XY model. Topological phase transition.

1. Topological quantities. Consider continuous 2-component unit vector functions $\mathbf{n}(x)$ on a circle $0 < x \leq 2\pi$. For any such function one can count the number of rotations of $\mathbf{n}$ upon a cyclic change of x from 0 to 2π. We would like to show that this integer number is a topological characteristic of the function, i.e. it does not change when the function is deformed in a continuous way.

This problem can be analyzed using a complex number representation of 2-component unit vectors, $z(x) = n_1(x) + in_2(x)$, and interpreting the functions as mappings from the unit circle parameterized by x to the unit circle $|z| = 1$ in the complex z plane.

a) Define topological charge as

$$q = \frac{1}{2\pi i} \oint \frac{z'}{z} dx \,, \quad z' \equiv dz/dx. \tag{1}$$

Mathematically, this quantity is known as the degree of a mapping. Show that q is an integer number. (Hint: In terms of the phase, $z = e^{i\theta}$, what is the meaning of the quantity z'/z?)

b) For any integer n construct a function $z(x)$ with topological charge $q = n$.

c) Show that q does not change when the function $z(x)$ is replaced by $z'(x)$ such that z' is related to z by a small deformation: $|z(x) - z'(x)| < \epsilon$, where ϵ is a small positive number (e.g. 10^{-2}).

2. Correlation function in the XY model. Consider the XY model on a 2D square lattice:

$$\mathcal{H} = -J \sum_{|\mathbf{x}-\mathbf{x}'|=1} \mathbf{n}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}') = -J \sum_{|\mathbf{x}-\mathbf{x}'|=1} \cos(\theta_{\mathbf{x}} - \theta_{\mathbf{x}'}) \tag{2}$$

where the angles $\theta_{\mathbf{x}}$ are defined as in Problem 1, $n_1 + in_2 = e^{i\theta}$, or $n_1 = \cos\theta$, $n_2 = \sin\theta$. We are interested in the asymptotic behavior of the pair correlation function

$$C_2(\mathbf{x} - \mathbf{y}) = \langle \mathbf{n}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{y}) \rangle \tag{3}$$

at large distances $|\mathbf{x} - \mathbf{y}|$.

a) At high temperature $T \gg J$, the correlation between different phases is weak, and one can evaluate $C_2(\mathbf{x} - \mathbf{y})$ as

$$\langle e^{i(\theta_{\mathbf{x}}-\theta_{\mathbf{y}})} \rangle = \langle e^{i(\theta_{\mathbf{x}}-\theta_{\mathbf{x}'})} e^{i(\theta_{\mathbf{x}'}-\theta_{\mathbf{x}''})} ... e^{i(\theta_{\mathbf{x}^{(n)}}-\theta_{\mathbf{y}})} \rangle \approx \langle e^{i(\theta_{\mathbf{x}}-\theta_{\mathbf{x}'})} \rangle \langle e^{i(\theta_{\mathbf{x}'}-\theta_{\mathbf{x}''})} \rangle ... \langle e^{i(\theta_{\mathbf{x}^{(n)}}-\theta_{\mathbf{y}})} \rangle \tag{4}$$

where the points $\mathbf{x}'$, $\mathbf{x}''$, ..., $\mathbf{x}^{(n)}$ are taken on the shortest path connecting $\mathbf{x}$ and $\mathbf{y}$. Show that the correlation function, evaluated using (4), does not depend on the path between $\mathbf{x}$ and $\mathbf{y}$, and falls off exponentially with distance.

b) At low temperature $T \ll J$, the correlations are strong and the phase difference between neighboring sites is typically very small. This allows to expand cosine in the hamiltonian (2) and rewrite it as

$$\mathcal{H}_{sw} = \sum_{|\mathbf{x}-\mathbf{x}'|=1} \frac{1}{2} J \left(\theta_{\mathbf{x}} - \theta_{\mathbf{x}'}\right)^2 \tag{5}$$

(we dropped constant term as well as the terms higher order in $\theta_{\mathbf{x}} - \theta_{\mathbf{x}'}$). Expanding the XY hamiltonian in small phase difference is called a *spin wave approximation*. To evaluate the correlation function (3), one can perform a gaussian averaging with the distribution $e^{-\beta \mathcal{H}_{sw}}$.

Find the asymptotic behavior of the correlation function at large distances. (Hint: How does the hamiltonian (3) look in Fourier representation?)

c) To characterize quantitatively the difference between the two phases, consider an abelian analog of *Wilson loop* used in gauge theories. Define topological charge of a finite domain $\mathcal{D}$ as a sum of topological charges of the plaquettes p within the domain:

$$Q(\mathcal{D}) = \sum_{p_j \in \mathcal{D}} q(p_j)\,, \quad q(p_j) = \frac{1}{2\pi} \sum_{n=1,\dots,4} \arg w^j_{n,n+1}\,, \quad w^j_{n,n+1} = e^{i(\theta_{\mathbf{x}^j_n} - \theta_{\mathbf{x}^j_{n+1}})} \tag{6}$$

with the sum over n taken over four square plaquette corner sites x^j_1, x^j_2, x^j_3, x^j_4, and the argument function taking values $-\pi < \arg w \le \pi$.

In the high temperature phase the fluctuations are strong, and typical values of $Q(\mathcal{D})$ grow with domain size. Show that $\langle Q(\mathcal{D})\rangle = 0$ and estimate $\langle Q^2(\mathcal{D})\rangle$ for a large domain of area A. Compare to the low temperature phase.

3. Vortex confinement. Consider the XY model (2) perturbed by a term $\lambda \sin^2\theta$, which introduces a weak anisotropy with respect to spin rotation. It is interesting to find out how the anisotropy changes the phase diagram. Let us study the ground states, vortices, and vortex-vortex interaction. At low temperature, the problem can be treated in the spin wave approximation:

$$\mathcal{H}_{sw} = \sum_{|\mathbf{x}-\mathbf{x}'|=1} \frac{1}{2} J \left(\theta_{\mathbf{x}} - \theta_{\mathbf{x}'}\right)^2 + \sum_{\mathbf{x}} \lambda \sin^2\theta_{\mathbf{x}} \tag{7}$$

The perturbation removes the degeneracy due to rotational symmetry: at $T = 0$ there are 2 ground states, $\theta = 0, \pi$.

a) Consider a spin distribution uniform within two domains: $\theta(x, y) \to 0$ at large negative x, and $\theta(x, y) \to \pi$ at large positive x. To find $\theta(\mathbf{x})$ at the domain interface, make a gradient expansion of the first term in (7) and, assuming that θ is a function of x only, look for a variational solution of the energy functional

$$\mathcal{H} = \int \frac{1}{2} J (d\theta/dx)^2 + \lambda \sin^2\theta_x \tag{8}$$

with asymptotic values 0 and π at $x = \pm\infty$. Show that the width of the domain wall is of the order of $(J/\lambda)^{1/2}$ (which justifies using gradient expansion at small λ). Find the surface tension of the wall, i.e. the energy per unit length.

b) Show that the spin distribution far away from a vortex with $q = \frac{1}{2\pi}\oint \nabla\theta dl = 1$ can be described by 2 sectors, each being a domain with constant $\theta = 0, \pi$. The sector regions are separated by the domain walls similar to the one described in part a). The distortion field $\nabla\theta$ is nearly zero in the domains and is mainly concentrated in the walls. Estimate vortex energy in a system of size L.

c) Consider a vortex and an anti-vortex a large distance L apart. Describe spin distribution in the plane. How does the energy of their interaction vary asymptotically at large L?

d) Consider the system (7) at finite temperature. Using the above results for vortex interaction, discuss the possibility of a topological transition due to vortex pairs unbinding and vortices proliferation.