8.334: Statistical Mechanics II Problem Set # 5 Due: **3/12/03**

Fluctuations. Scaling. Gaussian statistics.

1. The fluctuation region width.

Thermodynamic fluctuations control the behavior near the critical point. However, the size of the temperature interval, in which fluctuations are large, depends on microscopic details, and in some cases can be quite narrow. In Lecture 5 we discussed the mean field theory validity criterion, and used it to estimate the critical region. It is interesting to look a bit closer at several examples.

a) For the Ising ferromagnet in space dimension $d = 3$, described by the mean field theory developed in Problem 1, PS#2 (see also Lecture 2), estimate the width of the temperature interval near critical point, $\tau = (T_c - T)/T_c$, where the contribution of fluctuations to the thermodynamic quantities is dominant.

b) Consider a weakly interacting Bose gas (space dimension $d = 3$) with density n, particle mass m, and a short-range interparticle interaction

$$U(\mathbf{r} - \mathbf{r}') = g\delta(\mathbf{r} - \mathbf{r}')\,, \quad g = 4\pi\hbar^2 a/m \tag{1}$$

The interaction is called weak, when atom scattering length a, related to the interaction constant by[1] Eq.(1, is much less then the interparticle distance, $a \ll n^{-1/3}$.

In this weakly nonideal gas, the Bose-Einstein condensation transition temperature T_c is accurately described by the ideal Bose gas theory. The mean field energy for the condensate amplitude ψ can be written in the form of Eq.(1), PS#1, with n being the temperature-dependent condensate density of an ideal gas. Estimate the fluctuation region τ_* for this system, using the parameters for magnetically trapped Rubidium from the experiment in JILA: $n \approx 10^{14}\,\mathrm{cm}^{-3}$, $a \approx 100\,a_B$, where $a_B = 0.5 \times 10^{-8}\,\mathrm{cm}$ is Bohr's radius.

2. A relation between scaling dimensions of a field and an operator.

Consider an RG transformation of a hamiltonian near critical point, $\mathcal{H} = \mathcal{H}_0 + \delta\mathcal{H}$, where the hamiltonian $\mathcal{H}_0$ is invariant under renormalization, $\mathcal{R}(\mathcal{H}_0) = \mathcal{H}_0$, and $\delta\mathcal{H}$ is a perturbation. Suppose that $\delta\mathcal{H}$ is of the form

$$\delta\mathcal{H} = \int h(\mathbf{x})\phi(\mathbf{x})d^3x \tag{2}$$

where $\phi(\mathbf{x})$ is some density-type variable associated with the system (e.g. spin density, energy, charge or current density, etc.) and h is a field conjugate to ϕ. Suppose also that ϕ is a *scaling variable.* This means that the RG transforms the ϕ into itself,

$$\phi'(\mathbf{x}') = b^{y_\phi}\phi(\mathbf{x})\,, \quad \mathbf{x}' = \mathbf{x}/b, \tag{3}$$

i.e. the perturbation $\delta\mathcal{H} \propto \int \phi(\mathbf{x})d^dx$ is an eigenvector of the RG transformation linearized at the fixed point $\mathcal{H}_0$.

a) Show that the scaling dimensions of ϕ and h are related as

$$y_h + y_\phi = d \tag{4}$$

[1]This corresponds to the low energy limit of particle scattering. Although the real interaction between atoms is certainly not short-ranged, at low energy it can be effectively replaced by a δ–function *pseudopotential* expressed in terms of the scattering length by Eq.(1 — this is discussed in more detail, e.g., in "Statistical Mechanics," by Parthia.

b) For the Ising model, by choosing ϕ and h to be spin density and magnetic field, and using Eq.(4), derive the RG equation for the pair spin correlation function (Eq.(17), Lecture 7) and the scaling form (Eq.(21), Lecture 7).

c) How can one use the result (4), with properly chosen ϕ and h, to obtain the scaling equation for the free energy (Eqs.(5,6), Lecture 7)?

3. Gaussian statistics.

Prove the basic facts about gaussian statistics of one variable, many variables, and fields (Eqs.(5,6,7,8), Lecture 8).

4. RG for quadratic hamiltonians.

a) Apply the field-theoretic RG transformation described in Lecture 8 (coarse grain + rescale + renormalize hamiltonian) to a gaussian field problem with a quadratic hamiltonian

$$\mathcal{H} = \int \frac{1}{2} K \left(\partial_\mu \phi\right)^2 d^d x \tag{5}$$

Show that, depending on the dimensionality d, the rigidity constant K grows or decreases upon RG transformation, with the critical dimension $d = 2$.

b) Consider thermal fluctuations of the variable $\delta\phi_{12} = \phi(\mathbf{x}_1) - \phi(\mathbf{x}_2)$. Show that the distribution is gaussian and relate its variance to the correlator

$$\langle \delta\phi_{12}^2 \rangle = \langle \left(\phi(\mathbf{x}_1) - \phi(\mathbf{x}_2)\right)^2 \rangle \tag{6}$$

c) How does the quantity (6) depend on the points 1 and 2 separation at $|\mathbf{x}_1 - \mathbf{x}_2| \to \infty$?