Massachusetts Institute of Technology
Physics Department

Physics 8.20 — **IAP 2003**
Introduction to Special Relativity — **January 17, 2003**

Assignment 3
Due January 24, 2003

Announcements

- Please remember to put your name at the top of your paper.
- Problem Sets can also be downloaded from http://web.mit.edu/8.20/
- Depending on our rate of progress, some of the problems at the end of the problem set may be delayed until Problem Set #4.
- It is possible that some problems will be omitted if we do not get to cover the necessary material in class. Please watch for announcements!

Topics for this period

- The Doppler Effect
- Kinematics and "paradoxes'
- Relativistic energy and momentum
- Relativistic collisions and decays '

Reading Assignment 3

- Resnick and Halliday, §3, Supplements A and B, (Supplement A was also assigned on Problem Set 1, now would be a good time to study it further)
- French, §5 pages 134-159, §6.

Problem Set 3

1. **A very large Doppler shift**

   Resnick and Halliday §2, page 88, Problem 64.

2. **Longitudinal and transverse Doppler shifts in terms of wavelengths**

   (a) Show that the Doppler shift formulas can be written in the following forms:

   $$\begin{aligned} \lambda &= \lambda_0(1-\beta+\frac{1}{2}\beta^2+\ldots) \quad \text{approaching} \\ \lambda &= \lambda_0(1+\beta+\frac{1}{2}\beta^2+\ldots) \quad \text{receeding} \\ \lambda &= \lambda_0(1+\frac{1}{2}\beta^2+\frac{3}{8}\beta^4+\ldots) \quad \text{transverse} \end{aligned}$$

   where λ_0 is the wavelength measured by an observer at rest with respect to the source.

   (b) Derive a similar expression valid through order β^2 when the source makes an angle θ relative to the direction away from the observer.

3. **Won't this excuse get you in worse trouble?**

   The wavelength of red light is 600 nm and the wavelength of green light is 500 nm. You run a red light, and a policeman pulls you over. You tell him that because of the Doppler shift, the light looked green to you. If the policeman believes you and has taken 8.20, how fast does he conclude you were driving?

4. **The Most Distant Quasar Known:**

   The most distant quasar which has been observed is described as "having a redshift of $z = 6.28$". In this problem, we explore what this means. (This quasar was discovered by Xiaohui Fan and collaborators in 2001.)

   Hot hydrogen gas emits light with a particular set of frequencies, referred to as emission lines. The spectrum of emission lines serves as a finger print, allowing an astronomer who observes it to deduce that she is seeing hot hydrogen gas. The "Lyman-α line" is a prominent feature of the hydrogen spectrum which has a wavelength of 121.6 nm, or 1.216×10^{-7} m. This is the wavelength measured in a laboratory experiment, in which both the hot hydrogen and the detection apparatus (the "telescope") are at rest. [Note that this wavelength is deep in the ultraviolet. The human eye is sensitive to light with wavelengths from about 400 nm (violet light) to about 700 nm (red light).]

   Fan and his collaborators found a quasar in which the Lyman-α line has wavelength 885 ± 3 nm. The Doppler shift is so great that ultraviolet light has become infrared! By convention, the redshift z is defined as $1 + z =$(observed wavelength/emitted wavelength). In this case, $1 + z = 885/121.6$, yielding $z = 6.28 \pm .02$.

Use the formula for the special relativistic Doppler shift to deduce the relative velocity between the earth and the distant quasar, assuming that the quasar is receding directly away from the earth. (This is a good approximation; any transverse velocity would be much smaller than the recession velocity you have calculated.)

Cosmological Aside, beyond the scope of 8.20 (by Krishna Rajagopal) A more complete analysis of the implications of the redshifts of distant galaxies and quasars is beyond the scope of 8.20. For those of you interested, I thought I'd make a few comments.

In the 1930's, Edwin Hubble discovered that the more rapidly a galaxy is receding from us, the farther away from us it is. Furthermore, since light from more distant galaxies takes longer to reach us, we see these distant galaxies as they looked long ago, when the universe was younger than it is today.

You'll just have to take my word for the following implications of a redshift $z = 6.28$, as a derivation requires general relativity. The precise relation between redshift and distance is best stated as follows. Consider two galaxies or quasars — for example, our galaxy and the quasar observed by Fan — which are separated by a distance $R(t)$. Suppose that $R = R_0$ today. At the time when the light which Fan detected was emitted, R was only $R_0/(1 + z)$. Between that long ago time and today, the distance between any two galaxies in the universe has expanded by a factor $(1+z)$. Einstein's theory of general relativity (analyzed beyond the level which we will attempt later in 8.20) relates the behavior of $R(t)$ to the density of matter (i.e. galaxies) in the universe. This density is not known well, but is not too far from the "critical density", for which Einstein's equations make the particularly simple prediction that $R(t)$ is proportional to $t^{2/3}$. This means

$$\frac{R(t)}{R_0} = \left(\frac{t}{t_0}\right)^{2/3}$$

where t_0 is the present age of the universe. Putting it all together, when the light from the distant quasar was emitted, the universe was only $t_0/(1 + z)^{3/2}$ years old, or about 1/20 of its present age!

5. Visual appearance of a rod

This problem essentially recapitulates the discussion in lecture.

Suppose a rod of rest length ℓ_0 is oriented along the x' axis in the frame Σ'. The frame Σ' moves to the right along the x-axis when viewed from the frame Σ. The origins of Σ and Σ' coincide at $t = t' = 0$ and their axes are parallel.

An observer sits at the origin of Σ and watches the rod by looking at the light omitted by it.

(a) What is the *apparent* length of the rod?

(b) How do you reconcile this result with Lorentz Contraction?

6. Another version of the famous polevaulter problem

The frames Σ' and Σ are related in the standard fashion (as in the previous problem). A (one dimensional) garage is at rest in Σ. Its front end is at $x = 0$; its rear end at $x = L_0$. The garage has doors at both ends. Initially the front door is open and the rear door closed.

A Stacy Dragila (the world's woman's pole vault record holder) runs with velocity v up the x-axis as viewed from Σ. Of course, she is at rest in Σ'. She holds her pole, which is of rest length ℓ_0, parallel to the x-axis. [ℓ_0 is much greater than the rest length of the garage.] The right end of her pole reaches the left end of the garage at $t = t' = 0$. She runs so fast that Lorentz contraction shortens her pole to $\ell = L_0/2$ as viewed in Σ.

In the rest frame of the garage, it is clear that her pole fits into the garage, making the following sequence of events possible:

- **Event A** At the instant that the back end of the pole reaches the front of the garage, the front door of the garage is closed.
- **Event B** At the instant that the front end of the pole reaches the back end of the garage, the back door of the garage is opened.

Note that Event B occurs after Event A in Σ.

(a) What is Dragila's velocity in terms of L_0, ℓ_0 and c?

(b) Find the position and time of Events A and B in Σ.

(c) Now consider the events in Stacy's rest frame. Transform Events A and B to Σ' and show that the back door opens before the front door closes, making it possible for Stacy and the pole to get in and out of the garage without being crushed.

(d) After getting bored opening and closing doors at pre-assigned times during a practice session, the door operators (in Σ) propose another scheme: The front door closer proposes to send a signal to the back door opener as soon as he closes the front door (Event A) telling him to open the back door. Will this signal reach the back door in time to be effective?

7. The airplane falling through the ice

In lecture I asserted that a rod (an "airplane") drifting down upon an ice sheet with a hole in it, sees the plane of the ice rotated in its rest frame. The aim of this problem is to establish this effect.

This is a conceptually challenging problem

(a) First, let Σ and Σ' be frames related in the standard fashion (see problem 5) with relative speed v. Suppose a rod at rest in Σ' and of rest length ℓ_0 makes an angle θ' with the x' axis. Find the length of the rod (ℓ) and the angle (θ) that it makes with the x axis as observed in Σ.

(b) Now introduce new (rotated) Cartesian coordinate axes $\tilde{x}'$ and $\tilde{y}'$ in Σ' so that the rod lies along the $\tilde{x}'$ axis. Let the $\tilde{x}$ and $\tilde{y}$ axes be defined to be parallel to the $\tilde{x}'$ and $\tilde{y}'$ axes when the origins of Σ and Σ' coincide. Show that the origin of Σ' has velocity components $(v_{\tilde{x}}, v_{\tilde{y}}) = (v\cos\theta', -v\sin\theta')$ viewed from Σ.

(c) Show that the rod, which lies along the $\tilde{x}'$ coordinate axis in Σ', is observed at an angle to the $\tilde{x}$ coordinate axis in Σ. What is the angle?

8. Paradox of the Fast Walker:

In lecture we studied four classes of "paradoxes" that arise in special relativity: i) the pole vaulter; ii) the drifting airplane; iii) the sliding ice boat; and iv) the twins. The same effects have been restated in different forms over the years. Here is another paradox. Relate it to one of the four we discussed and resolve the problem. No calculations are necessary.

This paradox was invented by Wolfgang Rindler in 1961.

A man walks very fast over a rectangular grid, of the type used in some bridge roadways. The rest length of the walker's foot is equal to the spacing between the grid elements. In the rest frame of the grid, his Lorentz contraction makes him narrower than the grid spacing; observers in that frame expect him to fall in. In the rest frame of the walker, in contrast, the grid spacing is contracted and he should pass over the grid without any difficulty. These two predictions are contradictory. Which is correct?

9. Proper acceleration

Let a particle be moving along the x-axis when viewed in the frame Σ. The particle's *proper acceleration*, α, is defined as its acceleration measured in its instantaneous rest frame. Specifically, suppose at time t the particle has velocity v. Then it is instantaneously at rest in the frame Σ' moving with velocity v relative to Σ. Then $\alpha = dv'/dt'$.

Show that the acceleration observed in Σ is related to α by

$$\frac{dv}{dt} = \gamma^{-3}(v)\alpha$$

where $\gamma(v) = 1/\sqrt{1 - v^2/c^2}$ as usual, and v is the instantaneous velocity.

10. Constant proper acceleration

Suppose a particle experiences constant proper acceleration, $\alpha = \alpha_0$. Suppose it starts out at rest in Σ at $t = 0$.

(a) Use the result of the previous problem to show that its velocity (as measured in Σ is given by

$$v(t) = \frac{\alpha_0 t}{\sqrt{1 + (\frac{\alpha_0 t}{c})^2}}$$

(b) Let $\alpha_0 = g = 9.8\text{m}/\text{sec}^2$. How long would it take the particle to reach $v = 0.99c$?, $v = 0.9999c$? according to an observer in Σ?

(c) How long would it take to reach these speeds according to an observer on the particle?

11. Calibrating Minkowski space

French, §3, Problem 3-8, page 87.

12. Measuring time and distance in different frames

Resnick and Halliday, Supplement A, problems 8 and 9, page 279. Note, these problems refer to Figure A-4 which would be good to understand carefully.

13. An aging astronaut

French §5, PRoblem 5-21, page 164. Note the discussion of multiple reference frames and the effects of general relativity (which are negligible).

14. Keeping careful track of twins

French, §5, Problem 5-20, page 164.

15. Long lived muons

Laboratory experiments on muons at rest show that they have an average proper lifetime of about 2.3×10^{-6} seconds. An experiment has recently been completed at Brookhaven National Laboratory, which has measured the magnetic moment of the muon. In order to measure subtle magnetic effects, it is necessary to observe the muons for as long as possible. With this in mind the experimenters store the muons in a ring (with a diameter of 10 meters) with the aide of intense magnetic fields. The muons move in circulate orbits at very high speed. An average muon is found to make about 400 orbits before it decays.

- Compute the speed and γ-factor for the muons.
- How many proper lifetimes does the average muon go through as seen in the laboratory?
- Relate this all to the twin paradox.

Note, the comments in French, Problem 5-20 about accelerating frames apply to this problem as well.

16. Relativistic energy conservation

French, §1, Problem 1-1, page 29.

17. Relativistic kinetic energy and momentum

French §1, Problem 1-15, page 33.

18. Non-relativistic limit

The correct relativistic formula for the energy of a particle of rest mass m and speed v is, of course, $E(v) = m\gamma(v)c^2$.

(a) Expand $E(v)$ to order v^4. Identify the first correction to the usual non-relativistic formula for the kinetic energy, $T = \frac{1}{2}mv^2$.

(b) How large is the correction of part a) for the Earth relative to the Sun? In other words, approximately how much heavier is the Earth on account of its motion around the Sun? You will need the Earth's mass and its orbital speed to complete this estimate.

(c) How large a percentage error did you make by using the approximation of part a) instead of the exact result in the case of the Earth's motion around the Sun?

19. Hard work

An electron initially moves at $v = 0.9c$ in some reference frame.

(a) How much work must be done on the electron to increase its speed from $0.9c$ to $0.99c$?

(b) How much work must be done to increase its speed from $0.99c$ to $0.999c$?

20. J/ψ decays

Some say that the modern era in particle physics began with the discovery of the particle known as J/ψ. It is a bound state of a charm quark and its corresponding antiquark and was the first direct evidence for this species of quark. It was discovered simultaneously by Professors Samuel Ting, Ulrich Becker, and Min Chen and their collaborators of MIT and by Professor Burton Richter and his collaborators at the Stanford Linear Accelerator Center (hence the double name!). The J/ψ undergoes many interesting decays into other particles. The rest mass of the J/ψ is 3097MeV/c^2.

(a) Sometimes the J/ψ decays to an electron and (its antiparticle) a positron. The rest mass of the electron (and positron) is 0.5110Mev/c^2. Calculate the energy and γ factor of the electron emitted by a $J\psi$ decaying at rest.

(b) Sometimes the J/ψ decays to three photons. In the rest frame of the J/ψ, what is the maximum possible energy of one of the decay photons? Why do the three photons lie in a plane (in the rest frame of the J/ψ)? Suppose the photons are emitted symmetrically. What is the angle between adjacent photons? What are the energies of the photons?