It is obvious that $\Phi(P)$ is a standard decomposition tableau and

$$
\begin{aligned}
\Phi(P) \ &\in\ \mathrm{SDT}(ws_0s_1\cdots s_{n-1}) \\
&=\ \mathrm{SDT}(v)
\end{aligned}
$$

Clearly, all $P \in \mathrm{SDT}_n(w)$ are determined by $\Phi(P)$. So, Φ is the desired injection.

Suppose $\forall a \in R(v)$ that the length of the longest unimodal subsequence is strictly less than n. Let $P' \in \mathrm{SDT}(v)$. Define $\Psi(P')$ by adding $n-1\ n-2\cdots 1\ 0$ to P'.

Since the longest unimodal subsequence in P' has length strictly less than n, the longest unimodal sequence in $\pi_{\Psi(P')}$ has to be the top row of $\Psi(P')$. Hence, $\Psi(P')$ is a standard decomposition tableau and

$$\begin{aligned}\Psi(P') \ &\in \ \mathrm{SDT}_n(vs_{n-1}\cdots s_1 s_0)\\ &= \ \mathrm{SDT}_n(w)\end{aligned}$$

Clearly, Φ and Ψ are inverses of each other and therefore Φ is a bijection between the 2 sets. $\qquad\square$

The following case is interesting. It appears in [1] and it shows that any Schur P-function can be written as a B_n stable Schubert polynomial.

Corollary 3.5 ([1, Proposition 3.14])
Let $w = \bar{\lambda_1}\bar{\lambda_2}\cdots\bar{\lambda_l}123\cdots\widehat{\lambda_l}\cdots\widehat{\lambda_{l-1}}\cdots\cdots\widehat{\lambda_1}\cdots$ where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ is a shifted shape. Then,

$$G_w(\boldsymbol{x}) = P_\lambda(\boldsymbol{x})$$

Here, $\hat{k}$ means omitting k.

Proof: We will prove by induction on l.

Case: $l = 1$. Then, $w = \bar{\lambda_1}123\cdots\widehat{\lambda_1}\cdots$ Clearly, $\mathrm{SDT}(w)$ contains only

λ_1-1	λ_1-2	$\bullet\,\bullet\,\bullet$	1	0

Hence, $G_w(\boldsymbol{x}) = P_\lambda(\boldsymbol{x})$.

Case: $l > 1$. It can be shown that $\mathrm{SDT}_n(w) = \mathrm{SDT}(w)$. Then using the previous theorem, Φ is a bijection between $\mathrm{SDT}(w)$ and $\mathrm{SDT}(v)$ where

$$v = \bar{\lambda_2}\cdots\bar{\lambda_l}123\cdots\widehat{\lambda_l}\cdots\widehat{\lambda_{l-1}}\cdots\cdots\widehat{\lambda_2}\cdots$$

By the induction hypothesis, $\mathrm{SDT}(v)$ contains only

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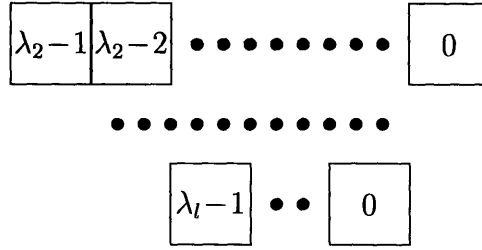
Hence, SDT(w) contains only
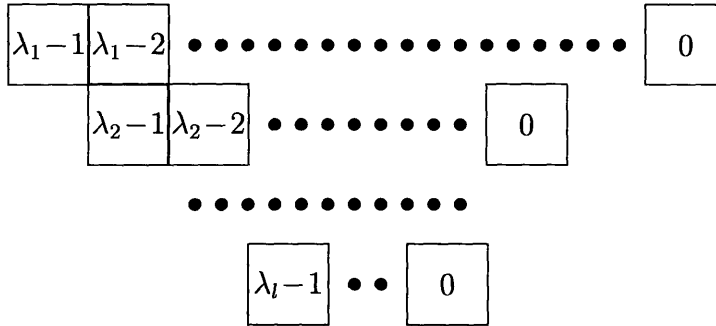
and

$$G_w(x) = P_\lambda(x)$$

□

Using the property of the decreasing parts, we arrive at a generalization of Corollary 1.23 on the length of a unimodal subsequence.

Theorem 3.6 *Let $a = a_1 a_2 \cdots a_m \in R(w)$. Suppose*

$$a \xrightarrow{K} (P, Q)$$

If a_m is in box $(1, k)$ in P then the longest unimodal subsequence ending in a_m has length k.

Proof: Assuming that a_m is in box $(1, k)$ in P, we will use induction on k.

Case: $k = 1$. This forces $a = a_1$ and the result is trivial.

Case: $k > 1$. Let the longest unimodal subsequence of a ending in a_m have length g. Suppose $a_{i_1} a_{i_2} \cdots a_{i_{g-1}} a_m$ is one such subsequence. Then $a_{i_1} a_{i_2} \cdots a_{i_{g-1}}$ is

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a unimodal subsequence ending in $a_{i_{g-1}}$ of longest length. By induction hypothesis, when we apply the Kraśkiewicz insertion on $a_1 a_2 \cdots a_{i_{g-1}}$, $a_{i_{g-1}}$ is the entry in box $(1, g-1)$.

If $a_{i_{g-1}} < a_m$, when we continue the insertion with $a_{i_{g-1}+1} \cdots a_m$, the entry in $(1, g-1)$ will be replaced by smaller numbers unless $(1, g-1)$ joins the decreasing part of the first row. In either case, $k \geq g$.

If $a_{i_{g-1}} > a_m$, this means that $a_{i_1} a_{i_2} \cdots a_{i_{g-1}}$ is a decreasing sequence. Let P' be the insertion tableau of $a_1 a_2 \cdots a_{m-1}$. By Theorem 3.2, this means that $|P'_1 \downarrow| \geq g-1$. This in turn would mean that $k \geq g$.

Now, let the entry in box $(1, k-1)$ of P be b. We examine the insertion of $a_1 a_2 \cdots b$. By induction hypothesis, the longest unimodal subsequence of $\boldsymbol{a}$ ending in b has length $k-1$. Pick such a subsequence $a_{j_1} a_{j_2} \cdots a_{j_{k-2}} b$. If $b < a_m$, we can append a_m and get a unimodal subsequence ending in a_m. This shows that $g \geq k$. If $b > a_m$, this means that

$$|P_1 \downarrow| \geq k \;\; \Rightarrow \;\; |P'_1 \downarrow| \geq k-1$$

By Theorem 3.2, $a_1 a_2 \cdots a_{m-1}$ contains a strictly decreasing subsequence of length $k-1$. Clearly, by appending a_m to it, we would get a unimodal subsequence of length k. This implies $g \geq k$.

Combining the above two inequalities on g and k, we get $g = k$ as desired. $\qquad \square$

3.2 The Symmetric Group

The symmetric group S_n can be considered as a subgroup of B_n generated by $s_i, i > 0$. In 1-line notation, S_n is just the subgroup of signed permutations with no signs. Hence, we can both apply the Edelman-Greene insertion and the Kraśkiewicz insertion on any reduced words of elements in S_n. We wish to explore the connections between these two insertions. Through them, we hope to relate the stable Schubert polynomials for S_n and B_n. We will assume that the reader is familiar with the

Edelman-Greene insertion. A description can be found in [3]. We will also assume all the results about the Edelman-Greene insertion that are used here. To distinguish the tableaux obtained by Edelman-Greene insertion and that by Kraśkiewicz insertion, we will denote the insertion tableau of the Edelman-Greene insertion by $\tilde{P}$ and the recording tableau by $\tilde{Q}$. So,

$$a \overset{EG}{\Rightarrow} (\tilde{P}, \tilde{Q})$$

represents the Edelman-Greene insertion of a and

$$a \overset{K}{\Rightarrow} (P, Q)$$

represents the Kraśkiewicz insertion. The set of tableaux $\tilde{P}$ that is obtained from the Edelman-Greene insertion of all reduced words of $w \in S_n$ is denoted by $\text{SDT}_S(w)$.

We will use $F_w(x)$ to denote the S_n stable Schubert polynomial for $w \in S_n$. This is denoted as $G_w(x)$ in [5]. We will follow the definition in that paper.

Theorem 3.7

$$F_w(x) = \sum_{\tilde{P} \in \text{SDT}_S(w)} s_{\text{sh}(\tilde{P})}(x)$$

For all $w \in S_n$, since $l_0(w) = 0$, we can describe the G_w directly in terms of generalized sequences; that is

$$G_w(x) = \sum_{a \in R(w)} \sum_{f \in K'(a)} x_{|f_1|} x_{|f_2|} \cdots x_{|f_m|}$$

Now we need to introduce the superfication operation. The *superfication* of a Schur function is defined as follows:

$$s_\lambda(x/x) = \omega_y s_\lambda(x, y)|_{y=x}$$

Here, $s_\lambda(x, y)$ is the Schur function in two set of variables. ω_y acts on the y variables only and it is an involution on Λ defined by

$$\omega_y e_k(y) = h_k(y)$$

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The superfication is then extended to any symmetric function by linearity. The following alternate description of $s_\lambda(\pmb{x}/\pmb{x})$ is very useful.

Theorem 3.8

$$s_\lambda(\pmb{x}/\pmb{x}) = \sum_T \pmb{x}^T$$

where the sum is over all Q-semistandard Young tableau.

The reader can refer to [16, Section 2.7] for more details. Also, it is not difficult to show using this description that

$$s_{\lambda^t}(\pmb{x}/\pmb{x}) = s_\lambda(\pmb{x}/\pmb{x})$$

where λ^t is the transpose of λ.

We will also need some results about the Edelman-Greene insertion.

Theorem 3.9 ([3, Lemma 6.28]) *Let $\pmb{a} \in R(w)$ for some $w \in S_n$. Let a_j, a_{j+1} be 2 consecutive elements in $\pmb{a}$. We apply the Edelman-Greene insertion on $\pmb{a}$.*

1. *If $a_j > a_{j+1}$, then the insertion of a_{j+1} will end in a strictly lower row than that of a_j*

2. *If $a_j < a_{j+1}$, the insertion of a_{j+1} will end in a higher row or the same row as that of a_j.*

The reader is asked to refer to [3] for a proof. The next result is known to Stembridge. It also appears in [4, Corollary 8.1] and [1, Proposition 3.17]. We give a direct combinatorial proof.

Theorem 3.10 *Let $w \in S_n$, then*

$$G_w(\pmb{x}) = F_w(\pmb{x}/\pmb{x})$$

Proof: We will use the tableau description of the superfication of a Schur function. We will also need the Edelman-Greene insertion to give a bijection, Φ between the

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$\{(\boldsymbol{a}, \boldsymbol{f}) : \boldsymbol{a} \in R(w), \boldsymbol{f} \in K'(\boldsymbol{a})\}$ and $\{(\tilde{P}, T) : \tilde{P} \in \mathrm{SDT}_S(w), T$ is a Q-semistandard Young tableau, $\mathrm{sh}(T) = \mathrm{sh}(\tilde{P})\}$.

Let $w \in S_n$ and $\boldsymbol{a} \in R(w)$. Given $(\boldsymbol{a}, \boldsymbol{f})$ where $\boldsymbol{f}$ is an $\boldsymbol{a}$-compatible generalized sequence, we apply the Edelman-Greene insertion on $\boldsymbol{a}$ to get

$$\boldsymbol{a} \stackrel{EG}{\Longrightarrow} (\tilde{P}, \tilde{Q})$$

Construct a Young diagram with the same shape as $\tilde{Q}$ and fill each box with f_j when the corresponding box in $\tilde{Q}$ has entry j. This will be our T and we let $\Phi(\boldsymbol{a}, \boldsymbol{f}) = (\tilde{P}, T)$.

We claim that T is Q-semistandard. Clearly, T is weakly increasing in both columns and rows since

$$j < k \;\Rightarrow\; f_j \le f_k$$

Suppose $f_j = f_{j+1} = \cdots = f_k = \bar{l}$. Then $a_j a_{j+1} \cdots a_k$ is a decreasing sequence. By Theorem 3.9, their insertion will end in different rows and in $\tilde{Q}$, the boxes with entries $j, j+1, \cdots, k$ will form a vertical strip. This shows that T is row strict with respect to barred numbers. Similarly, let $f_j = f_{j+1} = \cdots = f_k = l$. Then, $a_j a_{j+1} \cdots a_k$ is an increasing sequence. By Theorem 3.9, the boxes with the entries $j, j+1, \cdots, k$ form a horizontal strip in $\tilde{Q}$. So, T is Q-semistandard.

Now given $(\tilde{P}, T)$, we first construct $\tilde{Q}$ by taking a Young diagram with shape same as T and filling each box with consecutive numbers starting from 1 in the following manner:

1. the entries preserve the order of the numbers in T

2. when more than 1 box in T has the same barred number, we fill these boxes consecutively from top to bottom

3. when more than 1 box in T has the same unbarred number, we fill the boxes consecutively from left to right

The description is unambiguous as the boxes with the same barred number must form a vertical strip and those that contain the same unbarred number must form a

horizontal strip. This new tableau is a standard Young tableau and we denote it as $\tilde{Q}$. Now, applying the inverse Edelman-Greene insertion, we have

$$(\tilde{P}, \tilde{Q}) \overset{EG^{-1}}{\longrightarrow} \boldsymbol{a}$$

where $\boldsymbol{a} \in R(w)$. Let $\boldsymbol{f}$ be the content of T laid out in weakly increasing order. Then, define $\Psi(\tilde{P}, T) = (\boldsymbol{a}, \boldsymbol{f})$.

We have to show that $\boldsymbol{f}$ is $\boldsymbol{a}$-compatible. Let $f_j = f_{j+1} = \cdots = f_k = \bar{l}$. In T, these entries are in a vertical strip and in $\tilde{Q}$ the corresponding entries $j, j+1, \cdots k$ are also in a vertical strip with j in the top box and $j+1$ in the second box and so on. So, by Theorem 3.9 the corresponding subsequence $a_j a_{j+1} \cdots a_k$ has to be strictly decreasing.

Similarly, if $f_j = f_{j+1} = \cdots = f_k = l$. The same argument would show that $a_j a_{j+1} \cdots a_k$ is a strictly increasing sequence. Hence, $\boldsymbol{f}$ is indeed $\boldsymbol{a}$-compatible.

Clearly, Φ and Ψ are inverses of each other and $\text{con}(T) = \boldsymbol{f}$. So,

$$
\begin{aligned}
G_w(\boldsymbol{x}) &= \sum_{\boldsymbol{a} \in R(w)} \sum_{\boldsymbol{f} \in K'(\boldsymbol{a})} x_{|f_1|} x_{|f_2|} \cdots x_{|f_m|} \\
&= \sum_{\tilde{P} \in \text{SDT}_S(w)} \sum_{\text{sh}(T) = \text{sh}(P)} \boldsymbol{x}^T \\
&= F_w(\boldsymbol{x}/\boldsymbol{x})
\end{aligned}
$$

$\square$

Next, we are going to present an application of results in the S_n stable Schubert polynomials into B_n stable Schubert polynomials. In [10], some focus is given to answering the question when S_n stable Schubert polynomials F_w is also a Schur function. We state the following result without proof. Given a permutation, $w \in S_n$, we can define its inversion table, $I(w)$ to be the set $\{(i,j) : i < j, w_i > w_j\}$. Next, let $r(w)$ to be sequence of numbers, $(r_1, r_2, \cdots, r_{n-1})$ where r_k is the number of $(i,j) \in I(w)$ such that $i = k$. If we rearrange $r(w)$ into a decreasing sequence, we call it the shape of w and denote it by $\tilde{r}(w)$.

Theorem 3.11 ([14, Theorem 4.1]) *Let $w \in S_n$.*

$$F_w(x) = s_\lambda(x)$$

for some normal shape λ iff w is 2143-avoiding. Furthermore,

$$\tilde{r}(w) = \lambda$$

Conversely, given any Schur function, s_λ, there is a 2143-avoiding permutation w such that

$$F_w(x) = s_\lambda(x)$$

Please see [11, 1.27] and [14, Theorem 4.1] for a proof of this result. Combining this result and Theorem 3.10, we get

$$
\begin{aligned}
F_w(x) &= s_\lambda(x) \\
\Rightarrow \quad G_w(x) &= s_\lambda(x/x) \\
\Rightarrow \quad s_\lambda(x/x) &= \sum_{R \in \mathrm{SDT}(w)} Q_{\mathrm{sh}(R)}(x)
\end{aligned}
$$

This gives us an alternate proof of the result:

Corollary 3.12 ([16, Section 7.3]) *$s_\lambda(x/x)$ can always be expressed as a nonnegative integer combination of Q_μ.*

3.3   More Edelman-Greene Insertion

In the previous section, we have shown a connection between the Edelman-Greene insertion and the Kraśkiewicz insertion. In this section, we will show another. Let w_S denote the longest element of S_n which is $n\ n-1 \cdots 2\ 1$ in 1-line notation. We need the standard decomposition tableau, U of the signed permutation $\bar{n} \cdots \overline{32}\bar{1}$. As have been shown earlier in Theorem 3.3, $\mathrm{SDT}(\bar{n} \cdots \overline{32}\bar{1})$ contains only

$$U = \begin{array}{|c|c|c|c|c|}
\hline
n-1 & n-2 & \bullet\bullet\bullet\bullet & & 0 \\
\hline
 & n-2 & \bullet\bullet\bullet\bullet & & 0 \\
\hline
\end{array}$$

(diagram of the tableau U with entries $n-1$, $n-2$, $\bullet\bullet\bullet\bullet$, 0 in the first row; $n-2$, $\bullet\bullet\bullet\bullet$, 0 in the second row; $\bullet\bullet\bullet\bullet$; 1, 0; 0)

Let

$$\pi_U \overset{K}{\to} (U,V)$$

It is easy to verify that V is column-wise.

Definition 3.13 *Let Q be a shifted Young tableau and S be a shifted Young tableau that is contained in Q. By this we mean that all the entries in S also appear in the same boxes in Q. Then, we define $Q - S$ to be the skew shifted Young tableau that is obtained by deleting all the boxes in Q that are also in S.*

Theorem 3.14 *Let $a = a_1 a_2 \cdots a_m$ be a reduced word of w in S_n and let*

$$a \overset{EG}{\to} (\tilde{P}, \tilde{Q})$$

Suppose the Kraśkiewicz insertion of a into (U,V) gives (P,Q). Then,

1. $P{\Downarrow} = U$

2. The increasing parts of each row of P form an unshifted tableau, denoted by $P{\Uparrow}$. If we add $i - 1$ to each box in the ith row of $P{\Uparrow}$, we get $\tilde{P}$.

3. If we and subtract $n(n-1)/2$ from every box in $Q - V$, we get $\tilde{Q}$.

Proof: First, we note that

$$\pi_U a \;\in\; R(\bar{n} \cdots \bar{2}\bar{1}\, w)$$

$$= R(\overline{n - w_1 + 1}\; \overline{n - w_2 + 1} \cdots \overline{n - w_n + 1})$$

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So, the Kraśkiewicz insertion makes sense. We will use induction on m, length of the word $\boldsymbol{a}$, to prove the theorem.

Case: $m = 1$. In the Kraśkiewicz insertion, a_1 is appended to the first row of U and the result is trivial.

Case: $m > 1$. Let (P', Q') be the shifted tableaux that are obtained from the Kraśkiewicz insertion of $a_1 a_2 \cdots a_{m-1}$ and let $(\tilde{P}', \tilde{Q}')$ be the unshifted tableaux that are obtained from the Edelman-Greene insertion of $a_1 a_2 \cdots a_{m-1}$. By the induction hypothesis, (P', Q') and $(\tilde{P}', \tilde{Q}')$ are related as described above. Now compare the Edelman-Greene insertion of a_m into $\tilde{P}'$ and the Kraśkiewicz insertion of a_m into P'. Let us denote the entries in $\tilde{P}'$ by $b_{i,j}$ and $\mathrm{sh}(\tilde{P}') = \lambda$. Beginning with the first row, let us look at the possible cases.

Subcase: a_m gets appended to $\tilde{P}'_1$. In this case, a_m also gets appended to the first row of P' under Kraśkiewicz insertion.

Subcase: $\tilde{P}'_1$ does not contain any number equal to a_m. Then, for the Edelman-Greene insertion, we get

$$b_{1,1} b_{1,2} \cdots b_{1,j} \cdots b_{1,\lambda_1} \overset{in}{\leftarrow} a_m$$
$$= \quad b_{1,j} \overset{out}{\leftarrow} b_{1,1} b_{1,2} \cdots a_m \cdots b_{1,\lambda_1}$$

where $b_{1,j}$ is the smallest element bigger than a_m. Now, in the Kraśkiewicz insertion, we find

$$n-1 \; n-2 \cdots 1 \; 0 | b_{1,1} b_{1,2} \cdots b_{1,j} \cdots b_{1,\lambda_1} \overset{in}{\leftarrow} a_m$$
$$\overset{b_{1,j}}{\leftarrow}$$
$$= \quad n-1 \; n-2 \cdots 1 \; 0 \mid b_{1,1} b_{1,2} \cdots a_m \cdots b_{1,\lambda_1}$$
$$= \quad b_{1,j} - 1 \overset{out}{\leftarrow} n-1 \; n-2 \cdots 1 \; 0 | b_{1,1} b_{1,2} \cdots a_m \cdots b_{1,\lambda_1}$$

Subcase: $\tilde{P}'_1$ contains a number same as a_m. For the Edelman-Greene insertion,

$$b_{1,1} b_{1,2} \cdots b_{1,j} \cdots b_{1,\lambda_1} \overset{in}{\leftarrow} a_m$$
$$= \quad a_m + 1 \overset{out}{\leftarrow} b_{1,1} b_{1,2} \cdots b_{1,j} \cdots b_{1,\lambda_1}$$

where $b_{1,j} = a_m$. The Kraśkiewicz insertion will give

$$n - 1 \; n - 2 \cdots 1 \; 0 | b_{1,1} b_{1,2} \cdots b_{1,j} \cdots b_{1,\lambda_1} \overset{in}{\leftarrow} a_m$$
$$a_{\underleftarrow{m}} + 1$$
$$= \; n - 1 \; n - 2 \cdots 1 \; 0 \mid b_{1,1} b_{1,2} \cdots b_{1,j} \cdots b_{1,\lambda_1}$$
$$= \; a_m \overset{out}{\leftarrow} n - 1 \; n - 2 \cdots 1 \; 0 | b_{1,1} b_{1,2} \cdots a_m \cdots b_{1,\lambda_1}$$

So the number to be inserted into P_2' is always 1 less than the number to be inserted into $\tilde{P}_2'$. $P_1{\uparrow} = \tilde{P}_1$ and $P_1{\downarrow}$ is unchanged. By repeating the process, we see that $P{\Downarrow} = U$ remains unchanged and the two different insertions of a_m will end up in the same row such that $\tilde{P}$ and $P{\Uparrow}$ have the same shape. This means that $Q - V$ is just $\tilde{Q}$ with all its entries increased by $n(n-1)/2$. But as the insertion gets to a lower row, the difference between the numbers to be inserted into the next row under the Edelman-Greene insertion and under Kraśkiewicz insertion gets bigger. This will result in that the entries in $\tilde{P}_i$ are $i - 1$ more than the corresponding entries in $(P{\Uparrow})_i$ as stated in the theorem. $\qquad \square$

This theorem allows us to apply results about the Edelman-Greene insertion into the Kraśkiewicz insertion. For starters, we are able to prove a conjecture by Stembridge([15]). This has also been proved by Billey and Haiman in [1, Equation (3.15)].

Theorem 3.15 *Let $w \in S_n$. Denote $\bar{w}$ as the element of B_n obtained from w by putting a bar over all w_i. There exists a bijection Φ from $\mathrm{SDT}_S(w_S w)$ to $\mathrm{SDT}(\bar{w})$. Furthermore,*

$$G_{\bar{w}}(x) \; = \; \sum_{\tilde{P} \in \mathrm{SDT}_S(w_S w)} P_{\delta_n + \mathrm{sh}(\tilde{P})}(x)$$

where $\delta_n = (n, n - 1, \cdots, 2, 1)$.

Proof: We will now describe a map Ψ from $\mathrm{SDT}(\bar{w})$ to $\mathrm{SDT}_S(w_S w)$. Let $P \in \mathrm{SDT}(\bar{w})$. Then any reduced word of $\bar{w}$ must contain n 0's. By Theorem 3.1, $P{\Downarrow}$ has to be the tableau, U. This means that $P \in \mathrm{SDT}(\bar{w})$ is uniquely determined by the increasing parts of every row and they form an unshifted tableau $P{\Uparrow}$. Let $\Psi(P)$ be the unshifted tableau that is obtained by adding $i - 1$ to each box in the ith row of $P{\Uparrow}$. From

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the description, it is not obvious that $\Psi(P) \in \text{SDT}_S(w_S w)$. Let Q be the standard Young tableau such that

1. $\text{sh}(Q) = \text{sh}(P)$

2. Q contains V and the other boxes have their entries row-wise

Then, by using the inverse Kraśkiewicz insertion,

$$(P, Q) \overset{K^{-1}}{\rightarrow} \pi_U \boldsymbol{a}$$

$\boldsymbol{a}$ is reduced and does not contain any 0.

$$
\begin{aligned}
\boldsymbol{a} &\in R(\bar{n} \cdots \bar{2}\bar{1}\ \bar{w}) \\
&= R(n - w_1 + 1\ n - w_2 + 1 \cdots n - w_n + 1) \\
&= R(w_S w)
\end{aligned}
$$

By Theorem 3.14, the Edelman-Greene insertion tableau of $\boldsymbol{a}$ is $\Psi(P)$. So, $\Psi(P) \in \text{SDT}_S(w_S w)$.

Conversely, let $\tilde{P} \in \text{SDT}_S(w_S w)$. We define a map $\Phi : \text{SDT}_S(w_S w) \to \text{SDT}(\bar{w})$. We first subtract $i - 1$ from each box in the ith row of $\tilde{P}$. Then, we append this to U to get the shifted tableau $\Phi(\tilde{P})$. Again from Theorem 3.14, $\Phi(\tilde{P})$ is the standard decomposition tableau that is obtained by the Kraśkiewicz insertion of $\pi_U \pi_{\tilde{P}}$. So, $\Phi(\tilde{P}) \in \text{SDT}(\bar{w})$.

Clearly, Φ and Ψ are inverses of each other. Furthermore, $\text{sh}(\Phi(\tilde{P})) = \delta_n + \text{sh}(\tilde{P})$ and $l_0(\bar{w}) = l(\Phi(\tilde{P})) = n$. Hence, we get

$$G_{\bar{w}}(\boldsymbol{x}) = \sum_{\tilde{P} \in \text{SDT}_S(w_B w)} P_{\delta_n + \text{sh}(\tilde{P})}(\boldsymbol{x})$$

$\square$

Example: Consider the permutation $w = 21453$. Then $\bar{w} = \bar{2}\bar{1}\bar{4}\bar{5}\bar{3}$ and $w_S w = 45213$. $\text{SDT}_S(45213)$ contains only

$$
\begin{array}{|c|c|c|}
\hline
1 & 2 & 4 \\
\hline
2 & 3 \\
\cline{1-2}
3 & 4 \\
\cline{1-2}
\end{array}
$$

Therefore, SDT($\bar{2}\bar{1}4\bar{5}\bar{3}$) contains only

$$
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
4 & 3 & 2 & 1 & 0 & 1 & 2 & 4 \\
\hline
  & 3 & 2 & 1 & 0 & 1 & 2 \\
\cline{2-7}
  &   & 2 & 1 & 0 & 1 & 2 \\
\cline{3-7}
  &   &   & 1 & 0 \\
\cline{4-5}
  &   &   &   & 0 \\
\cline{5-5}
\end{array}
$$

The reader can verify that

$$G_{\tilde{w}}(x) = P_{(8,6,5,2,1)}(x)$$

3.4 The Recording Tableau

In this section, we will explore the properties about the recording tableau of the Kraśkiewicz insertion. There is an operation called the evacuation which can be applied on a standard shifted Young tableau. This operation is studied in detail in [6] and in [7]. We will show how this relates to the recording tableau of the Kraśkiewicz insertion. The main results are Theorem 3.24 and Corollary 3.28.

For a start, we would like to characterize the recording tableaux of π_P where P is a standard decomposition tableau. But first, some definitions and notations.

Definition 3.16 *Let Q be a shifted Young tableau. Denote by $Q|_j$ the shifted Young tableau that is obtained from Q by deleting all the boxes that have entries strictly bigger than j.*

Recall from Section 1.2, the definition a rim hook.

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Lemma 3.17 *Let $P \in \mathrm{SDT}(w)$ and*

$$\pi_P \overset{K}{\hookrightarrow} (P, Q)$$

with $\mathrm{sh}(P) = (\lambda_1, \lambda_2, \cdots, \lambda_l)$. *Then for all* $i \geq 1$,

1. $\mathrm{sh}(Q|_{\lambda_i + \lambda_{i+1} + \cdots + \lambda_l}) = (\lambda_i, \lambda_{i+1}, \cdots, \lambda_l)$

2. $Q|_{\lambda_i + \lambda_{i+1} + \cdots + \lambda_l} - Q|_{\lambda_{i+1} + \cdots + \lambda_l}$ *is a connected rim hook with the entries increasing down the vertical part of the rim hook and then across the horizontal part*

Proof: Use induction on l.

Case: $l = 1$. Obvious.

Case: $l > 1$. Let P' be the standard decomposition tableau obtained from P by deleting the first row, P_1. Clearly,

$$\pi_P = \pi_{P'} P_1$$

and

$$\pi_{P'} \overset{K}{\hookrightarrow} (P', Q')$$

where $\mathrm{sh}(P') = (\lambda_2, \lambda_3, \cdots, \lambda_l)$ and $Q' = Q|_{\lambda_2 + \lambda_3 + \cdots \lambda_l}$. Since, $l(P') = l(P) - 1$, by the induction hypothesis, for $i > 1$,

$$
\begin{aligned}
\mathrm{sh}(Q|_{\lambda_i + \lambda_{i+1} + \cdots + \lambda_l}) &= \mathrm{sh}(Q'|_{\lambda_i + \lambda_{i+1} + \cdots + \lambda_l}) \\
&= (\lambda_i, \lambda_{i+1}, \cdots, \lambda_l)
\end{aligned}
$$

and $Q'|_{\lambda_i + \lambda_{i+1} + \cdots + \lambda_l} - Q'|_{\lambda_{i+1} + \cdots + \lambda_l} = Q|_{\lambda_i + \lambda_{i+1} + \cdots + \lambda_l} - Q|_{\lambda_{i+1} + \cdots + \lambda_l}$ is a connected rim hook with the entries increasing down the vertical part of the rim hook and then across the horizontal part. Now, it remains to show that $Q - Q'$ is a rim hook with the desired property. Note that

$$
\begin{aligned}
\mathrm{sh}(Q - Q') &= \lambda / (\lambda_2, \lambda_3, \cdots, \lambda_l) \\
&= \{(i, j) : 1 \leq i \leq l, \lambda_{i+1} + i \leq j \leq \lambda_i + i - 1\}
\end{aligned}
$$

Note that the first box in row i of $Q - Q'$ is $(i, \lambda_{i+1} + i)$ and the last box in row $i+1$ is $(i+1, \lambda_{i+1} + i)$. Hence, $Q - Q'$ is a connected rim hook. Now P_1 is a unimodal sequence and by Theorem 2.12, the corresponding entries $\lambda_1 + \lambda_2 + \cdots + \lambda_{l-1} + 1, \lambda_1 + \lambda_2 + \cdots + \lambda_{l-1} + 2, \cdots, \lambda_1 + \lambda_2 + \cdots + \lambda_l$ in Q has to form a rim hook and they increase down the vertical part and then across the horizontal part of the rim hook. $\square$

Given $\boldsymbol{a} \in R(w)$, we know that $\boldsymbol{a}^r \in R(w^{-1})$. We would like to see how the Kraśkiewicz insertion of these two words are related with respect to the recording tableaux. The following lemma provides a nice description for the recording tableau of $(\pi_P)^r$.

Lemma 3.18 *Let $P \in \mathrm{SDT}(w), w \in B_n$. Let*

$$\pi_P \xrightarrow{K} (P, Q)$$
$$(\pi_P)^r \xrightarrow{K} (R, S)$$

Then, S is row-wise. Furthermore, $\mathrm{sh}(P) = \mathrm{sh}(Q) = \mathrm{sh}(R) = \mathrm{sh}(S)$.

Proof: We will do this by induction on $l = l(P)$.

Case: $l = 1$. This is obvious.

Case: $l > 1$. Let $\mathrm{sh}(P) = \lambda$. Let us denote row j of P by P_j. Abusing notation, we will also use P_j to denote the reduced word made up of numbers in P_j. So,

$$\pi_P = P_l P_{l-1} \cdots P_2 P_1$$
$$(\pi_P)^r = P_1^r P_2^r \cdots P_l^r$$

Let us apply the Kraśkiewicz insertion on $(\pi_P)^r$ but let us restrict ourselves to the changes in the first row. Since P_1^r is a unimodal sequence of longest length in $(\pi_P)^r$, none of the numbers in $P_2^r P_3^r \cdots P_l^r$ are appended onto the first row.

$$\emptyset \xleftarrow{in} P_1^r P_2^r \cdots P_l^r = P_1^r \xleftarrow{in} P_2^r \cdots P_l^r$$
$$= A_2 \cdots A_l \xleftarrow{out} R_1$$

where A_i is the sequence of numbers that are bumped out when P_i^r is inserted and

R_1 is the first row of R. From Theorem 2.12, each A_i is a unimodal sequence. Let

$$A_l^r \cdots A_3^r A_2^r \to (U, V)$$

We claim that $\pi_U = A_l^r \cdots A_3^r A_2^r$. That is to say that $A_l^r \cdots A_3^r A_2^r$ is the reduced word of a standard decomposition tableau. To show this, we claim that for all $i > 1$, A_i^r is a unimodal subsequence of maximum length in $A_l^r \cdots A_i^r$. Suppose this is not so. Let

$$A_l^r \cdots A_i^r \to (U', V')$$

and let $\pi_{U'} = U_k' \cdots U_1'$. $|U_1'| > |A_i|$ and $U_k' \cdots U_1' \sim A_l^r \cdots A_i^r$. If we compare the Kraśkiewicz insertions, again restricting ourselves to the first row,

$$\emptyset \overset{in}{\leftarrow} R_1^r A_l^r A_{l-1}^r \cdots A_i^r \;=\; R_1^r \overset{in}{\leftarrow} A_l^r A_{l-1}^r \cdots A_i^r$$
$$=\; P_l P_{l-1} \cdots P_i \overset{out}{\leftarrow} X$$

and

$$\emptyset \overset{in}{\leftarrow} R_1^r U_k' U_{k-1}' \cdots U_1' \;=\; R_1^r \overset{in}{\leftarrow} U_k' U_{k-1}' \cdots U_1'$$
$$=\; W_k W_{k-1} \cdots W_1 \overset{out}{\leftarrow} X$$

where $|W_1| = |U_1'| > |P_i|$. This shows that the resulting insertion tableaux are different. But $R_1^r U_k' \cdots U_1' \sim R_1^r A_l^r \cdots A_i^r$. This is a contradiction. Hence, $A_l^r A_{l-1}^r \cdots A_2^r$ is the reading word of some standard decomposition tableau. So, by the induction hypothesis,

$$A_2 A_3 \cdots A_l \to (R', S')$$

where $R' = R - R_1$ and S' is a row-wise standard Young tableau and $\mathrm{sh}(S') = \mathrm{sh}(V)$. Hence, indeed S is row-wise and $\mathrm{sh}(Q) = \mathrm{sh}(S)$. $\qquad\square$

Since the Kraśkiewicz insertion is a bijection, From the above two lemmas and using the fact that the Kraśkiewicz insertion is a bijection, we conclude that:

Theorem 3.19 *Let $a \in R(w)$.*

1. a is the reading word of a standard decomposition tableau, $P \in \mathrm{SDT}(w)$ iff the recording tableau corresponding to a satisfies the properties described in

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Lemm 3.17.

2. a^r *is the reading word of a standard decomposition tableau,* $P \in \mathrm{SDT}(w^{-1})$ *iff the recording tableau corresponding to* a *is row-wise.*

Theorem 3.20 *Let* $a \in R(w)$. *Suppose*

$$a \overset{K}{\rightsquigarrow} (P, Q)$$
$$a^r \overset{K}{\rightsquigarrow} (R, S)$$

Then,

$$\mathrm{sh}(Q) = \mathrm{sh}(S)$$

Proof:

$$\pi_P \sim a \ \Rightarrow \ (\pi_P)^r \sim a^r$$

Hence, from the previous lemma, $\mathrm{sh}(P) = \mathrm{sh}(R)$. Therefore, $\mathrm{sh}(Q) = \mathrm{sh}(P) = \mathrm{sh}(R) = \mathrm{sh}(S)$ □

We know that the map given by $a \to a^r$ is a bijection between $R(w)$ and $R(w^{-1})$ for all w in B_n. This induces a bijection between $\mathrm{SDT}(w)$ and $\mathrm{SDT}(w^{-1})$ which leads naturally to the next result.

Corollary 3.21 *Let* $w \in B_n$. *Then*

$$G_w(x) = G_{w^{-1}}(x)$$

Proof: Define $\Psi : \mathrm{SDT}(w) \to \mathrm{SDT}(w^{-1})$ by letting $\Psi(P)$ be the insertion tableau of $(\pi_P)^r$. This is a bijection since

$$\pi_{\Psi(P)} \ \sim \ (\pi_P)^r$$
$$\Rightarrow \ (\pi_{\Psi(P)})^r \ \sim \ \pi_P$$

From Theorem 3.20, $\mathrm{sh}(\Psi(P)) = \mathrm{sh}(P)$. Hence,

$$G_w(x) \ = \ \sum_{R \in \mathrm{SDT}(w)} 2^{l(P) - l_0(w)} P_{\mathrm{sh}(R)}(x)$$

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$$= \sum_{R \in \text{SDT}(w)} 2^{l(\Psi(R)) - l_0(w^{-1})} P_{\text{sh}(\Psi(R))}(\boldsymbol{x})$$

$$= G_{w^{-1}}(\boldsymbol{x})$$

□

This corollary can be proved using the bijection of the reduced words themselves and noting that $G_w(\boldsymbol{x})$ is symmetric. The reader can refer to [1, Corollary 3.5] for this alternate method.

The next lemma is long but crucial. It shows how the shape of the recording tableau changes when we remove a number from the beginning of the insertion.

Lemma 3.22 *Let $\boldsymbol{a} \in R(w)$ and*

$$a_1 a_2 \cdots a_{m-1} a_m \overset{K}{\to} (P, Q)$$

$$a_1 a_2 \cdots a_{m-1} \overset{K}{\to} (P', Q')$$

$$a_2 \cdots a_{m-1} a_m \overset{K}{\to} (R, S)$$

$$a_2 \cdots a_{m-1} \overset{K}{\to} (R', S')$$

Then,

$$\begin{array}{ccc}
\text{sh}(P') & \subset & \text{sh}(P) \\
\cup & & \cup \\
\text{sh}(R') & \subset & \text{sh}(R)
\end{array}$$

Furthermore, let $(p, q) = \text{sh}(P) - \text{sh}(P')$ and $(p', q') = \text{sh}(P') - \text{sh}(R')$. Then $\text{sh}(R) \neq \text{sh}(P')$ iff $\text{sh}(P) - \text{sh}(R') = \{(p, q), (p', q')\}$ is not connected.

Proof: Clearly, $\text{sh}(P') \subset \text{sh}(P)$ and $\text{sh}(R') \subset \text{sh}(R)$. Let

$$a_m a_{m-1} \cdots a_2 a_1 \overset{K}{\to} (U, V)$$

$$a_m a_{m-1} \cdots a_2 \overset{K}{\to} (U', V')$$

From Theorem 3.20,

$$\text{sh}(R) = \text{sh}(U') \subset \text{sh}(U) = \text{sh}(P)$$

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This gives one inclusion and we can get the other inclusion by the same method. Next, we will prove the last statement.

($\Rightarrow$) By the inclusion of shapes, $\text{sh}(R) \neq \text{sh}(P')$ forces

$$\text{sh}(P) - \text{sh}(R) = (p', q')$$

Since (p, q) and (p', q') are corner boxes of $\text{sh}(P)$, they cannot be connected.

($\Leftarrow$) We will begin with some simplifications. Without loss of generality, we can assume $p < p'$. Since all the insertion tableaux are not changed when we replace $a_2 a_3 \cdots a_{m-1}$ by $\pi_{R'}$, we can assume that $a_2 a_3 \cdots a_{m-1} = \pi_{R'} = R'_l R'_{l-1} \cdots R'_2 R'_1$. We then have

$$a_1 R'_l R'_{l-1} \cdots R'_2 R'_1 a_m \overset{K}{\to} (P, Q)$$
$$a_1 R'_l R'_{l-1} \cdots R'_2 R'_1 \overset{K}{\to} (P', Q')$$
$$R'_l R'_{l-1} \cdots R'_2 R'_1 a_m \overset{K}{\to} (R, S)$$
$$R'_l R'_{l-1} \cdots R'_2 R'_1 \overset{K}{\to} (R', S')$$

Compare the insertion tableaux P' and R'. Note that

$$p' > 1$$
$$\Rightarrow |P'_1| = |R'_1|$$
$$\Rightarrow P'_1 = R'_1$$

since R'_1 is a unimodal subsequence of longest length in $a_1 R'_l R'_{l-1} \cdots R'_2 R'_1$. Next we proceed to use induction on p.

Case: $p = 1$. So, $|P_1| = |P'_1| + 1$ which implies that $P'_1 a_m$ is unimodal. Therefore,

$$P_1 = P'_1 a_m = R'_1 a_m = R_1$$
$$\Rightarrow |R_1| = |P'_1| + 1$$
$$\Rightarrow \text{sh}(R) \neq \text{sh}(P')$$

Case: $p > 1$. So, $|P_1| = |P'_1|$ which implies that $P'_1 a_m$ is not unimodal. Therefore,

$$P'_1 \overset{in}{\leftarrow} a_m = a'_m \overset{out}{\leftarrow} P_1$$

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Now, $R_1' = P_1'$ and R_1 is also obtained from R_1' by inserting a_m. Hence, $R_1 = P_1$. We then have

$$a_1 R_l' R_{l-1}' \cdots R_2' a_m' \overset{K}{\twoheadrightarrow} (\hat{P}, \hat{Q})$$

$$a_1 R_l' R_{l-1}' \cdots R_2' \overset{K}{\twoheadrightarrow} (\hat{P}', \hat{Q}')$$

$$R_l' R_{l-1}' \cdots R_2' a_m' \overset{K}{\twoheadrightarrow} (\hat{R}, \hat{S})$$

$$R_l' R_{l-1}' \cdots R_2' \overset{K}{\twoheadrightarrow} (\hat{R}', \hat{S}')$$

where $\hat{P}, \hat{P}', \hat{R}, \hat{R}'$ are standard decomposition tableaux obtained from the corresponding tableaux by deleting the first row. Since $\mathrm{sh}(\hat{P}) - \mathrm{sh}(\hat{R}')$ is not connected, by induction hypothesis,

$$\mathrm{sh}(\hat{P}') \neq \mathrm{sh}(\hat{R})$$

$$\Rightarrow \mathrm{sh}(P') \neq \mathrm{sh}(R)$$

$\square$

Next, we will follow some notation in [13, Section 3.11] to define the notion of evacuation. First, we define the *delta operator*, Δ.

Definition 3.23 *Let Q be a standard shifted Young tableau. Define $\Delta(Q)$ to be the resulting tableau after applying the following operations:*

1. remove the entry 1 from Q

2. apply jeu de taquin into this box

3. deduct 1 from each of the remaining boxes

This is essentially the same as [13, Definition 3.11.1]. Note that here, we are applying Δ on shifted Young tableaux. In the notation of [7], $\Delta(Q)$ is the tableau that is obtained by subtracting 1 from every box in $Q(1 \to \infty)$.

Theorem 3.24 *Let $a = a_1 a_2 \cdots a_m \in R(w)$ and suppose*

$$a_1 a_2 \cdots a_m \overset{K}{\twoheadrightarrow} (P, Q)$$

$$a_2 \cdots a_m \overset{K}{\twoheadrightarrow} (R, S)$$

Then,

$$S = \Delta(Q)$$

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Proof: Induct on m.

Case: $m = 1$. Trivial.

Case: $m > 1$. Let

$$a_1 a_2 \cdots a_{m-1} \xrightarrow{K} (P', Q')$$

$$a_2 \cdots a_{m-1} \xrightarrow{K} (R', S')$$

Let $(p,q) = \mathrm{sh}(Q) - \mathrm{sh}(Q')$ and $(p', q') = \mathrm{sh}(Q') - \mathrm{sh}(S')$. This is the same as the setup is Lemma 3.22. By induction hypothesis, $\Delta(Q') = S'$. Note that $Q' = Q|_{m-1}$ and $S' = S|_{m-2}$.

Subcase: $\mathrm{sh}(S) = \mathrm{sh}(Q')$. So, box (p', q') holds the entry $m - 1$ in S and box (p, q) holds the entry m in Q. From Lemma 3.22, $\{(p,q), (p', q')\}$ are connected. When we apply Δ on Q, the jeu de taquin process can be split into two parts. The first part consists of jeu de taquin moves inside $Q|_{m-1}$. It is the same as the jeu de taquin moves when we compute $\Delta(Q|_{m-1})$. Therefore, the box (p', q') is vacated. The second part slides the entry of box (p, q) into (p', q'). It is not difficult to see then that $\Delta(Q) = S$.

Subcase: $\mathrm{sh}(S) \neq \mathrm{sh}(Q')$. Box (p, q) holds the entry $m - 1$ in S and box (p, q) holds the entry m in Q. From Lemma 3.22, $\{(p,q), (p', q')\}$ are not connected. Using the same reasoning as above, we see that when we apply Δ on Q, the jeu de taquin process vacates box (p', q') and does not affect box (p, q). Hence $\Delta(Q) = S$. $\square$

Example: Consider the reduced word 241230 of the signed permutation $\bar{3}2514$. The Kraśkiewicz insertion of 241230 gives

$$\left(\begin{array}{c} \begin{array}{|c|c|c|c|} \hline 4 & 2 & 0 & 3 \\ \hline \end{array} \\ \begin{array}{|c|c|} \hline 2 & 1 \\ \hline \end{array} \end{array} \; , \; \begin{array}{c} \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 5 \\ \hline \end{array} \\ \begin{array}{|c|c|} \hline 3 & 6 \\ \hline \end{array} \end{array} \right)$$

and the insertion applied on 41230 gives

$$\left(\begin{array}{c} \begin{array}{|c|c|c|c|} \hline 4 & 2 & 0 & 3 \\ \hline \end{array} \\ \begin{array}{|c|} \hline 1 \\ \hline \end{array} \end{array} \; , \; \begin{array}{c} \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} \\ \begin{array}{|c|} \hline 5 \\ \hline \end{array} \end{array} \right)$$

The reader can check that the second recording tableau can be obtained by applying Δ on the previous recording tableau.

We take a detour here for an application of Δ. In Section 3.2, we have shown a relation between the Edelman-Greene insertion of w where $w \in S_n$ and the Kraśkiewicz insertion of w. Making use of the previous result, we present a connection between the recording tableaux of the Edelman-Greene insertion and the Kraśkiewicz insertion. Recall that any unshifted Young tableau can be considered as a shifted tableau.

Theorem 3.25 *Let $w \in S_n$ and $\boldsymbol{a} \in R(w)$. Suppose*

$$\boldsymbol{a} \overset{EG}{\to} (\tilde{P}, \tilde{Q})$$
$$\boldsymbol{a} \overset{K}{\to} (P, Q)$$

Then Q is jeu de taquin equivalent to $\tilde{Q}$.

Proof: Recall in Theorem 3.14, we considered the Kraśkiewicz insertion of the reduced word $\pi_U \boldsymbol{a}$ of $\bar{n} \cdots \bar{2}\bar{1}w$ where

$$U = \begin{array}{|c|c|c|c|c|c|}
\hline
n-1 & n-2 & \bullet\ \bullet\ \bullet\ \bullet & \multicolumn{1}{c}{} & 0 \\
\cline{1-2}\cline{5-5}
 & n-2 & \bullet\ \bullet\ \bullet\ \bullet & & 0 \\
\cline{2-2}\cline{5-5}
\end{array}$$

is the only standard decomposition tableau of $\bar{n} \cdots \bar{2}\bar{1}$. Let

$$\pi_U \boldsymbol{a} \overset{K}{\to} (P', Q')$$

Then, we know that Q' contains V where V is the standard Young tableau labelled column-wise and $\text{sh}(V) = \text{sh}(U) = (n, \cdots, 2, 1)$. Also, $\tilde{Q}$ can be obtained from Q' by deleting V and subtracting $n(n-1)/2$ from every remaining box. Now, applying Theorem 3.24 to get rid of π_{U_n} from the insertion, we find that

$$Q = \Delta^{n(n-1)/2}(Q')$$

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This is equivalent to applying jeu de taquin to convert $\tilde{Q}$ into a normal shifted shape. Hence, Q is jeu de taquin equivalent to $\tilde{Q}$. $\qquad\square$

Now, we define evacuation.

Definition 3.26 *Let Q be a standard shifted Young tableau with $|Q| = m$. We define* $\mathrm{ev}(Q)$ *to be a shifted Young tableau of the same shape and box (i,j) has entry $m-k+1$ if $\mathrm{sh}(\Delta^k(Q))$ and $\mathrm{sh}(\Delta^{k-1}(Q))$ differ in box (i,j).*

Again, this is essentially the same as [13, Definition 3.11.1] but we are using it on shifted Young tableaux instead.

[7, Section 8] contains an alternate definition of $\mathrm{ev}(Q)$ and more information on other properties of evacuation. We state a simple lemma without proof.

Lemma 3.27 *Let Q be a standard shifted Young tableau of size m. Then*

$$\mathrm{ev}(\Delta(Q)) = \mathrm{ev}(Q)|_{m-1}$$

This evacuation enables us to give a refinement of Theorem 3.20.

Corollary 3.28 *Let $\boldsymbol{a} \in R(w)$ and*

$$\boldsymbol{a} \;\overset{K}{\leadsto}\; (P, Q)$$
$$\boldsymbol{a}^{\mathrm{r}} \;\overset{K}{\leadsto}\; (R, S)$$

Then,

$$S = \mathrm{ev}(Q)$$

Proof: Let $\boldsymbol{a} = a_1 a_2 \cdots a_m$. We use induction on m.

Case: $m = 1$. Trivial.

Case: $m > 1$. Consider the Kraśkiewicz insertion of the reduced word $a_2 a_3 \cdots a_m$. From Lemma 3.24,

$$a_2 a_3 \cdots a_m \;\overset{K}{\leadsto}\; (P', \Delta Q)$$

Let $(p,q) = \mathrm{sh}(Q) - \mathrm{sh}(\Delta Q)$. By definition, box (p,q) in $\mathrm{ev}(Q)$ has entry m. Applying the induction hypothesis on $a_2 a_3 \cdots a_m$, we get

$$a_m a_{m-1} \cdots a_3 a_2 \overset{K}{\dashrightarrow} (R', S')$$

where $S' = \mathrm{ev}(\Delta(Q)) = \mathrm{ev}(Q)|_{m-1}$. The last equality is from the previous lemma. Now,

$$R = R' \overset{in}{\leftharpoonup} a_1$$

This shows that

$$S|_{m-1} = S' = \mathrm{ev}(Q)|_{m-1}$$

From Theorem 3.20, $\mathrm{sh}(S) = \mathrm{sh}(Q)$ and $\mathrm{sh}(S') = \mathrm{sh}(Q')$. This means that the entry m in S is in the box (p,q) and hence $S = \mathrm{ev}(Q)$. $\qquad\square$

Example: Let $a = 214203$. It is a reduced word of $w = \bar{3}2514$. Applying the Kraśkiewicz insertion on a gives

$$\left(
\begin{array}{c}
\begin{array}{|c|c|c|c|}
\hline
4 & 2 & 0 & 3 \\
\hline
\end{array}
\\
\begin{array}{|c|c|}
\hline
2 & 1 \\
\hline
\end{array}
\end{array}
\;,\;
\begin{array}{c}
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & 6 \\
\hline
\end{array}
\\
\begin{array}{|c|c|}
\hline
4 & 5 \\
\hline
\end{array}
\end{array}
\right)$$

and on $a^r = 302412$, it gives

$$\left(
\begin{array}{c}
\begin{array}{|c|c|c|c|}
\hline
4 & 2 & 1 & 2 \\
\hline
\end{array}
\\
\begin{array}{|c|c|}
\hline
0 & 3 \\
\hline
\end{array}
\end{array}
\;,\;
\begin{array}{c}
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & 4 \\
\hline
\end{array}
\\
\begin{array}{|c|c|}
\hline
5 & 6 \\
\hline
\end{array}
\end{array}
\right)$$

a^r is a reduced word of $w^{-1} = 42\bar{1}53$. It can be verified that the recording tableau can be obtained from each other by applying the evacuation operator. We have hoped that there is a similar operation on the insertion tableau. However, we have failed to find one.

3.5 Promotion Sequence

In [6, Theorem 5.12], the short promotion sequence of a shifted tableau is used to give a bijection between reduced words of w_B, the longest element in B_n and standard shifted Young tableau of shape $(2n - 1, 2n - 3, \cdots, 3, 1)$. The same method gives a bijection between reduced words of w_S, the longest element of S_n and standard Young tableau of shape $(n - 1, n - 2, \cdots, 1)$. In [3, Theorem 7.18], it is shown that the Edelman-Greene insertion on $R(w_S)$ is the inverse operation of the short promotion sequence. The proof is involved but an easier proof has since been found by Fomin and Greene. In this section, we will give a proof that shows that the Kraśkiewicz insertion applied on $R(w_B)$ is the inverse operation of the promotion for the standard shifted Young tableau of shape $(2n - 1, 2n - 3, \cdots, 3, 1)$.

Lemma 3.29 *Let $N = n^2$ and $\boldsymbol{a} = a_1 a_2 \cdots a_{N-1} a_N \in R(w_B)$. If a_0 is a number such that $a_0 a_1 a_2 \cdots a_{N-1} \in R(w_B)$, then*

$$a_0 = a_N$$

Proof:

$$a_0 a_1 a_2 \cdots a_{N-1} = w_B$$
$$\Rightarrow \quad a_0 w_B a_N = w_B$$
$$\Rightarrow \quad a_0 = w_B a_N w_B$$
$$= a_N$$

$\square$

The definition of the promotion operator is in [6, Section 4] and the definition for the short promotion sequence is in [6, Section 5]. We will restate these definitions here but we will restrict ourselves to the case of standard shifted Young tableau of shape $(2n - 1, 2n - 3, \cdots, 3, 1)$.

Definition 3.30 *Let $N = n^2$. Given T a standard shifted Young tableau of shape $(2n - 1, 2n - 3, \cdots, 3, 1)$, we define the promotion operator, $p(T)$ as follows:*

1. delete the largest entry in T

2. apply jeu de taquin into that box

3. put 0 into the box $(1, 1)$

4. add 1 to every box

Give the last box $(i, 2n - 2i + 1)$ *of each row* i, *the label* $n - i$. *Then, the short promotion sequence* $\hat{p}(T) = (r_1, r_2, \cdots, r_N)$ *is the sequence of numbers where* r_i *is the label of the box with the largest entry in the tableau* $p^{N-i}(T)$.

The promotion operation is almost the inverse of the Δ operation. If we take $p(T)$ and apply the Δ operation on it, we get

$$\Delta(p(T)) = T|_{N-1}$$

Note that the largest entry of any standard shifted Young tableau has to be in the last box of some row. Hence, r_i is well defined. Note also that the entry in box $(i, 2n - 2i + 1)$ of the standard decomposition tableau of w_B is $n - i$ which is the label we assigned.

Let us consider the Kraśkiewicz insertion of reduced words of w_B only. Since $\mathrm{SDT}(w_B)$ contains only 1 tableau P, we can define a map Ψ from $R(w)$ to the set of standard shifted Young tableaux of shape $(2n - 1, 2n - 3, \cdots, 3, 1)$ by letting

$$\Psi(\boldsymbol{a}) = Q$$

where Q is the recording tableau obtained when we apply the insertion on $\boldsymbol{a}$. We claim that $\hat{p}$ and Ψ are inverses of each other.

Theorem 3.31 *Let* $\boldsymbol{a} \in R(w_B)$ *and* $\boldsymbol{a} \to (P, Q)$ *Then,*

$$\hat{p}(Q) = \boldsymbol{a}$$

Proof: Let $\boldsymbol{a} = a_1 a_2 \cdots a_N$.

$$a_1 a_2 \cdots a_{N-1} a_N \xrightarrow{K} (P, Q)$$

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From Lemma 3.29, $a_N a_1 a_2 \cdots a_{N-1} \in R(w_B)$. So,

$$a_N a_1 a_2 \cdots a_{N-1} \overset{K}{\rightsquigarrow} (P, S)$$

But, from Theorem 3.24, the recording tableau for $a_1 a_2 \cdots a_{N-1}$ is

$$Q|_{N-1} = \Delta(S)$$

From the previous remarks,

$$\Delta(p(Q)) = Q|_{N-1} = \Delta(S)$$

and since $\mathrm{sh}(Q) = \mathrm{sh}(S)$,

$$p(Q) = S$$

This shows that the promotion operator, p acting on Q corresponds to moving the last number in the reduced word $\boldsymbol{a}$ to the first position.

The largest entry N in Q is obtained when we insert a_N. Suppose it is in box $(i, 2n - 2i + 1)$ of Q. If we observe the insertion procedure of a_N, we note that the first $i - 1$ rows are of the form, $n - j \ n - j - 1 \ \cdots 101 \cdots n - j - 1 \ n - j$ where $1 \leq j < i$. So, each time the number to be inserted into the next row remains a_N. This shows that $a_N = n - i$ which is the label of the box $(i, 2n - 2i + 1)$. Hence,

$$r_N = a_N$$

We can then repeat the procedure on $a_N a_1 a_2 \cdots a_{N-1}$ to show that $r_{N-1} = a_{N-1}$ and so on. Therefore,

$$\hat{p}(Q) = \boldsymbol{a}$$

$\square$

All this can be revised to show that corresponding case for the Edelman-Greene insertion on reduced words of w_S and the short promotion sequence. We will not

provide the details here but just mentioned that there are analogues of Lemma 3.24 and Lemma 3.29 for the Edelman-Greene insertion.

3.6 Shifted Mixed Insertion

Haiman introduced mixed insertion and its shifted analogue in [7]. Its relation to the Worley-Sagan insertion (see [13], [16]) was explored in detail there. In this section, we will show that the Kraśkiewicz insertion is closely related to shifted mixed insertion.

The shifted mixed insertion is an algorithm that maps a permutation a of S_n into pairs of tableaux, (T, T') where T is a P-semistandard shifted tableau with distinct numbers and T' is a standard shifted Young tableau of the same shape. Let us denote the set of such T by $\mathcal{T}_n$. For details, please see [7, Definition 6.7]. We denote it as

$$a \xrightarrow{mixed} (T, T')$$

We now look at the Kraśkiewicz insertion on reduced words that are made up of the numbers $1, 2, \cdots, n$ and each of them appearing only once. In this way, we can treat the Kraśkiewicz insertion as an insertion algorithm on S_n. Let us call this the *restricted Kraśkiewicz insertion*. By Theorem 1.26, the restricted Kraśkiewicz insertion gives a bijection between S_n and pairs of tableaux (P, Q) where P is a standard decomposition tableau with distinct entries and Q is a standard shifted Young tableau. Denote by $\mathcal{P}_n$ the set of such standard decomposition tableaux.

Theorem 3.32 *Let $a \in S_n$ and*

$$a \xrightarrow{K} (P, Q)$$
$$a \xrightarrow{mixed} (T, T')$$

Then $Q = T'$.

Proof: First, we make use of [7, Theorem 6.10]. It states that T' is the Worley-Sagan insertion tableau for a^{-1}. In [7, Corollary 6.3], it is shown that this is the same as

applying shifted jeu de taquin on the diagonal tableau with reading word the same as a^{-1}. We intend to show that Q is obtained in this manner.

To do this, we will make use of the short promotion sequence $\hat{p}$. Let $N = n^2 - n$. Let $b = a^{-1}$ and c be the sequence such that $c_i = N + b_i$. Let Q' be a standard shifted Young tableau of shape $(2n - 1, 2n - 3, \cdots, 3, 1)$ such that the reading word of the last top-right to bottom-left diagonal is c. This diagonal consists of boxes with coordinates $(i, 2n - 2i + 1)$ filled with entry c_{n-i+1}.

The short promotion sequence of Q' is

$$\hat{p}(Q') = r_1 r_2 \cdots r_{N+1} r_{N+2} \cdots r_{N+n}$$

By definition of the short promotion sequence,

$$
\begin{aligned}
r_{c_{n-i+1}} &= n - i \\
\Rightarrow \quad r_{N+b_i} = r_{c_i} &= i - 1 \\
\Rightarrow \quad r_{N+i} &= a_i - 1
\end{aligned}
$$

By Theorem 3.31,

$$\hat{p}(Q') = r_1 r_2 \cdots r_{N+1} r_{N+2} \cdots r_{N+n} \xrightarrow{K} (P', Q')$$

where P' is the unique standard decomposition tableau of $\mathrm{SDT}(w_B)$. By Lemma 3.24,

$$r_{N+1} r_{N+2} \cdots r_{N+n} \xrightarrow{K} (R, S)$$

where $S = \Delta^N(Q')$. But this is equivalent to applying shifted jeu de taquin on the last diagonal with the box $(i, 2n - 2i + 1)$ filled with b_{n-i+1}. Therefore,

$$S = T'$$

Since $r_{N+1} \cdots r_{N+n}$ is basically a shifted by 1 and the steps in the Kraśkiewicz inser-

tion for both are the same, we can conclude that

$$Q = S = T'$$

$\square$

For the restricted Kraśkiewicz insertion, since all the numbers are distinct, we avoid the special cases of the insertion algorithm that involves repeated numbers. Correspondingly, the B-Coxeter-Knuth relations reduces to($a < b < c < d$):

Definition 3.33 (Elementary Restricted B-Coxeter-Knuth Relations)

$$abdc \sim adbc \quad (1)$$
$$acdb \sim acbd \quad (2)$$
$$adcb \sim dacb \quad (3)$$
$$badc \sim bdac \quad (4)$$

and their reverses.

This is the same as the list in [6, Corollary 3.2]. Thus we have arrived at analogues of Knuth relations for shifted mixed insertion. Since the standard decomposition tableaux are representatives of the restricted B-Coxeter-Knuth relations, this gives

Theorem 3.34 *There exists a shape preserving bijection, Φ from $\mathcal{T}_n$ to $\mathcal{P}_n$. Furthermore, if $\boldsymbol{a} \in S_n$ the respective insertions map*

$$\boldsymbol{a} \quad \xrightarrow{K} \quad (P, Q)$$
$$\boldsymbol{a} \quad \xrightarrow{mixed} \quad (T, Q)$$

Then, $\Phi(T) = P$.

Next, we can interpret properties of the restricted Kraśkiewicz insertion in the context of shifted mixed insertion and vice versa.

Theorem 3.35 *Let $\boldsymbol{a} \in S_n$ and*

$$\boldsymbol{a} \xrightarrow{K} (P, Q)$$

and let

$$\boldsymbol{a}^{-1} \overset{WS}{\to} (R, S)$$

denote the Worley-Sagan insertion of $\boldsymbol{a}^{-1}$. Then, $Q = R$ and S is the shifted mixed insertion tableau of $\boldsymbol{a}$.

A description of the Worley-Sagan insertion can be found in [16] or [12, Section 3]. This theorem is just a rehash of [7, Theorem 6.10].

Theorem 3.36 *Let $\boldsymbol{a} \in S_n$ and suppose*

$$\boldsymbol{a} \overset{mixed}{\Longrightarrow} (T, T')$$

where $sh(T) = \lambda$. Then λ_1 is the length of the longest unimodal sequence in $\boldsymbol{a}$.

This theorem can be generalized to give interpretations to λ_i where $i > 1$. We quote a result from [12].

Theorem 3.37 ([12, Corollary 5.2]) *Let $\boldsymbol{a} \in S_n$. Suppose that the shape of the Worley-Sagan insertion tableau is $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$, then for $k \leq l$ the maximum length of a strictly k-decreasing subsequence in $\boldsymbol{a}^r \boldsymbol{a}$ is*

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k + \binom{k}{2}$$

A strictly k-decreasing subsequence of $\boldsymbol{a}^r \boldsymbol{a}$ is a union of k disjoint strictly decreasing subsequences of $\boldsymbol{a}^r \boldsymbol{a}$.

Let $\boldsymbol{b}$ be a strictly decreasing subsequence in $\boldsymbol{a}^r \boldsymbol{a}$. It is made up of distinct numbers in $\boldsymbol{a}$. If we look at these numbers in $\boldsymbol{a}$, they appear in a decreasing manner starting from the right towards the left and then from left to right. Conversely, any set of numbers in $\boldsymbol{a}$ that can be obtained in this manner can be written as a strictly decreasing subsequence of $\boldsymbol{a}^r \boldsymbol{a}$.

For example, if $\boldsymbol{a} = 52314$ then $\boldsymbol{a}^r \boldsymbol{a} = 4132552314$. Let $\boldsymbol{b} = 4321$. It is a strictly decreasing subsequence of $\boldsymbol{a}^r \boldsymbol{a}$ and it appears in $\boldsymbol{a}$ in a decreasing manner from right to left and left to right.

Now, let $c = a^{-1}$. If we look at the sequence of inverse images of numbers of b, they appear as a unimodal subsequence in c. Conversely, a unimodal subsequence in c gives rise to such a b in a. Using the previous example, $a^{-1} = 42351$. The subsequence corresponding to b is 4235.

Making use of this and Theorem 3.35 and noting that the shape of the insertion tableau and recording tableau are always the same, we can reinterpret the Theorem 3.37 in terms of k-unimodal subsequence.

Definition 3.38 *Let d be a sequence of distinct numbers. A k-unimodal subsequence is a union of k unimodal subsequences of d. Any number can appear in at most 2 of the unimodal subsequences and any pair of unimodal subsequences can only have at most 1 number in common.*

Corollary 3.39 *Let $a \in S_n$. If*

$$a \overset{K}{\to} (P, Q)$$

where $\mathrm{sh}(P) = \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$, then for $k \leq l$, the maximum length of a k-unimodal subsequence is

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k + \binom{k}{2}$$

This result is also true if we replace the Kraśkiewicz insertion by the shifted mixed insertion.

Example: Take the permutation $a = 57286431$. Under the Kraśkiewicz insertion, it maps to the tableau

8	6	4	3	1
	7	2		
		5		

The shape is $(5, 2, 1)$. $\{728, 86431\}$ is a 2-unimodal subsequence of a of length 8. $\{526, 728, 86431\}$ is a 3-unimodal subsequence of length 11. The reader can verify that these subsequences attain the maximun length.

Remarks: It would be nice if the definition of a k-unimodal subsequence can be altered to disjoint unimodal subsequence and change the maximum length to $\lambda_1 + \lambda_2 + \cdots + \lambda_k$. Unfortunately, this is not possible. For example, the permutation $b = 57628431$ also maps to the same tableau above but there exists a 2-unimodal subsequence $\{7628, 5431\}$ which have distinct numbers.

Chapter 4

Analogues for D_n

In this chapter, we aim to find D_n analogues of all the results obtained so far for B_n. In Section 4.1, we will give the definitions and some results about the reduced words of D_n. An analogue of the Kraśkiewicz insertion algorithm is described in Section 4.2. It turns out that this D-Kraśkiewicz insertion does not apply on the reduced words directly. The D-Coxeter-Knuth relations will be introduced in Section 4.3. Most of the results of the D-Kraśkiewicz insertion are proven here. In Section 4.4, we introduce the D_n stable Schubert polynomials and using the D-Kraśkiewicz insertion, we are able to show that they can be expressed as nonnegative integer combinations of Schur P-functions.

4.1 D_n

Consider a standard basis $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_n\}$ for a vector space of dimension n over the real numbers. The simple roots for D_n are

$$\alpha_0 = \epsilon_1 + \epsilon_2$$
$$\alpha_1 = \epsilon_2 - \epsilon_1$$
$$\alpha_2 = \epsilon_3 - \epsilon_2$$
$$\cdots$$
$$\alpha_{n-1} = \epsilon_n - \epsilon_{n-1}$$

Let s_i denote the simple reflection in the hyperplane perpendicular to α_i. These generate the Weyl group of the root system which we denote by D_n as well. In what follows, we will just write the subscripts of the reflections. The simple reflections satisfy the Coxeter relations for D_n. They are

$$
\begin{aligned}
01 &\sim 10 \\
020 &\sim 202 \\
ab &\sim ba & b > a+1, (a,b) \neq (0,2) \\
a\,a+1\,a &\sim a+1\,a\,a+1 & a \neq 0
\end{aligned}
$$

D_n is a subgroup of B_n and we can represent its elements as signed permutations with even number of signs. Under this notation,

$$
\begin{aligned}
s_0 &= \bar{2}\bar{1}34\cdots n \\
s_i &= 12\cdots i-1\,i+1\,i\,i+2\cdots n \quad \text{for } 1 \leq i < n
\end{aligned}
$$

Let $w \in D_n$, we can express it as a product of s_i's. Those of shortest length are called reduced words. This shortest length is called the length of w and denoted by $l_D(w)$. We will use $R_D(w)$ to denote the set of all reduced words of w. However, we will drop the subscript D if there is no confusion with S_n or B_n.

There is an automorphism Γ on the group D_n that switches s_0 and s_1.

$$
\Gamma(s_i) = \begin{cases} s_0 & \text{if } i = 1 \\ s_1 & \text{if } i = 0 \\ s_i & \text{otherwise} \end{cases}
$$

It is not difficult to show that

$$
\Gamma(w) = v^{-1}wv
$$

where $v^{-1} = v = 1\bar{2}\bar{3}\cdots\bar{n}$ in 1-line notation. Note that v is an element of B_n but not D_n when n is even.

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Let $w = w_1 w_2 \cdots w_n \in D_n$. We write $\bar{w}_i$ to mean

$$\bar{w}_i = \begin{cases} \bar{k} & \text{if } w_i = k \\ k & \text{if } w_i = \bar{k} \end{cases}$$

With some more work, we can show that

Lemma 4.1

$$\Gamma(w) = \begin{cases} \bar{w}_1 w_2 \cdots \bar{1} \cdots w_n & w_1 \neq 1 \text{ and } 1 \text{ is not barred in } w \\ \bar{w}_1 w_2 \cdots 1 \cdots w_n & w_1 \neq 1 \text{ and } 1 \text{ is barred in } w \\ w & \text{otherwise} \end{cases}$$

In particular, $\Gamma(w) = w$ iff $w_1 = 1$ or $\bar{1}$.

We can also turn Γ into a map on reduced words by simply letting $\Gamma(\boldsymbol{a})$ be the reduced word that is obtained by turning all the 1's in $\boldsymbol{a}$ into 0's and 0's into 1's. The motivation behind these results on Γ is that the D-Kraśkiewicz insertion that we will introduce in the next section treats the simple reflections s_0 and s_1 equally. To further demonstrate this point, we need to introduce flattened words.

Definition 4.2 *Let $\boldsymbol{a} = a_1 a_2 \cdots a_m$ be a reduced word in D_n. Convert all the 0's that appear in $\boldsymbol{a}$ to 1's and denote this new word by $\check{\boldsymbol{a}}$. We call $\check{\boldsymbol{a}}$ a flattened word. Denote the set of flattened words of w by $\check{R}(w)$.*

Under this new scheme, a flattened word will only contain positive integers and can never have three consecutive 1's. This idea of flattened words first appeared in [6] but in another form called winnowed words. Later, it was independently used by Billey and Haiman(see [1]) in their investigations for D analogues of stable Schubert polynomials.

Let $\boldsymbol{a}$ be a reduced word of w. We can treat the flattened word $\check{\boldsymbol{a}}$ as a word of some permutation v of S_n. But in this case, $\check{\boldsymbol{a}}$ need not be reduced. It is not difficult to show that v is just the signed permutation w without the bars.

Example: 2041 is a reduced word of $\bar{1}3254$. 2141 is the corresponding flattened word. It is also a word, but not reduced, for 13254.

Obviously, more than one reduced word can be flattened into the same flattened word. Also, they need not be reduced words of the same signed permutation.

Definition 4.3 *Let c be a flattened word. Define $N(c, w)$ to be the number of reduced words $a \in R(w)$ such that $\check{a} = c$.*

Lemma 4.4 *Let c be a flattened word of w of length m and $c_m = 1$. There exist $a, b \in R(w)$ such that $\check{a} = \check{b} = c$, $a_m = 0$ and $b_m = 1$ iff c factors into de, $e \neq \emptyset$, where the permutation v of S_n corresponding to e is such that*

$$v_1 = 1$$

Proof: ($\Rightarrow$) Let us calculate the 1-line notation of w^{-1} from the reduced words a^r and b^r using the method described in Section 1.3. Examine the symbol 1 closely. $a_m = 0$ causes the symbol 1 to have a bar and $b_m = 1$ lets the symbol 1 be without a bar. Since eventually we should get the same 1-line notation, at some point, the 2 reduced words would give the symbol 1 the same parity. But, for this to happen, the symbol 1 must arrive at the first position. Hence, we can factor c into de and where the permutation v corresponding to e has the property

$$\begin{aligned} v_1^{-1} &= 1 \\ \Rightarrow \quad v_1 &= 1 \end{aligned}$$

($\Leftarrow$) Let $a \in R(w)$ such that $\check{a} = c$. Suppose $a_m = 0$. Then we can write a as fg where $\check{f} = d$ and $\check{g} = e$. From Lemma 4.1, $\Gamma(g)$ represents the same permutation as g. This means that $b = f\Gamma(g)$ is a reduced word of w that ends in 1 and $\check{b} = c$. $\quad \square$

Theorem 4.5 *Let c be a flattened word and $w \in D_n$ such that $N(c, w)$ is non-zero. Then, $N(c, w)$ is a power of 2. Furthermore, it does not depend on the choice of w.*

Proof: We will use induction on the length m of c.

Case: $m = 1$. If $N(c, w)$ is non-zero, it can only take the value 1.

Case: $m > 1$. Let $c' = c_1 c_2 \cdots c_{m-1}$.

Subcase: $c_m \neq 1$. Then

$$N(c, w) = N(c', w s_{c_m})$$

Hence, by induction hypothesis, $N(c, w)$ is a power of 2. Furthermore, $N(c, w)$ is independant of w since $N(c', w s_{c_m})$ is independant of $w s_{c_m}$.

Subcase: $c_m = 1$. If there is a factorization of c into de where $e \neq \emptyset$ and the permutation v corresponding to e has the property

$$v_1 = 1$$

then by Lemma 4.4, for any w such that $N(c, w) \neq 0$, w has reduced words a and b such that

$$\check{a} = \check{b} = c \quad \text{and} \quad a_m = 0, b_m = 1$$

By induction hypothesis, $N(c', w s_0) = N(c', w s_1)$ is a power of 2. So,

$$N(c, w) = N(c', w s_0) + N(c', w s_1) = 2N(c', w s_0)$$

is also a power of 2.

If c does not factor in such a manner then for any w such that $N(c, w) \neq 0$, all reduced words $a \in R(w)$ such that $\check{a} = c$, must all end with $a_m = 0$ or all end with $a_m = 1$. Without loss of generality, assume $a_m = 0$. Then $N(c, w) = N(c', w s_0)$ is a power of 2. Note that the calculations depend only on c. Hence $N(c, w)$ is independant of w. $\qquad \square$

We will now write $N(c)$ instead of $N(c, w)$ since the number is independent of w as long as it is non-zero.

Remarks: In [1, Proposition 3.7], an explicit formula for $N(c)$ is given.

$$N(c) = 2^{m-k+1}$$

where m is the number of 1's in c and k is the number of distinct values of $s_{c_1} \cdots s_{c_j}(1)$ for $0 \leq j \leq l(w)$. This can be proved using the previous theorem.

4.2 D-Kraśkiewicz Insertion

In this section, we will give a definition of unimodal sequences for reduced words and flattened words. They differ slightly from the definition of unimodal sequences for reduced words in B_n. Then we will describe the insertion algorithm.

Definition 4.6 *A sequence of nonnegative integers* $a = a_1 a_2 \cdots a_m$ *is said to be* unimodal *if for some* $j \leq m$

  1. $a_1 > a_2 > \cdots > a_{j-1}$

  2. $a_j < a_{j+1} < \cdots < a_m$

  3. $a_{j-1} > a_j$ *or* $a_{j-1} = 0, a_j = 1$

The decreasing part *of the unimodal sequence is defined to be*

$$a\!\downarrow = a_1 a_2 \cdots a_j$$

The increasing part *is defined to be*

$$a\!\uparrow = a_{j+1} a_{j+2} \cdots a_m$$

Let $c = c_1 c_2 \cdots c_m$ *be a flattened word. It is said to be* unimodal *if*

  1. $c_1 > c_2 > \cdots > c_{j-1}$

  2. $c_j < c_{j+1} < \cdots < c_m$

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3. $c_{j-1} > c_j$ *or* $c_{j-1} = c_j = 1$

The decreasing part *of the unimodal flattened word is defined by*

$$\mathbf{c}\!\downarrow = c_1 c_2 \cdots c_j$$

The increasing part *is defined by*

$$\mathbf{c}\!\uparrow = c_{j+1} c_{j+2} \cdots c_m$$

The definition of a unimodal reduced word for B_n and D_n are the same. However, the definition for the decreasing part is different. For example, 2013 is a unimodal reduced word. When considered as a reduced word of the signed permutation $3\bar{1}42$ in B_n, the decreasing part is 20. When considered as a reduced word of the signed permutation $\bar{1}\bar{3}42$ in D_n, the decreasing part is 201.

Here are some examples of unimodal flattened words. 5431127 is unimodal with decreasing part 54311 and increasing part 27. It is a flattened word of $\bar{1}\bar{6}234587$. 1245 is unimodal with decreasing part 1 and increasing part 245. It is a flattened word for both 231564 and $\bar{2}3\bar{1}564$.

Definition 4.7 *Let P be a tableau with l rows such that*

1. $\pi_P = P_l P_{l-1} \cdots P_2 P_1$ is a flattened word of w

2. P_i is a unimodal subsequence of maximum length in $P_l P_{l-1} \cdots P_{i+1} P_i$

Then, P is called a standard decomposition tableau *of w and we denote the set of such tableaux by* $\mathrm{SDT}_D(w)$.

For convenience, we will write $\mathrm{SDT}(w)$ in place of $\mathrm{SDT}_D(w)$ when no confusion arises.

Let $\boldsymbol{a}$ be a flattened word. Now, we will describe the *D-Kraśkiewicz insertion*. First we construct a sequence of pairs of tableaux

$$(\emptyset, \emptyset) = (P^{(0)}, Q^{(0)}), (P^{(1)}, Q^{(1)}), \cdots, (P^{(m)}, Q^{(m)}) = (P, Q)$$

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with $\text{sh}(P^{(i)}) = \text{sh}(Q^{(i)})$. We obtain $(P^{(i)}, Q^{(i)})$ from $(P^{(i-1)}, Q^{(i-)})$ using the algorithm shown below.

Insertion Algorithm:

Input: a_i and $(P^{(i-1)}, Q^{(i-1)})$. Output: $(P^{(i)}, Q^{(i)})$.

Step 1: Let $a = a_i$ and $R = $ 1st row of $P^{(i-1)}$.

Step 2: Insert a into R as follows:

- Case 0: $R = \emptyset$. If the empty row is the kth row, we write a indented $k - 1$ boxes away from the left margin. This new tableau is $P^{(i)}$. For $Q^{(i)}$, we add i to $Q^{(i-1)}$ so that $P^{(i)}$ and $Q^{(i)}$ have the same shape. Stop.

- Case 1: Ra is unimodal. Append a to R and let $P^{(i)}$ be this new tableau. To get $Q^{(i)}$, we add i to $Q^{(i-1)}$ so that $P^{(i)}$ and $Q^{(i)}$ have the same shape. Stop.

- Case 2: Ra is not unimodal. Let b be the smallest number in $R{\uparrow}$ bigger than or equal to a.

  - Case 2.0.1: $a = 1$ and R contains 2112. Leave R unchanged and go to Step 2 with $a = 1$ and R equal to the next row.

  - Case 2.0.2: $a = 1$ and R contains 212. Change the last 2 into a 1 and go to Step 2 with $a = 1$ and R equal to the next row.

  - Case 2.0.3: $a = 1$ and R contains 112 but not 2112. Change the first 1 into a 2 and go to Step 2 with $a = 1$ and R equal to the next row.

  - Case 2.1.1: $b \neq a$. We put a in b's position and let $c = b$.

  - Case 2.1.2: $b = a$. We leave $R{\uparrow}$ unchanged and let $c = a + 1$.

  We insert c into $R{\downarrow}$. Let d be the biggest number in $R{\downarrow}$ which is smaller than or equal to c. If this number is 1 and there are two 1's in $R{\downarrow}$, we let d be the left 1.

  - Case 2.1.3: $d \neq c$. We put c in d's place and let $a' = d$.

  - Case 2.1.4: $d = c$. We leave $R{\downarrow}$ unchanged and let $a' = c - 1$.

Step 3: Repeat Step 2 with $a = a'$ and R equal to the next row.

$\square$

We will denote this insertion operation given above by:

$$P^{(i-1)} \leftarrow a_i = P^{(i)}$$

The D-Kraśkiewicz insertion associates $\check{a}$ with the last pair of tableaux (P, Q) and we denote this operation by

$$a \xrightarrow{D-K} (P, Q)$$

As in B_n, we will call P the *insertion tableau* and Q the *recording tableau*. We will omit the $D - K$ above the arrow if it no confusion arises.

Example: Let $a = 3021202 \in R(1\bar{2}\bar{4}3)$. Then $\check{a} = 3121212$.

$$P^{(3)} = \boxed{3\,|\,1\,|\,2} \qquad Q^{(3)} = \boxed{1\,|\,2\,|\,3}$$

$$P^{(3)} \leftarrow 1 = \;\boxed{3\;1\;|\;2}\; \leftarrow 1$$

$$= \;\boxed{3\;1\;|\;1}\;\overset{2}{\leftarrow}$$

$$= \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\\hline \end{array} \;\underset{\leftarrow 1}{}$$

$$= \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\\hline 1 \\\cline{1-1} \end{array} \;= P^{(4)}$$

$$Q^{(4)} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\\hline 4 \\\cline{1-1} \end{array}$$

$$P^{(5)} = \begin{array}{|c|c|c|c|}\hline 3 & 2 & 1 & 2 \\\hline 1 \\\cline{1-1} \end{array} \qquad Q^{(5)} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\\hline 4 \\\cline{1-1} \end{array}$$

$$P^{(5)} \leftarrow 1 \;=\; \begin{array}{|c|c|c|c|} \hline 3 & 2 & 1 & 2 \\ \hline & 1 \\ \cline{2-2} \end{array} \;\leftarrow 1$$

$$=\; \begin{array}{|c|c|c|c|} \hline 3 & 2 & 1 & 1 \\ \hline & 1 & 1 \\ \cline{2-3} \end{array} \;=\; P^{(6)}$$

$$Q^{(6)} \;=\; \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline & 4 & 6 \\ \cline{2-3} \end{array}$$

$$P^{(7)} \;=\; \begin{array}{|c|c|c|c|c|} \hline 3 & 2 & 1 & 1 & 2 \\ \hline & 1 & 1 \\ \cline{2-3} \end{array} \qquad Q^{(7)} \;=\; \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 7 \\ \hline & 4 & 6 \\ \cline{2-3} \end{array}$$

The reader can verify that $P^{(7)}$ is a standard decomposition tableau for $1\bar{2}\bar{4}3$. We will prove in the next section that the insertion tableau is always a standard decomposition tableau.

4.3   D-Coxeter-Knuth Relations

As in the B_n case, the D-Kraśkiewicz insertion can be imitated by certain relations. We call them the D-Coxeter-Knuth relations. However, they are not relations between reduced words but relations between flattened words. In the table below, $a < b < c < d$ or $1 = a = b < c < d$ unless otherwise stated.

Definition 4.8 (Elementary D-Coxeter-Knuth Relations)

$$1121 \quad \sim \quad 1212 \tag{1}$$

$$a\ b\ b+1\ b \quad \sim \quad a\ b+1\ b\ b+1 \qquad a < b \tag{2}$$

$$b\ a\ b+1\ b \quad \sim \quad b\ b+1\ a\ b \qquad a < b \tag{3}$$

$$a\ a+1\ b\ a \quad \sim \quad a\ a+1\ a\ b \qquad a+1 < b \tag{4}$$

$$a+1\ a\ b\ a+1 \quad \sim \quad a+1\ b\ a\ a+1 \quad a+1 < b \tag{5}$$

$$abdc \quad \sim \quad adbc \tag{6}$$

$$acdb \quad \sim \quad acbd \tag{7}$$

$$adcb \quad \sim \quad dacb \tag{8}$$

$$badc \quad \sim \quad bdac \tag{9}$$

and their reverses.

Definition 4.9 *Let a, b be 2 flattened words for some $w \in D_n$. If $a = cxd$ and $b = cyd$ where both x and y have length 4 and (x, y) appeared in the list above, we say a is* elementary D-Coxeter-Knuth related *to b.*

Let $e, f \in \check{R}(w)$. If there exist $e = a_1, a_2, \cdots, a_k = f \in \check{R}(w)$ such that each pair (a_i, a_{i+1}) is elementary D-Coxeter-Knuth related, we say e and f are D-Coxeter-Knuth related. *We denote this as* $e \overset{D}{\sim} f$.

This set of relations appeared in [6, Table 5] as C_l-relations. We will write $e \sim f$ if no ambiguity arises. As mentioned earlier, the D-Coxeter-Knuth relations are relations between flattened words. But, we can translate these relations to reduced words in the following sense.

Theorem 4.10 *Let $a \in R(w)$. Suppose c is a flattened word such that $c \sim \check{a}$. Then, there exists $b \in R(w)$ such that*

$$\check{b} = c$$

Proof: It suffices to check this when c and $\check{a}$ differ by an elementary D-Coxeter-Knuth relation. The result is clear for the D-Coxeter-Knuth relations (2) to (9). For (1), we

exhibit the table

1121	1212
0121	0212
1021	0212
0120	1202
1020	1202

Each pair of entries in each row are related by the Coxeter relations for D_n. □

We now proceed to give analogues of lemmas and theorems and Section 1.4. This will eventually lead us to the proof that the D-Kraśkiewicz insertion is a bijection between flattened words and pairs of tableaux.

Lemma 4.11 *Let R be a unimodal flattened word and $a, 0 < a < n$ such that Ra is also flattened. If*

$$R \overset{in}{\leftarrow} a = a' \overset{out}{\leftarrow} R'$$

then

$$Ra \sim a'R'$$

Proof: The proof is similar to the B_n case. We imitate the insertion by D-Knuth-Coxeter relations. However, we will only do this for the special Cases 2.0.1, 2.0.2, 2.0.3. The reader can refer to Theorem 1.16 for the other cases. Let $R = r_1 r_2 \cdots r_m$ where $R{\downarrow} = r_1 \cdots r_k, k \leq m$.

Case 2.0.1: $a = 1$ and R contains 2112.

$$
\begin{aligned}
Ra &= r_1 \cdots r_{k-3} 2112 \cdots r_m \underline{1} \\
&\sim r_1 \cdots r_{k-3} 2112 r_{k+2} \underline{1} \cdots r_m && \text{by } (8^r)\text{'s} \\
&\sim r_1 \cdots r_{k-3} 2112 \underline{1} r_{k+2} \cdots r_m && \text{by } (4) \\
&\sim r_1 \cdots r_{k-3} 2\underline{1} 212 r_{k+2} \cdots r_m && \text{by } (1) \\
&\sim r_1 \cdots r_{k-3} \underline{1} 2112 r_{k+2} \cdots r_m && \text{by } (1^r) \\
&\sim r_1 \cdots \underline{1} r_{k-3} 2112 r_{k+2} \cdots r_m && \text{by } (4^r) \\
&\sim \underline{1} r_1 \cdots r_{k-3} 2112 r_{k+2} \cdots r_m && \text{by } (8)\text{'s} \\
&= a'R'
\end{aligned}
$$

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Case 2.0.2: $a = 1$ and R contains 212.

$$
\begin{aligned}
Ra &= r_1 \cdots r_{k-3}212r_{k+2} \cdots r_m \underline{1} \\
&\sim r_1 \cdots r_{k-3}212\underline{1}r_{k+2} \cdots r_m \\
&\sim r_1 \cdots r_{k-3}\underline{1}211r_{k+2} \cdots r_m \quad \text{by } (1^r) \\
&\sim r_1 \cdots \underline{1}r_{k-3}211r_{k+1} \cdots r_m \quad \text{by } (4^r) \\
&\sim \underline{1}r_1 \cdots r_{k-3}211r_{k+1} \cdots r_m \\
&= a'R'
\end{aligned}
$$

Case 2.0.3: $a = 1$ and R contains 112 but not 2112.

$$
\begin{aligned}
Ra &= r_1 \cdots r_{k-3}112r_{k+2} \cdots r_m \underline{1} \\
&\sim r_1 \cdots r_{k-3}112\underline{1}r_{k+2} \cdots r_m \\
&\sim r_1 \cdots r_{k-3}\underline{1}212r_{k+2} \cdots r_m \quad \text{by } (1) \\
&\sim \underline{1}r_1 \cdots r_{k-3}212r_{k+1} \cdots r_m \\
&= a'R'
\end{aligned}
$$

$\square$

Theorem 4.12 Let $a \in R(w)$ and $\check{a} \xrightarrow{D-K} (P, Q)$. Then $\check{a} \sim \pi_P$.

Proof: This uses the previous lemma and we omit the proof as it is exactly the same in B_n case. $\square$

The above theorem together with Theorem 4.10 shows that there is a reduced word a of w such that $\check{a} = \pi_P$.

Lemma 4.13 Let $a, b \in \check{R}(w)$ and $a \sim b$. If c is a unimodal subsequence of length k in a, then there is a unimodal subsequence d of length k in b.

Proof: The proof is the same as the B_n case. We will just check the special D-Coxeter-Knuth relations.

$11\underline{2}1$	$\underline{1}212$
$1\underline{2}12$	$11\underline{2}1$
$1\underline{2}12$	1121

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The explanation for the table is the same as that in the proof of Theorem 1.20. □

Theorem 4.14 *Let $a \in R(w)$. If*

$$\check{a} \xrightarrow{D-K} (P, Q)$$

then $P \in STD(w)$ and Q is a standard shifted Young tableau.

Proof: The proof is essentially the same as the proof for the Kraśkiewicz insertion.

□

Corollary 4.15 *Let $a \in R(w)$. Let $\check{a} \to (P, Q)$ and λ_1 be the length of P_1. Then the lengths of the longest unimodal subsequences in $\check{a}$ and a are both λ_1.*

Proof: This follows from Lemma 4.13, Theorem 4.14 and that a unimodal subsequence of a corresponds to a unimodal subsequence of $\check{a}$ in the obvious way. □

Lemma 4.16 *Let R be a unimodal flattened word and $a, 0 < a < n$ be such that Ra is flattened but not unimodal. If*

$$R \xleftarrow{in} a = a' \xleftarrow{out} R'$$

then

$$R'^{r} \xleftarrow{in} a' = a \xleftarrow{out} R^{r}$$

Proof: The proof is the same as that for the B_n case except for the special insertions which we will verify below.

Case 2.0.1: $a = 1$ and R contains 2112.

$$\underbrace{\cdots 2112 \cdots}_{R} \xleftarrow{in} 1 = 1 \xleftarrow{out} \underbrace{\cdots 2112 \cdots}_{R = R'}$$

$$\Rightarrow \underbrace{\cdots 2112 \cdots}_{R^{r}} \xleftarrow{in} 1 = 1 \xleftarrow{out} \underbrace{\cdots 2112 \cdots}_{R^{r}}$$

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Case 2.0.2: $a = 1$ and R contains 212.

$$\underbrace{\cdots 212 \cdots}_{R} \overset{in}{\leftarrow} 1 \; = \; 1 \overset{out}{\leftarrow} \underbrace{\cdots 211 \cdots}_{R'}$$

$$\Rightarrow \underbrace{\cdots 112 \cdots}_{R'^r} \overset{in}{\leftarrow} 1 \; = \; 1 \overset{out}{\leftarrow} \underbrace{\cdots 212 \cdots}_{R^r}$$

Case 2.0.3: $a = 1$ and R contains 112.

$$\underbrace{\cdots 112 \cdots}_{R} \overset{in}{\leftarrow} 1 \; = \; 1 \overset{out}{\leftarrow} \underbrace{\cdots 212 \cdots}_{R'}$$

$$\Rightarrow \underbrace{\cdots 212 \cdots}_{R'^r} \overset{in}{\leftarrow} 1 \; = \; 1 \overset{out}{\leftarrow} \underbrace{\cdots 211 \cdots}_{R^r}$$

$\square$

Lemma 4.17 *Given (P, Q), $P \in \mathrm{SDT}(w)$ and Q a standard shifted Young tableau, let Q' be the standard shifted Young tableau obtained by removing the largest entry from Q. There exists a unique $a > 0$ and a unique standard decomposition tableau P' such that*

$$P' \leftarrow a = P$$

and $\mathrm{sh}(P') = \mathrm{sh}(Q')$.

Proof: Let $l(w) = m$. Suppose m is in box (j, k) of Q. Let e be the entry of box (j, k) in P. As in the proof of Theorem 1.25, we want to reverse the insertion procedure. We insert e into P^r_{j-1} to get P'^r_{j-1} and a_{j-1}. We then insert a_{j-1} into P^r_{j-2} and so on. Suppose a_i is empty for some i. That is to say, the insertion ends with

$$P^r_i \leftarrow a_{i+1} = P'^r_i$$

Consider the tableau R formed by rows $i, i+1, \cdots, j$ of P. It is a standard decomposition tableau and $\pi_R \sim P'_j P'_{j-1} \cdots P'_i$ So, the length of the longest unimodal subsequence is $|P_i|$. But P'_i is a unimodal subsequence of length $|P_i| + 1$ in $P'_j P'_{j-1} \cdots P'_i$. This contradicts Lemma 4.13. Hence, a_i always exists. Let $a = a_1$ and P' to be the tableau with $P'_i, 1 \le i < j$ as the first $j - 1$ rows and $P_i, i \ge j$ as the succeeding rows.

By Lemma 4.16, the insertion step at each row is invertible and we get

$$P' \leftarrow a = P$$

Furthermore, each a_i is uniquely determined. Hence, a and P' are unique. Next, we have to show that $\pi_{P'}$ is a flattened word. From Lemma 4.11,

$$\pi_{P'}a \sim \pi_P$$

By Theorem 4.10, $\exists \boldsymbol{b} = b_1 b_2 \cdots b_m \in R(w)$ such that

$$\check{\boldsymbol{b}} = \pi_{P'}a$$

Let $\boldsymbol{b}' = b_1 b_2 \cdots b_{m-1}$. Then, $\pi_{P'} = \check{\boldsymbol{b}}'$ and $\boldsymbol{b}' \in R(ws_{b_m})$. Hence, $\pi_{P'}$ is a flattened word and P' is a standard decomposition tableau. $\square$

Theorem 4.18 *The D-Kraśkiewicz insertion is a bijection between $\check{R}(w)$ and pairs of tableaux (P, Q) where $P \in \mathrm{SDT}(w)$ and Q is a standard shifted Young tableau.*

Proof: The proof is the same as that for the B_n case. It makes use of Theorem 4.14 and Lemma 4.17. $\square$

Example: For the signed permutation $\bar{2}\bar{3}14$, $\check{R}(\bar{2}\bar{3}14) = \{1211, 2121\}$. Under the D-Kraśkiewicz insertion, 1211 maps to

$$\left(\begin{array}{c}
\begin{array}{|c|c|c|}
\hline
2 & 1 & 1 \\
\hline
\end{array} \\
\begin{array}{|c|}
\hline
1 \\
\hline
\end{array}
\end{array} \;,\; \begin{array}{c}
\begin{array}{|c|c|c|}
\hline
1 & 2 & 4 \\
\hline
\end{array} \\
\begin{array}{|c|}
\hline
3 \\
\hline
\end{array}
\end{array} \right)$$

and 2121 maps to

$$\left(\begin{array}{c}
\begin{array}{|c|c|c|}
\hline
2 & 1 & 1 \\
\hline
\end{array} \\
\begin{array}{|c|}
\hline
1 \\
\hline
\end{array}
\end{array} \;,\; \begin{array}{c}
\begin{array}{|c|c|c|}
\hline
1 & 2 & 3 \\
\hline
\end{array} \\
\begin{array}{|c|}
\hline
4 \\
\hline
\end{array}
\end{array} \right)$$

Remarks: The D-Kraśkiewicz insertion has a few differences from the Kraśkiewicz insertion.

1. It does not apply on reduced words but on flattened words.

2. There exists w, v such that $\text{SDT}_D(v) = \text{SDT}_D(w)$.

3. It is not a bijection between the set $\cup_{w \in D_n} \check{R}(w)$ and pairs of tableaux.

Example: $\check{R}(\bar{3}\bar{2}1) = \{121\} = \check{R}(3\bar{2}\bar{1})$. Actually, we can say more about when $\check{R}(w) = \check{R}(v)$.

Theorem 4.19 *Let $v, w \in D_n$, $v \neq w$. $w = \Gamma(v)$ iff $\check{R}(w) = \check{R}(v)$.*

Proof: ($\Rightarrow$) $w = \Gamma(v)$ implies that there is a bijection between $R(v)$ and $R(w)$. If $\boldsymbol{a} \in R(v)$, the bijection is given by changing all the 1's in $\boldsymbol{a}$ into 0's and all the 0's into 1's. Let this new reduced word be $\boldsymbol{b}$. Clearly, $\check{\boldsymbol{a}} = \check{\boldsymbol{b}}$. Therefore, $\check{R}(w) = \check{R}(v)$.
($\Leftarrow$) We will use induction on $l(w) = l(v) = m$.

Case: $m = 1$. Trivial.

Case: $m > 1$.

Subcase: If $\exists \boldsymbol{c} \in \check{R}(w)$ such that $c_m \neq 1$, then we know that

$$\check{R}(ws_{c_m}) = \{\boldsymbol{c}' : \boldsymbol{c}'c_m \in \check{R}(w)\} = \check{R}(vs_{c_m})$$

and by induction hypothesis,

$$
\begin{aligned}
ws_{c_m} &= \Gamma(vs_{c_m}) = \Gamma(v)s_{c_m} \\
\Rightarrow \quad w &= \Gamma(v)
\end{aligned}
$$

Subcase: All flattened words of w and v end in 1. This means that $l(ws_i) > l(w)$ for all $i > 1$. In other words, we can always switch adjacent symbols, other than the first pair, in w and increase its length. So, $w = w_1 w_2 \cdots w_n$ has to be in the form: There exists $k \leq n$ such that

1. w_i are barred for all $1 < i \leq k$ and $|w_2| > |w_3| > \cdots > |w_k|$

2. w_i are unbarred for all $i > k$ and $w_{k+1} < w_{k+2} < \cdots < w_n$

This goes for v too.

Let $c \in \check{R}(w)$. Treat c as a word of some permutation u of S_n. Clearly, both w and v differ from u by placement of the bars. That is $|w_i| = |v_i| = u_i$ for all $1 \leq i \leq n$. Let u_j be the smallest number in $u_2 u_3 \cdots u_n$. Then from above,

$$u_2 > u_3 > \cdots > u_j < u_{j+1} < \cdots < u_n$$

It is not difficult to see that there are exactly two possible signed permutations of D_n that have the prescribed form. Without loss of generality, let

$$w_i = \begin{cases} \bar{u}_i & 2 \leq i \leq j \\ u_i & i > j \end{cases}$$

$$v_i = \begin{cases} \bar{u}_i & 2 \leq i < j \\ u_i & i \geq j \end{cases}$$

Since there must be even number of bars in w and v, w_1 and v_1 must have opposite parity.

If $u_1 \neq 1$, then $u_j = 1$. By Lemma 4.1, $\Gamma(v) = w$. If $u_1 = 1$, then $u_j = 2$. It can be shown that in this case $l(w) \neq l(v)$. But this is not possible and we are done. $\square$

Corollary 4.20 *Let $v, w \in D_n n$. $w = \Gamma(v)$ iff* $\mathrm{SDT}(v) = \mathrm{SDT}(w)$.

Proof: This follows from $\mathrm{SDT}(v) = \mathrm{SDT}(w)$ iff $\check{R}(v) = \check{R}(w)$. $\square$

The next two theorems are analogues of Theorem 2.10 and Theorem 2.12. Their proofs are basically the same as those of their analogues. That also means that they are long and tedious. We omit them.

Theorem 4.21 *Let $c = c_1 c_2 \cdots c_m \in \check{R}(w)$ and*

$$c \to (P, Q)$$

If $c_i c_{i+1} \cdots c_k$, a subsequence of c is unimodal, then the boxes in Q with the entries $i, i+1, \cdots, k$ form a rim hook. Moreover, the way the entries appear in the rim hooks

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is as follows:

1. *the entries $i, i+1, \cdots, j$ form a vertical strip where j is the entry in the leftmost and lowest box of the rim hook*

2. *these entries are increasing down the vertical strip*

3. *the entries $j+1, \cdots, k-1, k$ form a horizontal strip*

4. *these entries are increasing from left to right*

Theorem 4.22 *Let $c \to (P, Q)$. Let the boxes with entries $i, i+1, \cdots, k$ form a rim hook in Q. Furthermore, the entries increases down a vertical strip and then along a horizontal strip from left to right. Then, the corresponding subsequence, $c_i c_{i+1} \cdots c_k$ is a unimodal sequence.*

4.4   The NilCoxeter Algebra for D_n

Definition 4.23 *Let $\mathcal{D}_n$ be the* nilCoxeter *algebra for D_n. It is a non-commutative algebra generated by $u_0, u_1, \cdots, u_{n-1}$ with the relations:*

$$
\begin{aligned}
u_i^2 &= 0 & i \geq 0 \\
u_0 u_1 &= u_1 u_0 \\
u_0 u_2 u_0 &= u_2 u_0 u_2 \\
u_i u_{i+1} u_i &= u_{i+1} u_i u_{i+1} & i > 0 \\
u_i u_j &= u_j u_i & j > i+1, \text{ and } (i, j) \neq (0, 2)
\end{aligned}
$$

The first relation shows that the nilCoxeter algebra is spanned by reduced words of D_n. All the relations except the first listed above are exactly the Coxeter relations for reduced words of D_n. Hence, the nilCoxeter algebra has a vector space basis of elements of D_n.

Let x be an indeterminate and let

$$
\begin{aligned}
D(x) &= (1 + x u_{n-1})(1 + x u_{n-2}) \cdots (1 + x u_2)(1 + x u_1)(1 + x u_0) \\
&\quad (1 + x u_2)(1 + x u_3) \cdots (1 + x u_{n-1})
\end{aligned}
$$

Lemma 4.24

1. $(1 + xu_i)^{-1} = 1 - xu_i$

2. $(1 + xu_0)(1 + xu_1) = (1 + xu_1)(1 + xu_0)$

3. $D(x)D(y) = D(y)D(x)$

4. $D(-x)D(x) = 1$

Proof: We will only prove (3). The rest are obvious. We need a result on the nil-Coxeter algebra, $\mathcal{A}_{n-1}$ for the root system A_{n-1}. It is the subalgebra of $\mathcal{D}_n$ generated by $u_1, u_2 \cdots, u_{n-1}$. The subalgebra generated by $u_0, u_2, \cdots u_{n-1}$ is also isomorphic to $\mathcal{A}_{n-1}$. We quote this result without proof.

Lemma 4.25 ([5, Lemma 2.1])
Let
$$A_i(x) = (1 + xu_{n-1})(1 + xu_{n-2}) \cdots (1 + xu_{i+1})(1 + xu_i)$$

then
$$A_i(x)A_i(y) = A_i(y)A_i(x)$$

From (1), it is easy to see that

$$A_i(x)^{-1} = (1 - xu_i)(1 - xu_{i+1}) \cdots (1 - xu_{n-1})$$

Lemma 4.26 ([5, Lemma 4.1]) *Let* $\tilde{A}_i(x) = A_i(-x)^{-1} = (1 + xu_i)(1 + xu_{i+1}) \cdots (1 + xu_{n-1})$. *Then*
$$\tilde{A}_i(x)A_i(y) = A_i(y)\tilde{A}_i(x)$$

Now,

$$D(x)D(y)$$
$$= A_2(x)(1 + xu_1)(1 + xu_0)\tilde{A}_2(x)A_2(y)(1 + yu_0)(1 + yu_1)\tilde{A}_2(y)$$
$$= A_2(x)(1 + xu_1)A_2(y)(1 + yu_0)(1 + xu_0)\tilde{A}_2(x)(1 + yu_1)\tilde{A}_2(y)$$
$$= A_2(y)(1 + yu_1)A_2(x)(1 + xu_1)(1 - yu_1)(1 + yu_0)(1 + xu_0)$$
$$\quad (1 - xu_1)(1 + yu_1)\tilde{A}_2(y)(1 + xu_1)\tilde{A}_2(x)$$
$$= A_2(y)(1 + yu_1)A_2(x)(1 + xu_0)(1 + yu_0)\tilde{A}_2(y)(1 + xu_1)\tilde{A}_2(x)$$
$$= A_2(y)(1 + yu_1)(1 + yu_0)\tilde{A}_2(y)A_2(x)(1 + xu_0)(1 + xu_1)\tilde{A}_2(x)$$
$$= D(y)D(x)$$

$\square$

Definition 4.27 *Let $\mathcal{D}_n[x]$ be the polynomial ring in the indeterminates $x_1, x_2, \cdots$. Consider the following expansion*

$$D(x_1)D(x_2)\cdots = \sum_{w \in D_n} H_w(x)w$$

H_w *are called the* D_n *stable Schubert polynomials.*

Like the G_w's, from Lemma 4.24 it can be shown that the H_w's are symmetric functions and furthermore,

Theorem 4.28 $H_w(x) \in \tilde{\Lambda}$

In [1, Equation (3.10)], the $E_w(X)$ defined there is exactly the same as our $H_w(x)$. We are interested whether H_w's are linear combinations of P_λ with nonnegative integer coefficients.

Definition 4.29 *Let $a = a_1a_2 \cdots a_m \in R(w)$. We say that a sequence of positive integers $i = (i_1, i_2, \cdots, i_m)$ is an a-compatible sequence if*

1. $i_1 \le i_2 \le \cdots \le i_m$

2. $i_j = i_{j+1} = \cdots = i_k$ occurs only when $a_j a_{j+1} \cdots a_k$ is a unimodal sequence and does not contain 01.

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Denote the set of a-compatible sequence as $K(a)$.

Consider an a-compatible sequence i. Suppose $i_j = i_{j+1} = \cdots = i_k$ is a constant subsequence of i such that $i_{j-1} < i_j, i_k < i_{k+1}$. We give it a weight of 2 if the corresponding subsequence $a_j a_{j+1} \cdots a_k$ does not contain 0 or 1. Let $W(a, i)$ be the product of all these weights. $W(a, i)$ is a power of 2. This will lead us to a description of $H_w(x)$ in terms of a-compatible sequences.

Theorem 4.30

$$H_w(x) = \sum_{a \in R(w)} \sum_{i \in K(a)} W(a, i) x_{i_1} x_{i_2} \cdots x_{i_m}$$

Proof: Same as the proof of Theorem 2.7. □

For the rest of this chapter, we will show that H_w can be written as a nonnegative integer combination of Schur P-functions. The proof is more complicated than the B_n analogue. We have divided the proof into Lemma 4.33 and Lemma 4.34. Lemma 4.33 gives a bijection between Q-semistandard shifted tableaux and generalized sequences that are compatible flattened words. This bijection is almost the same as the bijection described in the proof of Theorem 2.13. Lemma 4.34 relates the a-compatible sequences to these $\check{a}$-generalized sequences. The definitions are given below. We also need the following result on $N(c)$. From Theorem 4.5, we know that $N(c)$ is a power of 2 and it depends only on c. In particular, if $P \in \mathrm{SDT}(w)$, $N(\pi_P)$ is a power of 2. For convenience, we write $N(P)$ instead of $N(\pi_P)$. The next lemma relates $N(c)$ to $N(P)$. Let $l_1(c)$ denote the number of 1's in c and $l_1(P)$ denote the number of 1's in P.

Lemma 4.31 *Let $P \in \mathrm{SDT}(w)$. For all $c \in \check{R}(w)$ such that $c \to (P, Q)$, then*

$$\frac{2^{l_1(c)}}{N(c)} = \frac{2^{l_1(P)}}{N(P)}$$

Proof: Since all the flattened words of w that map to the same insertion tableau are related by D-Coxeter-Knuth relations, it suffices to prove when b and c are two flattened words related by an elementary D-Coxeter-Knuth relation.

Case: (1) $1121 \sim 1212$. Let $\boldsymbol{b} = \boldsymbol{d}1121e$ and $\boldsymbol{c} = \boldsymbol{d}1212e$. We will pair up reduced words of w that flattened to $\boldsymbol{b}$. Then we give a bijection between these pairs and reduced words that flattened to $\boldsymbol{c}$. Let $\boldsymbol{b} = \boldsymbol{d}1121e$ and $\boldsymbol{c} = \boldsymbol{d}1212e$. The bijection is given by

$$(\boldsymbol{x}0120\boldsymbol{y}, \boldsymbol{x}1020\boldsymbol{y}) \;\leftrightarrow\; \boldsymbol{x}1202\boldsymbol{y}$$
$$(\boldsymbol{x}0121\boldsymbol{y}, \boldsymbol{x}1021\boldsymbol{y}) \;\leftrightarrow\; \boldsymbol{x}0212\boldsymbol{y}$$

where $\check{\boldsymbol{x}} = \boldsymbol{d}$ and $\check{\boldsymbol{y}} = e$. Since $l_1(\boldsymbol{b}) = l_1(\boldsymbol{c}) + 1$

$$
\begin{aligned}
\frac{2^{l_1(\boldsymbol{b})}}{N(\boldsymbol{b})} &= \frac{2^{l_1(\boldsymbol{c})+1}}{2N(\boldsymbol{c})} \\
&= \frac{2^{l_1(\boldsymbol{c})}}{N(\boldsymbol{c})}
\end{aligned}
$$

Case: (2)–(9). The number of 1's in $\boldsymbol{b}$ and $\boldsymbol{c}$ are the same. So, $l_1(\boldsymbol{b}) = l_1(\boldsymbol{c})$. The bijection is straightforward. Hence,

$$\frac{2^{l_1(\boldsymbol{b})}}{N(\boldsymbol{b})} = \frac{2^{l_1(\boldsymbol{c})}}{N(\boldsymbol{c})}$$

$\square$

Definition 4.32 *Let $\boldsymbol{c}$ be a flattened word of length m and $\boldsymbol{f}$ a generalized sequence of the same length as $\boldsymbol{c}$. We say that $\boldsymbol{f}$ is $\boldsymbol{c}$-compatible if*

1. $f_1 \le f_2 \le \cdots \le f_m$

2. $f_j = f_{j+1} = \cdots = f_k = \bar{l} \;\Rightarrow\; c_j > c_{j+1} > \cdots > c_k$

3. $f_j = f_{j+1} = \cdots = f_k = l \;\Rightarrow\; c_j < c_{j+1} < \cdots < c_k$

Denote the set of all $\boldsymbol{c}$-compatible sequences by $\check{K}(\boldsymbol{c})$.

This definition of generalized sequences that are compatible with $\boldsymbol{c}$ is the same as the definition of the generalized sequences used in the proof of Theorem 2.13.

Lemma 4.33 *Let $P \in \text{SDT}(w)$.*

$$\sum_{c \sim \pi_P} \sum_{f \in \check{K}(c)} x_{|f_1|} x_{|f_2|} \cdots x_{|f_m|} = Q_{\text{sh}(P)}(x)$$

Proof: Let $|P| = m$. We will exhibit a bijection Φ from $\{(c, f) : c \sim \pi_P, f \in \check{K}(c)\}$ to $\{T : Q\text{-semistandard Young tableau}, \text{sh}(T) = \text{sh}(P)\}$.

Step 1: Apply D-Kraśkiewicz insertion to c. We will get

$$c \to (P, Q)$$

Step 2: Take a Young diagram of the same shape as Q. We fill each box by $|f_j|$ when the corresponding box in Q has the entry j. Then, the new tableau is weakly increasing along the columns and rows since

$$j < k \implies |f_j| \le |f_k|$$

Step 3: For each constant subsequence, $f_j f_{j+1} \cdots f_k$ such that $|f_j| = |f_{j+1}| = \cdots = |f_k| = l$ with $|f_{j-1}| < |f_j|$ and $|f_k| < |f_{k+1}|$, we know that $c_j c_{j+1} \cdots c_k$ is unimodal. Let $h, j \le h \le k$ be the index of the smallest number in $c_j c_{j+1} \cdots c_k$ and if there are two 1's in $c_j c_{j+1} \cdots c_k$, let h be the index of the left 1.

Furthermore, from Theorem 4.21, the entries $j, j+1, \cdots k$ form a rim hook in Q. Let g be the entry of the box in the lowest row and leftmost column among all the boxes in the rim hook.

1. if f_h is unbarred, we add a bar to all the new entries from f_j to f_{g-1}

2. if f_h is barred, we add a bar to all the new entries from f_j to f_g

This will give us a Young tableau with barred and unbarred numbers. This will be our T. Clearly, $\text{sh}(T) = \text{sh}(Q) = \text{sh}(P)$. With the exact same reasoning as in the proof of Theorem 2.13, it can be shown that T is a Q-semistandard Young tableau. Let

$$\Phi(c, f) = T$$

Given T, we now construct the inverse map Φ^{-1}.

Step 1: Take a Young diagram of the same shape as T. We fill all the boxes with distinct numbers $1, 2, \cdots m$ as follows:

1. the entries in the Young diagram preserve the order of the entries in T.

2. for all the boxes in T with the same barred number, these form a vertical strip and we fill the corresponding boxes in an increasing order from top to bottom.

3. for all the boxes in T with the same unbarred number, these form a horizontal strip and we fill the corresponding boxes in an increasing order from left to right.

This will be our Q. It is clearly a standard shifted Young tableau with the same shape as P.

Step 2: Apply the inverse D-Kraśkiewicz insertion on (P, Q), we will get $\boldsymbol{c}$.

Step 3: To get $\boldsymbol{f}$, remove all the bars in T and let $\boldsymbol{i} = i_1 i_2 \cdots i_m$ be all the content of this new tableau laid out in weakly increasing order. Fix a number l, we know that in T, all the boxes with the entry $\bar{l}$ or l form a rim hook in T. The corresponding boxes in Q are filled with consecutive numbers, $j, j+1, \cdots, k$. Also, they satisfy the hypothesis in Theorem 4.22. Hence, $c_j c_{j+1} \cdots c_k$ is a unimodal sequence. Let h be the index of the smallest number in $c_j c_{j+1} \cdots c_k$ or the index of the left 1 if there are two 1's in $c_j c_{j+1} \cdots c_k$. Now, let (I, J) be the coordinates of the lowest and leftmost boxes in the rim hook which has entry $\bar{l}$ or l.

1. If (I, J) has the entry $\bar{l}$, then we add bars to $i_j, i_{j+1}, \cdots i_h$.

2. If (I, J) has the entry l, we add bars to $i_j, i_{j+1}, \cdots, i_{h-1}$.

This generalized sequence will be our $\boldsymbol{f}$. By construction, it is $\boldsymbol{c}$-compatible. So, $\Phi^{-1}(T) = (\boldsymbol{c}, \boldsymbol{f})$. Thus

$$
\begin{aligned}
\sum_{\boldsymbol{c} \sim \pi_P} \sum_{\boldsymbol{f} \in \check{K}(\boldsymbol{c})} x_{|f_1|} x_{|f_2|} \cdots x_{|f_m|} &= \sum_{\mathrm{sh}(T)=\mathrm{sh}(P)} \boldsymbol{x}^T \\
&= Q_{\mathrm{sh}(P)}(\boldsymbol{x})
\end{aligned}
$$

Before we state and prove Lemma 4.34, we look at an example to demonstrate the difficulties involved and give some insights to the construction of the maps that are described in the proof.

Example: Consider the signed permutation $w = \bar{1}234$.

$$R(\bar{1}234) = \{01, 10\}, \ \check{R}(\bar{1}234) = \{11\}$$

Below is a partial list of 10-compatible sequences, 01-compatible sequences and 11-compatible generalized sequences.

10	01	11
$11\ 22\cdots$		$\bar{1}1\ 11\ \bar{2}\bar{2}\ 22\cdots$
$12\ 23\cdots$	$12\ 23\cdots$	$\bar{1}\bar{2}\ \bar{1}2\ 1\bar{2}\ 12\ \bar{2}\bar{3}\ \bar{2}3\ 2\bar{3}\ 23\cdots$
$13\ 24\cdots$	$13\ 24\cdots$	$\bar{1}\bar{3}\ \bar{1}3\ 1\bar{3}\ 13\ \bar{2}\bar{4}\ \bar{2}4\ 2\bar{4}\ 24\cdots$
$\cdots$	$\cdots$	$\cdots$

This table shows that there is a possible two to one map from 11-compatible generalized sequences to 10-compatible sequences and 01-compatible sequences that covers each compatible sequence. Indeed this is essentially what we are going to do in the next lemma, albeit with some modifications. Denote the number of pairs of consecutive 1's in c by $l_{11}(c)$ and those in P by $l_{11}(P)$.

Lemma 4.34 *Let $c \in \check{R}(w)$ be a given flattened word.*

$$\frac{2^{l_1(c)}}{N(c)} \sum_{a:\check{a}=c} \sum_{i\in K(a)} W(a,i) x_{i_1} x_{i_2} \cdots x_{i_m} = \sum_{f\in\check{K}(c)} x_{|f_1|} x_{|f_2|} \cdots x_{|f_m|}$$

Proof: Consider $a, b \in R(w)$ and $\check{a} = \check{b} = c$. Let $c_k = c_{k+1} = 1$, then $a_k a_{k+1} = 01$ or 10. Suppose whenever this occurs, $a_k a_{k+1} = b_k b_{k+1}$. By Definition 4.29,

$$K(a) = K(b)$$

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Conversely, if $a, b \in R(w)$ are such that $K(a) = K(b)$ and $\check{a} = \check{b}$, then whenever the pair 01 or 10 occurs in a, it also appears in b in the same positions.

Also, it is easy to see that for a fixed a, the number of reduced words b such that $a_k a_{k+1} = b_k b_{k+1}$ whenever $c_k = c_{k+1} = 1$ is

$$\frac{N(c)}{2^{l_{11}(c)}}$$

Let us define a new operation on reduced words of w to reflect the significance of 01, 10 appearing in a reduced word. Given $a \in R(w)$, define $\hat{a}$ by changing all the 0's into 1's except when they appear in 01 or 10. We call $\hat{a}$ a *partially flattened word*. Define $K(\hat{a}) = K(a)$ and $W(\hat{a}, i) = W(a, i)$. Now, let $L(c)$ be the set of sequences where d is obtained from c as follows:

Whenever $c_k = c_{k+1} = 1$, let

$$d_k = 0, d_{k+1} = 1 \quad \text{or} \quad d_k = 1, d_{k+1} = 0$$

Clearly, $L(c) = \{\hat{a} : \check{a} = c\}$. Hence,

$$\sum_{a:\check{a}=c} \sum_{i \in K(a)} W(a, i) x_{i_1} x_{i_2} \cdots x_{i_m} = \frac{N(c)}{2^{l_{11}(c)}} \sum_{d \in L(c)} \sum_{i \in K(d)} W(d, i) x_{i_1} x_{i_2} \cdots x_{i_m}$$

Now, we have to verify that for a fixed flattened word c

$$\sum_{f \in \check{K}(c)} x_{|f_1|} x_{|f_2|} \cdots x_{|f_m|} = 2^{l_1(c)-l_{11}(c)} \sum_{d \in L(c)} \sum_{i \in K(d)} W(d, i) x_{i_1} x_{i_2} \cdots x_{i_m}$$

This can be done by defining a map Φ taking $f \in \check{K}(c)$ to (d, i) where $i \in K(d)$ and $d \in L(c)$. Let $i = |f|$.

Let $|f_j| = |f_{j+1}| = \cdots = |f_k| = l$ be a constant subsequence and $|f_{j-1}| < l < |f_{k+1}|$. We know that $c_j c_{j+1} \cdots c_k$ is unimodal.

1. If $c_j c_{j+1} \cdots c_k$ does not contain 1 or 11, then we let $d_j d_{j+1} \cdots d_k = c_j c_{j+1} \cdots c_k$.

   In this case, there are two choices for $f_j f_{j+1} \cdots f_k$. This will be accounted by

the weight of 2 assigned to $i_j i_{j+1} \cdots i_k$ in $W(\boldsymbol{d}, \boldsymbol{i})$.

2. If $c_j c_{j+1} \cdots c_k$ contains 11, say $c_h = c_{h+1} = 1$, then we let $d_h = 1, d_{h+1} = 0$ and for $g, j \leq g < h, h+1 < g \leq k$, $d_g = c_g$ There are 2 choices for $f_j f_{j+1} \cdots f_k$.

3. If $c_{j-1} = c_j = 1$, we consider two subcases.

    (a) If f_{j-1} is barred, let $d_{j-1} = 1, d_j = 0$. There are 2 choices for $f_j f_{j+1} \cdots f_k$.

    (b) If f_{j-1} is unbarred, we let $d_{j-1} = 0, d_j = 1$. There are 2 choices for $f_j f_{j+1} \cdots f_k$.

4. If $c_j c_{j+1} \cdots c_k$ contains a 1 which is not next to another 1, we let $d_j d_{j+1} \cdots d_k = c_j c_{j+1} \cdots c_k$. We have 2 choices for $f_j f_{j+1} \cdots f_k$.

Using the above procedures, we define $\Phi(\boldsymbol{f}) = (\boldsymbol{d}, \boldsymbol{i})$. It can be verified that $\boldsymbol{i}$ is $\boldsymbol{d}$-compatible.

For a fixed $(\boldsymbol{d}, \boldsymbol{i})$, we see that there are $W(\boldsymbol{d}, \boldsymbol{i}) 2^{l_1(\boldsymbol{C}) - l_{11}(\boldsymbol{C})}$ choices of $\boldsymbol{f} \in \check{K}(\boldsymbol{c})$ that map to $(\boldsymbol{d}, \boldsymbol{i})$. Hence,

$$\sum_{\boldsymbol{f} \in \check{K}(\boldsymbol{c})} x_{|f_1|} x_{|f_2|} \cdots x_{|f_m|} = 2^{l_1(\boldsymbol{C}) - l_{11}(\boldsymbol{C})} \sum_{\boldsymbol{d} \in L(\boldsymbol{c}} \sum_{\boldsymbol{i} \in K(\boldsymbol{d})} W(\boldsymbol{d}, \boldsymbol{i}) x_{i_1} x_{i_2} \cdots x_{i_m}$$

Therefore, for a fixed flattened word $\boldsymbol{c}$

$$\frac{2^{l_1(\boldsymbol{C})}}{N(\boldsymbol{c})} \sum_{\boldsymbol{a}: \check{\boldsymbol{a}} = \boldsymbol{c}} \sum_{\boldsymbol{i} \in K(\boldsymbol{a})} W(\boldsymbol{a}, \boldsymbol{i}) x_{i_1} x_{i_2} \cdots x_{i_m} = \sum_{\boldsymbol{f} \in \check{K}(\boldsymbol{c})} x_{|f_1|} x_{|f_2|} \cdots x_{|f_m|}$$

$\square$

Combining all the results together, we get

Theorem 4.35 *For all $w \in D_n$,*

$$H_w(\boldsymbol{x}) = \sum_{R \in \mathrm{SDT}(w)} 2^{l(R) - l_1(R)} N(R) P_{\mathrm{sh}(R)}(\boldsymbol{x})$$

and H_w is a nonnegative integer linear combination of Schur P-functions.

Proof:

$$H_w(x) = \sum_{R\in \mathrm{SDT}(w)} \sum_{C\sim_{\pi_R}} \sum_{\check{a}=c} \sum_{i\in K(a)} W(a,i) x_{i_1} x_{i_2} \cdots x_{i_m}$$

$$= \sum_{R\in \mathrm{SDT}(w)} \sum_{C\sim_{\pi_R}} \frac{N(c)}{2^{l_1(C)}} \sum_{f\in \check{K}(C)} x_{|f_1|} x_{|f_2|} \cdots x_{|f_m|}$$

$$= \sum_{R\in \mathrm{SDT}(w)} \frac{N(R)}{2^{l_1(R)}} Q_{\mathrm{sh}(R)}(x)$$

$$= \sum_{R\in \mathrm{SDT}(w)} \frac{N(R) 2^{l(R)}}{2^{l_1(R)}} P_{\mathrm{sh}(R)}(x)$$

The first line is a modification of Theorem 4.30 by splitting the sum according to the standard decomposition tableaux of w. The second equality is from Lemma 4.34, the third from Lemma 4.31 and Lemma 4.33. The last equality is a simple application of turning the Schur Q-functions into Schur P-functions.

From Theorem 4.5, $N(R)$ is a power of 2 and

$$N(R) \geq 2^{l_{11}(R)}$$

and

$$l_{11}(R) + l(R) \geq l_{11}(R) + l_1(R) - l_{11}(R) = l_1(R)$$

This implies that

$$2^{l(R)-l_1(R)} N(R)$$

is a nonnegative integer power of 2. Hence, H_w is a nonnegative integer linear combination of Schur P-functions. $\square$

Example: $\bar{1}342 \in D_4$.

$$R(\bar{1}342) = \{2301, 2310, 2031, 2130, 2103, 2013\}$$
$$\check{R}(\bar{1}342) = \{2311, 2131, 2113\}$$

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$$\mathrm{SDT}(\bar{1}342) = \left\{ \begin{array}{c} \begin{array}{|c|c|c|} \hline 3 & 1 & 1 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \quad, \quad \begin{array}{|c|c|c|c|} \hline 2 & 1 & 1 & 3 \\ \hline \end{array} \end{array} \right\}$$

The reader can verify that

$$H_{(\bar{1}342)}(\boldsymbol{x}) = 2P_{(3,1)}(\boldsymbol{x}) + P_{(4)}(\boldsymbol{x})$$

By comparing the coefficients of the monomial $x_1 x_2 \cdots x_m$ in this expansion of $H_w(\boldsymbol{x})$, we immediately arrive at

Corollary 4.36 *The number of reduced words of w is*

$$\sum_{R \in \mathrm{SDT}(w)} \frac{N(R)2^{l(R)}}{2^{l_1(R)}} g^{\mathrm{sh}(R)}$$

where g^λ is the number of standard shifted Young tableau of shape λ.

A similar result can be obtained for flattened words.

Corollary 4.37 *The number of flattened words of w is*

$$\sum_{R \in \mathrm{SDT}(w)} 2^{l(R) - l_1(R)} g^{\mathrm{sh}(R)}$$

Chapter 5

Properties of the D-Kraśkiewicz

Insertion

In this chapter, we will describe some properties of the D-Kraśkiewicz insertion. They are analogues of properties of the Kraśkiewicz insertion. We will state some of the results without proof since most of them are basically the same except for some special cases that can be verified easily. Section 5.1 covers some bijections, one of which relates to the automorphism Γ. We will describe some properties of the insertion tableau in Section 5.2. Section 5.3 looks into relations with the Edelman-Greene insertion. Section 5.4 covers the properties of the recording tableau and the short promotion sequence.

5.1   Some Bijections

Recall from the previous chapter that there is an automorphism Γ that changes 1 to 0 and 0 to 1 in a reduced word. It gives a bijection from $R(w)$ to $R(v^{-1}wv)$ where $v = 1\bar{2}\bar{3}\cdots\bar{n}$. Abusing notations, we write $\Gamma(w)$ instead of $v^{-1}wv$. Clearly, $\check{R}(w) = \check{R}(\Gamma(w))$. This leads us to conclude that:

Theorem 5.1 $\mathrm{SDT}(w) = \mathrm{SDT}(\Gamma(w))$ *and* $H_w(\boldsymbol{x}) = H_{\Gamma(w)}(\boldsymbol{x})$.

Next, we will look into the signed permutations with when $w_n = \bar{n}$. In what follows, we will use the notation:

$$\bar{w}_i = \begin{cases} \bar{k} & \text{if } w_i = k \\ k & \text{if } w_i = \bar{k} \end{cases}$$

Theorem 5.2 *Let $w = w_1 w_2 \cdots w_{n-1} \bar{n}$. There exists a bijection between $\mathrm{SDT}(w)$ and $\mathrm{SDT}(v)$ where $v = \bar{w}_1 w_2 \cdots w_{n-1} n$.*

$$H_w(\boldsymbol{x}) = \sum_{R \in \mathrm{SDT}(v)} \frac{N(R) 2^{l(R)}}{2^{l_1(R)}} P_{(2n-2, \mathrm{sh}(R))}(\boldsymbol{x})$$

where $(2n - 2, \lambda) = (2n - 2, \lambda_1, \lambda_2, \cdots)$.

Proof: Given any $\boldsymbol{a} \in R(w)$, we try to find the longest unimodal subsequence in $\check{\boldsymbol{a}}$. This is given by the simple reflections that affect the symbol n. Clearly, this has to be $n-1\, n-2\, \cdots 2112 \cdots n-1$. Let $P \in \mathrm{SDT}(w)$. Then $P_1 = n-1\, n-2\, \cdots 2112 \cdots n-1$. Deleting the first row gives a new standard decomposition tableau of the signed permutation v. Denote this operation by Φ. Clearly, Φ is reversible and gives a bijection between $\mathrm{SDT}(w)$ and $\mathrm{SDT}(v)$. Moreover,

$$\begin{aligned} l(\Phi(P)) &= l(P) - 1 \\ l_1(\Phi(P)) &= l_1(P) - 2 \\ N(\Phi(P)) &= \frac{N(P)}{2} \\ \mathrm{sh}(\Phi^{-1}(R)) &= (2n - 2, \mathrm{sh}(R)) \end{aligned}$$

Therefore,

$$H_w(\boldsymbol{x}) = \sum_{R \in \mathrm{SDT}(v)} \frac{N(R) 2^{l(R)}}{2^{l_1(R)}} P_{(2n-2, \mathrm{sh}(R))}(\boldsymbol{x})$$

$\square$

By repeated applications of the above result, we immediately get the following:

Corollary 5.3 *Let w_D denote the longest word in D_n.*

$$w_D = \begin{cases} 1\bar{2}\bar{3} \cdots \bar{n} & \text{if } n \text{ is odd} \\ \bar{1}\bar{2}\bar{3} \cdots \bar{n} & \text{if } n \text{ is even} \end{cases}$$

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Then

$$H_{w_D}(x) = P_{(2n-2,2n-4,\cdots,4,2)}(x)$$

Proof: Using the previous theorem, we can show that $\mathrm{SDT}(w_D)$ contains just one tableau, P where the ith row P_i is $n-i\ n-i-1\cdots 2112\cdots\ n-i$. Furthermore,

$$\frac{N(P)2^{l(P)}}{2^{l_1(P)}} = 1$$

Hence

$$H_{w_D}(x) = P_{(2n-2,2n-4,\cdots,4,2)}(x)$$

$\square$

We can generalize this result to

Corollary 5.4 *Let $i_1, i_2, \cdots, i_k$ be a sequence of numbers such that $1 < i_1 < i_2 < \cdots < i_k \leq n$. Let*

$$w = \begin{cases} 12\cdots\bar{i}_1\cdots\bar{i}_2\cdots\cdots\bar{i}_k\cdots & \text{if } k \text{ is even} \\ \bar{1}2\cdots\bar{i}_1\cdots\bar{i}_2\cdots\cdots\bar{i}_k\cdots & \text{if } k \text{ is odd} \end{cases}$$

Then,

$$H_w(x) = P_{(2i_k-2,\cdots,2i_2-2,2i_1-2)}(x)$$

5.2 Descreasing Parts

Recall from the previous chapter, a flattened word is said to be unimodal if it is strictly decreasing and then strictly increasing but allowing for two 1's in the middle. Let us call a sequence, $c = c_1c_2\cdots c_m$ *1-weakly decreasing* if

1. $c_1 > c_2 > \cdots > c_{m-2} > c_{m-1} > c_m$ or

2. $c_1 > c_2 > \cdots > c_{m-2} > c_{m-1} = c_m = 1$.

Using almost the same method, we get an analogue of Theorem 3.1.

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Theorem 5.5 *Let P be a standard decomposition tableau and let $P \Downarrow$ be the tableau that is obtained when we delete the increasing parts of each row of P. Then, $P \Downarrow$ is a shifted tableau which is 1-weakly decreasing in each row and in each top-left to bottom-right diagonal.*

With some extra work, we can conclude a similar result where the condition 1-weakly decreasing is replaced by strictly decreasing.

Theorem 5.6 *Let $P \in \text{SDT}(w)$. Let T denote the tableau obtained from $P \Downarrow$ by removing a 1 from a row if it contains two 1's. Then, T is a shifted tableau which is strictly decreasing in rows and top-left to bottom-right diagonals.*

Theorem 5.7 *Let $c \to (P, Q)$ and let $\text{sh}(P \Downarrow) = (\mu_1, \mu_2, \cdots, \mu_l)$. Then, the longest 1-weakly decreasing subsequence in c has length μ_1.*

Proof: It can be verified that the elementary D-Coxeter-Knuth relations preserves the length of the longest decreasing 1-weakly decreasing subsequence. We will only do this for the special elementary D-Coxeter-Knuth relations.

11$\underline{21}$	21$\underline{21}$
1$\underline{211}$	21$\underline{21}$

Next, we can show that any 1-weakly decreasing subsequence d of P must have length less than μ_1. The argument is basically the same as in the proof of Theorem 3.2. □

Using the above results we can give analogues of Theorem 3.4.

Theorem 5.8 *Let $w, v \in D_n$ where either*

$$w = \bar{n} w_2 w_3 \cdots w_n, \quad v = \bar{w}_2 w_3 \cdots w_n n$$

or

$$w = n w_2 w_3 \cdots w_n, \quad v = w_2 w_3 \cdots w_n n$$

Let $\mathrm{SDT}_{n-1}(w) = \{P \in \mathrm{SDT}(w) : |P_1| = n - 1\}$. *Then, there exists an injection* Φ *from* $\mathrm{SDT}_{n-1}(w)$ *into* $\mathrm{SDT}(v)$. *Furthermore, if* $\forall \breve{a} \in \breve{R}(v)$, *the length of the longest unimodal in* $\breve{a}$ *is strictly less than* $n - 1$, *then* Φ *is a bijection between* $\mathrm{SDT}_{n-1}(w)$ *and* $\mathrm{SDT}(v)$.

Proof: We omit the proof here. It is almost the same as its B_n analogue. □

Using induction and the above bijection, we can get the analogue of Theorem 3.3.

Corollary 5.9 *Let*

$$
w = \begin{cases}
\left. \begin{array}{c} \bar{n}\overline{n-1}\cdots\bar{2}\bar{1} \\[4pt] n\overline{n-1}\cdots\bar{2}1 \end{array} \right\} & \textit{if } n \textit{ is even} \\[24pt]
\left. \begin{array}{c} \bar{n}\overline{n-1}\cdots\bar{2}1 \\[4pt] n\overline{n-1}\cdots\bar{2}\bar{1} \end{array} \right\} & \textit{if } n \textit{ is odd}
\end{cases}
$$

Then

$$
H_w(\boldsymbol{x}) = P_{(n-1,\cdots,2,1)}(\boldsymbol{x})
$$

Proof: Let w be the signed permutation above. It suffices to prove that $\mathrm{SDT}(w)$ contains only
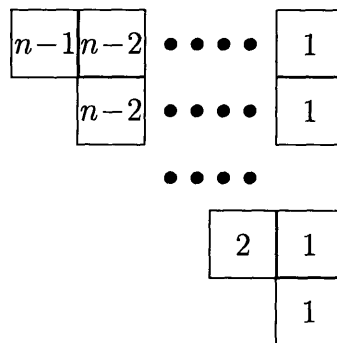
This is an easy application of induction and the previous theorem. We omit the details. □

The next result is an analogue of Corollary 3.5.

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Corollary 5.10 ([1, Proposition 3.13]) *Let*

$$w = \begin{cases} \bar{\lambda_1}\bar{\lambda_2}\cdots\bar{\lambda_l}123\cdots\widehat{\lambda_l}\cdots\widehat{\lambda_{l-1}}\cdots\cdots\widehat{\lambda_1}\cdots & \text{if } l \text{ is even} \\ \bar{\lambda_1}\bar{\lambda_2}\cdots\bar{\lambda_l}\bar{1}23\cdots\widehat{\lambda_l}\cdots\widehat{\lambda_{l-1}}\cdots\cdots\widehat{\lambda_1}\cdots & \text{if } l \text{ is odd} \end{cases}$$

where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ is a shifted shape. Then,

$$H_w(\boldsymbol{x}) = P_\lambda(\boldsymbol{x})$$

Here, $\hat{k}$ means omitting k.

Proof: The argument is a slight generalization of the proof in the previous corollary. We omit the details. □

This corollary shows that any Schur P-function has a description as a D_n stable Schubert polynomial.

5.3 The Symmetric Group and Edelman-Greene Insertion

Just as S_n is a subgroup of B_n, we can consider S_n as a subgroup of D_n. If $w \in S_n$, all reduced words $\boldsymbol{a}$ do not contain the symbol 0. Clearly, $R_S(w) = R_B(w) = R_D(w) = \check{R}(w)$. Here, the subscripts denote the reduced words of w in the various Coxeter groups. This means that the flattened words are the reduced words themselves and $N(P) = 1$. Furthermore, when we apply the D-Kraśkiewicz insertion on these words, all the special cases involving two 1's are avoided. This shows that the insertion procedure is exactly the same as the Kraśkiewicz insertion. If we denote the set of standard decomposition tableau obtained by the Kraśkiewicz insertion as $\text{SDT}_B(w)$ and that by the D-Kraśkiewicz insertion by $\text{SDT}_D(w)$, we find that $\text{SDT}_B(w) = \text{SDT}_D(w)$. We formulate some of the results in the next theorem.

Theorem 5.11 *Let $w \in S_n$. Then $\mathrm{SDT}_D(w) = \mathrm{SDT}_B(w)$ and*

$$H_w(\boldsymbol{x}) \;=\; \sum_{R \in \mathrm{SDT}_B(w)} 2^{l(R) - l_1(R)} P_{\mathrm{sh}(R)}(\boldsymbol{x})$$

Furthermore, if $w_1 = 1$,

$$H_w(\boldsymbol{x}) = G_w(\boldsymbol{x}) = F_w(\boldsymbol{x}/\boldsymbol{x})$$

Proof: The first formula for H_w is a consequence of the discussion above. As for the second statement, $w_1 = 1$ implies that $l_1(\boldsymbol{a}) = 0$ for any reduced word $\boldsymbol{a}$ of w. So this means that the $H_w = G_w$ and the other equality is from Theorem 3.10. $\qquad\square$
The second statement appears as part of Proposition 3.17 in [1]. The homomorphism ϕ defined there is exactly superfication.

Now using the previous theorem and Theorem 3.25, we get

Theorem 5.12 *Let $w \in S_n$ and $\boldsymbol{c} \in \check{R}(w)$. Suppose*

$$\boldsymbol{c} \xrightarrow{EG} (\tilde{P}, \tilde{Q})$$

$$\boldsymbol{c} \xrightarrow{D\text{-}K} (P, Q)$$

Then Q is jeu de taquin equivalent to $\tilde{Q}$.

Let $w \in S_n$ and $w_S = n \cdots 321$. In B_n, we have shown that the Edelman-Greene insertion of $w_S w$ is related to the Kraśkiewicz insertion of $\bar{w}$. In D_n, to get a similar result, we have to define $\bar{w}$ in a suitable manner. Denote by $\bar{w}$ the element of D_n obtained from w by putting a bar over all w_i when n is even and all w_i except the symbol 1 when n is odd. Also, we need the following result.

Lemma 5.13 *Let $w \in S_n$. If $P \in \mathrm{SDT}(\bar{w})$, then P contains the tableau*

$$\begin{array}{|c|c|c c c c|c|}
\hline
n-1 & n-2 & \bullet & \bullet & \bullet & \bullet & 1 \\
\hline
 & n-2 & \bullet & \bullet & \bullet & \bullet & 1 \\
\cline{2-2}\cline{7-7}
\end{array}$$

$$\bullet\ \bullet\ \bullet\ \bullet$$

$$\begin{array}{|c|c|}
\hline
2 & 1 \\
\hline
 & 1 \\
\cline{2-2}
\end{array}$$
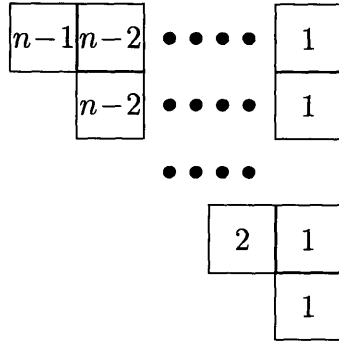
Proof: We will use induction on n.

Case: $n = 1$. Trivial.

Case: $n > 1$. Let $\check{a} \in \check{R}(w)$. The simple reflections that affect the symbol n will yield a decreasing subsequence $n - 1\ n - 2\ \cdots 21$. In particular, if $P \in \mathrm{SDT}(w)$, π_P must contain this subsequence. Hence, $P_1 \!\downarrow\, = n - 1 \cdots 21$ or $n - 1 \cdots 211$. Suppose $P_1 = n - 1\ n - 2\ \cdots 21 a_1 a_2 \cdots a_k$. Let P' be the standard decomposition tableau that is obtained by removing the P_1. We now have to split into the case when n is even and when n is odd.

Subcase: n is even. Let $v = w s_{a_k} \cdots s_{a_1} s_0 s_2 \cdots s_{n-1}$. Then $v_n = n$ and v_1 is unbarred. Clearly, $P' \in \mathrm{SDT}(v)$. Treat v as a signed permutation in D_{n-1}. Consider $\Gamma(v)$. From Theorem 5.1, $P' \in \mathrm{SDT}(\Gamma(v))$ as well. Now, $\Gamma(v) = \bar{u}$ for some $u \in S_{n-1}$. Hence, by induction hypothesis P' must contain

$$\begin{array}{|c|c|c c c c|c|}
\hline
n-2 & n-3 & \bullet & \bullet & \bullet & \bullet & 1 \\
\hline
 & n-3 & \bullet & \bullet & \bullet & \bullet & 1 \\
\cline{2-2}\cline{7-7}
\end{array}$$

$$\bullet\ \bullet\ \bullet\ \bullet$$

$$\begin{array}{|c|c|}
\hline
2 & 1 \\
\hline
 & 1 \\
\cline{2-2}
\end{array}$$
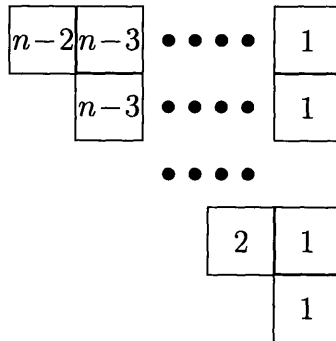
Subcase: n is odd. Consider the signed permutation $\Gamma(w)$. From Lemma 4.1, we know that the first number in $\Gamma(w)$ is unbarred. From Theorem 5.1, $P \in \Gamma(w)$. Now, let $v = \Gamma(w) s_{a_k} \cdots s_{a_1} s_0 s_2 \cdots s_{n-1}$. As before, $v_n = n$ but v_1 is barred. Also,

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$P' \in \mathrm{SDT}(v)$. If we consider v to be a signed permutation of D_{n-1}, then it is easy to see that $v = \bar{u}$ for some $u \in S_{n-1}$. Hence, by induction hypothesis, P' contains the tableau as in the even case. Therefore, P must contain the tableau shown in the theorem. □

Theorem 5.14 *Let $w \in S_n$. There exists a bijection Φ from $\mathrm{SDT}_S(w_S w)$ to $\mathrm{SDT}(\bar{w})$ where $w_S = n\ n-1\ \cdots 321$. Furthermore,*

$$H_{\bar{w}}(x) = \sum_{\tilde{P} \in \mathrm{SDT}_S(w_S w)} P_{\delta_{n-1} + \mathrm{sh}(\tilde{P})}(x)$$

where $\delta_{n-1} = (n-1, n-2, \cdots, 2, 1)$.

Proof: Using the previous lemma, we can imitate the proof of Theorem 5.14 to show a bijection Φ between $\mathrm{SDT}_S(w_S w)$ and $\mathrm{SDT}_D(\bar{w})$. We omit the details of the bijection but again, we observe that the Edelman-Greene insertion appears as part of the D-Kraśkiewicz insertion. Next, we note that

$$
\begin{aligned}
l(\Phi(\tilde{P})) &= n-1 \\
l_1(\Phi(\tilde{P})) &= n-1 + l_1(\tilde{P}) \\
N(\Phi(\tilde{P})) &= 2^{l_1(\tilde{P})} \\
\mathrm{sh}(\Phi(\tilde{P})) &= \delta_{n-1} + \mathrm{sh}(\tilde{P})
\end{aligned}
$$

Hence,

$$H_{\bar{w}}(x) = \sum_{\tilde{P} \in \mathrm{SDT}_S(w_S w)} P_{\delta_{n-1} + \mathrm{sh}(\tilde{P})}(x)$$

□

This theorem has also been independently proved in [1, Equation (3.16)]

5.4 Recording Tableau and Promotion Sequence

In this section, we give analogues of results of Section 3.4. The proofs are omitted because they are basically the same as those in that section.

Lemma 5.15 *Let* $P \in \mathrm{SDT}(w)$ *and*

$$(\pi_P)^{\mathrm{r}} \to (R, S)$$

Then S *is row-wise and* $\mathrm{sh}(P) = \mathrm{sh}(R) = \mathrm{sh}(S)$.

Theorem 5.16 *Let* $c \in \check{R}(w)$ *and* $c \to (P, Q)$ *and* $c^{\mathrm{r}} \to (R, S)$ *Then,*

$$\mathrm{sh}(Q) = \mathrm{sh}(S)$$

Proof: Follows easily from the previous lemma. $\qquad\square$

Corollary 5.17 *Let* $w \in D_n$. *Then*

$$H_w(\boldsymbol{x}) = H_{w^{-1}}(\boldsymbol{x})$$

Proof: There is an obvious bijection from $R(w)$ to $R(w^{-1})$ given by reversing the reduced words. This induces a bijection on $\check{R}(w)$ and $\check{R}(w^{-1})$ and also a bijection between $\mathrm{SDT}(w)$ and $\mathrm{SDT}(w^{-1})$. From Theorem 5.16, this bijection is shape-preserving. Furthermore, from Lemma 4.31,

$$\frac{N(P)}{2^{l_1(P)}} = \frac{N(R)}{2^{l_1(R)}}$$

since $\pi_P \sim \pi_R^{\mathrm{r}}$ and keeping in mind that $N(\boldsymbol{c}) = N(\boldsymbol{c}^{\mathrm{r}})$ for any flattened word $\boldsymbol{c}$. Therefore,

$$H_{w^{-1}}(\boldsymbol{x}) = H_w(\boldsymbol{x})$$

$\qquad\square$

Recall the definition of the delta operator, Δ in Definition 3.23. Given a standard shifted Young tableau Q, $\Delta(Q)$ is obtained by deleting the entry 1 and then applying jeu de taquin to fill in this empty box and finally changing all the entries accordingly to get a standard shifted Young tableau again.

Theorem 5.18 *Let* $c = c_1 c_2 \cdots c_m \in \check{R}(w)$ *and*

$$c_1 c_2 \cdots c_m \to (P, Q)$$

$$c_2 \cdots c_m \to (R, S)$$

Then,

$$S = \Delta(Q)$$

From here, we recall the definition of evacuation. From Definition 3.26, $\mathrm{ev}(Q)$ encodes the shapes of successive applications of delta operator on Q. This enables us to refine Theorem 5.16.

Corollary 5.19 *Let* $c \in \check{R}(w)$ *and*

$$\begin{aligned} c &\to (P, Q) \\ c^r &\to (R, S) \end{aligned}$$

Then,

$$S = \mathrm{ev}(Q)$$

For the rest of the section, we aim to give an analogue of Theorem 3.31. The analogue of Lemma 3.29 is:

Lemma 5.20 *Let* $N = n^2 - n$ *and* $a = a_1 a_2 \cdots a_{N-1} a_N \in R(w_D)$ *and* a_0 *be a number such that* $a_0 a_1 a_2 \cdots a_{N-1} \in R(w_D)$. *When* n *is even,*

$$a_0 = a_N$$

and when n *is odd,*

$$a_0 = \begin{cases} a_N & \text{if } a_N > 1 \\ 0 & \text{if } a_N = 1 \\ 1 & \text{if } a_N = 0 \end{cases}$$

Proof:

$$a_0 a_1 a_2 \cdots a_{N-1} = w_D$$
$$\Rightarrow \quad a_0 w_D a_N = w_D$$
$$\Rightarrow \quad a_0 = w_D a_N w_D$$

Using this formula, it is an easy verification of the result. $\qquad\square$

When we translate this result to flattened words of w_D, we get:

Lemma 5.21 *Let $N = n^2 - n$ and $\mathbf{c} = c_1 c_2 \cdots c_{N-1} c_N \in \check{R}(w_D)$. If c_0 is a number such that $c_0 c_1 c_2 \cdots c_{N-1} \in \check{R}(w_D)$, then $c_0 = c_N$.*

Recall Definition 3.30. We have to alter the definition of the short promotion sequence slightly but the basics are still the same.

Definition 5.22 *Let $N = n^2 - n$. Given T a standard shifted Young tableau of shape $(2n - 2, 2n - 4, \cdots 4, 2)$, we define the promotion operator acting on T as follows:*

1. *delete the largest entry in T*

2. *apply jeu de taquin into that box*

3. *put 0 into the box $(1,1)$*

4. *add 1 to every box*

Give the last box $(i, 2n - 2i)$, of each row i, the label $n - i$. Then, the short promotion sequence $\hat{p}(T) = (r_1, r_2, \cdots, r_N)$ is the sequence of numbers where r_i is the label of the box with the largest entry in the tableau $p^{N-i}(T)$.

With this new definition, it was shown in [6, Theorem 5.16] that $\hat{p}$ gives a bijection between standard shifted Young tableau of shape $(2n-2, 2n-4, \cdots 4, 2)$ and flattened words of w_D. Note that the labels given to box $(i, 2n - 2i)$ is exactly the entry of the same box in P, the unique standard decomposition tableau of w_D.

With slight changes in the proof, we get the analogue of Theorem 3.31.

Theorem 5.23 *Let $\mathbf{c} \in \check{R}(w_D)$. If $\mathbf{c} \to (P, Q)$, then $\hat{p}(Q) = \mathbf{c}$.*

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Appendix A

Open Problems

We will list a few open questions that lead from here.

1. There is a theory on projective representations of S_n. The relation between the Schur P-function and irreducible projective representations mirror that of the relation between the Schur functions and irreducible representations for S_n. This means that the B_n and D_n stable Schubert polynomials correspond to some projective representations. What are these?

2. In the theory of stable Schubert polynomials, there is a nice characterization on those that are themselves Schur functions. Is there a corresponding characterization in the B and D analogues?

3. Is there a connection between the Kraśkiewicz insertion and shifted mixed insertion. On one hand, can we describe an analogue of the shifted mixed insertion for reduced words of B_n and D_n? On the other hand, is there a generalized Kraśkiewicz insertion that applies onto sequence of numbers that allow repetition? Also, we still seek a nicer interpretation of the shape of the insertion tableau in terms of unimodal subsequences.

4. How would these stable Schubert polynomials be helpful in the search for Schubert polynomials analogues for B and D?

5. Is there something similar to the stable Schubert polynomials out there for the exceptional Coxeter groups or for any Coxeter group?

Appendix B

Proof of Theorem 1.18

Recall the table of elementary B-Coxeter-Knuth relations:

Elementary B-Coxeter-Knuth Relations

$$0101 \sim 1010 \tag{1}$$

$$ab(b+1)b \sim a(b+1)b(b+1) \tag{2}$$

$$ba(b+1)b \sim b(b+1)ab \tag{3}$$

$$a(a+1)ba \sim a(a+1)ab \qquad a+1 < b \tag{4}$$

$$(a+1)ab(a+1) \sim (a+1)ba(a+1) \quad a+1 < b \tag{5}$$

$$abdc \sim adbc \tag{6}$$

$$acdb \sim acbd \tag{7}$$

$$adcb \sim dacb \tag{8}$$

$$badc \sim bdac \tag{9}$$

and the reverse of these. Here $a < b < c < d$ unless otherwise stated.

We intend to prove the more difficult direction of Theorem 1.18:

Theorem B.1 *Let $a, b \in R(w)$. If $a \sim b$ then they have the same insertion tableau.*

Some simplifications of the proof:

1. It suffices to show this when a and b differ by an elementary B-Coxeter-Knuth relation.

2. Also, we can use induction on $l(w)$ to reduce to the case where $\boldsymbol{a}$ and $\boldsymbol{b}$ differ in the last 4 letters. This means that we can write $\boldsymbol{a} = \boldsymbol{cd}$ and $\boldsymbol{b} = \boldsymbol{ce}$ where $\boldsymbol{d} \sim \boldsymbol{e}$ is one of the elementary B-Coxeter-Knuth relation listed above.

3. Let R be the insertion tableau of $\boldsymbol{c}$. Since $\boldsymbol{c} \sim \pi_R$, by induction, we can replace $\boldsymbol{c}$ by π_R.

4. Again, using induction, we can reduced to the case where R consists of only one row and $\boldsymbol{c} = \pi_R$ is just a unimodal sequence. Now, we just have to prove:

Proposition B.2 *Let $\boldsymbol{d}, \boldsymbol{e}$ be two sequences of length 4 that are elementary B-Coxeter-Knuth related. Let R be a unimodal sequence such that $R\boldsymbol{d}$ is reduced. Suppose*

$$R \overset{in}{\longleftarrow} \boldsymbol{d} \;=\; \boldsymbol{f} \overset{out}{\longleftarrow} R'$$
$$R \overset{in}{\longleftarrow} \boldsymbol{e} \;=\; \boldsymbol{g} \overset{out}{\longleftarrow} R''$$

Then $R' = R''$ and $\boldsymbol{f} \sim \boldsymbol{g}$.

5. We can assume that when we insert $\boldsymbol{d}$ or $\boldsymbol{e}$ into R, all the numbers cause bumpings.

To explain this last simplification, consider the following situation of inserting the number a into R which does not cause bumping:

$$R \overset{in}{\longleftarrow} a = Ra$$

We make some changes to R as follows: Take two large numbers $x < y$ which are bigger than a and all the numbers in R. Replace R by xRy. Then the previous insertion becomes:

$$xRy \overset{in}{\longleftarrow} a = x \overset{out}{\longleftarrow} yRa$$

When we delete x and y, we get back the original result. This means that we can turn every insertion into one that causes bumping.

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So, suppose we have proved Proposition B.2 whenever $|f| = |g| = 4$. Consider the insertion when either $|f|$ or $|g| < 4$.

$$R \overset{in}{\leftarrow} d \;=\; f \overset{out}{\leftarrow} R'$$
$$R \overset{in}{\leftarrow} e \;=\; g \overset{out}{\leftarrow} R''$$

Note that $|R'| = |R''|$ since by Corollory 1.23, $|R'|$ and $|R''|$ are the lengths of the longest unimodal subsequences in Rd and Re respectively and by Lemma 1.20, these lengths are invariant under B-Coxeter-Knuth relations. This also means that $|f| = |g|$.

Now, extend R on both sides with sequences x and y of suitable large numbers and apply the insertions again.

$$xRy \overset{in}{\leftarrow} d \;=\; f' \overset{out}{\leftarrow} zR'$$
$$xRy \overset{in}{\leftarrow} e \;=\; g' \overset{out}{\leftarrow} wR''$$

By our assuption, $zR' = wR''$ and $f' \sim g'$. Since z and w contain the large numbers that have been attached to R, deleting them would yield $R' = R''$.

Clearly, f' contains f as a subsequence and g' contains g as a subsequence too. If $f' = g'$, then it is easy to see that $f = g$. If $f' \sim g'$, we note that they have to be one of the elementary relations (3) − (9) or their reverses. A quick check will show that deleting the largest number in any of these elementary relations will yield two equal subsequences. Hence again, $f = g$.

Proof of Proposition B.2

Now, we will do a case by case checking of the various elementary relations. Through out the rest of this proof, we will assume that all the unimodal sequences are reduced words and the numbers that are to be inserted preserve this property. We will use "|" to split R into decreasing and increasing parts. To simplify some notation, we introduce the following terminology.

1. If during an insertion, the length of the decreasing part increases, we say that a *hijack* has occured. The number that is moved from the increasing part to the

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decreasing part is said to be hijacked.

2. If 0 is to be inserted into a unimodal sequence that contains the subsequence 101, we will call this a *zero insertion*. This is Case 2.0 of the Kraśkiewicz insertion algorithm described in Section 1.3.

Case: (1) $d = 0101, e = 1010$.

Since $R0101$ has to be reduced, we can divide this case into four subcases.

Subcase: $R = \cdots 21012 \cdots$.

$$\cdots 21012 \cdots \xleftarrow{in} 0101 = 0101 \xleftarrow{out} \cdots 21012 \cdots$$
$$\cdots 21012 \cdots \xleftarrow{in} 1010 = 1010 \xleftarrow{out} \cdots 21012 \cdots$$

Subcase: $R = \cdots e1012 \cdots, e > 2$.

$$\cdots e1012 \cdots \xleftarrow{in} 0101 = 0101 \xleftarrow{out} \cdots e2101 \cdots$$
$$\cdots e1012 \cdots \xleftarrow{in} 1010 = 1010 \xleftarrow{out} \cdots e2101 \cdots$$

In the previous two subcases, f and g are exactly the sequences appearing as (1).

Subcase: $R = \cdots 1|2cd \cdots$ and 1 is the smallest number in R.

$$
\begin{aligned}
\cdots 12cd \cdots \xleftarrow{in} 0101 &= 1 \xleftarrow{out} \cdots 20cd \cdots \xleftarrow{in} 101 \\
&= 1c' \xleftarrow{out} \cdots c \cdots 01d \cdots \xleftarrow{in} 01 \\
&= 1c'0d' \xleftarrow{out} \cdots d \cdots 101 \cdots \\
\cdots 12cd \cdots \xleftarrow{in} 1010 &= 1 \xleftarrow{out} \cdots 21cd \cdots \xleftarrow{in} 010 \\
&= 1c' \xleftarrow{out} \cdots c \cdots 10d \cdots \xleftarrow{in} 10 \\
&= 1c'd'0 \xleftarrow{out} \cdots d \cdots 101 \cdots
\end{aligned}
$$

$1 < c' < d'$ and $1c'0d' \sim 1c'd'0$ by $(8)^r$.

Subcase: $R = \cdots e|fg \cdots, 1 < e < f < g$ and e is the smallest number in R.

$$\cdots efg \cdots \xleftarrow{in} 0101 = f''g''0h'' \xleftarrow{out} \cdots e101 \cdots$$
$$\cdots efg \cdots \xleftarrow{in} 1010 = f''g''h''0 \xleftarrow{out} \cdots e101 \cdots$$

$f'' < g'' < h''$ and $f''g''0h'' \sim f''g''h''0$ by (4) or $(8)^r$.

Case: (2) $\boldsymbol{d} = ab(b+1)b, \boldsymbol{e} = a(b+1)b(b+1)$.

In the next three subcases, we consider the particular situation when $\boldsymbol{d} = 0121, \boldsymbol{e} = 0212$.

Subcase: $R = \cdots e10123 \cdots, e > 2$.

$$\cdots 10123 \cdots \xleftarrow{in} 0121 \;=\; 0120 \xrightleftharpoons{out} \cdots 32123 \cdots$$
$$\cdots 10123 \cdots \xleftarrow{in} 0212 \;=\; 0102 \xrightleftharpoons{out} \cdots 32123 \cdots$$

$0120 \sim 0102$ by (4).

Subcase: $R = \cdots e210123 \cdots, e > 3$.

$$\cdots 210123 \cdots \xleftarrow{in} 0121 \;=\; 0121 \xrightleftharpoons{out} \cdots 320123 \cdots$$
$$\cdots 210123 \cdots \xleftarrow{in} 0212 \;=\; 0212 \xrightleftharpoons{out} \cdots 320123 \cdots$$

$0121 \sim 0212$ by (2).

Subcase: $R = \cdots 3210123 \cdots$.

$$\cdots 3210123 \cdots \xleftarrow{in} 0121 \;=\; 0121 \xrightleftharpoons{out} \cdots 3210123 \cdots$$
$$\cdots 3210123 \cdots \xleftarrow{in} 0212 \;=\; 0212 \xrightleftharpoons{out} \cdots 3210123 \cdots$$

$0121 \sim 0212$ by (2).

After dealing with this particular case, we let $\boldsymbol{d}$ and $\boldsymbol{e}$ be other sequences of the form $ab(b+1)b$ and $a(b+1)b(b+1)$ respectively. In the next three subcases, the results are still true when inserting $a = 0$ is a zero insertion. We just have to observe that $a'' = 0$ and at every step, a' and a are actually one number.

Subcase: $R = \cdots (b+1)b \cdots (b+1)(b+2) \cdots$.

$$\cdots (b+1)b \cdots (b+1)(b+2) \cdots \overset{in}{\Leftarrow} ab(b+1)b$$
$$= \quad a'' \overset{out}{\Leftarrow} \cdots (b+1)b \cdots a' \cdots a \cdots (b+1)(b+2) \cdots \overset{in}{\Leftarrow} b(b+1)b$$
$$= \quad a''b(b+1)b \overset{out}{\Leftarrow} \cdots (b+2)(b+1) \cdots a' \cdots a \cdots b(b+1) \cdots$$
$$\cdots (b+1)b \cdots (b+1)(b+2) \cdots \overset{in}{\Leftarrow} a(b+1)b(b+1)$$
$$= \quad a'' \overset{out}{\Leftarrow} \cdots (b+1)b \cdots a' \cdots a \cdots (b+1)(b+2) \cdots \overset{in}{\Leftarrow} (b+1)b(b+1)$$
$$= \quad a''(b+1)b(b+1) \overset{out}{\Leftarrow} \cdots (b+2)(b+1) \cdots a' \cdots a \cdots b(b+1) \cdots$$

$a'' < b$ and $a''b(b+1)b \sim a''(b+1)b(b+1)$ by (2).

Subcase: $R = \cdots (b+1)(b+2) \cdots$ but $R\!\downarrow$ does not contain $(b+1)b$.

$$\cdots (b+1)(b+2) \cdots \overset{in}{\Leftarrow} ab(b+1)b$$
$$= \quad a''b'' \overset{out}{\Leftarrow} \cdots b+1 \cdots a' \cdots a \cdots b(b+2) \cdots \overset{in}{\Leftarrow} (b+1)b$$
$$= \quad a''b''(b+1) \overset{out}{\Leftarrow} \cdots b+2 \cdots a' \cdots a \cdots b(b+1) \cdots \overset{in}{\Leftarrow} b$$
$$= \quad a''b''(b+1)d'' \overset{out}{\Leftarrow} \cdots (b+2)(b+1) \cdots a' \cdots a \cdots b(b+1) \cdots$$
$$\cdots (b+1)(b+2) \cdots \overset{in}{\Leftarrow} a(b+1)b(b+1)$$
$$= \quad a''b'' \overset{out}{\Leftarrow} \cdots b+2 \cdots a' \cdots a \cdots (b+1)(b+2) \cdots \overset{in}{\Leftarrow} b(b+1)$$
$$= \quad a''b''d'' \overset{out}{\Leftarrow} \cdots (b+2)(b+1) \cdots a' \cdots a \cdots b(b+2) \cdots \overset{in}{\Leftarrow} b+1$$
$$= \quad a''b''d''(b+1) \overset{out}{\Leftarrow} \cdots (b+2)(b+1) \cdots a' \cdots a \cdots b(b+1) \cdots$$

$a'' < b'' < b+1, d'' < b+1$ and $a''b''(b+1)d'' \sim a''b''d''(b+1)$ by $(4), (7)$ or $(8)^{\mathrm{r}}$.

Subcase: R does not contain $b(b+1)$.

$$R \overset{in}{\Leftarrow} ab(b+1)b$$
$$= \quad a''b''c'' \overset{out}{\Leftarrow} \cdots c' \cdots b' \cdots a' \cdots a \cdots b(b+1) \cdots \overset{in}{\Leftarrow} b$$
$$= \quad a''b''c''d'' \overset{out}{\Leftarrow} \cdots c' \cdots b+1 \cdots a' \cdots a \cdots b(b+1) \cdots$$
$$R \overset{in}{\Leftarrow} a(b+1)b(b+1)$$
$$= \quad a''b'' \overset{out}{\Leftarrow} \cdots b' \cdots a' \cdots a \cdots b+1 \cdots \overset{in}{\Leftarrow} b(b+1)$$
$$= \quad a''b''d'' \overset{out}{\Leftarrow} \cdots b' \cdots b+1 \cdots a' \cdots a \cdots b \cdots \overset{in}{\Leftarrow} b+1$$
$$= \quad a''b''d''c'' \overset{out}{\Leftarrow} \cdots c' \cdots b+1 \cdots a' \cdots a \cdots b(b+1) \cdots$$

$a'' < b'' < c'', d'' < b'' < c''$ and $a''b''c''d'' \sim a''b''d''c''$ by $(4),(7)$ or $(8)^{\mathrm{r}}$.

For the following subcases, the zero insertion does not arise. That is we can assume $a \neq 0$ or if $a = 0$ that R does not contain the subsequence 101.

Subcase: $R = \cdots a''|a' \cdots$.

$$
\begin{aligned}
& R \xleftarrow{in} ab(b+1)b \\
= \; & a'' \xleftarrow{out} \cdots a'a| \cdots \xleftarrow{in} b(b+1)b \\
= \; & a''a' \xleftarrow{out} \cdots b'a|b \cdots \xleftarrow{in} (b+1)b \\
= \; & a''a'b' \xleftarrow{out} \cdots c'a|b(b+1) \cdots \xleftarrow{in} b \\
= \; & a''a'b'a \xleftarrow{out} \cdots c'(b+1)b|b+1 \cdots \\
& R \xleftarrow{in} a(b+1)b(b+1) \\
= \; & a'' \xleftarrow{out} \cdots a'a| \cdots \xleftarrow{in} (b+1)b(b+1) \\
= \; & a''a' \xleftarrow{out} \cdots b'a|b+1 \cdots \xleftarrow{in} b(b+1) \\
= \; & a''a'a \xleftarrow{out} \cdots b'(b+1)b| \cdots \xleftarrow{in} b+1 \\
= \; & a''a'ab' \xleftarrow{out} \cdots c'(b+1)b|b+1 \cdots
\end{aligned}
$$

$a'' < a' < b', a < a' < b'$ and $a''a'b'a \sim a''a'ab'$ by $(4),(7)$ or $(8)^{\mathrm{r}}$.

Subcase: $R = \cdots a''e|a' \cdots, e < a$.

$$
\begin{aligned}
& R \xleftarrow{in} ab(b+1)b \\
= \; & a'' \xleftarrow{out} \cdots a'e|a \cdots \xleftarrow{in} b(b+1)b \\
= \; & a''a' \xleftarrow{out} \cdots b'e|a \cdots b \cdots \xleftarrow{in} (b+1)b \\
= \; & a''a'b' \xleftarrow{out} \cdots c'e|a \cdots b(b+1) \cdots \xleftarrow{in} b \\
= \; & a''a'b'e \xleftarrow{out} \cdots c'(b+1)a| \cdots b(b+1) \cdots \\
& R \xleftarrow{in} a(b+1)b(b+1) \\
= \; & a'' \xleftarrow{out} \cdots a'e|a \cdots \xleftarrow{in} (b+1)b(b+1) \\
= \; & a''a' \xleftarrow{out} \cdots b'e|a \cdots b+1 \cdots \xleftarrow{in} b(b+1) \\
= \; & a''a'e \xleftarrow{out} \cdots b'(b+1)a| \cdots b \cdots \xleftarrow{in} b+1 \\
= \; & a''a'eb' \xleftarrow{out} \cdots c'(b+1)a| \cdots b(b+1) \cdots
\end{aligned}
$$

$a'' < a' < b', e < a''$ and $a''a'b'e \sim a''a'eb'$ by (7) or $(8)^{\mathrm{r}}$.

Case: $(2)^r$ $\quad d = b(b+1)ba, e = (b+1)b(b+1)a$

Let us take care of the particular situation $d = 2120, e = 1210$.

Subcase:

$$\cdots 210 \cdots \overset{in}{\longleftarrow} 2120$$
$$= \ b'' \overset{out}{\longleftarrow} \cdots b' \cdots 2102 \cdots \overset{in}{\longleftarrow} 120$$
$$= \ b''1 \overset{out}{\longleftarrow} \cdots b' \cdots 2101 \cdots \overset{in}{\longleftarrow} 20$$
$$= \ b''1c''0 \overset{out}{\longleftarrow} \cdots c' \cdots b' \cdots 21012 \cdots$$

$$\cdots 210 \cdots \overset{in}{\longleftarrow} 1210$$
$$= \ b'' \overset{out}{\longleftarrow} \cdots b' \cdots 2101 \cdots \overset{in}{\longleftarrow} 210$$
$$= \ b''c'' \overset{out}{\longleftarrow} \cdots c' \cdots b' \cdots 2101 \cdots \overset{in}{\longleftarrow} 10$$
$$= \ b''c''10 \overset{out}{\longleftarrow} \cdots c' \cdots b' \cdots 21012 \cdots$$

$1 < b'' < c''$ and $b''1c''0 \sim b''c''10$ by $(6)^r$.

Next, we want to deal with the subcases where there is a zero insertion. This can only occur when $a = 0$.

Subcase: $R = \cdots (b+1)b \cdots 101 \cdots (b+1)(b+2) \cdots$.

$$\cdots (b+1)b \cdots 101 \cdots (b+1)(b+2) \cdots \overset{in}{\longleftarrow} b(b+1)b0$$
$$= \ b(b+1)b0 \overset{out}{\longleftarrow} \cdots (b+2)(b+1) \cdots 101 \cdots b(b+1) \cdots$$
$$\cdots (b+1)b \cdots 101 \cdots (b+1)(b+2) \cdots \overset{in}{\longleftarrow} (b+1)b(b+1)0$$
$$= \ (b+1)b(b+1)0 \overset{out}{\longleftarrow} \cdots (b+2)(b+1) \cdots 101 \cdots b(b+1) \cdots$$

Subcase: $R = \cdots 101 \cdots (b+1)(b+2) \cdots$ but $R{\downarrow}$ does not contain $(b+1)b$.

$$\cdots 101 \cdots (b+1)(b+2) \cdots \overset{in}{\longleftarrow} b(b+1)b0$$
$$= \ b''c''(b+1)0 \overset{out}{\longleftarrow} \cdots (b+2)(b+1) \cdots 101 \cdots b(b+1) \cdots$$
$$\cdots 101 \cdots (b+1)(b+2) \cdots \overset{in}{\longleftarrow} (b+1)b(b+1)0$$
$$= \ b''(b+1)c''0 \overset{out}{\longleftarrow} \cdots (b+2)(b+1) \cdots 101 \cdots b(b+1) \cdots$$

Subcase: R contains 101 but not $(b+1)(b+2)$.

$$\cdots 101 \cdots \overset{in}{\Leftarrow} b(b+1)b0$$
$$= b''a''c''0 \overset{out}{\Leftarrow} \cdots b' \cdots b+1 \cdots 101 \cdots b(b+1) \cdots$$
$$\cdots 101 \cdots \overset{in}{\Leftarrow} (b+1)b(b+1)0$$
$$= b''c''a''0 \overset{out}{\Leftarrow} \cdots b' \cdots b+1 \cdots 101 \cdots b(b+1) \cdots$$

In the next few subcases, the zero insertion does not arise.

Subcase: $R = \cdots | b'c' \cdots, b' > b+1$.

$$\cdots | b'c' \cdots \overset{in}{\Leftarrow} b(b+1)ba$$
$$= b'' \overset{out}{\Leftarrow} \cdots b' \cdots b | c' \cdots \overset{in}{\Leftarrow} (b+1)ba$$
$$= b''c'' \overset{out}{\Leftarrow} \cdots c' \cdots b' \cdots b | (b+1) \cdots \overset{in}{\Leftarrow} ba$$
$$= b''c''b \overset{out}{\Leftarrow} \cdots c' \cdots b' \cdots (b+1)b | \cdots \overset{in}{\Leftarrow} a$$
$$= b''c''ba'' \overset{out}{\Leftarrow} \cdots a' \cdots c' \cdots b' \cdots (b+1)ba \cdots$$
$$\cdots | b'c' \cdots \overset{in}{\Leftarrow} (b+1)b(b+1)a$$
$$= b'' \overset{out}{\Leftarrow} \cdots b' \cdots b+1 | c' \cdots \overset{in}{\Leftarrow} b(b+1)a$$
$$= b''c'' \overset{out}{\Leftarrow} \cdots c' \cdots b' \cdots (b+1)b | \cdots \overset{in}{\Leftarrow} (b+1)a$$
$$= b''c''a'' \overset{out}{\Leftarrow} \cdots a' \cdots c' \cdots b' \cdots (b+1)b(b+1) \cdots \overset{in}{\Leftarrow} a$$
$$= b''c''a''b \overset{out}{\Leftarrow} \cdots a' \cdots c' \cdots b' \cdots (b+1)ba \cdots$$

$b'' < c'' < a''$, $b < c'' < a''$ and $b''c''ba'' \sim b''c''a''b$ by (4), (7) or $(8)^{\mathrm{r}}$.

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Subcase: $R = \cdots |(b+1)(b+2)\cdots$.

$$
\begin{aligned}
&\quad \cdots |(b+1)(b+2)\cdots \xleftarrow{in} b(b+1)ba\\
={}& b'' \xleftarrow{out} \cdots (b+1)b|b+2\cdots \xleftarrow{in} (b+1)ba\\
={}& b''(b+1) \xleftarrow{out} \cdots (b+2)b|b+1\cdots \xleftarrow{in} ba\\
={}& b''(b+1)b \xleftarrow{out} \cdots (b+2)(b+1)b|\cdots \xleftarrow{in} a\\
={}& b''(b+1)ba'' \xleftarrow{out} \cdots a' \cdots (b+2)(b+1)ba|\cdots\\
&\quad \cdots |(b+1)(b+2)\cdots \xleftarrow{in} (b+1)b(b+1)a\\
={}& b'' \xleftarrow{out} \cdots (b+2)(b+1)|b+2 \xleftarrow{in} b(b+1)a\\
={}& b''(b+1) \xleftarrow{out} \cdots (b+2)(b+1)b|\cdots \xleftarrow{in} (b+1)a\\
={}& b''(b+1)a'' \xleftarrow{out} \cdots a' \cdots (b+2)(b+1)b|b+1\cdots \xleftarrow{in} a\\
={}& b''(b+1)a''b \xleftarrow{out} \cdots a' \cdots (b+2)(b+1)b|a\cdots
\end{aligned}
$$

$b'' < b+1 < a''$ and $b''(b+1)ba'' \sim b''(b+1)a''b$ by (4), (7) or (8)$^{\mathrm{r}}$.

Subcase: $R = \cdots (b+1)b \cdots (b+1)(b+2)\cdots$.

$$
\begin{aligned}
&\quad \cdots (b+1)b \cdots (b+1)(b+2)\cdots \xleftarrow{in} b(b+1)ba\\
={}& b(b+1)ba'' \xleftarrow{out} \cdots (b+2)(b+1)\cdots a' \cdots a \cdots b(b+1)\cdots\\
&\quad \cdots (b+1)b \cdots (b+1)(b+2)\cdots \xleftarrow{in} (b+1)b(b+1)a\\
={}& (b+1)b(b+1)a'' \xleftarrow{out} \cdots (b+2)(b+1)\cdots a' \cdots a \cdots b(b+1)\cdots
\end{aligned}
$$

$a'' < b$ and $b(b+1)ba'' \sim (b+1)b(b+1)a''$ by (2)$^{\mathrm{r}}$.

Subcase: $R = \cdots (b+1)(b+2)\cdots$ but $R{\downarrow}$ does not contain $(b+1)b$.

$$
\begin{aligned}
&\quad \cdots (b+1)(b+2)\cdots \xleftarrow{in} b(b+1)ba\\
={}& b'' \xleftarrow{out} \cdots b+1 \cdots b(b+2)\cdots \xleftarrow{in} (b+1)ba\\
={}& b''(b+1)c''a'' \xleftarrow{out} \cdots (b+2)(b+1)\cdots a' \cdots a \cdots b(b+1)\cdots\\
&\quad \cdots (b+1)(b+2)\cdots \xleftarrow{in} (b+1)b(b+1)a\\
={}& b'' \xleftarrow{out} \cdots b+2 \cdots (b+1)(b+2)\cdots \xleftarrow{in} b(b+1)a\\
={}& b''c''(b+1)a'' \xleftarrow{out} \cdots (b+2)(b+1)\cdots a' \cdots a \cdots b(b+1)\cdots
\end{aligned}
$$

$a'' < c'' < b'' < b+1$ and $b''(b+1)c''a'' \sim b''c''(b+1)a''$ by (6)$^{\mathrm{r}}$.

Subcase: $R\uparrow$ does not contain $(b+1)(b+2)$.

$$
\begin{aligned}
R &\xleftarrow{in} b(b+1)ba \\
&= b''c'' \xleftarrow{out} \cdots c' \cdots b' \cdots b(b+1) \cdots \xleftarrow{in} ba \\
&= b''c''d''a'' \xleftarrow{out} \cdots c' \cdots b' \cdots b+1 \cdots a' \cdots a \cdots b(b+1) \cdots \\
R &\xleftarrow{in} (b+1)b(b+1)a \\
&= b'' \xleftarrow{out} \cdots b' \cdots | \cdots b+1 \cdots \xleftarrow{in} b(b+1)a \\
&= b''d'' \xleftarrow{out} \cdots b' \cdots b+1 \cdots | \cdots b \cdots \xleftarrow{in} (b+1)a \\
&= b''d''c'' \xleftarrow{out} \cdots c' \cdots b' \cdots b+1 \cdots | \cdots b(b+1) \cdots \xleftarrow{in} a \\
&= b''d''c''a'' \xleftarrow{out} \cdots c' \cdots b' \cdots b+1 \cdots a' \cdots a \cdots b(b+1) \cdots
\end{aligned}
$$

$a'' < d'' < b'' < c''$ and $b''c''d''a'' \sim b''d''c''a''$ by $(6)^{\mathrm{r}}$.

We will now give a more unified approach to the remaining elementary B-Coxeter-Knuth relations (3) – (9) and their reverses.

Observe that in all these relations d and e differ only by switching two adjacent numbers x and y. Moreover, $x < y - 1$. This leads us to examine the insertions of xy and yx into a unimodal sequence S in detail. Through out the rest of this, we will assume that $x < y - 1$.

We already have some results about inserting xy and yx from Section 2.2. We will built on this. From the proof of Lemma 2.8, by re-examining the cases where two numbers are bumped out, we conclude that:

Lemma B.3 *Let $x < y$ and suppose*

$$
\begin{aligned}
S \xleftarrow{in} xy &= x'' \xleftarrow{out} S' \xleftarrow{in} y \\
&= x''y'' \xleftarrow{out} S''
\end{aligned}
$$

then $x'' < y''$ and during $S' \xleftarrow{in} y$, no hijack occurs.

Proof: We only need to show that no hijack occurs during the insertion $S' \xleftarrow{in} y$.

Case: $S \xleftarrow{in} x$ is a zero insertion. In this case, S' contains the subsequence 101 and during $S' \xleftarrow{in} y$, no hijack occurs.

Case: $S \overset{in}{\leftarrow} x$ is not a zero insertion and x is in $S' \downarrow$. Now, x is the smallest number in S' and

$$S' = \cdots x' \cdots x | \cdots$$

When we insert y into S', it bumps some number z to the right of x. Clearly, $z > x'$ and when it is inserted into $S' \downarrow$ it bumps x' or some number to the left. Therefore, x is left untouched and no hijack occurs.

Case: x is in $S' \uparrow$. Note that x' is in $S' \downarrow$ and $x' > x$. As in the case above, during $S' \overset{in}{\leftarrow} y$, y bumps some number z to the right of x. So, $z > x'$ and the same argument applies. $\qquad \square$

Next, we need to find out how inserting xy relates to inserting yx. To do this, we divide the investigation into two big parts, depending on whether $S \overset{in}{\leftarrow} x$ is a zero insertion.

Suppose $S \overset{in}{\leftarrow} x$ is a zero insertion. Then S contains the subsequence 101 and $x = 0$. Let

$$S \overset{in}{\leftarrow} y \;=\; y'' \overset{out}{\leftarrow} \underbrace{\cdots y' \cdots y \cdots}_{S}$$

where y', y'' are the numbers that are bumped out of $S \uparrow, S \downarrow$ respectively. Recall from Section 1.3 that $y' = y + 1$ when the increasing part of S contains $y(y+1)$ and otherwise y' is the smallest number in $S \uparrow$ which is bigger than y. Similarly, $y'' = y' - 1$ if $S \downarrow$ contains $y'(y'-1)$.

So, comparing the insertions of $0y$ and $y0$ into S, we have

$$
\begin{aligned}
S \overset{in}{\leftarrow} 0y &= 0 \overset{out}{\leftarrow} S \overset{in}{\leftarrow} y \\
&= 0y'' \overset{out}{\leftarrow} S'
\end{aligned}
$$

$$S \overset{in}{\leftarrow} y0 \;=\; y'' \overset{out}{\leftarrow} S' \overset{in}{\leftarrow} 0$$

The outcome of $S \overset{in}{\leftarrow} y0$ depend on whether S' contains the subsequence 101.

Lemma B.4 *Let x, y, S, S' be as above.*

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1. If S' contains 101, then

$$S \overset{in}{\twoheadleftarrow} 0y \;=\; 0y'' \overset{out}{\twoheadleftarrow} S'$$

$$S \overset{in}{\twoheadleftarrow} y0 \;=\; y''0 \overset{out}{\twoheadleftarrow} S'$$

where $y'' > 0$.

2. If S' does not contain 101, then when we insert $y0vu$ and $0yvu$ into S where $u < v < y$, we get

$$S \overset{in}{\twoheadleftarrow} 0yvu \;=\; 0101 \overset{out}{\twoheadleftarrow} S'''$$

$$S \overset{in}{\twoheadleftarrow} y0vu \;=\; 1010 \overset{out}{\twoheadleftarrow} S'''$$

Proof: If S' contains 101, then $S' \overset{in}{\twoheadleftarrow} 0 = 0 \overset{out}{\twoheadleftarrow} S'$ and the result follows.

If S' does not contain 101, then $y'' = 1$ and

$$S' = \cdots y'0|1 \cdots$$

So,

$$
\begin{aligned}
S \overset{in}{\twoheadleftarrow} 0yvu &= 01 \overset{out}{\twoheadleftarrow} \cdots y'0|1 \cdots \overset{in}{\twoheadleftarrow} vu \\
&= 010 \overset{out}{\twoheadleftarrow} \cdots y'v'1| \cdots v \cdots \overset{in}{\twoheadleftarrow} u \\
&= 0101 \overset{out}{\twoheadleftarrow} \cdots y'v'u' \cdots u \cdots v \cdots
\end{aligned}
$$

$$
\begin{aligned}
S \overset{in}{\twoheadleftarrow} y0vu &= 1 \overset{out}{\twoheadleftarrow} \cdots y'0|1 \cdots \overset{in}{\twoheadleftarrow} 0vu \\
&= 10 \overset{out}{\twoheadleftarrow} \cdots y'10| \cdots \overset{in}{\twoheadleftarrow} vu \\
&= 101 \overset{out}{\twoheadleftarrow} \cdots y'v'0| \cdots v \cdots \overset{in}{\twoheadleftarrow} u \\
&= 1010 \overset{out}{\twoheadleftarrow} \cdots y'v'u' \cdots u \cdots v \cdots
\end{aligned}
$$

$\square$

Suppose $S \overset{in}{\twoheadleftarrow} x$ is not a zero insertion. Let

$$S \overset{in}{\twoheadleftarrow} x \;=\; x'' \overset{out}{\twoheadleftarrow} \cdots x' \cdots x \cdots$$

$$S \overset{in}{\twoheadleftarrow} y \;=\; y'' \overset{out}{\twoheadleftarrow} \cdots y' \cdots y \cdots$$

where x', y' are the numbers that are bumped out of $S{\downarrow}$ by x, y respectively and x'', y'' are the numbers that are bumped out of $S{\downarrow}$ by x', y' respectively.

There are several different outcomes. We have divided these into six lemmas.

Lemma B.5 *If $x < y - 1, x' < y' - 1, x'' < y'' - 1$ and provided that during the insertion $S \xleftarrow{in} y$, x' is not hijacked, then*

$$S \xleftarrow{in} xy = x''y'' \xleftarrow{out} S'$$
$$S \xleftarrow{in} yx = y''x'' \xleftarrow{out} S'$$

where $x'' < y''$.

Proof: It suffices to see that

$$S = \cdots y'' \cdots x'' \cdots x' \cdots y' \cdots$$

and verify the insertions. $\square$

Lemma B.6 *Suppose $x < y - 1, x' < y' - 1, x'' = y''$ and suppose also that during the insertion $S \xleftarrow{in} y$, x' is not hijacked. Then, when we insert xyv into S or yxv into S where $x < v < y$, we get*

$$S \xleftarrow{in} xyv = x''x'v'' \xleftarrow{out} S'$$
$$S \xleftarrow{in} yxv = x''v''x' \xleftarrow{out} S'$$

where $v'' < x'' < x'$.

Proof: As in the proof of the previous lemma, we find that

$$S = \cdots x''v'' \cdots x' \cdots y' \cdots$$

where $v'' < x''$. When we insert xyv and yxv into S, we get

$$
\begin{aligned}
S \xleftarrow{in} xyv &= x'' \xleftarrow{out} \cdots x'v'' \cdots x \cdots y' \cdots \xleftarrow{in} yv \\
&= x''x' \xleftarrow{out} \cdots y'v'' \cdots x \cdots y \cdots \xleftarrow{in} v \\
&= x''x'v'' \xleftarrow{out} \cdots y'v' \cdots x \cdots v \cdots
\end{aligned}
$$

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$$S \overset{in}{\leftarrow} yxv \;=\; x'' \overset{out}{\leftarrow} \cdots y'v'' \cdots x' \cdots y \cdots \overset{in}{\leftarrow} xv$$
$$=\; x''v'' \overset{out}{\leftarrow} \cdots y'x' \cdots x \cdots y \cdots \overset{in}{\leftarrow} v$$
$$=\; x''v''x' \overset{out}{\leftarrow} \cdots y'v' \cdots x \cdots v \cdots$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Lemma B.7 *If $x < y - 1, x' < y' - 1, x'' = y'' - 1$ and if during the insertion $S \overset{in}{\leftarrow} y$, x' is not hijacked, then $x' = x'' + 1$. Furthermore, when we insert xyv into S or yxv into S where $x < v < y$,*

$$S \overset{in}{\leftarrow} xyv \;=\; (x'-1)x'(x'-1) \overset{out}{\leftarrow} S'$$
$$S \overset{in}{\leftarrow} yxv \;=\; x'(x'-1)x' \overset{out}{\leftarrow} S'.$$

Proof: In order that $x'' = y'' - 1$, $S\!\downarrow$ must contain $(x'' + 1)x''$. Since x' must bump x'', we are forced to have $x' = x'' + 1$ and

$$S = \cdots x'(x'-1) \cdots x' \cdots y' \cdots$$

So, applying the insertions,

$$S \overset{in}{\leftarrow} xyv \;=\; x'-1 \overset{out}{\leftarrow} \cdots x'(x'-1) \cdots x \cdots y' \cdots \overset{in}{\leftarrow} yv$$
$$=\; (x'-1)x' \overset{out}{\leftarrow} \cdots y'(x'-1) \cdots x \cdots y \cdots \overset{in}{\leftarrow} v$$
$$=\; (x'-1)x'(x'-1) \overset{out}{\leftarrow} \cdots y'v' \cdots x \cdots v \cdots$$

$$S \overset{in}{\leftarrow} yxv \;=\; x' \overset{out}{\leftarrow} \cdots y'(x'-1) \cdots x' \cdots y \cdots \overset{in}{\leftarrow} xv$$
$$=\; x'(x'-1) \overset{out}{\leftarrow} \cdots y'x' \cdots x \cdots y \cdots \overset{in}{\leftarrow} v$$
$$=\; x'(x'-1)x' \overset{out}{\leftarrow} \cdots y'v' \cdots x \cdots v \cdots$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Lemma B.8 *If $x < y - 1, x' = y'$ and y not hijacked when $R \overset{in}{\leftarrow} y$, then $x'' = y''$. Furthermore, when we insert xyv into S or yxv into S where $x < v < y$, we get*

$$S \overset{in}{\leftarrow} xyv \;=\; x''v''w'' \overset{out}{\leftarrow} S'$$
$$S \overset{in}{\leftarrow} yxv \;=\; x''w''v'' \overset{out}{\leftarrow} S'$$

where $w'' < x'' < v''$

Proof: It can be shown that

$$S = \cdots x'' \cdots x'v' \cdots$$

with $x' < v'$. Next,

$$
\begin{aligned}
S \xleftarrow{in} xyv &= x'' \xleftarrow{out} \cdots x' \cdots xv' \cdots \xleftarrow{in} yv \\
&= x''v'' \xleftarrow{out} \cdots v' \cdots x' \cdots xy \cdots \xleftarrow{in} v \\
&= x''v''w'' \xleftarrow{out} \cdots v' \cdots y \cdots xv \cdots
\end{aligned}
$$

$$
\begin{aligned}
S \xleftarrow{in} yxv &= x'' \xleftarrow{out} \cdots x' \cdots yv' \cdots \xleftarrow{in} xv \\
&= x''w'' \xleftarrow{out} \cdots x' \cdots y \cdots xv' \cdots \xleftarrow{in} v \\
&= x''w''v'' \xleftarrow{out} \cdots v' \cdots y \cdots xv \cdots
\end{aligned}
$$

where $w'' < x'' < v''$. $\qquad\square$

The previous lemmas dealt with the cases when some number is not hijacked. The next two are when some number is hijacked.

Lemma B.9 *If $x < y-1$, $x' < y'-1$ and during $S \xleftarrow{in} y$, x' is hijacked, then $x'' = y''$. Furthermore, when we insert $xyvu$ or $yxvu$ into S where $x < v < y$, $u < v < y$, and inserting u does not involve the special insertion, we get*

$$
\begin{aligned}
S \xleftarrow{in} xyvu &= x''x'xw \xleftarrow{out} S' \\
S \xleftarrow{in} yxvu &= x''x'wx \xleftarrow{out} S'
\end{aligned}
$$

where $x < x' < w$ and $x'' < x' < w$.

Proof: In order that x' be hijacked when we insert y into S, we need

$$S = \cdots x''|x' \cdots y' \cdots$$

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$$S \xleftarrow{in} xyvu = x'' \xleftarrow{out} \cdots x'x|w\cdots y' \cdots \xleftarrow{in} yvu$$

$$= x''x' \xleftarrow{out} \cdots y'x|w\cdots y \cdots \xleftarrow{in} vu$$

$$= x''x'x \xleftarrow{out} \cdots y'v'w| \cdots v \cdots y \cdots \xleftarrow{in} u$$

$$= x''x'xw \xleftarrow{out} \cdots y'v'vu| \cdots v \cdots y \cdots$$

$$S \xleftarrow{in} yxvu = x'' \xleftarrow{out} \cdots y'x'|w \cdots y \cdots \xleftarrow{in} xvu$$

$$= x''x' \xleftarrow{out} \cdots y'w|x \cdots y \cdots \xleftarrow{in} vu$$

$$= x''x'w \xleftarrow{out} \cdots y'v'x| \cdots v \cdots y \cdots \xleftarrow{in} u$$

$$= x''x'wx \xleftarrow{out} \cdots y'v'vu| \cdots v \cdots y \cdots$$

$\square$

Lemma B.10 *If $x < y - 1, x' = y'$ and if during $S \xleftarrow{in} y$, y is hijacked, then $x'' = y''$. Furthermore, when we insert $xyvu$ or $yxvu$ into S where $x < v < y$, $u < v < y$ and inserting u does not involve the special insertion, we get*

$$S \xleftarrow{in} xyvu = x''w''xv'' \xleftarrow{out} S'$$
$$S \xleftarrow{in} yxvu = x''w''v''x \xleftarrow{out} S'$$

where $x'' < w'' < v'', x < w'' < v''$.

Proof: During $S \xleftarrow{in} y$, y is hijacked. Since $x < y$, during the insertion of x into S, x is also hijacked. There are two possible scenarios. Either x is smaller than all other numbers in S, so that

$$S = \cdots x'' \cdots |x'w \cdots$$

or the smallest number in S is bumped out during $S \xleftarrow{in} x$ which means

$$S = \cdots x''|x'w \cdots$$

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For the first case,

$$
\begin{aligned}
S \xleftarrow{in} xyvu &= x'' \xleftarrow{out} \cdots x' \cdots x | w \cdots \xleftarrow{in} yvu \\
&= x''w'' \xleftarrow{out} \cdots w \cdots x' \cdots x | y \cdots \xleftarrow{in} vu \\
&= x''w''x \xleftarrow{out} \cdots w \cdots x' \cdots yv | v \cdots \xleftarrow{in} u \\
&= x''w''xv'' \xleftarrow{out} \cdots v \cdots w \cdots x' \cdots yvu | \cdots
\end{aligned}
$$

$$
\begin{aligned}
S \xleftarrow{in} yxvu &= x'' \xleftarrow{out} \cdots x' \cdots y | w \cdots \xleftarrow{in} xvu \\
&= x''w'' \xleftarrow{out} \cdots w \cdots x' \cdots yx | v \cdots \xleftarrow{in} vu \\
&= x''w''v'' \xleftarrow{out} \cdots v \cdots w \cdots x' \cdots yx | v \cdots \xleftarrow{in} u \\
&= x''w''v''x \xleftarrow{out} \cdots v \cdots w \cdots x' \cdots yvu | \cdots
\end{aligned}
$$

For the second case, the results are the exactly the same except that

$$
S' = \cdots v \cdots w \cdots x' \cdots yvu | \cdots
$$

$\square$

We are now ready to apply these lemmas.

Case: $(4), (7)$ and $(8)^{r}$

Let $\boldsymbol{d} = uvxy$ and $\boldsymbol{e} = uvyx$ be related by $(4), (7)$ or $(8)^{r}$. These relations share the inequalities $u < v < y$ and $x < v < y$.

$$
R \xleftarrow{in} uv = u''v'' \xleftarrow{out} \underbrace{\cdots v' \cdots u \cdots v \cdots}_{S}
$$

v appears in $S\!\uparrow$ and v' appears in $S\!\downarrow$. From Lemma B.3, $u'' < v'' < y''$ and no hijack occurs when we insert y into S.

If $S \xleftarrow{in} x$ is a zero insertion, then from Lemma B.4(1), $\boldsymbol{f} = u''v''0y'', \boldsymbol{g} = u''v''y''0$ and they are related by $(4), (7)$ or $(8)^{r}$.

If $S \xleftarrow{in} x$ is not a zero insertion, we find $x < v < y$, $x' < v' < y'$ and $x'' < v'' < y''$. This fits the hypothesis of Lemma B.5. So, $\boldsymbol{f} = u''v''x''y''$ and $\boldsymbol{g} = u''v''y''x''$ where $u'' < v'' < y''$ and $x'' < v'' < y''$. Clearly, this is an elementary relation of type $(4), (7)$

or $(8)^r$.

Case: $(3), (5), (6), (9)$ and $(9)^r$. Let $\boldsymbol{d} = uxyv$ and $\boldsymbol{e} = uyxv$. We observe that in all these relations, $u < y$ and $x < v < y$. Let S be the resulting sequence after inserting u into R. From Lemma B.3, no hijack occurs when we insert y into S since $u < y$. So, if $S \overset{in}{\leftarrow} x$ is not a zero insertion, we can apply Lemmas B.5, B.6, B.7 and B.8 to cover all the possible scenarios. The sequences that are bumped out in each case are:

- From Lemma B.5, $u''x''y''v'' \sim u''y''x''v''$ by $(3), (5), (6), (9)$ or $(9)^r$ as $u'' < y''$ and $x'' < v'' < y''$.

- From Lemma B.6, $u''x''x'v'' \sim u''x''v''x'$ by $(4), (7)$ or $(8)^r$ since $u'' < x'' < x'$ and $v'' < x'' < x'$.

- From Lemma B.7, $u''x'(x'-1)x' \sim u''(x'-1)x'(x'-1)$ by (2) since $u'' < x'-1$.

- From Lemma B.8, $u''x''v''w'' \sim u''x''w''v''$ by $(4), (7)$ or $(8)^r$ as $u'' < x''$ and $w'' < x'' < v''$.

If $S \overset{in}{\leftarrow} x$ is a zero insertion, then from Lemma B.4(1), $\boldsymbol{f} = u''0y''v''$ and $\boldsymbol{g} =$

Case: $(6)^r$ $\qquad \boldsymbol{d} = cdba, \boldsymbol{e} = cbda$.

We return to the case by case analysis. We first deal with those that contains a zero insertion.

Subcase: $\boldsymbol{d} = cdb0, \boldsymbol{e} = cbd0, b > 1$

$$
\begin{aligned}
&\cdots 101 \cdots \overset{in}{\leftarrow} cdb0 \\
=\ &c''d''b''0 \overset{out}{\leftarrow} \cdots d' \cdots c' \cdots b' \cdots 101 \cdots b \cdots c \cdots d \cdots \\
&\cdots 101 \cdots \overset{in}{\leftarrow} cbd0 \\
=\ &c''b''d''0 \overset{out}{\leftarrow} \cdots d' \cdots c' \cdots b' \cdots 101 \cdots b \cdots c \cdots d \cdots
\end{aligned}
$$

$0 < b'' < c'' < d''$ and $c''d''b''0 \sim c''b''d''0$ by $(6)^r$.

Subcase: $\boldsymbol{d} = cd10, \boldsymbol{e} = c1d0$

$$\cdots 10b' \cdots \xleftarrow{in} cd10$$
$$= c''d''b''0 \xleftarrow{out} \cdots d' \cdots c' \cdots b' \cdots 101 \cdots c \cdots d \cdots$$
$$\cdots 10b' \cdots \xleftarrow{in} c1d0$$
$$= c''b''d''0 \xleftarrow{out} \cdots d' \cdots c' \cdots b' \cdots 101 \cdots c \cdots d \cdots$$

$0 < b'' < c'' < d''$ and $c''d''b''0 \sim c''b''d''0$ by $(6)^r$.

In the next few subcases, c is hijacked when it is inserted.

Subcase:
$$R \xleftarrow{in} cdba$$
$$= c'' \xleftarrow{out} \cdots c' \cdots c| \cdots \xleftarrow{in} dba$$
$$= c''c' \xleftarrow{out} \cdots d' \cdots c| \cdots d \cdots \xleftarrow{in} ba$$
$$= c''c'b'' \xleftarrow{out} \cdots d'b' \cdots cb| \cdots d \cdots \xleftarrow{in} a$$
$$= c''c'b''b' \xleftarrow{out} \cdots d'a' \cdots cba \cdots d \cdots$$
$$R \xleftarrow{in} cbda$$
$$= c''c' \xleftarrow{out} \cdots b' \cdots cb| \cdots \xleftarrow{in} da$$
$$= c''c'b' \xleftarrow{out} \cdots d' \cdots cb| \cdots d \cdots \xleftarrow{in} a$$
$$= c''c'b'b'' \xleftarrow{out} \cdots d'a' \cdots cba \cdots d \cdots$$

$b'' < c'' < c' < b'$ and $c''c'b''b' \sim c''c'b'b''$ by $(8)^r$.

Subcase:

$$R \xleftarrow{in} cdba$$
$$= c'' \xleftarrow{out} \cdots c' \cdots c| \cdots \xleftarrow{in} dba$$
$$= c''d'' \xleftarrow{out} \cdots d' \cdots c' \cdots c| \cdots d \cdots \xleftarrow{in} ba$$
$$= c''d''b'' \xleftarrow{out} \cdots d' \cdots b' \cdots cb| \cdots d \cdots \xleftarrow{in} a$$
$$= c''d''b''a'' \xleftarrow{out} \cdots d' \cdots a' \cdots b' \cdots c' \cdots cba \cdots d \cdots$$
$$R \xleftarrow{in} cbda$$
$$= c''b'' \xleftarrow{out} \cdots b' \cdots c' \cdots cb| \cdots \xleftarrow{in} da$$
$$= c''b''d'' \xleftarrow{out} \cdots d' \cdots b' \cdots c' \cdots cb| \cdots d \cdots \xleftarrow{in} a$$
$$= c''b''d''a'' \xleftarrow{out} \cdots d' \cdots a' \cdots b' \cdots c' \cdots cba \cdots d \cdots$$

$c'' < b'' < a'' < d''$ and $c''d''b''a'' \sim c''b''d''a''$ by (6).

Subcase:

$$R \xleftarrow{in} cdba$$
$$= c''b'' \xleftarrow{out} \cdots d' \cdots c' \cdots c| \cdots d \cdots \xleftarrow{in} ba$$
$$= c''b''a'' \xleftarrow{out} \cdots d'b' \cdots cb| \cdots d \cdots \xleftarrow{in} a$$
$$= c''b''a''b' \xleftarrow{out} \cdots d'a' \cdots c' \cdots cba \cdots d \cdots$$

$$R \xleftarrow{in} cbda$$
$$= c''b'' \xleftarrow{out} \cdots b' \cdots c' \cdots cb| \cdots \xleftarrow{in} da$$
$$= c''b''b' \xleftarrow{out} \cdots d' \cdots c' \cdots cb| \cdots d \cdots \xleftarrow{in} a$$
$$= c''b''b'a'' \xleftarrow{out} \cdots d'a' \cdots c' \cdots cba \cdots d \cdots$$

$c'' < a'' < b'' < b'$ and $c''b''a''b' \sim c''b''b'a''$ by (7).

Subcase:

$$\cdots c'(c'-1) \cdots \xleftarrow{in} cdba$$
$$= c'-1 \xleftarrow{out} \cdots c'(c'-1) \cdots c| \cdots \xleftarrow{in} dba$$
$$= (c'-1)c' \xleftarrow{out} \cdots d'(c'-1) \cdots c| \cdots d \cdots \xleftarrow{in} ba$$
$$= (c'-1)c'(c'-1) \xleftarrow{out} \cdots d'b' \cdots cb| \cdots d \cdots \xleftarrow{in} a$$
$$= (c'-1)c'(c'-1)b' \xleftarrow{out} \cdots d'a' \cdots c' \cdots cba \cdots d \cdots$$
$$\cdots c'(c'-1) \cdots \xleftarrow{in} cbda$$
$$= (c'-1)c' \xleftarrow{out} \cdots b'(c'-1) \cdots cb| \cdots \xleftarrow{in} da$$
$$= (c'-1)c'b' \xleftarrow{out} \cdots d'(c'-1) \cdots cb| \cdots d \cdots \xleftarrow{in} a$$
$$= (c'-1)c'b'(c'-1) \xleftarrow{out} \cdots d'a' \cdots c' \cdots cba \cdots d \cdots$$

$b' > c'$ and $(c'-1)c'(c'-1)b' \sim (c'-1)c'b'(c'-1)$ by (4).

Subcase: c appears in the increasing part after insertion.

It can be verified that $b' < c' < d'$ and $b'' < c'' < d''$. We can apply Lemma B.5 and $f = c''d''b''a''$, $g = c''b''d''a''$ where $a'' < d''$. So, $f \sim g$ by $(3), (5), (6), (6)^r, (9)$ or $(9)^r$.

Case: $(4)^r, (7)^r, (8)$

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Now, $\boldsymbol{d} = xyvu, \boldsymbol{e} = yxvu$ where $x < v < y$ and $u < v$. We let $S = R$. If $S \overset{in}{\leftarrow} x$ is not a zero insertion, the various situations that can arise are covered by Lemmas B.5, B.6, B.7, B.8, B.9 and B.10.

- From Lemma B.5, $\boldsymbol{f} = x''y''v''u''$ and $\boldsymbol{g} = y''x''v''u''$. By inspection, we observe that $x'' < v'' < y''$ and $u'' < v''$. Therefore, $\boldsymbol{f} \sim \boldsymbol{g}$ by $(4)^{\mathrm{r}}, (7)^{\mathrm{r}}$ or (8).

- From Lemma B.6, $\boldsymbol{f} = x''x'v''u''$ and $\boldsymbol{g} = x''v''x'u''$. Since $v'' < x'' < x'$ and $u'' < v''$, $\boldsymbol{f} \sim \boldsymbol{g}$ by $(3), (5), (6), (6)^{\mathrm{r}}, (9)$ or $(9)^{\mathrm{r}}$.

- From Lemma B.7, $x'(x'-1)x'u'' \sim (x'-1)x'(x'-1)u''$ by $(2)^{\mathrm{r}}$ as $u'' < x'$.

- From Lemma B.8, $x''v''w''u'' \sim x''w''v''u''$ by $(3), (5), (6), (6)^{\mathrm{r}}, (9)$ or $(9)^{\mathrm{r}}$ since $w'' < x'' < v''$ and $u'' < v''$.

- From Lemma B.9, $x''x'xw \sim x''x'wx$ by $(4), (7)$ or $(8)^{\mathrm{r}}$

- From Lemma B.10, $x''w''xv'' \sim x''w''v''x$ by $(4), (7)$ or $(8)^{\mathrm{r}}$

If $S \overset{in}{\leftarrow} x$ is a zero insertion, then from Lemma B.4, we either have

- $\boldsymbol{f} = 0y''v''u'', \boldsymbol{g} = y''0v''u''$ where $u'' < v'' < y''$. So, $\boldsymbol{f} \sim \boldsymbol{g}$ by $(4)^{\mathrm{r}}$ or (8) or

- $\boldsymbol{f} = 0101, \boldsymbol{g} = 1010$.

This ends the proof of Theorem 1.18.

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