Price Theory Encompassing

MIT 14.126, Fall 2001

Road Map

- Traditional Doctrine: Convexity-Based
- Disentangling Three Big Ideas
  - Convexity: Prices & Duality
  - Order: Comparative Statics, Positive Feedbacks, Strategic Complements
  - Value Functions: Differentiability and Characterizations, Incentive Equivalence Theorems

Convexity and Traditional Price Theory

Convexity at Every Step

- Global or local convexity conditions imply
  - Existence of prices
  - Comparative statics, using *second-order conditions*
  - Dual representations, which lead to…
    - Hotelling's lemma
    - Shephard's lemma
    - Samuelson-LeChatelier principle
- "Convexity" is at the core idea on which the whole analysis rests.

Samuelson-LeChatelier Principle

- Idea: Long-run demand is "more elastic" than short-run demand.
- Formally, the statement applies to smooth demand functions for sufficiently small price changes.
- Let p=(p_x,w,r) be the current vector of output and input prices and let p' be the long-run price vector that determined the current choice of a fixed input, say capital.
- Theorem: If the demand for labor is differentiable at this point, then:

$$\left.\frac{\partial l^L}{\partial w}\right|_p \leq \left.\frac{\partial l^S}{\partial w}\right|_{p,p'=p} \leq 0$$

Varian's Proof

- Long- and short-run profit functions defined:

$$\pi^L(p) = \max_{k,l} p_x f(k,l) - wl - rk$$

$$\pi^S(p,p') = \max_{l} p_x f\left(k^*(p'),l\right) - wl - rk^*(p')$$

- Long-run profits are higher:

$$\pi^L(p) \geq \pi^S(p,p') \text{ for all } p,p' \text{ and } \pi^L(p) = \pi^S(p,p)$$

- So, long-run demand "must be" more elastic:

$$\left.\frac{\partial^2 \pi^L}{\partial w^2}\right|_p \geq \left.\frac{\partial^2 \pi^S}{\partial w^2}\right|_{p,p'=p} \geq 0 \text{ and } \left.\frac{\partial l^L}{\partial w}\right|_p \leq \left.\frac{\partial l^S}{\partial w}\right|_{p,p'=p} \leq 0$$

A Robust "Counterexample"

- The production set consists of the convex hull of these three points, with free disposal allowed:

Capital	Labor	Output
0	0	0
1	1	1
0	2	1

- Fix the price of output at 9 and the price of capital at 3, and suppose the wage rises from w=2 to w=5. Demands are:

$$l^L(2) = 2, l^S(5,2) = 0, l^L(5) = 1$$

- *Long run labor demand falls less than short-run labor demand.*
- Robustness: Tweaking the numbers or "smoothing" the production set does not alter this conclusion.

An Alternative Doctrine

Disentangling Three Ideas

Separating the Elements

- Convexity
  - Proving existence of prices
  - Dual representations of convex sets
  - Dual representations of optima
- Order
  - Comparative statics
  - Positive feedbacks (LeChatelier principle)
  - Strategic complements
- Envelopes
  - Useful with dual functions
  - Multi-stage optimizations
  - Characterizing information rents

Invariance Chart

Conclusions about $\max_{x \in S} f(x,t)$	**Transformations of Choice Variable**
Supporting ("Lagrangian") prices exist	Linear ("convexity preserving")
Optimal choices increase in parameter	Order-preserving
Long-run optimum change is larger, same direction	Order-preserving
Value function derivative formula: $V'(t) = f_2(x^*(t),t)$	One-to-one

Convexity Alone

Pure Applications of Convexity

- Separating Hyperplane Theorem
  - Existence of prices
  - Existence of probabilities
  - Existence of Dual Representations
    - Example: Bondavera-Shapley Theorem
    - Example: Linear programming duality
- "Alleged" Applications of Duality
  - Hotelling's lemma
  - Shephard's lemma

Separating Hyperplane Theorem

- Theorem. Let S be a non-empty, closed convex set in R^N and $x \notin S$. Then there exists $p \in R^N$ such that

$$p \cdot x > \max\{p \cdot y \mid y \in S\}$$

- Proof. Let $y \in S$ be the nearest point in S to x. Let

$$p = (x - y)/\|x - y\|$$

  - Argue that such a point y exists.
  - Argue that p·x>p·y.
  - Argue that if $z \in S$ and p·z>p·y, then for some small positive t, tz+(1-t)y is closer to x than y is.

Dual Characterizations

- Corollary. If S is a closed convex set, then S is the intersection of the closed "half spaces" containing it.
  - Defining

$$\pi(p) = \max\{p \cdot x \mid x \in S\}$$

  - it must be true that

$$S = \bigcap_{p \in R^N} \{x \mid p \cdot x \leq \pi(p)\}$$

Convexity and Quantification

- The following conditions on a closed set S in $\mathbf{R}^N$ is are equivalent
  - S is convex
  - For every x on the boundary of S, there is a supporting hyperplane for S through x.
  - For every concave objective function f there is some λ such that the maximizers of f(x) subject to $x \in S$ are maximizers of $f(x)+\lambda \cdot x$ subject to $x \in \mathbf{R}^N$.

Order Alone

"Order" Concepts & Results

- Order-related definitions
- Optimization problems
  - Comparative statics for separable objectives
  - An improved LeChatelier principle
  - Comparative statics with non-separable "trade-offs"
- Equilibrium w/ Strategic Complements
  - Dominance and equilibrium
  - Comparative statics
  - Adaptive Learning
  - LeChatelier principle for equilibrium

Two Aspects of Complements

- Constraints
  - Activities are complementary if doing one enables doing the other…
  - …or at least doesn't prevent doing the other.
    - This condition is described by sets that are sublattices.
- Payoffs
  - Activities are complementary if doing one makes it weakly more profitable to do the other…
    - This is described by supermodular payoffs.
  - …or at least doesn't change the other from being profitable to being unprofitable
    - This is described by payoffs satisfying a single crossing condition.

Definitions: "Lattice"

- Given a partially ordered set $(X,\geq)$, define
  - The "*join*": $x \vee y = \inf\{z \in X \mid z \geq x, z \geq y\}$.
  - The "*meet*": $x \wedge y = \sup\{z \in X \mid z \leq x, z \leq y\}$.
- $(X,\geq)$ is a "*lattice*" if

$$(\forall x, y \in X)\, x \wedge y, x \vee y \in X$$

- Example: $X=\mathbf{R}^N$,

$$x \geq y \text{ if } x_i \geq y_i, i = 1,\ldots,N$$
$$(x \wedge y)_i = \min(x_i, y_i); i = 1,\ldots,N$$
$$(x \vee y)_i = \max(x_i, y_i); i = 1,\ldots,N$$

Definitions, 2

- $(X,\geq)$ is a "*complete lattice*" if for every non-empty subset S, a greatest lower bound $\inf(S)$ and a least upper bound $\sup(S)$ exist in X.
- A function f: X→**R** is "*supermodular*" if

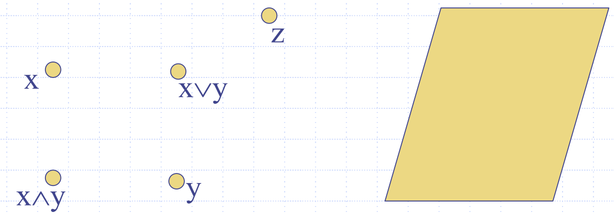
$$(\forall x, y \in X)\, f(x) + f(y) \leq f(x \wedge y) + f(x \vee y)$$

- A function f is "*submodular*" if $-f$ is supermodular.

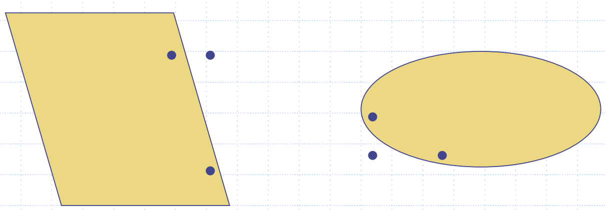
Definitions, 3

- Given two subsets S,T⊂X, "S is *as high as* T," written S≥T, means
$$[\,x \in S \text{ and } y \in T\,]$$
$$\Rightarrow [\,x \vee y \in S \text{ and } x \wedge y \in T\,]$$
- A function x* is "*isotone*" (or "*weakly increasing*") if
$$t \geq t' \Rightarrow x^*(t) \geq x^*(t')$$
  - "Nondecreasing" is not used because…
- A set S is a "*sublattice*" if S≥S.

Sublattices of $\mathbf{R}^2$



Not Sublattices



- Convexity, order and topology are mostly independent concepts. However, in **R**, these concepts coincide
  - Topology: S = compact set with boundary $\{a,b\}$
  - Convexity: $S = \{\alpha a + (1-\alpha)b \mid \alpha \in [0,1]\}$
  - Order: $S = [a,b] = \{x \mid a \leq x \leq b\}$

"Pairwise" Supermodularity

- Theorem (Topkis). Let $f:\mathbf{R}^N \to \mathbf{R}$. The following are equivalent:
  - f is supermodular
  - For all n≠m and x_{-nm}, the restriction $f(.,.,x_{-nm}):\mathbf{R}^2 \to \mathbf{R}$ is supermodular.

Proof of Pairwise Supermodularity

- $\Rightarrow$ This direction follows from the definition.
- $\Leftarrow$ Given $x \neq y$, suppose for notational simplicity that

$$x_i = \begin{cases} \max(x_i, y_i) \text{ for } i = 1,\dots,n \\ \min(x_i, y_i) \text{ for } i = n+1,\dots,N \end{cases}$$

- Then,

$$f(x \vee y) - f(y) = \sum_{i=1}^{n} \left(f(x_1,\dots,x_i,y_{i+1},\dots,y_N) - f(x_1,\dots,x_{i-1},y_i,\dots,y_N)\right)$$
$$\geq \sum_{i=1}^{n} [f(x_1,\dots,x_i,y_{i+1},\dots y_n,x_{n+1},\dots x_N)$$
$$- f(x_1,\dots,x_{i-1},y_i,\dots y_n,x_{n+1},\dots x_N)]$$
$$= f(x) - f(x \wedge y)$$

QED

"Pairwise" Sublattices

- Theorem (Topkis). Let S be a sublattice of $\mathbf{R}^N$. Define

$$S_{ij} = \left\{ x \in \Re^N \mid (\exists z \in S)\, x_i = z_i, x_j = z_j \right\}$$

Then, $S = \bigcap_{i,j} S_{ij}$.

- Remark. Thus, a sublattice can be expressed as a collection of constraints on pairs of arguments. In particular, undecomposable constraints like

$$x_1 + x_2 + x_3 \leq 1$$

can never describe in a sublattice.

Proof of Pairwise Sublattices

It is immediate that $S \subset \bigcap_{i,j} S_{ij}$. For the reverse,

suppose $x \in \bigcap_{i,j} S_{ij}$. Then, $\left(\exists z^{ij} \in S\right) z_i^{ij} = x_i$ and $z_j^{ij} = x_j$.

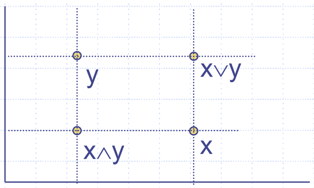
Define $z^i = \vee_j z^{ij} \in S$. For all j, $z_j^i \geq x_j, z_i^i = x_i$.

So, $z = \wedge_i z^i \in S$ satisfies $z_i = x_i$ for all i.

QED

Complementarity

- Complementarity/supermodularity has equivalent characterizations:
  - Higher marginal returns
  $$f(x \vee y) - f(x) \geq f(y) - f(x \wedge y)$$
  - Nonnegative mixed second differences
  $$[f(x \vee y) - f(x)] - [f(y) - f(x \wedge y)] \geq 0$$
  - For smooth objectives, non-negative mixed second derivatives:
  $$\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0 \text{ for } i \neq j$$



Monotonicity Theorem

- Theorem (Topkis). Let f:X×**R** →**R** be a supermodular function and define

$$x^*(t) \equiv \arg\max_{x \in S(t)} f(x,t).$$

If $t \geq t'$ and $S(t) \geq S(t')$, then $x^*(t) \geq x^*(t')$.

- Corollary. Let f:X×**R** →**R** be a supermodular function and suppose $S(t)$ is isotone. Then, for each t, $S(t)$ and $x^*(t)$ are sublattices.
- Proof of Corollary. Trivially, $t \geq t$, so $S(t) \geq S(t)$ and $x^*(t) \geq x^*(t)$. **QED**

Proof of Monotonicity Theorem

- Suppose that f is supermodular

  and that $x \in x^*(t), x' \in x^*(t'), t > t'$.
- Then, $(x \wedge x') \in S(t'), (x \vee x') \in S(t)$

  So, $f(x,t) \geq f(x \vee x',t)$ and $f(x',t') \geq f(x \wedge x',t')$.
- If either any of these inequalities are strict then their sum contradicts supermodularity:

$$f(x,t) + f(x',t') > f(x \wedge x',t') + f(x \vee x',t).$$

QED

Necessity for Separable Objectives

- Theorem (Milgrom). Let f:**R**^{N}×**R** →**R** be a supermodular function and suppose S is a sublattice.

Let $x^*_{g,S}(t) \equiv \arg\max_{x \in S} f(x,t) + \sum_{n=1}^{N} g_n(x_n)$.

- Then, the following are equivalent:
  - f is supermodular
  - For all $g_1,\ldots,g_N : \Re \to \Re$, $x^*_{g,S}(t)$ is isotone.
- Remarks:
  - This is a "robust monotonicity" theorem.
  - The function $g(x) \equiv \sum g_n(x_n)$ is "modular":

    $g(x) + g(y) = g(x \wedge y) + g(x \vee y)$.

Proof

- $\Rightarrow$ Follows from Topkis's theorem.
- $\Leftarrow$ It suffices to show "pairwise supermodularity." Hence, it is sufficient to show that supermodularity is necessary when N=2. We treat the case of two choice variables; the treatment of a choice variable and parameter is similar.
- Let $x, y \in \Re^2$ be unordered: $x_1 > y_1, x_2 < y_2$
- Fix

$$g_i(z_i) = \begin{cases} -\infty \text{ if } z_i \notin \{x_i, y_i\} \\ f(x \wedge y) - f(x) \text{ if } z_i = x_i, i = 1 \\ f(x \wedge y) - f(y) \text{ if } z_i = y_i, i = 2 \\ 0 \text{ otherwise} \end{cases}$$

- If $f(x) + f(y) > f(x \wedge y) + f(x \vee y)$, then $x^*_g = \{x, y, x \wedge y\}$ is not a sublattice, so $\neg\left(x^*(t) \geq x^*(t)\right)$. QED

Application: Production Theory

- Problem:

$$\max_{k,l} pf(k,l) - L(l,w) - K(k,r)$$

- Suppose that L is supermodular in the natural order, for example, $L(l,w)=wl$.
  - Then, $-L$ is supermodular when the order on l is reversed.
  - $l^*(w)$ is nonincreasing in the natural order.
- If f is supermodular, then $k^*(w)$ is also nonincreasing.
  - That is, capital and labor are "price theory complements."
- If f is supermodular with the reverse order, then capital and labor are "price theory substitutes."

Application: Pricing Decisions

- A monopolist facing demand $D(p,t)$ produces at unit cost c.

$$p^*(t) = \arg\max_p (p-c)D(p,t)$$

$$= \arg\max_{p>c} \log(p-c) + \log(D(p,t))$$

- $p^*(c,t)$ is always isotone in c. It is also isotone in t if $\log(D(p,t))$ is supermodular in (p,t), which is the same as being supermodular in $(\log(p),t)$, which means that increases in t make demand less elastic:

$$\frac{\partial \log D(p,t)}{\partial \log(p)} \text{ nondecreasing in } t$$

Application: Auction Theory

- A firm's value of winning an item at price p is U(p,t), where t is the firm's type. (Losing is normalized to zero.) A bid of p wins with probability F(p).
- Question: Can we conclude that p(t) is nondecreasing, without knowing F?

$$p_F^*(t) = \arg\max_p U(p,t)F(p)$$

$$= \arg\max_p \log(U(p,t)) + \log(F(p))$$

- Answer: Yes, if and only if $\log(U(p,t))$ is supermodular.

Long v Short-Run Demand

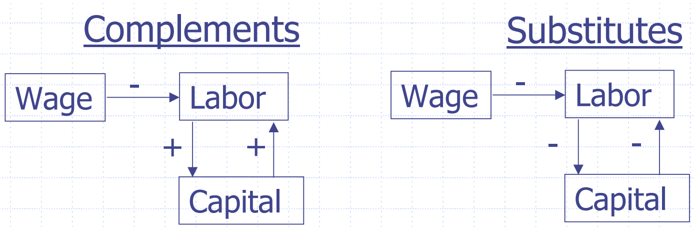
- Notation. Let $l^S(w,w')$ be the short-run demand for labor when the current wage is w and the wage determining fixed inputs is w'.
- Setting $w=w'$ in l^S gives the long run demands.
- Samuelson-LeChatelier principle:

$$0 \geq l_1(w,w) \geq \frac{d}{dw} l(w,w).$$

  - which can be restated revealingly as:

$$0 \geq l_2(w,w).$$

Milgrom-Roberts Analysis



- Remarks:
  - This analysis involves no assumptions about convexity, divisibility, etc.
  - For smooth demands, symmetry of the substitution matrix implies that, locally, one of the two cases above applies.

Improved LeChatelier Principle

- Let $H(x,y,t)$ be supermodular and S a sublattice.
- Let $\left(x^*(t),y^*(t)\right)=\max\arg\max_{(x,y)\in S} H(x,y,t)$
- Let $x^*(t,t')=\max\arg\max_{x\in\{x'|(x',y^*(t'))\in S\}} H(x,y^*(t'),t)$
- Theorem (Milgrom & Roberts). x^* is isotone in both arguments. In particular, if $t>t'$, then

$$x^*(t)=x^*(t,t)\geq x^*(t,t')\geq x^*(t',t')=x^*(t')$$

Proof

- By the Topkis Monotonicity Theorem, $y^*(t)\geq y^*(t')$
- Applying the same theorem again, for all $t,t''>t'$

$$x^*(t',t')=\max\arg\max_{x\in\{x'|(x',y^*(t'))\in S\}} H(x,y^*(t'),t')$$

$$\leq x^*(t,t')=\max\arg\max_{x\in\{x'|(x',y^*(t'))\in S\}} H(x,y^*(t'),t)$$

$$\leq x^*(t,t'')=\max\arg\max_{x\in\{x'|(x',y^*(t''))\in S\}} H(x,y^*(t''),t)$$

- Setting $t=t''$ completes the proof.

Long v Short-Run Demand

- Theorem. Let w>w'. Suppose capital and labor are ***complements***, i.e., $f(k,l)$ is supermodular in the natural order. If demand is single-valued at w and w', then

$$l^S(w,w)\leq l^S(w,w')\leq l^S(w',w')$$

- Theorem. Let w>w'. Suppose capital and labor are ***substitutes***, i.e., $f(k,l)$ is supermodular when capital is given its reverse order. If demand is single-valued at w and w', then

$$l^S(w,w)\leq l^S(w,w')\leq l^S(w',w')$$

Non-separable Objectives

- Consider an optimization problem featuring "trade-offs" among effects.
  - x is the real-valued choice variable
  - $B(x)$ is the "benefits production function"
  - Optimal choice is

$$x_B^*(t) = \arg\max_{x \in X} \pi(x, B(x), t)$$

Robust Monotonicity Theorem

- Define: $x_B^*(t) = \arg\max_{x \in X} \pi(x, B(x), t)$
- Theorem. Suppose π is continuously differentiable and π_2 is nowhere 0. Then:

$$\left[(\forall x, y) \frac{\pi_1(x,y,t)}{|\pi_2(x,y,t)|} \text{ is increasing in } t\right]$$

$$\Rightarrow \left[\text{For all } B,\ x_B^*(t) \text{ is isotone}\right]$$

$$\Rightarrow \left[(\forall x, y) \frac{\pi_1(x,y,t)}{|\pi_2(x,y,t)|} \text{ is nondecreasing in } t\right]$$

Application: Savings Decisions

- By saving x, one can consume $F(x)$ in period 2.

$$V(w) = \max_{0 \le x \le w} U(w - x, F(x))$$

$$x_F^*(w) = \max \arg\max_{0 \le x \le w} U(w - x, F(x))$$

- Define: $\pi(x, y, t) = U(t - x, y)$
- Analysis. If MRS_{xy} increases with x, then optimal savings are isotone in wealth:

$$\left[\frac{U_1(x,y)}{U_2(x,y)} \text{ increasing in } x\right] \Rightarrow x_F^*(w) \text{ isotone}$$

- This is the same condition as found in price theory, when F is restricted to be linear. Here, F is unrestricted.
  - Also applies to Koopmans consumption-savings model.

Introduction to Supermodular Games

Formulation

- N players (infinite is okay)
- Strategy sets X_n are complete sublattices
  - $\underline{x}_n = \min X_n, \overline{x}_n = \max X_n$
- Payoff functions $U_n(x)$ are
  - Continuous
  - "Supermodular with isotone differences"

$$(\forall n)(\forall x_n, x_n' \in X_n)(\forall x_{-n} \geq x_{-n}' \in X_{-n})$$
$$U_n(x) + U_n(x') \leq U_n(x \wedge x') + U_n(x \vee x')$$

Bertrand Oligopoly Models

- Linear/supermodular Oligopoly:

Demand: $Q_n(x) = A - ax_n + \sum_{j \neq n} b_j x_j$

Profit: $U_n(x) = (x_n - c_n) Q_n(x)$

$\frac{\partial U_n}{\partial x_m} = b_m (x_n - c_n)$ which is increasing in x_n

- Log-supermodular Oligopoly:

$$\log U_n(x) = \log(x_n - c_n) + \log Q_n(x)$$
$$\frac{\partial^2 U_n}{\partial x_m \partial x_n} \geq 0 \Leftrightarrow \frac{\partial^2 \log Q_n(x)}{\partial \log x_n \partial \log x_m} \geq 0$$

Linear Cournot Duopoly

- Inverse Demand: $P(x) = A - x_1 - x_2$

$U_n(x) = x_n P(x) - C_n(x_n)$

$\frac{\partial U_n}{\partial x_m} = -x_n$

- Linear Cournot duopoly (but not more general oligopoly) is supermodular if one player's strategy set is given the reverse of its usual order.

Analysis of Supermodular Games

- Extremal Best Reply Functions

$$B_n(x) = \max\left(\underset{x_n' \in X_n}{\operatorname{argmax}} U_n(x_n', x_{-n})\right)$$
$$b_n(x) = \min\left(\underset{x_n' \in X_n}{\operatorname{argmax}} U_n(x_n', x_{-n})\right)$$

  - By Topkis's Theorem, these are isotone functions.
- Lemma:

$\neg[x_n \geq b_n(\underline{x})] \Rightarrow [x_n$ is strictly dominated by $b_n(\underline{x}) \vee x_n]$

- Proof. If $\neg[x_n \geq b_n(\underline{x})]$, then

$$U_n(x_n \vee b_n(\underline{x}), x_{-n}) - U_n(x_n, x_{-n}) \geq U_n(b_n(\underline{x}), \underline{x}_{-n}) - U_n(x_n \wedge b_n(\underline{x}), \underline{x}_{-n}) > 0$$

Rationalizability & Equilibrium

- Theorem (Milgrom & Roberts): The smallest rationalizable strategies for the players are given by
$$\underline{z} = \lim_{k\to\infty} b^k(\underline{x})$$
Similarly the largest rationalizable strategies for the players are given by
$$\overline{z} = \lim_{k\to\infty} B^k(\overline{x})$$
Both are Nash equilibrium profiles.

Proof

- Notice that $b^k(\underline{x})$ is an isotone, bounded sequence, so its limit $\underline{z}$ exists.
- By continuity of payoffs, its limit is a fixed point of b, and hence a Nash equilibrium.
- Any strategy less than $\underline{z}_n$ is less than some $b^k{}_n(\underline{x})$ and hence is deleted during iterated deletion of dominated strategies.
- **QED**

Comparative Statics

- Theorem. (Milgrom & Roberts) Consider a family of supermodular games with payoffs parameterized by t. Suppose that for all n, x_{-n}, $U_n(x_n,x_{-n};t)$ is supermodular in (x_n,t). Then
$\overline{z}(t), \underline{z}(t)$ are isotone.
- Proof. By Topkis's theorem, $b_t(x)$ is isotone in *t*. Hence, if $t > t'$,
$$b_t^k(\underline{x}) \geq b_{t'}^k(\underline{x})$$
$$\underline{z}(t) = \lim_{k\to\infty} b_t^k(\underline{x}) \geq \lim_{k\to\infty} b_{t'}^k(\underline{x}) \geq \underline{z}(t')$$
and similarly for $\overline{z}$. **QED**

Adaptive Learning

- Player n's behavior is called "*consistent with adaptive learning*" if for every date *t* there is some date *t'* after which n does not play a strategy that is strictly dominated in the game in which others are restricted to play only strategies they have played since date *t*.
- Theorem (Milgrom & Roberts). In a finite strategy game, if every player's behavior is consistent with adaptive learning, then all eventually play only rationalizable strategies.

Equilibrium LeChatelier Principle

- Formulation
  - Consider a parameterized family of supermodular games with payoffs parameterized by t. Suppose that for all n, x_{-n}, $U_n(x_n, x_{-n}; t)$ is supermodular in (x_n, t).
  - Fixing player 1's strategy at $\underline{z}_1(t')$ induces a supermodular game among the remaining players. Let $\underline{y}(t,t')$ be the smallest Nash equilibrium in the induced game, with $\underline{y}_1(t,t')=\underline{z}_1(t')$.
- Theorem.
  - If $t > t'$, then $\underline{z}(t) \geq \underline{y}(t,t') \geq \underline{z}(t')$.
  - If $t < t'$, then $\underline{z}(t) \leq \underline{y}(t,t') \leq \underline{z}(t')$.

...and a similar conclusion applies to the maximum equilibrium.

Proof

- Observe (exercise) that
  $$\underline{z}(t) = \underline{y}(t,t), \underline{z}(t') = \underline{y}(t',t').$$
- Suppose t>t'.
  - By the comparative statics theorem, $\underline{z}$ is isotone, so:
    $$\underline{z}(t) \geq \underline{z}(t').$$
  - Hence, by the comparative statics theorem applied again, $\underline{y}$ is isotone, so:
    $$\underline{y}(t,t) \geq \underline{y}(t,t') \geq \underline{y}(t',t').$$

QED

Envelope Functions Alone

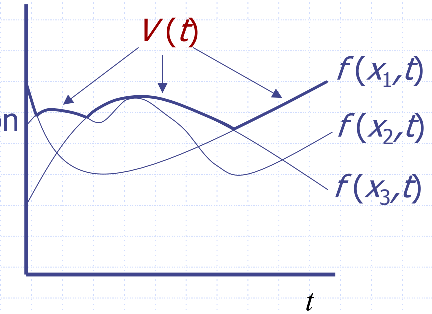
Based on "Envelope Theorems for Arbitrary Choice Sets" by Paul Milgrom & Ilya Segal

What are "Envelope Theorems"?

- Envelope theorems deal with the properties of the value function: $V(t) \equiv \max_{x \in X} f(x,t)$
- Answer questions about...
  - when V is differentiable, directionally differentiable, Lipschitz, or absolutely continuous
  - when V satisfies the "envelope formula"
    $V'(t) = f_t(x,t)$ for $x \in x^*(t)$
- "Traditional" envelope theorems assume that set X is convex and the objective $f(\cdot,t)$ is concave and differentiable.

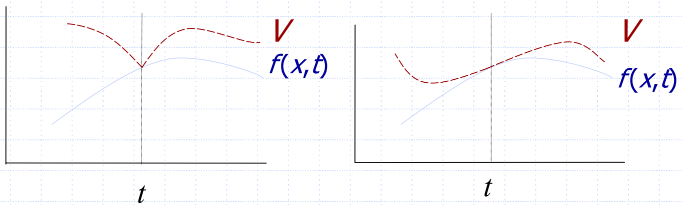
Intuitive Argument

- When $X=\{x_1,x_2,x_3\}$...
  - V is left- and right-differentiable everywhere
  - if $f_t(x,t)$ is constant on $x\in x^*(t)$, then V is differentiable at t
  - envelope formulas apply for
    - $V'(t)=f_t(x^*(t),t)$
    - $V'(t+)$ and $V'(t-)$



Envelope Derivative Formula

- Theorem 1. Take $t\in[0,1]$ and $x\in x^*(t)$, and suppose that $f_t(x,t)$ exists.
  - If t<1 and $V'(t+)$ exists, then $V'(t+)\geq f_t(x,t)$.
  - If t>0 and $V'(t-)$ exists, then $V'(t-)\leq f_t(x,t)$.
  - If $t\in(0,1)$ and $V'(t)$ exists, then $V'(t) = f_t(x,t)$.
- Proof:



Absolute Continuity

- Theorem 2(A). Suppose that
  - $f(x,\cdot)$ is is differentiable (or just absolutely continuous) for all $x\in X$ with derivative (or density) f_t.
  - there exists an integrable function $b(t)$ such that $|f_t(x,\cdot)| \leq b(t)$ for all $x\in X$ and almost all $t\in[0,1]$.

Then V is absolutely continuous with density satisfying $|V'(t)|\leq b(t)$.

Proof of Theorem 2(A)

- Define

$$B(t)=\int_0^t b(s)ds$$

- Then for $t''> t'$:

$$|V(t'')-V(t')|\leq \sup_{x\in X}|f(x,t'')-f(x,t')|$$

$$=\sup_{x\in X}\left|\int_{t'}^{t''} f_t(x,t)dt\right| \leq \int_{t'}^{t''}\sup_{x\in X}|f_t(x,t)|dt$$

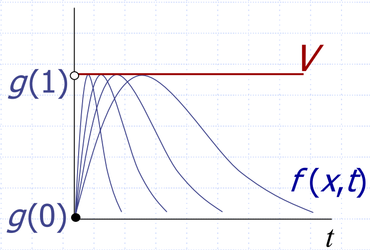
$$\leq \int_{t'}^{t''} b(t)dt = |B(t'')-B(t')|$$

- It suffices to prove the theorem for intervals, because open intervals are a basis for the open sets. **QED**

Why do we need $b(\cdot)$?

- Let $X=(0,1]$ and $f(x,t)=g(t/x)$, where g is smooth and single-peaked with unique maximum at 1.
  - $V(0)=g(0)$, $V(t)=g(1)$: V is discontinuous at 0.
  - This example has no integrable bound $b(t)$:

$$\sup_{x\in(0,\infty)}\left|f_t(x,t)\right| = \sup_{x\in(0,\infty)}\left|\tfrac{1}{t}\left(\tfrac{t}{x}g'(\tfrac{t}{x})\right)\right| = \tfrac{1}{t}\sup_{x\in(0,\infty)}\left|xg'(x)\right|$$



Envelope Integral Formula

- Theorem 2(B). Suppose that, in addition to the assumptions of 2(A), the set of optimizers $x^*(t)$ is non-empty for all t. Then for any selection $x(t)\in x^*(t)$,

$$V(s)=V(0)+\int_0^s f_t(x(t),t)dt.$$

Equi-differentiability

- Definition. A family of functions $\{f(x,\cdot)\}_{x\in X}$ is "*equi-differentiable*" at $t\in(0,1)$ if

$$\lim_{t'\to t}\sup_x\left|\frac{f(x,t')-f(x,t)}{t'-t}-f_t(x,t)\right|=0$$

- If X is finite, this is the same as simple differentiability.

Directional Differentiability

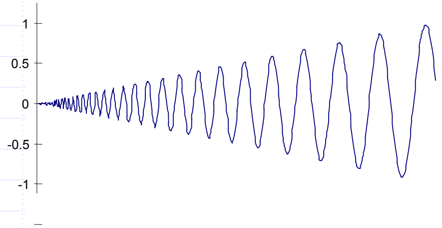
- Theorem 3. If
  - (i) $\{f(x,\cdot)\}_{x\in X}$ is equi-differentiable at t_0,
  - (ii) $x^*(t)$ is non-empty for all t, and
  - (iii) $\sup_x|f_t(x,t_0)|<\infty$,

  then for any selection $x(t)\in x^*(t)$, V is left- and right-differentiable at $t_0\in(0,1)$ and the derivatives satisfy

$$V'(t_0+)=\lim_{t\to t_0+}f_t(x(t),t_0)$$
$$V'(t_0-)=\lim_{t\to t_0-}f_t(x(t),t_0)$$

Role of "Equi-differentiability"

- Simple differentiability (rather than equi-differentiability) is not enough for V to have left- and right-derivatives:
  - Let $g(t) = t \sin \log(t), f(x,t) = g(t)$ if $t > \exp(-\pi/2 - 2\pi x)$, $f(x,t) = -t$ otherwise.
  - Then, $V(t) = g(t)$



Continuous Problems

- Theorem 4. Suppose X is a non-empty compact space, f is upper semi-continuous on X and f_t is continuous in (x,t). Then,
  - V is directionally differentiable

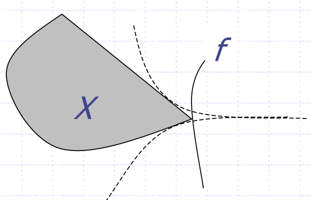
    $$V'(t+) = \max_{x \in x^*(t)} f_t(x,t) \text{ for } t \in [0,1)$$

    $$V'(t-) = \min_{x \in x^*(t)} f_t(x,t) \text{ for } t \in (0,1]$$

  - In particular, $V'(t+) \geq V'(t-)$.
  - V is differentiable at t if any of the following hold:
    - V is concave (because $V'(t+) \leq V'(t-)$)
    - t is a maximum of $V(\cdot)$ (because $V'(t+) \leq V'(t-)$)
    - $x^*(t)$ is a singleton (because $V'(t+) = V'(t-)$)

Contrast to a "Traditional" Approach

- In some approaches, the differentiability of x^* is used in the argument. However, V can be differentiable even when x^* is not. This often happens, for example, in strictly convex problems:



Applications

Hotelling's Lemma

- Define:

$$\pi(p) = \max_{x \in X} p \cdot x$$

$$x^*(p) = \arg\max_{x \in X} p \cdot x$$

- Theorem. Suppose X is compact. Then, $\pi'(p)$ exists if and only if $x^*(p)$ is a singleton, and in that case $\pi'(p) = x^*(p)$.

Shephard's Lemma

- Define:

$$C(y,p) = \min_{x \in X, x_1 = y} -p_{-1} \cdot x_{-1}$$

$$x^*(p) = \arg\min_{x \in X, x_1 = y} -p_{-1} \cdot x_{-1}$$

- Remark: The variable x_1 represents "output" and the other variables represent inputs, measured as negative numbers.
- Theorem. Suppose X is compact. Then, $\partial C/\partial p$ exists if and only if $x^*(p)$ is a singleton, and in that case $\partial C/\partial p = x^*(p)$.

Multi-Stage Maximization

- Stage 1: choose investment $t \geq 0$.
- Stage 2: choose action vector $x \in X \neq \emptyset$
- Assume:
  - f(x,t) is equidifferentiable in t and $t^*>0$
  - f(x,t) is u.s.c. in x and X is compact
- Conclusion: the value function $V(t)$ is differentiable at t^* and $V'(t^*)=0$.
  - Proof: Apply theorem 4.

Mechanism Design

- Y=set of outcomes
- Agent's type is t, utility is $f(x,t)$.
- M=message space. h:M→Y is outcome function.
- X=h(M) is set of "accessible outcomes."
- Assume that each type has an optimal choice

$$x(t) \in \arg\max_{x \in X} f(x,t)$$

Analysis

- Corollary 1. Suppose that the agent's utility function $f(x,t)$ is differentiable and absolutely continuous in t for all $x \in Y$, and that $\sup_{x \in Y} f_t(x,t)$ is integrable on [0,1]. Then the agent's equilibrium utility V in any mechanism implementing a given choice rule x must satisfy the following integral condition.

$$V(t) = V(0) + \int_0^t f_t(x(s), s)\,ds.$$

  - This had previously been shown only with (sometimes "weak") additional conditions.

Mechanism Design Applications

- Models in which payoffs are $v \cdot p - \pi$, so

$$U(v) = U(0) + \int_0^1 v \cdot p^*(sv)\,ds.$$

- Theorems
  - Green-Laffont Theorem
    - Uniqueness of Dominant Strategy Mechanisms
  - Holmstrom-Williams Theorem
    - Bayesian Revenue Equivalence
  - Myerson-Satterthwaite Theorem
    - Necessity of Bargaining Inefficiency
  - Jehiel-Moldovanu Theorem
    - Impossibility of Efficiency with Value Interdependencies

Green-Laffont Theorem

- "Uniqueness of dominant strategy implementation."
- Theorem (Holmstrom's variation). Suppose that
  - M is a direct mechanism to implement the efficient outcome in dominant strategies
  - the type space is smoothly path-connected.
- Then,
  - the payment function for player j in mechanism M is equal to the payment function of the Vickrey-Clarke-Groves pivot mechanism plus some function $g_j(v_{-j})$ (which depends only on the other player's types).

Green-Laffont Theorem

- Given any value vector v, let $\{v_j(t) | t \in [0,1]\}$ be a smooth path connecting some fixed value $\underline{v}_j$ to $v_j = v_j(1)$. By the Envelope Theorem applied to the path parameter t,

$$U_j(v_j(t), v_{-j}) = p_j(v_j(t), v_{-j}) \cdot v_j - X_j(v_j(t), v_{-j})$$

$$= U_j(\underline{v}_j, v_{-j}) + \int_0^t p_j(v_j(s), v_{-j}) \cdot v_j'(s)\,ds$$

$$\therefore X_j(v_j(1), v_{-j}) = f_j(v_{-j}) + p_j(v_j(1), v_{-j}) \cdot v_j - \int_0^1 p_j(v_j(s), v_{-j}) \cdot v_j'(s)\,ds$$

where $f(v_{-j}) = -U_j(\underline{v}_j, v_{-j})$

- So, X_j is fully determined by the functions p and f_j.

Holmstrom-Williams' Theorem

- Theorem: *Any mechanism that Bayes-Nash implements efficient outcomes on a smoothly path-connected type space entails the same expected payments as the Vickrey mechanism, plus some bidder-specific constant.*
- Proof. Let $\{v_j(s), s\in[0,1]\}$ be a path from some fixed value vector to any other value vector. By the Envelope Theorem,

$$U_j\left(v_j(t)\right) = p_j\left(v_j(t)\right)\cdot v_j(t) - X_j(v_j(t))$$
$$= U_j\left(v_j(0)\right) + \int_0^t p_j\left(v_j(s)\right)\cdot v_j'(s)ds$$

- Hence, $X_j(v)$ is uniquely determined by $U_j(0)$. It is equal to $U_j(0)$ plus the expected payment in the Vickrey mechanism.

Two-Person Bargaining

- Assume
  - there is a buyer with value v distributed on [0,1]
  - there is a seller with cost c distributed on [0,1]
- The Vickrey-Clarke-Groves mechanism
  - has each party report its value
  - entails p*(v,c)=1 if v>c and p*(v,c)=0 otherwise
  - payments are
    - if p*(v,c)=0, no payments
    - if p*(v,c)=1, buyer pays c and the seller receives v

Myerson-Sattherthwaite Theorem

- Expected profits are:
  - $U_B(v)=E[(v-c)1_{\{v>c\}}|v]$, so $E[U_B(v)]=E[(v-c)1_{\{v>c\}}]$
  - $U_S(c)=E[(v-c)1_{\{v>c\}}|s]$, so $E[U_S(c)]=E[(v-c)1_{\{v>c\}}]$
  - each bidder expects to receive the ***entire social surplus***.
- Apply Holmstrom-Williams theorem:
- **Theorem (Myerson-Satterthwaite)**. There is no mechanism and Bayesian Nash equilibrium such that the mechanism implements for all v,c with v>c and
  - $U_B(0)=U_S(1)=0$ ("voluntary participation by worst type")
  - $E[U_B(v)]+E[U_S(c)]\leq E[(v-c)1_{\{v>c\}}]$ ("balanced expected budget")

Subtleties

- Consider a model in which:

$$\Pr\{v>1\} = \Pr\{c<1\} = 1$$

- Q: Why doesn't simply trading at price $p=1$ violate the theorem in this model?
- A: Because it prescribes trade even when c>v!