Landau theory, susceptibility & fluctuations

1. Ising ferromagnet.

a) Consider the Landau functional for an Ising ferromagnet near transition. Evaluate the free energy by minimizing the functional and find entropy and specific heat near the transition. Compare with the result of Problem #1, PS2.

b) Using Landau theory of a ferromagnet in an external field, consider magnetization as a function of temperature. Make a plot showing $M(T)$ for different values of magnetic field, positive and negative.

c) Find magnetic susceptibility $\chi = dM/dH$ in zero field separately above and below the transition. Compare the two results.

2. Ising antiferromagnet, Landau theory.

Antiferromagnet on a cubic lattice is described by spin Hamiltonian with negative exchange interaction between nearest neighbors:

$$\mathcal{H} = \frac{1}{2} J \sum_{|r-r'|=1} \sigma_r \sigma_{r'} \,, \quad J > 0 \tag{1}$$

with $\sigma_r = \pm 1$ Ising spins. This interaction favors opposite spins on neighboring lattice sites, which results in a ground state with spins opposite on each of the two sublattices. In fact, there are two equivalent ground states related by an overall spin sign change. (Compare with the two ground states in a binary alloy pictured in figure (1), Lecture 2.)

a) Develop Landau theory for the phase transition in Ising antiferromagnet. Phenomenologically, choose the order parameter to be staggered magnetization, i.e. the difference of magnetization on two sublattices. Derive the form of Landau free energy from a symmetry argument, show that it is the same as for the ferromagnet.

b) Now, describe an antiferromagnet near the phase transition in the presence of an external magnetic field. Using symmetry argument, show that there is no coupling of uniform field to the staggered magnetization *linear* in the order parameter.

c) Generalize this approach by including the average magnetization in the Landau functional. Write the most general free energy form, allowed by symmetry, in the presence of an external field. Find magnetic susceptibility χ above and below the transition. Determine the form of singularity in $\chi(T)$ at $T = T_c$.

3. Curie and Neel susceptibilities.

Here we consider ferromagnet and antiferromagnet in magnetic field, starting with the microscopic hamiltonian and solving the problem using the mean field approach.

a) From the mean field equation for the Ising and Heisenberg ferromagnet magnetization derived in class (Eqs.(11),(17), Lecture 2), derive susceptibility in zero field at $T > T_c$. Express the resulting $\chi(T)$ in terms of T/T_c (Curie law).

b) From the mean field approach, find the zero field susceptibility of an Ising antiferromagnet at $T > T_c$ (Neel law).

c) Using the mean field approach, find the zero field susceptibility χ of an Ising antiferromagnet at $T < T_c$. Does the behavior near T_c agree with the Landau theory result of Problem 2?

4. Ornstein-Zernicke formula.

a) Use Landau theory to study the correlation function $\langle m(\mathbf{x})m(\mathbf{x}')\rangle$ in an Ising ferromagnet near T_c. For the order parameter Fourier harmonics $m_{\mathbf{k}} = \int e^{-i\mathbf{k}\cdot\mathbf{x}} m(\mathbf{x}) d^3x$, derive the formula

$$\langle m_{\mathbf{k}} m_{-\mathbf{k}}\rangle = \frac{A}{\xi^2\mathbf{k}^2 + 1} \tag{2}$$

Find the correlation length ξ and the constant A at $T > T_c$.

b) What is the relationship of this correlator with the susceptibility χ?

c) Consider the correlation fucntion $C_2(\mathbf{x}-\mathbf{x}') = \langle \delta m(\mathbf{x})\delta m(\mathbf{x}')\rangle$ in real space. Find its functional form in $D = 3$ using the above formula.

d) Consider the correlation fucntion $C_2(\mathbf{x}-\mathbf{x}')$ in the ordered state, at $T < T_c$. Use equipartition theorem to show that the structure factor form (2) does not change (up to numerical factors in A and ξ). From that, derive a real space form of C_2.

5. Goldstone mode fluctuation suppressed by an external field.

a) Scattering of light in a dielectric medium takes place due to spatial fluctuations of refraction index, i.e. the dielectric permeability $\delta\epsilon_{ij}$. The angular distribution of scattered light is determined by the correlator $\langle\delta\epsilon_{ij}\delta\epsilon_{i'j'}\rangle$ of the fluctuating part of permeability. This quantity is called the scattering tensor[1]. Show that when an electromagnetic plane wave with wavevector $\mathbf{q}$ and polarization $\mathbf{E}$ is scattered on the fluctuations in the medium, the scattered field $\mathbf{E}'$ is determined by the correlator of permeability $\delta\epsilon$ Fourier harmonics

$$\langle E'_i(E'_j)^*\rangle \propto V\langle\delta\epsilon_{im}(\mathbf{k})\delta\epsilon^*_{jn}(-\mathbf{k})\rangle E_m E^*_n\,, \quad \mathbf{k} = \mathbf{q}' - \mathbf{q} \tag{3}$$

where V is the system volume, $\delta\epsilon_{ij}(\mathbf{k}) = \int \delta\epsilon_{ij}(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}d^3x$ and $\mathbf{q}'$ is the wavevector of a scattered wave.

b) It is interesting to apply this result to scattering in a nematic liquid crystal[2] caused by fluctuations of the nematic order parameter $\mathbf{n}$. In this case, the fluctuating part of dielectric permeability $\delta\epsilon_{ij}$ is proportional to the traceless tensor $n_i(\mathbf{x})n_j(\mathbf{x}) - \frac{1}{3}\delta_{ij}$.

Consider a liquid crystal in the presence of a uniform static electric field. The polarizability of anisotropic rod-like molecules is much higher along the molecule axis than in the transverse direction, $\alpha_\parallel \gg \alpha_\perp$. Hence the field is trying to align the molecules, which is described by adding a term $-\frac{\alpha}{2}(\mathbf{E}\cdot\mathbf{n})^2$ to the liquid crystal free energy (Eq.(13), lecture 1). (Here $\alpha = \alpha_\parallel - \alpha_\perp$ is a positive constant.) Show that with the field applied, the lowest energy state is indeed $\mathbf{n} \parallel \mathbf{E}$.

c) Consider the fluctuations $\mathbf{n} = \mathbf{n}^{(0)} + \delta\mathbf{n}$, with $\delta\mathbf{n} \perp \mathbf{n}^{(0)}$, by expanding the free energy to the lowest nonvanishing order in $\delta\mathbf{n}$. Find the correlator $\langle\delta\mathbf{n}_{\mathbf{k}}\delta\mathbf{n}_{-\mathbf{k}}\rangle$ and show that applied electric field suppresses scattering. The possibility to change the medium from opaque to transparent by an external field has various practical applications (such as the liquid crystal displays).

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[1] Definition and discussion of the scattering tensor can be found in: Landau & Lifshits, Electrodynamics of Continuous Media, §117

[2] Scattering in nematic liquid crystals is discussed in: Landau & Lifshits, Electrodynamics of Continuous Media, §122