# Learning 2:

# Replicator dynamics & Evolutionary stability

## Road Map

1. Evolutionarily stable strategies

2. Replicator dynamics

## 1 Notation

- $G = (S, A)$ a symmetric, 2-player game where

- $S$ is the strategy space;

- $A_{i,j} = u_1(s_i, s_j) = u_2(s_j, s_i)$;

- $x, y \in \Delta$ are mixed strategies; $u(x, y) = x^T Ay$;

- $u(ax + (1 - a)y, z) = au(x, z) + (1 - a)u(y, z)$.

# 2 Evolution stability

- Each player is endowed with a strategy (population/mutant strategy).
- Does not explain how a population arrives at such a strategy.
- Ask whether a strategy is robust to evolutionary pressures.
- Disregards effects on future actions.

# 3 ESS

Definition: A (mixed) strategy $x$ is said to be *evolutionarily stable* iff, given any $y \neq x$, there exists $\epsilon_y > 0$ s.t.

$$u(x,(1-\varepsilon)x+\varepsilon y) > u(y,(1-\varepsilon)x+\varepsilon y),$$

for each $\varepsilon$ in $(0,\epsilon_y]$.

Fact: $x$ is evolutionarily stable iff, $\forall y \neq x$,

1. $u(x,x) \geq u(y,x)$, and
2. $u(x,x) = u(y,x) \Longrightarrow u(x,y) > u(y,y)$.

Proof: Define *score function*

$$\begin{aligned} F_x(\varepsilon,y) &= u(x,(1-\varepsilon)x+\varepsilon y) - u(y,(1-\varepsilon)x+\varepsilon y) \\ &= u(x-y,x) + \varepsilon u(x-y,y-x). \\ \text{ESS} \iff & F_x(\varepsilon,y) > 0 \text{ for } \varepsilon \in (0,\epsilon_y]. \end{aligned}$$

# 4 ESS vs NE

- If $x \in \Delta^{ESS}$ then $(x, x)$ is NE ($x \in \Delta^{NE}$).

  In fact: $(x, x)$ is proper NE.

- $(x, x)$ is strict NE $\Longrightarrow x$ is ESS by default.

- Interior NE may not be ESS.

# 5 Hawk-Dove game

| |  |  |
|---|---|---|
|  | $\left(\frac{V-c}{2}, \frac{V-c}{2}\right)$ | $(V,0)$ |
|  | $(0,V)$ | $(V/2,V/2)$ |

Example: For $V = 4$, $c = 6$; $x = \left(\frac{2}{3}, \frac{1}{3}\right)$—NE; $\forall y \in \Delta$, $y \in BR(x)$.

$$u(x - y, y) = (x_1 - y_1)(2 - 3y_1) = \frac{1}{3}(2 - 3y_1)^2,$$

so, $x$ is ESS.

# 6 Rock-Scissors-Paper game

| | R | S | P |
|---|---|---|---|
| R | 0,0 | 1,-1 | -1,1 |
| S | -1,1 | 0,0 | 1,-1 |
| P | 1,-1 | -1,1 | 0,0 |

- Unique Nash Equilibrium $(s^*, s^*)$, where $s^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

- $s^*$ is not ESS. $(u(s^* - R, R) = 0)$.

# 7 ESS in role-playing games

- Given $(S^1, S^2, u_1, u_2)$, consider symmetric game $(S, u)$, where
  - $S = S^1 \times S^2$;
  - for $x = (x_1, x_2), y = (y_1, y_2) \in S$

$$u(x, y) = \frac{1}{2}[u_1(x_1, y_2) + u_2(x_2, y_1)].$$

Theorem: $x$ is ESS of $(S, u)$ iff $x$ is a strict NE of $(S^1, S^2, u_1, u_2)$.

# 8 Replicator dynamics

- Selection mechanism.
- $p_i(t) = \#$people who plays $s_i$ at $t$.
- $p(t) =$ total population at $t$.
- $x_i(t) = \frac{p_i(t)}{p(t)}$; $x(t) = (x_1(t), \ldots x_k(t))$.
- $u(x,x) = \sum_i x_i u(s_i, x)$.
- *Replicators* are pure strategies
$$\dot{p}_i = [\beta + u(s_i, x) - \delta]\, p_i.$$
- $$\dot{x}_i = [u(s_i, x) - u(x, x)]\, x_i = u(s_i - x, x) x_i.$$

# 9 Observations

- $$\begin{aligned} \frac{d}{dt}\left[\frac{x_i}{x_j}\right] &= \frac{\dot{x}_i}{x_j} - \frac{x_i}{x_j}\frac{\dot{x}_j}{x_j} \\ &= \Big[u(s_i, x) - u(s_j, x)\Big]\frac{x_i}{x_j}. \end{aligned}$$
- If $u$ becomes $u' = au + b$, then Replicator dynamics becomes
$$\dot{x}_i = au(s_i - x, x) x_i.$$

## 9.1 $2 \times 2$ games

Consider $(S, A)$, where $A = \begin{bmatrix} a_1, a_1 & 0, 0 \\ 0, 0 & a_2, a_2 \end{bmatrix}$.

We have

$$\begin{aligned} u(s_i, x) &= a_i x_i; \\ u(x, x) &= (x_1, x_2)A(x_1, x_2)^T = a_1 x_1^2 + a_1 x_2^2; \\ u(s_1 - x, x) &= (a_1 x_1 - a_2 x_2) x_2. \end{aligned}$$

and so

$$\dot{x}_1 = (a_1 x_1 - a_2 x_2) x_1 x_2.$$

## 9.2 Classification

1. $a_1 a_2 < 0$. Then
   - $x_1 \rightarrow_t 0$ when $a_1 < 0$;
   - $x_1 \rightarrow_t 1$ when $a_1 > 0$.

2. $a_1 a_2 > 0$; define $\lambda = \frac{a_2}{a_1 + a_2}$, $(\lambda, 1 - \lambda)$ is NE. Then,
   - $x_1 = \lambda$ is stable if $a_1 < 0$;
   - $x_1 = \lambda$ is unstable if $a_1 > 0$.

Compare with ESS.

Examples: Prisoner's dilemma, Chicken, Coordination game, Battle of the sexes, ...

## 10 Rationalizability

- $\boldsymbol{\xi}(t, x_0)$ is the solution to replicator dynamics starting at $x_0$.

Theorem: If a pure strategy $i$ is strictly dominated (by $y$), then $\lim_t \xi_i(t, x_0) = 0$ for any interior $x_0$.

Proof: Define $v_i(x) = \log(x_i) - \sum_j y_j \log(x_j)$. Then,

$$\begin{aligned} \frac{dv_i(x(t))}{dt} &= \frac{\dot{x}_i}{x_i} - \sum_j y_j \frac{\dot{x}_j}{x_j} \\ &= u(s_i - x, x) - \sum_j y_j u(s_j - x, x) \\ &= u(s_i - y, x) \leq -\epsilon < 0. \end{aligned}$$

Hence, $v_i(\boldsymbol{\xi}(t, x_0)) \to -\infty$, so $\xi_i(t, x_0) \to 0$.

Theorem: If $i$ is not rationalizable, then $\lim_t \xi_i(t, x_0) = 0$ for any interior $x_0$.

## 11 Theorems

Theorem: Every ESS $x$ is an asymptotically stable steady state of replicator dynamics.

(If the individuals can inherit the mixed strategies, the converse is also true.)

Theorem: If $x$ is an asymptotically stable steady state of replicator dynamics, and can be reached from an interior $x_0$, then $(x, x)$ is a perfect Nash equilibrium.