8.333 Problem Set 3

1. (From Reif Chapter 1) Consider N similar antennas emitting linearly polarized light of wavelength λ and velocity c. The antennas are located along the x-axis at a separation λ from each other. An observer is located on the x-axis at a great distance from the antennas. When a *single* antenna radiates, the observer measures an *intensity* equal to I.

   (i) If all the antennas are driven in phase by the same generator of frequency $\nu = c/\lambda$, what is the total intensity measured by the observer?

   (ii)If the antennas all radiate at the same frequency ν, but with completely random phases, what is the mean intensity measured by the observer?

2. In this problem, you will formulate the central limit theorem more precisely than was done in class and prove it. Consider N real independant random variables x_i with the joint probability distribution function $p(x_1, x_2, ...x_N) = p_1(x_1)p_2(x_2)...p_N(x_N)$. The variable

$$u = \frac{\sum_i x_i}{N} \tag{1}$$

   has mean $< u >$ of order 1 and variance $\Delta =< u^2 > - < u >^2$ of order $1/N$ (assuming as we did in the class that the variance of each x_i is finite). Consider the "scaled" variable

$$v = \frac{u- < u >}{\sqrt{\Delta}} \tag{2}$$

   which is expected to have fluctuations of order 1. This clearly has mean zero and variance 1.

   (i) First show that v has the distribution

$$P(v) = \sqrt{\Delta} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{i\lambda u} \int \prod_i dx_i e^{-i\lambda(\frac{x_1+x_2+.....x_N}{N} >)} p(x_1, x_2, ...x_N). \tag{3}$$

   (ii) Now argue that the integral over the x_i may be written

$$Q(\lambda, N) \equiv \int \prod_i dx_i e^{-i\lambda(\frac{x_1+x_2+.....x_N}{N} -)} p(x_1, x_2, ...x_N) \tag{4}$$

$$= \left\langle e^{-i\lambda(u)} \right\rangle \tag{5}$$

   In terms of Q

$$P(v) = \sqrt{\Delta} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \exp\left(i\lambda u + \ln Q(\lambda, N)\right) \tag{6}$$

(iii) Write a formal power series expansion for $\ln Q$:

$$\ln Q(\lambda, N) = \sum_{n=0}^{\infty} c_n (-i\lambda)^n. \tag{7}$$

Show that $c_0 = 0$, $c_1 =< u >$, $c_2 = \Delta/2$. Find also an expression for c_3 in terms of the averages of u. Show that in general $c_n = a_n/N^{n-1}$ (for $n \geq 1$) where a_n is finite as N goes to infinity.

(iv) Use the results of (ii) and (iii) to calculate $P(v)$. You will find it convenient to change the variable of integration in Eqn. 6 from λ to $\sqrt{\Delta}\lambda$. Show that

$$P(v) = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \exp\left(i\lambda v - \frac{\lambda^2}{2} + \sum_{n=3}^{\infty} c_n \Delta^{-\frac{n}{2}} (-i\lambda)^n \right) \tag{8}$$

Show that in the limit of large N it is sufficient to keep just the first two terms in the exponential. Evaluate the resulting integral to obtain the universal Gaussian distribution for v.

(v) Examine the leading correction to the universal distribution by keeping the $n = 3$ term. Expand the integral to leading order in this term, and show explicitly that the correction vanishes as N goes to infinity.