1 Lecture 7: Scaling theory.

1.1 General features of an RG flow

The renormalization scheme[1] we considered can be improved in many ways. Typically one would like to use blocking instead of bond moving. This allows to successively add more couplings and thus control the accuracy of the calculation. It also often allows to preserve lattice symmetry. A very good example of a not-to-difficult RG calculation is the work by Th. Niemeijer and J. M. J. van Leeuwen on the Ising problem on a triangular lattice [Phys. Rev. Lett., v. 31, p.1411 (1973)] cited in Lecture 5.

One disadvantage of the real space schemes is that the rescaling factor is discrete: $b = 2, 3, \ldots$. This is clearly an artifact of using the lattice. Later on we shall discuss other renormalization schemes in which the rescaling factor is a continuous variable. We shall consider *infinitesimal rescaling transformations*, when $b = 1 + \delta l$, with $\delta l \ll 1$. The couplings will also transform infinitesimally,

$$K_i \to K_i + (dK_i/dl)\delta l + O(\delta l^2) \tag{1}$$

and the RG equations will take the differential form

$$dK_i/dl = -\beta_i(\{K_i\}) \tag{2}$$

The functions β_i are called the RG beta functions. (The minus sign is added to make contact with the conventional definition of the beta function in quantum field theory, $\beta_i = \kappa\partial K_i/\partial\kappa$, where κ is a wave number. In this formulation the RG flow will look more like a flow: instead of a sequence of points we shall have stream lines.

Besides being a powerful computational tool, the real space RG is very good for building intuition about what kind of physics is captured by RG and what is left out. Many general features of the renormalization approach are much more transparent in this language than, say, in the field-theoretic language that we will adopt later.

The RG scheme can be generalized to include magnetic field, after which the RG flow acquires a second dimension. The K_* fixed point, with $h = 0$, is preserved. It is unstable with respect to perturbation in the h direction. Schematic RG flow is shown in a figure. The meaning of this instability is similar to the instability along the coupling strength (or effective temperature) direction. The transition is a sharp singularity only at $h = 0$. A finite h, no matter how weak, has a drastic effect: it polarizes the system and destroys the critical point. In the presence of a finite $\langle m\rangle$ both above and below $T_c(h = 0)$, there is no disordered state, but also there is no symmetry breaking in the polarized state.

Also, one can add other couplings, such as between the next nearest neighbors, etc., and enlarge the dimension of parameter space in which RG flow is defined. However, in

[1] Moved here from Lecture 6 and expanded

contrast with magnetic coupling, this does not add new unstable directions to the fixed point. The qualitative features of the RG flow in the coupling parameter space, even when there are many couplings in the play, is similar to the flow in the (J, h) plane: the RG flow linearized near the unstable fixed point K_* has two positive eigenvalues y_t and y_h, while other eigenvalues are negative. The two eigenvectors, u_t and u_h, describe perturbation away from the critical point due to temperature and magnetic field variation. The exact form of these vectors, if many couplings are considered may be more complicated than in the above example.

In all other directions, no matter how many, the fixed point is stable (i.e. the eigenvalues of linearized flow are negative). Such perturbations are called *irrelevant*. If they are present in the microscopic Hamiltonian, after several RG steps, the system gets back to the fixed point with irrelevant perturbations reduced. Thus the irrelevant perturbations do not affect the macroscopic behavior.

To make the role of irrelevant couplings more clear, consider a schematic RG flow in a large parameter space of all couplings. At $h = 0$ and temperature varying, initial microscopic Hamiltonian is represented by a 1D line in this space. Generally speaking, this line will miss the renormalization flow fixed point K_*. Why then the phase transition is sharp? The answer is that the deviation from K_* in all but two directions is not essential: the RG flow will bring the system back to the fixed point after several RG steps. This is the underlying reason for phase transitions in many systems being isomorphic, despite very different microscopic properties. In the literature, this is sometimes called *universality of critical phenomena.*

1.2 Scaling of thermodynamic quantities

Let us now discuss how the information obtained from the RG flow can be used to estimate the thermodynamic quantities. The RG transformation changes the free energy as

$$f(\{K\}) = g(\{K\}) + b^{-d} f(\{K'\}) \tag{3}$$

Here b is the block size, and d is space dimension. From this inhomogeneous relation one can extract a relation for the singular part[2] of the free energy which is homogeneous:

$$f_s(\{K\}) = b^{-d} f_s(\{K'\}) \tag{4}$$

Close to the fixed point we write this in terms of scaling variables:

$$f_s(u_t, u_h) = b^{-d} f_s(e^{y_t} u_t, e^{y_h} u_h) = b^{-nd} f_s(e^{n y_t} u_t, e^{n y_h} u_h) \tag{5}$$

with the irrelevant variables ignored. Let us choose to stop the interaction at $e^{n y_t} u_t = u_t^{(0)}$ with $u_t^{(0)}$ arbitrary but fixed (and sufficiently small, so that the linear approximation is valid). Solving for n, obtain

$$f_s(u_t, u_h) = |u_t/u_t^{(0)}|^{d/y_t} f_s(\pm u_t^{(0)}, u_h |u_t/u_t^{(0)}|^{-y_h/y_t}) \tag{6}$$

Rewriting this in terms of the nondimensionalized physical variables τ and $h = H/T_c$, obtain

$$f_s(u_t, u_h) = |\tau|^{d/y_t}\Phi(h/|\tau|^{y_h/y_t}) \tag{7}$$

where Φ is a *scaling function.*

The scaling exponents for thermodynamic quantities, introduced in Lecture 5, can now be obtained as follows.

- Specific heat $C = d^2f/d\tau^2|_{h=0} \propto |\tau|^{d/y_t-2}$, thus $\underline{\alpha = 2 - d/y_t}$.
- Order parameter $m = (df/dh)_{h=0} \propto \tau^{(d-y_h)/y_t}$, and so $\underline{\beta = (d - y_h)/y_t}$.
- Susceptibility $\chi = (d^2f/dh^2)_{h=0} \propto \tau^{(d-2y_h)/y_t}$, thus $\underline{\gamma = (2y_h - d)/y_t}$
- Exponent δ is bit more tricky:

$$M = \frac{df}{dh} = |\tau|^{(d-y_h)/y_t}\Phi'(h/|\tau|^{y_h/y_t}) \tag{8}$$

  For M to have a finite limit as $\tau \to 0$, the function $\Phi'(x)$ must behave as x^{d/y_h-1} at $x \to \infty$. Thus, setting $\tau = 0$, obtain $M \propto h^{d/y_h-1}$, i.e. $\underline{\delta = y_h/(d - y_h)}$.

This gives an expression of the critical indices through the RG eigenvalues $y_{t,h}$, and thus allows to use the results of RG calculation to predict the values of the indices.

Also, from a more general point of view, the scaling argument provides relations of different indices irrespective of the availability of a method to evaluate the eigenvalues y_t and y_h. These *scaling relations* are completely general, i.e. they are based only on the scaling hypothesis, and do not involve the information about specifics of the RG flow. Examples are:

$$\alpha + 2\beta + \gamma = 2 \tag{9}$$
$$\alpha + \beta(1 + \delta) = 2 \tag{10}$$

The relations of this form were postulated before the renormalization group theory was developed. The above discussion shows how RG arguments can be used to prove such relations.

1.3 Scaling for the correlation functions

Since the RG preserves not just the Hamiltonian, but the whole probability measure and statistics of coarse-grained fields, we can use it to study correlation functions. As a simple example illustrating the main points, consider spin-spin correlation function in the Ising model,

$$C_2(r_1 - r_2, \mathcal{H}) = \langle (s_{r_1} - \bar{s}_{r_1})(s_{r_2} - \bar{s}_{r_2})\rangle_{\mathcal{H}} = \langle s_{r_1}s_{r_2}\rangle_{\mathcal{H}} - \langle s_{r_1}\rangle_{\mathcal{H}}\langle s_{r_2}\rangle_{\mathcal{H}} \tag{11}$$

where $\bar{s} \equiv \langle s \rangle_{\mathcal{H}}$. We write $\mathcal{H}$ explicitly in order to keep track of the ensemble over which the averaging is performed.

The correlation function C_2 can be conveniently represented as a derivative of the partition function with respect to an auxiliary magnetic field:

$$C_2(r_1 - r_2, \mathcal{H}) = \frac{\partial^2}{\partial h(r_1) \partial h(r_2)} \ln Z(h)|_{h(r)=0} \tag{12}$$

The proof of this relation can be obtained by generalizing the proof in the theorem about the relation between linear response and fluctuations, presented at the end of Lecture 5.

Suppose that the field $h(r)$ varies slowly, o that it is essentially constant on the scale of the RG block size ba, and imagine applying the RG procedure to the Hamiltonian $\mathcal{H} - \sum_r h(r)s(r)$. According to the above argument, the renormalized Hamiltonian is of the same form:

$$\mathcal{H}' - \sum_r h'(r)s'(r), \tag{13}$$

where $h' = b^{y_h} h$. Since the RG preserves the entire partition function, we can write

$$\frac{\partial^2 \ln Z'(h')}{\partial h'(r_1') \partial h'(r_2')} = \frac{\partial^2 \ln Z(h)}{\partial h(r_1') \partial h(r_2')} \tag{14}$$

Now, let us examine the two sides of this formula. Since the length scale is changed by a factor b, the LHS is simply

$$C_2((r_1 - r_2)/b, \mathcal{H}') \tag{15}$$

The RHS is more tricky. An infinitesimal change $h' \to h' + \delta h'$ corresponds to a change $\delta h = b^{-y_h} \delta h'$ of the field $h(r_i)$ acting on *all* the spins within the block. The RHS is then equal to

$$b^{-2y_h} \langle (s_1^{(1)} + s_2^{(1)} + ...)(s^{(2)})_1 + s^{(2)})_2 + ...) \rangle_{\mathcal{H}} \tag{16}$$

which is $b^{2(d-y_h)} C_2(r_1 - r_2, \mathcal{H})$. Thus we obtain the transformation rule for the correlation function

$$C_2((r_1 - r_2)/b, \mathcal{H}') = b^{2(d-y_h)} C_2(r_1 - r_2, \mathcal{H}) \tag{17}$$

valid close to the fixed point.

For a symmetric Hamiltonian $\mathcal{H}$, which respects lattice symmetries, at large distances the correlation function is isotropic, it is a function only of the distance $r = |r_1 - r_2|$. In this situation, the correlation function has a power law form which can be obtained as follows. Let us set $h = 0$ near the critical point. Then

$$C_2(r, \tau) = b^{-2(d-y_h)} C_2(r/b, b^{y_t} \tau) \tag{18}$$

Now, we may iterate this equation n times, as we did for the free energy, stopping at a point where $b^{n y_t} \tau = 1$. After some algebra, it follows that the correlation function has a scaling form:

$$C_2(r, \tau) = |\tau|^{2(d-y_h)/y_t} \Psi(r/|\tau|^{-1/y_t}) \tag{19}$$

At large r we expect C_2 to decay as $e^{-r/\xi}$. Comparing with the scaling form of C_2, we obtain the scaling relation for the correlation length

$$\xi \propto |\tau|^{-1/y_t} \tag{20}$$

which provides a relation of the correlation length index (see Lecture 5) with an RG eigenvalue: $\underline{\nu = 1/y_t}$.

At the critical point $\tau = 0$ we should iterate the transformation law, stopping when $r/b^n = r_0$, where r_0 is some relatively large fixed distance. We then obtain

$$C_2(r) \propto r^{-2(d-y_h)} \tag{21}$$

so that $\underline{\eta = d + 2 - 2y_h}$.

We expressed the exponents ν and η, related to the spin-spin correlation function, through the RG eigenvalues y_t, y_h. This also provides a general relation of the correlation function exponents and the thermodynamic exponents:

$$\alpha = 2 - d\nu, \tag{22}$$

$$\gamma = \nu(2 - \eta) \tag{23}$$

The formula $\alpha = 2 - d\nu$ is called a *hyperscaling* relation, since it connects the singularity of the specific heat with the behavior of the correlation function.

To conclude, the relations between different critical indices are derived on the sole assumption of scale invariance, taking into account the symmetry change for the transition in the Ising model, but otherwise with minimal microscopic input. We considered the simplest case of a transition in the Ising problem. For transitions in different universality classes, the RG analysis may involve a different set of relevant fields, and give rise to different predictions for scaling at a critical point. The relations between different indices, connecting the scaling behavior of different observables, facilitate experimental tests of the scaling hypothesis.

1.4 Summarize

- **Scaling of thermodynamic quantities**, proposed in the early days of the theory of phase transition, can be demonstrated in the renormalization group approach.

- **The results for critical indices**, obtained from the RG theory are of two different kinds. First, the RG theory relates the indices with the eigenvalues of the RG transformation at a fixed point. This provides means of calculating the indices based on renormalization of a microscopic Hamiltonian. Second, the possibility to educe a large number of indices to a small number of relevant eigenvalues can be used to derive relations between different indices. These relations are exact, i.e. they are insensitive to the accuracy with which the RG eigenvalues are known, the only input required from the RG theory is how many of relevant eigenvalues the problem has, and what is their physical interpretation (temperature, field, etc.).