1 Lecture 10: Coulomb gas and Sine-Gordon problem. Villain duality. Clock models.

1.1 Sine-Gordon problem. RG analysis.

The Sine-Gordon hamiltonian is traditionally written as

$$\mathcal{H} = \int \left(\frac{1}{2}(\partial_\mu \phi)^2 - \lambda \cos \beta\phi \right) d^2x \quad (1)$$

where λ and β are coupling parameters. To connect with previous discussion, one can rescale $\phi \to \tilde{\phi}/\beta$, and rewrite the hamiltonian as $\mathcal{H} = \int \left(\frac{1}{2}K(\partial_\mu \tilde{\phi})^2 - \lambda \cos \tilde{\phi} \right) d^2x$ with $K = 1/\beta^2$. Classical ground states at zero temperature are given by the cosine minima $\phi = 2\pi n/\beta$.

The problem (1) arises in many different physical situations. To name just one, let us consider a solid crystal surface fluctuating in thermodynamic equilibrium. The surface can be described by a height function $h(\mathbf{x})$, measured with respect to a crystal plane. The free energy of the surface contains a gradient part that accounts for the effect of surface roughness, and also a periodic function of height, modelling the tendency of atoms to form complete layers in each crystal plane. In the simplest model, the periodic function is taken to be a cosine, which gives

$$\mathcal{H} = \int \left(\frac{1}{2}K(\partial_\mu h)^2 - \lambda \cos(2\pi h/a) \right) d^2x \quad (2)$$

where K is the surface tension constant scaled by temperature, and a is crystal lattice period. By a rescaling, $h \to h/K^{1/2}$, we obtain the Sine-Gordon problem (1) with the coupling constant $\beta = 2\pi/a\, K^{1/2}$. As a function of temperature, this system exhibits the so-called *roughening transition* from a rough state at high temperature, characterized by large amplitude of height fluctuations, to a smooth state at low temperature, in which the system is locked in one of the cosine minima.

To explore the effect of fluctuations, one can apply renormalization group the to problem (1). As always, we split $\phi(x)$ into a fast and a slow part, $\phi = \phi' + \delta\phi$, with Fourier harmonics $0 < k < \Lambda'$ and $\Lambda' < k < \Lambda$, respectively. We consider the problem at small $\lambda \ll 1$, with the fluctuations being mostly gaussian, and ask how the amplitude λ varies under the RG transformation. The quadratic part of the SG hamiltonian does not couple ϕ' and $\delta\phi$, the coupling arises only from the cosine term. Thus we need to calculate

$$\mathrm{tr}_{\delta\phi} \exp \left(- \int \left(\frac{1}{2}(\partial_\mu \phi)^2 - \lambda \cos \beta(\phi' + \delta\phi) \right) d^2x \right) \quad (3)$$

At small λ, we can replace this expression by

$$\exp \left(- \int \left(\frac{1}{2}(\partial_\mu \phi)^2 - \lambda \langle \cos \beta(\phi' + \delta\phi) \rangle_{\delta\phi} \right) d^2x \right) \quad (4)$$

The average over $\delta\phi$ is gaussian:

$$\langle \cos\beta(\phi' + \delta\phi)\rangle_{\delta\phi} = \frac{1}{2}\langle \exp\left(i\beta(\phi' + \delta\phi)\right)\rangle_{\delta\phi} + \text{c.c.} = \frac{1}{2} e^{i\beta\phi'} e^{-\beta^2\langle\delta\phi^2\rangle/2} + \text{c.c.} \quad (5)$$

$$\langle \delta\phi^2\rangle = \sum_{\mathbf{k}} \frac{1}{\mathbf{k}^2} = \frac{1}{2\pi}\int_{\Lambda'}^{\Lambda} \frac{dk}{k} = \frac{1}{2\pi}\ln\frac{\Lambda}{\Lambda'} \quad (6)$$

We see that the Sine-Gordon hamiltonian form is preserved,

$$\mathcal{H}(\phi') = \int \left(\frac{1}{2}(\partial_\mu\phi)^2 - \lambda_*\langle\cos(\beta\phi')\right) d^2x \quad (7)$$

with the coupling constant changed to $\lambda_* = (\Lambda'/\Lambda)^{\beta^2/4\pi}\lambda$.

Now we can ask whether the reduction of coupling by a factor $(\Lambda'/\Lambda)^{\beta^2/4\pi}$ is making it more or less important compared to the gradient term. Since for the fast field $\delta\phi$ the gradient ∂_μ is of order Λ, while for the slow field ϕ' it is of order Λ', the gradient term is effectively reduced by $(\Lambda'/\Lambda)^2$. The ratio of the cosine and gradient term is thus decreases or grows under RG depending on whether $\beta^2/4\pi$ is larger or smaller than 2. At

$$\beta^2 > 8\pi \quad (8)$$

the cosine term is suppressed by fluctuations so quickly that it is irrelevant compared to the gradient term. The system is driven in this case to the gaussian freely fluctuating field with the hamiltonian $\mathcal{H} = \frac{1}{2}\int(\partial_\mu\phi)^2 d^2x$. On the other hand, for

$$\beta^2 < 8\pi \quad (9)$$

the cosine term grows relative to the gradient term and, no matter how small the starting value λ, at some length scale takes over. The macroscopic state of the system in this case is locked in one of the cosine minima $\phi = 2\pi n/\beta$.

To put this analysis in the standard RG framework, we have to rescale $x = (\Lambda/\Lambda')x'$ at each RG step after averaging over $\delta\phi$. This shifts the reduced cutoff Λ' back to Λ, making the length unit invariant through the RG. Since the area element and the gradient are rescaled as $d^2x = (\Lambda/\Lambda')^2 d^2x'$ and $\partial'_\mu = (\Lambda'/\Lambda)\partial_\mu$, the gaussian part of the hamiltonian is invariant under RG, while the coupling constant λ changes to

$$\lambda' = (\Lambda/\Lambda')^{2-\beta^2/4\pi}\lambda \quad (10)$$

In terms of the RG time parameter $t = \ln(\Lambda/\Lambda')$, in the vicinity of the transition point $\beta^2 = 8\pi$, one can put the flow of λ in a differential form:

$$\delta\lambda = \lambda' - \lambda = \left(e^{(2-\beta^2/4\pi)\ln(\Lambda/\Lambda')} - 1\right)\lambda = (2 - \beta^2/4\pi)\,\lambda\,\delta t \quad (11)$$

which gives

$$\frac{d\lambda}{dt} = (2 - \beta^2/4\pi)\lambda \quad (12)$$

Interestingly, the solution of this differential equation reproduces the recursion relation (10) for all β.

Applying these results to the roughening problem (2), described by $\beta = 2\pi/a\, K^{1/2}$, we obtain a transition at $Ka^2 = \pi/2$. In the high temperature phase, the fluctuations of surface height diverge at large distances:

$$\langle (h(\mathbf{x}) - h(\mathbf{x}'))^2 \rangle = \sum_{\mathbf{k}} \frac{|e^{i\mathbf{k}\cdot\mathbf{x}} - e^{i\mathbf{k}\cdot\mathbf{x}'}|^2}{K\mathbf{k}^2} = \frac{1}{\pi K}\int_a^{|\mathbf{x}-\mathbf{x}'|} \frac{dk}{k} = \frac{1}{\pi K} \ln \frac{|\mathbf{x}-\mathbf{x}'|}{a} \tag{13}$$

In the low temperature phase, when the system is locked in one of the cosine minima, the fluctuations $\langle (h(\mathbf{x}) - h(\mathbf{x}'))^2 \rangle$ are finite.

1.2 Coulomb gas representation

We start by pointing out that changing cosine in the Sine-Gordon hamiltonian to another function with the same period,

$$F(\beta\phi) = \sum_{n=-\infty}^{+\infty} \lambda_n e^{in\beta\phi} \tag{14}$$

does not affect the transition at $\beta^2 = 8\pi$ and the two phases. This robustness can be demonstrated by a simple extension of the above RG analysis. Applied to the problem with interaction $F(\beta\phi)$, separately for each term in the Fourier expansion, the RG flow of couplings λ_n is

$$\frac{d\lambda_n}{dt} = (2 - \beta^2 n^2/4\pi)\lambda_n \tag{15}$$

We see that the couplings with $|n| > 1$ are always less relevant than the $n = 1$ coupling.

This observation allows to replace the cosine term by another function, making the problem more tractable (Villain, 1975). Here it will be convenient to consider the problem on a lattice,

$$\mathcal{H} = \frac{1}{2}\sum_{|\mathbf{x}-\mathbf{x}'|=1} (\phi_{\mathbf{x}} - \phi_{\mathbf{x}'})^2 + \sum_{\mathbf{x}} F(\beta\phi_{\mathbf{x}}) \tag{16}$$

where

$$e^{-F(\theta)} = \sum_{n=-\infty}^{+\infty} e^{in\theta + (\ln y)n^2} \tag{17}$$

with $0 < y < 1$. At small $y \ll 1$, we have

$$F(\theta) = -\ln\Big(1 + 2y\cos\theta + O(y^2)\Big) = -2y\cos\theta + O(y^2) \tag{18}$$

which means that $F(\theta)$ is a good replacement for the cosine term when $y = \lambda/2$. (More precisely, since in passing from a continual to a discrete problem we pick up a lattice plaquette area, the relation is $y = \lambda a^2/2$.)

To see the crucial simplification achieved by altering the Sine-Gordon problem, consider the partition function

$$Z = \sum_{\{n_{\mathbf{x}}\}} \sum_{\phi_{\mathbf{x}}} \exp\left(\frac{1}{2} \sum_{|\mathbf{x}-\mathbf{x}'|=1} (\phi_{\mathbf{x}} - \phi_{\mathbf{x}'})^2 + \sum_{\mathbf{x}} \left(in_{\mathbf{x}}\beta\phi_{\mathbf{x}} + (\ln y) n_{\mathbf{x}}^2\right)\right) \tag{19}$$

We note that at fixed integers $n_{\mathbf{x}}$, the problem is gaussian in the field $\phi_{\mathbf{x}}$, which can thus be summed over. This gives

$$Z = Z_0 \sum_{\{n_{\mathbf{x}}\}} \exp\left(-\frac{1}{2}\beta^2 \sum_{\mathbf{x},\mathbf{x}'} n_{\mathbf{x}} n_{\mathbf{x}'} \langle \phi_{\mathbf{x}} \phi_{\mathbf{x}'} \rangle + \sum_{\mathbf{x}} (\ln y) n_{\mathbf{x}}^2\right) \tag{20}$$

where Z_0 is the partition function at $n_{\mathbf{x}} = 0$ and

$$\langle \phi_{\mathbf{x}} \phi_{\mathbf{x}'} \rangle = \sum_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{2(1-\cos k_1) + 2(1-\cos k_2)} = \frac{1}{2\pi} \ln \frac{L}{|\mathbf{x}-\mathbf{x}'|} \tag{21}$$

Here the sum over $\mathbf{k}$ is taken over the reciprocal lattice period $-\pi/a < k_{1,2} < \pi/a$. The logarithmic form of the pair correlator is obtained for point separation $|\mathbf{x}-\mathbf{x}'|$ much larger than lattice constant a.

From this analysis we see that the Sine-Gordon problem is equivalent to the Coulomb gas problem

$$Z = {\sum_{\{n_{\mathbf{x}}\}}}' \exp\left(\frac{\beta^2}{4\pi} \sum_{\mathbf{x}\neq\mathbf{x}'} n_{\mathbf{x}} n_{\mathbf{x}'} \ln \frac{|\mathbf{x}-\mathbf{x}'|}{a} + \sum_{\mathbf{x}} (\ln y) n_{\mathbf{x}}^2\right) \tag{22}$$

Here the summation is constrained by charge neutrality, $\sum_{\mathbf{x}} n_{\mathbf{x}} = 0$. This problem is familiar from the Kosterlitz-Thouless theory (Lecture 9), in which a similar description of vortices in the XY model was employed. The gas (22) can be in two states, depending on the coupling strength $\beta^2/4\pi$. At strong coupling, the charges form tightly bound neutral pairs, so that in the macroscopic limit there are no free charges. At week coupling, entropic factors dominate, and charges are in the unbound state. Critical coupling can be determined by the energy of one charge $(\beta^2/4\pi)\ln(L/a)$ with its entropy $2\ln(L/a)$, which gives transition criterion

$$\beta^2 = 8\pi \tag{23}$$

identical to the prediction of the RG analysis above. The charge-neutral phase at $\beta^2 > 8\pi$ is thus associated with the gaussian phase of the Sine-Gordon problem, in which cosine coupling is made irrelevant due to strong fluctuations. The unbound charge phase at $\beta^2 < 8\pi$ corresponds to the strong coupling regime of the Sine-Gordon problem, with the field frozen in cosine minimum.

1.3 Clock models

are described by a hamiltonian

$$\mathcal{H} = \int \left(\frac{1}{2}K(\partial_\mu\theta)^2 - \lambda_p \cos(p\theta)\right) d^2x \tag{24}$$

with p an integer, and the variable θ taking values on a circle $0 < \theta < 2\pi$. Because of the topologically nontrivial order parameter space here we cannot eliminate p or K by a rescaling.

The models of this form arise as perturbations of the XY model. The case $p = 1$ simply represents the planar XY magnet in a parallel magnetic field. The XY magnet on a square or triangular lattice with a weak spin-lattice coupling corresponds to $p = 4, 6$, respectively. The name "clock models" refers to he ground states in the limit of large λ_p looking like a clock dial.

Modifying the cosine interaction in the same way as in the above discussion of the Sine-Gordon model, we obtain a partition function

$$Z = \sum_{\{n_{\mathbf{x}}\}} \sum_{\theta_{\mathbf{x}}} \exp\left(\sum_{|\mathbf{x}-\mathbf{x}'|=1} \frac{1}{2} K(\theta_{\mathbf{x}} - \theta_{\mathbf{x}'})^2 + \sum_{\mathbf{x}} \left(ip\, n_{\mathbf{x}} \theta_{\mathbf{x}} + (\ln y_p) n_{\mathbf{x}}^2 \right) \right) \tag{25}$$

with $y_p = \lambda_p a^2/2$. The hamiltonian is formally quadratic and it is now tempting to integrate over θ's. To do this, since θ lives in a topologically nontrivial space, we have to introduce vortices,

$$\theta_{\mathbf{x}} = \phi_{\mathbf{x}} + \sum_{\mathbf{x}' \neq \mathbf{x}} m_{\mathbf{x}'} \tan^{-1}\left(\frac{x_2 - x_2'}{x_1 - x_1'} \right) \tag{26}$$

where $m_{\mathbf{x}'}$ is vortex charge. The arctangent is the phase field at the point $\mathbf{x}$ of a vortex with a center at $\mathbf{x}'$. Now we can integrate over the smooth phase function $\phi_{\mathbf{x}}$. Evaluating gaussian integral, we obtain a partition function of vortices and charges:

$$Z(K, y_p, y) = {\sum_{\{n_{\mathbf{x}}\}}}' {\sum_{\{m_{\mathbf{x}}\}}}' e^{-\mathcal{H}} \tag{27}$$

where

$$\mathcal{H} = -\frac{p^2}{4\pi K} \sum_{\mathbf{x} \neq \mathbf{x}'} n_{\mathbf{x}} n_{\mathbf{x}'} \ln \frac{|\mathbf{x} - \mathbf{x}'|}{a} - \sum_{\mathbf{x}} (\ln y_p) n_{\mathbf{x}}^2 \tag{28}$$

$$-\pi K \sum_{\mathbf{x} \neq \mathbf{x}'} m_{\mathbf{x}} m_{\mathbf{x}'} \ln \frac{|\mathbf{x} - \mathbf{x}'|}{a} - \sum_{\mathbf{x}} (\ln y) m_{\mathbf{x}}^2 \tag{29}$$

$$+ip \sum_{\mathbf{x} \neq \mathbf{x}'} m_{\mathbf{x}} n_{\mathbf{x}'} \tan^{-1}\left(\frac{x_2 - x_2'}{x_1 - x_1'} \right) \tag{30}$$

Here the summation over $n_{\mathbf{x}}$ and $m_{\mathbf{x}}$ is constrained by vortex and charge neutrality, $\sum_{\mathbf{x}} n_{\mathbf{x}} = 0$, $\sum_{\mathbf{x}} m_{\mathbf{x}} = 0$.

The imaginary arctangent coupling defines a phase factor of the states in the partition function. It can be interpreted as an Aharonov-Bohm phase of charges $n_{\mathbf{x}}$ due to 2D monopoles with magnetic charge $p\, m_{\mathbf{x}}$.

From this representation of the clock problem as two scalar Coulomb gases, we obtain a duality theorem

$$Z(K, y_p, y) = Z\left(\frac{p^2}{4\pi^2 K}, y, y_p \right) \tag{31}$$

This relation is true up to a nonsingular prefactor arising from gaussian integral over smooth part of the phase field, which does not affect the critical behavior and macroscopic phases classification.

Now, let us discuss the phases of the clock model. The high temperature disordered state, associated with the free unbound Kosterlitz-Thouless vortices in the problem (28), occurs at $\pi K < 2$. The low temperature lock-in state with θ frozen in the cosine minimum, occurs when the charges in the Coulomb gas are unbound and free, i.e. when $p^2/4\pi K < 2$. For p large enough, there exist a range of intermediate temperatures,

$$\frac{2}{\pi} < K < \frac{p^2}{8\pi}\,, \tag{32}$$

such that *both* the cosine perturbation and vortices are unimportant. The state of the system is similar to the low temperature state of the XY model, with only gaussian fluctuations of θ remaining in this limit, a quasi-long-range order and power law correlations. The two phase transitions, at $K = 2/\pi$ and $K = p^2/8\pi$, are related by the duality theorem. Each of the transitions is of Kosterlitz-Thouless type.

This picture holds at $p > 4$. At $p = 4$, the two transitions coincide. By analogy with the Kosterlitz-Thouless theory, one can derive the RG flow equations

$$\frac{dK^{-1}}{dt} = 4\pi^3 y^3 - K^2 p^2 y_p^2\,, \quad \frac{dy}{dt} = (2-\pi K)y\,, \quad \frac{dy_p}{dt} = \left(2 - \frac{p^2}{4\pi K}\right) y_p \tag{33}$$

At $p = 4$, this gives three lines of fixed points, each with its own set of critical exponents. Schematic phase diagram is shown in the figure.

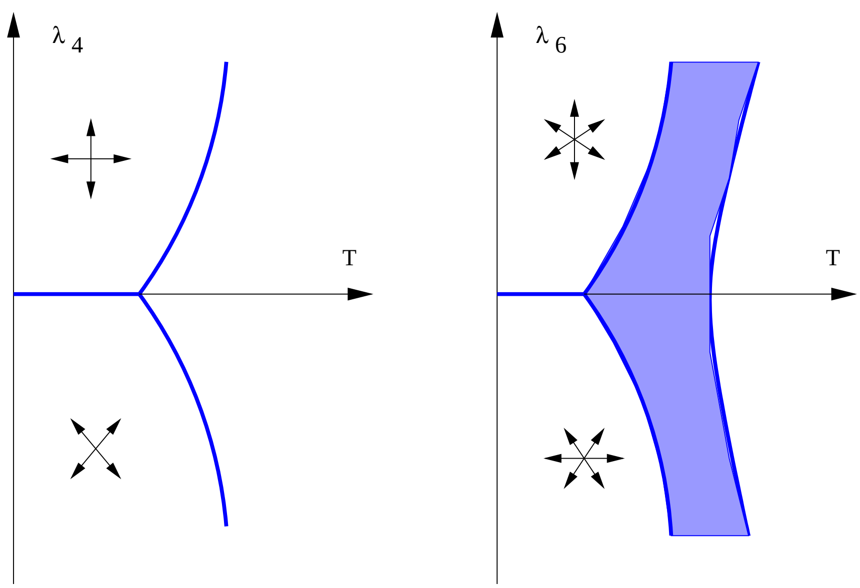

Figure 1: Phase diagram for clock models with $p = 4$ and $p = 6$. Critical lines and the region with quasi-long-range order are marked in blue.

1.4 Summarize

- **Mapping to a Coulomb gas** provides a relation between seemingly different systems. It is an efficient tool for solving 2D problems.

- **The Kosterlitz-Thouless scenario** of charge unbinding describes several different phase transitions: (i) the topological transition in the XY model; (ii) the lock-in transition in the Sine-Gordon model; (iii) the sequence of phase transitions in the clock models.