14.126 Game Theory
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Fall 2001

# A Synopsis of the Theory of Choice

This note summarizes the elements of the expected-utility theory. For a detailed exposition of the first four sections, see Kreps (1988); for the last section see Savage (1954). We will first define a choice function and present the necessary and sufficient conditions a choice function must satisfy in order to be represented by a preference relation — revealed preferences. We will then present the necessary and sufficient conditions that such a preference relation must satisfy in order to be represented by a utility function — ordinal representation. Next, we present the expected utility theories of Von Neuman and Morgenstern, Anscombe and Auman, and Savage, where the representing utility function takes some form of expectation — cardinal representation.

## 1 Revealed Preferences

We consider a set $X$ of alternatives. Alternatives are mutually exclusive in the sense that one cannot choose two distinct alternatives at the same time. We also take the set of feasible alternatives exhaustive so that a player's choices will always be defined.

**Definition 1** *By a* choice function*, we mean a function* $c : 2^X \setminus \{\emptyset\} \to 2^X \setminus \{\emptyset\}$ *such that*

$$c(A) \subseteq A \qquad \textit{for each } A \in 2^X \setminus \{\emptyset\}.$$

Here $c(A)$ consists of the alternatives the agent *may* choose if he is constrained to $A$; he will choose only one of them. Note that $c(A)$ is assumed to be non-empty.

Our second construct is a preference relation. Take a relation $\succeq$ on $X$, i.e., a subset of $X \times X$. A relation $\succeq$ is said to be *complete* if and only if, given any $x, y \in X$, either $x \succeq y$ or $y \succeq x$. A relation $\succeq$ is said to be *transitive* if and only if, given any $x, y, z \in X$,

$$[x \succeq y \text{ and } y \succeq z] \Rightarrow x \succeq z.$$

**Definition 2** *A relation is a* preference relation *if and only if it is complete and transitive.*

Given any preference relation $\succeq$, we can define strict preference $\succ$ by

$$x \succ y \iff [x \succeq y \text{ and } y \not\succeq x],$$

and the indifference $\sim$ by

$$x \sim y \iff [x \succeq y \text{ and } y \succeq x].$$

Now consider the choice function $c(\cdot;\succeq)$ of an agent who wants to choose the best available alternative with respect to a preference relation $\succeq$. This function is defined by

$$c(A;\succeq) = \{x \in A | x \succeq y \quad \forall y \in A\}.$$

Note that, since $\succeq$ is complete and transitive, $c(A;\succeq) \neq \emptyset$ whenever $A$ is finite. Consider a set $A$ with members $x$ and $y$ such that our agent may choose $x$ from $A$ (i.e., $x \succeq y$). Consider also a set $B$ from which he may choose $y$ (i.e., $y \succeq z$ for each $z \in B$). Now, if $x \in B$, then he may as well choose $x$ from $B$ (i.e., $x \succeq z$ for each $z \in B$). That is, $c(\cdot;\succeq)$ satisfies the following axiom by Hauthakker:

**Axiom 1** *(*Hauthakker*) Given any $A, B$ with $x, y \in A \cap B$, if $x \in c(A)$ and $y \in c(B)$, then $x \in c(B)$.*

It turns out that any choice function $c$ that satisfies Hauthakker's axiom can be considered coming from an agent who tries to choose the best available alternative with respect to some preference relation $\succeq_c$. Such a preference relation can be defined by

$$x \succeq_c y \iff x \in c(\{x, y\}).$$

**Theorem 1** *If $\succeq$ is a preference relation, then $c(\cdot;\succeq)$ satisfies Hauthakker's axiom. Conversely, if a choice function $c$ satisfies Hauthakker's axiom, then there exists a preference relation $\succeq_c$ such that $c = c(\cdot;\succeq_c)$.*

## 2 Ordinal representation

We are interested in preference relations that can be *represented* by a utility function $u : X \to \mathbb{R}$ in the following sense:

$$x \succeq y \Longleftrightarrow u(x) \geq u(y) \qquad \forall x, y \in X. \tag{OR}$$

Clearly, when the set $X$ of alternatives is countable, any preference relation can be represented in this sense. The following theorem states further that a relation needs to be a preference relation in order to be represented by a utility function.

**Theorem 2** *Let $X$ be finite (or countable). A relation $\succeq$ can be represented by a utility function $U$ in the sense of (OR) if and only if $\succeq$ is a preference relation. Moreover, if $U : X \to \mathbb{R}$ represents $\succeq$, and if $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, then $f \circ U$ also represents $\succeq$.*

By the last statement, we call such utility functions ordinal.

When $X$ is uncountable, some preference relations may not represented by any utility function, such as the lexicographic preferences on $\mathbb{R}^2$.[1] If the preferences are continuous they can be represented by a (continuous) utility function even when $X$ is uncountable.

**Definition 3** *Let $X$ be a metric space. A preference relation $\succeq$ is said to be* continuous *iff, given any two sequences $(x_n)$ and $(y_n)$ with $x_n \to x$ and $y_n \to y$,*

$$[x_n \succeq y_n \quad \forall n] \Longrightarrow x \succeq y.$$

**Theorem 3** *Let $X$ be a separable metric space, such as $\mathbb{R}^n$. A relation $\succeq$ on $X$ can be represented by some continuous utility function $U : X \to R$ in the sense of (OR) iff $\succeq$ is a continuous preference relation.*

When a player chooses between his strategies, he does not know which strategies the other players choose, hence he is uncertain about the consequences of his acts (namely, strategies). To analyze the players' decisions in a game, it would be useful then to have a theory of decision making that allows us to express an agent's preferences on the acts with uncertain consequences (strategies) in terms of his attitude towards the consequences.

[1] In fact, some form of countability is necessary for representability. $X$ must be separable with respect to the order topology of $\succeq$, i.e., it must contain a countable subset that is dense with respect to the order topology. (See Theorem 3.5 in Kreps, 1988.)

# 3 Cardinal representation

Consider a finite set $Z$ of consequences (or prizes). Let $S$ be the set of all states of the world. Take a set $F$ of acts $f : S \to Z$ as the set of alternatives (i.e., set $X = F$). Each state $s \in S$ describes all the relevant aspects of the world, hence the states are mutually exclusive. Moreover, the consequence $f(s)$ of act $f$ depends on the true state of the world, thus the agent may be uncertain about the consequences of his acts. We would like to represent the agent's preference relation $\succeq$ on $F$ by some $U : F \to \mathbb{R}$ such that

$$U(f) \equiv E[u \circ f]$$

(in the sense of (OR)) where $u : Z \to \mathbb{R}$ is a "utility function" on $Z$ and $E$ is an expectation operator on $S$. That is, we want

$$f \succeq g \iff U(f) \equiv E[u \circ f] \geq E[u \circ g] \equiv U(g). \tag{EUR}$$

In the formulation of Von Neumann and Morgenstern, the probability distribution (and hence the expectation operator $E$) is objectively given. In fact, they formulate acts as lotteries, i.e, probability distributions on $Z$. In such a world, they characterize the conditions (on $\succeq$) under which $\succeq$ is representable in the sense of (EUR).

For the cases of our concern, there is no objectively given probability distribution on $S$. For instance, the likelihood of the strategies that will be played by the other players is not objectively given. We therefore need to determine the agents' (subjective) probability assessments on $S$.

Anscombe and Aumann develop a tractable model in which the agent's subjective probability assessments are determined using his attitudes towards the lotteries (with objectively given probabilities) as well as towards the acts with uncertain consequences. To do this, they consider the agents' preferences on the set $P^S$ of all "acts" whose outcomes are lotteries on $Z$, where $P$ is the set of all lotteries (probability distributions on $Z$).

In this set up, it is straightforward to determine the agent's probability assessments. Consider a subset $A$ of $S$ and any two consequences $x, y \in Z$ with $x \succ y$. Consider the act $f_A$ that yields the sure lottery of $x$ on $A$,[2] and the sure lottery of $y$ on $S \backslash A$. (See Figure 1.) Under the sufficient continuity assumptions (which are also necessary for

[2] That is, $f_A(s) = \delta_x$ whenever $s \in A$ where $\delta_x$ assigns the probability 1 to the outcome $x$.

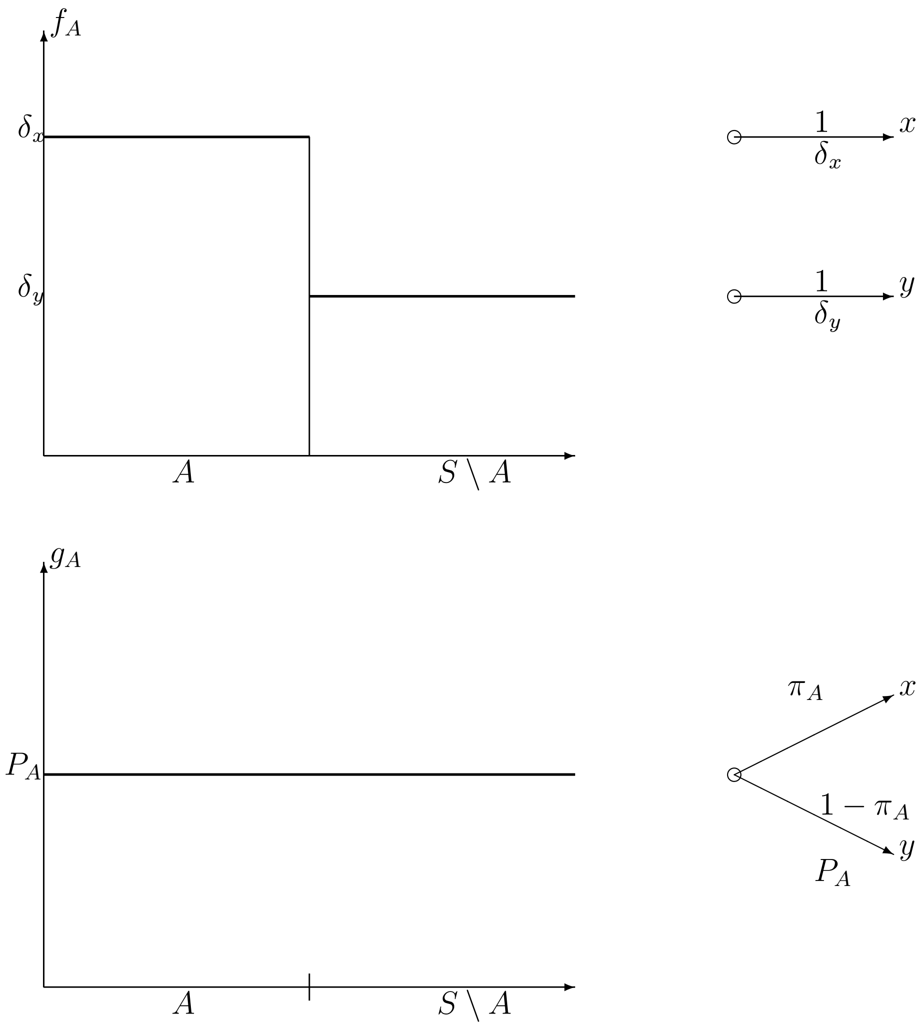


Figure 1: Figure for Anscombe and Aumann

representability), there exists some $\pi_A \in [0,1]$ such that the agent is indifferent between $f_A$ and the act $g_A$ that always yield the lottery $p_A$ that gives $x$ with probability $\pi_A$ and $y$ with probability $1-\pi_A$. Then, $\pi_A$ is the (subjective) probability the agent assigns to the event $A$ — under the assumption that $\pi_A$ does not depend on which alternatives $x$ and $y$ are used. In this way, we obtain a probability distribution on $S$. Using the theory of Von Neumann and Morgenstern, we then obtain a representation theorem in this extended space where we have both subjective uncertainty and objectively given risk.

Savage develops a theory with purely subjective uncertainty. Without using any objectively given probabilities, under certain assumptions of "tightness", he derives a unique probability distribution on $S$ that represent the agent's beliefs embedded in his preferences, and then using the theory of Von Neumann and Morgenstern he obtain a representation theorem — in which both utility function and the beliefs are derived from the preferences.

We will now present the theories of Von Neumann and Morgenstern and Savage.

# 4 Von Neumann and Morgenstern

We consider a finite set $Z$ of prizes, and the set $P$ of all probability distributions $p : Z \to [0,1]$ on $Z$, where $\sum_{z\in Z} p(z) = 1$. We call these probability distributions lotteries. We would like to have a theory that constructs a player's preferences on the lotteries from his preferences on the prizes. A preference relation $\succeq$ on $P$ is said to be represented by a von Neumann-Morgenstern utility function $u : Z \to \mathbb{R}$ if and only if

$$p \succeq q \iff U(p) \equiv \sum_{z\in Z} u(z)p(z) \geq \sum_{z\in Z} u(z)q(z) \equiv U(q) \tag{1}$$

for each $p, q \in P$. Note that $U : P \to \mathbb{R}$ represents $\succeq$ in ordinal sense. That is, the agent acts as if he wants to maximize the expected value of $u$.

The necessary and sufficient conditions for a representation as in (1) are as follows:

**Axiom 2** $\succeq$ *is complete and transitive.*

This is necessary by Theorem 2, for $U$ represents $\succeq$ in ordinal sense. The second condition is called *independence* axiom, stating that a player's preference between two


Figure 2: Two lotteries

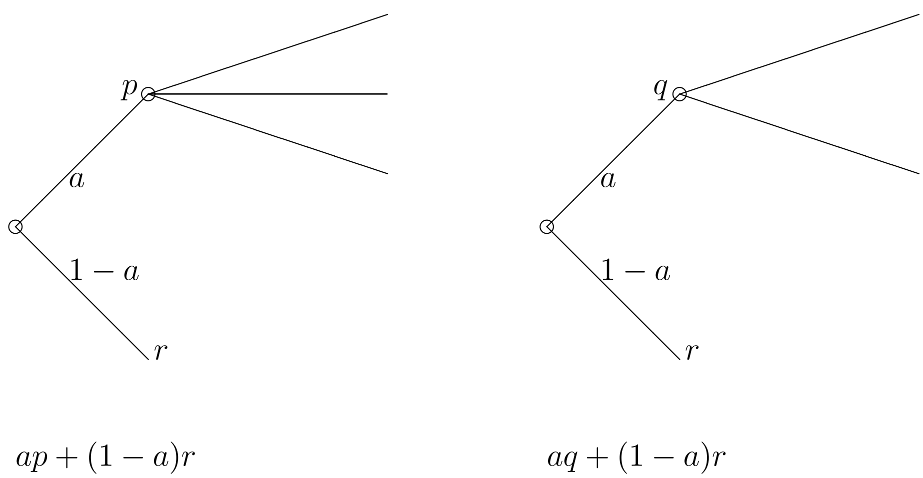


Figure 3: Two compound lotteries

lotteries $p$ and $q$ does not change if we toss a coin and give him a fixed lottery $r$ if "tail" comes up.

**Axiom 3** *For any $p, q, r \in P$, and any $a \in (0, 1]$, $ap + (1-a)r \succ aq + (1-a)r \iff p \succ q$.*

Let $p$ and $q$ be the lotteries depicted in Figure 2. Then, the lotteries $ap + (1-a)r$ and $aq + (1-a)r$ can be depicted as in Figure 3, where we toss a coin between a fixed lottery $r$ and our lotteries $p$ and $q$. Axiom 3 stipulates that the agent would not change his mind after the coin toss. Therefore, our axiom can be taken as an axiom of "dynamic consistency" in this sense.

The third condition is *continuity* axiom. It states that there are no "infinitely good"

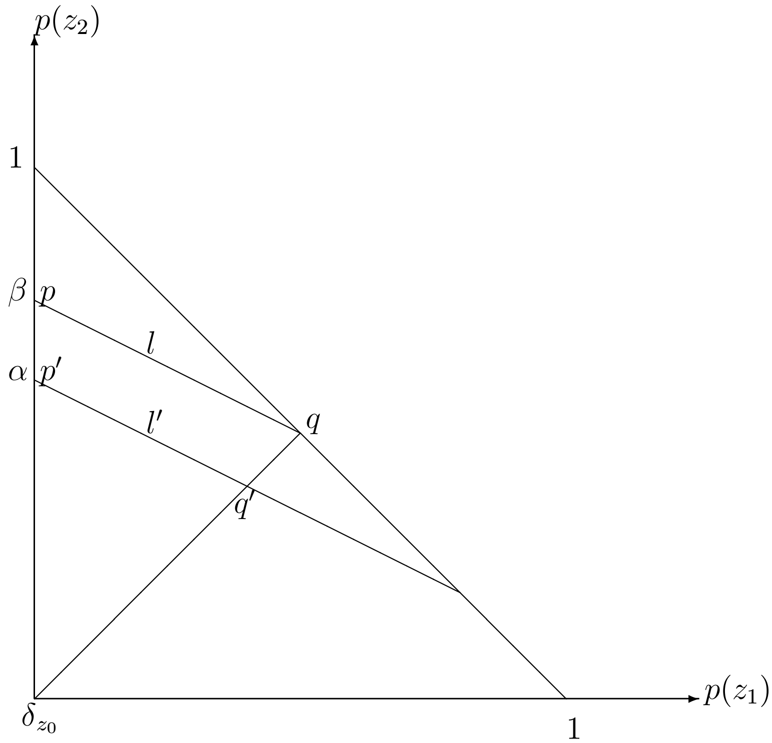


Figure 4: Indifference curves on the space of lotteries

or "infinitely bad" prizes. [Some degree of continuity is also required for ordinal representation.]

**Axiom 4** *For any $p, q, r \in P$, if $p \succ q \succ r$, then there exist $a, b \in (0, 1)$ such that $ap + (1-a)r \succ q \succ bp + (1-r)r$.*

Axioms 3 and 4 imply that, given any $p, q, r \in P$ and any $a \in [0, 1]$,

$$\text{if } p \sim q, \text{ then } ap + (1-a)\,r \sim aq + (1-a)r. \tag{2}$$

This has two implications:

1. The indifference curves on the lotteries are straight lines.

2. The indifference curves, which are straight lines, are parallel to each other.

To illustrate these facts, consider three prizes $z_0, z_1$, and $z_2$, where $z_2 \succ z_1 \succ z_0$. A lottery $p$ can be depicted on a plane by taking $p(z_1)$ as the first coordinate (on the horizontal axis), and $p(z_2)$ as the second coordinate (on the vertical axis). $p(z_0)$ is $1 - p(z_1) - p(z_2)$. [See Figure 4 for the illustration.] Given any two lotteries $p$ and $q$, the convex combinations $ap + (1-a)q$ with $a \in [0,1]$ form the line segment connecting $p$ to $q$. Now, taking $r = q$, we can deduce from (2) that, if $p \sim q$, then $ap + (1-a)q \sim aq + (1-a)q = q$ for each $a \in [0,1]$. That this, the line segment connecting $p$ to $q$ is an indifference curve. Moreover, if the lines $l$ and $l'$ are parallel, then $\alpha/\beta = |q'|/|q|$, where $|q|$ and $|q'|$ are the distances of $q$ and $q'$ to the origin, respectively. Hence, taking $a = \alpha/\beta$, we compute that $p' = ap + (1-a)\delta_{z_0}$ and $q' = aq + (1-a)\delta_{z_0}$, where $\delta_{z_0}$ is the lottery at the origin, and gives $z_0$ with probability 1. Therefore, by (2), if $l$ is an indifference curve, $l'$ is also an indifference curve, showing that the indifference curves are parallel.

Line $l$ can be defined by equation $u_1 p(z_1) + u_2 p(z_2) = c$ for some $u_1, u_2, c \in \mathbb{R}$. Since $l'$ is parallel to $l$, $l'$ can also be defined by equation $u_1 p(z_1) + u_2 p(z_2) = c'$ for some $c'$. Since the indifference curves are defined by equality $u_1 p(z_1) + u_2 p(z_2) = c$ for various values of $c$, the preferences are represented by

$$\begin{aligned} U(p) &= 0 + u_1 p(z_1) + u_2 p(z_2) \\ &\equiv u(z_0)p(z_0) + u(z_1)p(z_1) + u(z_2)p(z_2), \end{aligned}$$

where

$$\begin{aligned} u(z_0) &= 0, \\ u(z_1) &= u_1, \\ u(z_2) &= u_2, \end{aligned}$$

giving the desired representation.

This is true in general, as stated in the next theorem:

**Theorem 4** *A relation $\succeq$ on $P$ can be represented by a von Neumann-Morgenstern utility function $u : Z \to R$ as in (1) if and only if $\succeq$ satisfies Axioms 2-4. Moreover, $u$ and $\tilde{u}$ represent the same preference relation if and only if $\tilde{u} = au + b$ for some $a > 0$ and $b \in \mathbb{R}$.*

By the last statement in our theorem, this representation is "unique up to affine transformations". That this, an agent's preferences do not change when we change his von Neumann-Morgenstern (VNM) utility function by multiplying it with a positive number, or adding a constant to it; but they do change when we transform it through a non-linear transformation. In this sense, this representation is "cardinal". Recall that, in ordinal representation, the preferences wouldn't change even if the transformation were non-linear, so long as it was increasing.

# 5 Savage

Take a set $S$ of *states* $s$ of the world, a finite set $Z$ of *consequences* $(x, y, z)$, and take the set $F = Z^S$ of *acts* $f : S \to Z$ as the set of alternatives. Fix a relation $\succeq$ on $F$. We would like to find necessary and sufficient conditions on $\succeq$ so that $\succeq$ can be represented by some $U$ in the sense of (EUR); i.e., $U(f) = E[u \circ f]$. In this representation, both the utility function $u : Z \to R$ and the probability distribution $p$ on $S$ (which determines $E$) are derived from $\succeq$. Theorems 2 and 3 give us the first necessary condition:

**P 1** $\succeq$ *is a preference relation.*

The second condition is the central piece of Savage's theory:

**The Sure-thing Principle** *If an agent prefers some act $f$ to some act $g$ when he knows that some event $A \subset S$ occurs, and if he prefers $f$ to $g$ when he knows that $A$ does not occur, then he must prefer $f$ to $g$ when he does not know whether $A$ occurs or not.* This is the informal statement of the sure-thing principle. Once we determine the agent's probability assessments, it will give us the independence axiom, Axiom 3, of Von Neumann and Morgenstern. The following formulation of Savage, P2, not only implies this informal statement, but also allows us to state it formally, by allowing us to define conditional preferences. (The conditional preferences are also used to define the beliefs.)

**P 2** *Let $f, f', g, g' \in F$ and $B \subset S$ be such that*

$$f(s) = f'(s) \ \text{ and } g(s) = g'(s) \ \text{ at each } s \in B$$

*and*

$$f(s) = g(s) \ \textit{and}\ f'(s) = g'(s) \ \textit{at each}\ s \notin B.$$

*If* $f \succeq g$*, then* $f' \succeq g'$*.*

**Conditional preferences** Using P2, we can define the conditional preferences as follows. Given any $f, g, h \in F$ and $B \subset S$, define acts $f_{|B}^{h}$ and $g_{|B}^{h}$ by

$$f_{|B}^{h}(s) = \begin{cases} f(s) & \text{if } s \in B \\ h(s) & \text{otherwise} \end{cases}$$

and

$$g_{|B}^{h}(s) = \begin{cases} g(s) & \text{if } s \in B \\ h(s) & \text{otherwise} \end{cases}.$$

That is, $f_{|B}^{h}$ and $g_{|B}^{h}$ agree with $f$ and $g$, respectively, on $B$, but when $B$ does not occur, they yield the same default act $h$.

**Definition 4** ***(Conditional Preferences)*** $f \succeq g$ given $B$ *iff* $f_{|B}^{h} \succeq g_{|B}^{h}$.

P2 guarantees that $f \succeq g$ given $B$ is well-defined, i.e., it does not depend on the default act $h$. To see this, take any $h' \in F$, and define $f_{|B}^{h'}$ and $g_{|B}^{h'}$ accordingly. Check that

$$f_{|B}^{h}(s) \equiv f(s) \equiv f_{|B}^{h'}(s) \ \text{ and } \ g_{|B}^{h}(s) \equiv g(s) \equiv g_{|B}^{h'}(s) \ \text{ at each } s \in B$$

and

$$f_{|B}^{h}(s) \equiv h(s) \equiv g_{|B}^{h}(s) \ \text{ and } \ f_{|B}^{h'}(s) \equiv h'(s) \equiv g_{|B}^{h'}(s) \ \text{ at each } s \notin B.$$

Therefore, by P2, $f_{|B}^{h} \succeq g_{|B}^{h}$ iff $f_{|B}^{h'} \succeq g_{|B}^{h'}$.

Note that P2 precisely states that $f \succeq g$ given $B$ is well-defined. To see this, take $f$ and $g'$ arbitrarily. Set $h = f$ and $h' = g'$. Clearly, $f = f_{|B}^{h}$ and $g' = g_{|B}^{h'}$. Morevoer, the conditions in P2 define $f'$ and $g$ as $f' = f_{|B}^{h'}$ and $g = g_{|B}^{h}$. Thus, the conclusion of P2, "if $f \succeq g$, then $f' \succeq g'$", is the same as "if $f_{|B}^{h} \succeq g_{|B}^{h}$, then $f_{|B}^{h'} \succeq g_{|B}^{h'}$.

**Exercise 1** *Show that the informal statement of the sure-thing principle is formally true: given any* $f_1, f_2 \in F$*, and any* $B \subseteq S$*,*

$$[(f_1 \succeq f_2 \ \textit{given}\ B) \ \textit{and}\ (f_1 \succeq f_2 \ \textit{given}\ S \backslash B)] \Rightarrow [f_1 \succeq f_2].$$

*[Hint: define* $f := f_1 = f_{1|B}^{f_1} = f_{1|S\backslash B}^{f_1}$*,* $g' := f_2 = f_{2|B}^{f_2} = f_{2|S\backslash B}^{f_2}$*,* $f' := f_{1|B}^{f_2} = f_{2|S\backslash B}^{f_1}$*, and* $g := f_{2|B}^{f_1} = f_{1|S\backslash B}^{f_2}$*. Notice that you do not need to invoke P2 (explicitly).]*

Recall that our aim is to develop a theory that relates the preferences on the acts with uncertain consequences to the preferences on the consequences. (The preference relation $\succeq$ on $F$ is extended to $Z$ by embedding $Z$ into $F$ as constant acts. That is, we say $x \succeq x'$ iff $f \succeq f'$ where $f$ and $f'$ are constant acts that take values $x$ and $x'$, respectively.) The next postulate does this for conditional preferences:

**P 3** *Given any $f, f' \in F$, $x, x' \in Z$, and $B \subset S$, if $f \equiv x$, $f' \equiv x'$, and $B \neq \emptyset$, then*

$$f \succeq f' \text{ given } B \iff x \succeq x'.$$

For $B = S$, P3 is rather trivial, a matter of definition of a consequence as a constant act. When $B \neq S$, P3 is needed as an independent postulate. Because the conditinal preferences are defined by setting the outcomes of the acts to the same default act when the event does not occur, and two distinct constant acts cannot take the same value.

**Representing beliefs with qualitative probabilities** We want to determine our agent's beliefs embedded in $\succeq$. Towards this end, given any two events $A$ and $B$, we want to determine which event our agent thinks is more likely. To do this, let us take any two consequences $x, x' \in Z$ with $x \succ x'$. Our agent is asked to choose between the two gambles (acts) $f_A$ and $f_B$ with

$$\begin{aligned} f_A(s) &= \begin{cases} x & \text{if } s \in A \\ x' & \text{otherwise} \end{cases}, \\ f_B(s) &= \begin{cases} x & \text{if } s \in B \\ x' & \text{otherwise} \end{cases}. \end{aligned} \tag{3}$$

If our agent prefers $f_A$ to $f_B$, we can infer that he finds event $A$ more likely than event $B$, for he prefers to get the "prize" when $A$ occurs, rather than when $B$ occurs.

**Definition 5** *Take any $x, x' \in Z$ with $x \succ x'$. Given any $A, B \subseteq S$, we say that $A$* is at least as likely as *$B$ (and write $A \dot{\succeq} B$) iff $f_A \succeq f_B$, where $f_A$ and $f_B$ defined by (3).*

We want to make sure that this gives us well-defined beliefs. That is, it should not be the case that, when we use some $x$ and $x'$, we infer that agent finds $A$ strictly more likely than $B$, but when we use some other $y$ and $y'$, we infer that he finds $B$ strictly more likely than $A$. Our next assumption guaranties that $\dot{\succeq}$ is well-defined.

**P 4** *Given any $x, x', y, y' \in Z$ with $x \succ x'$ and $y \succ y'$, define $f_A, f_B, g_A, g_B$ by*

$$\begin{aligned} f_A(s) &= \begin{cases} x & \text{if } s \in A \\ x' & \text{otherwise} \end{cases}, \quad g_A(s) = \begin{cases} y & \text{if } s \in A \\ y' & \text{otherwise} \end{cases} \\ f_B(s) &= \begin{cases} x & \text{if } s \in B \\ x' & \text{otherwise} \end{cases}, \quad g_B(s) = \begin{cases} y & \text{if } s \in B \\ y' & \text{otherwise} \end{cases}. \end{aligned}$$

*Then,*

$$f_A \succeq f_B \iff g_A \succeq g_B.$$

Finally, make sure that we can find $x$ and $x'$ with $x \succ x'$:

**P 5** *There exist some $x, x' \in Z$ such that $x \succ x'$.*

We have now a well-defined relation $\dot{\succeq}$ that determines which of two events is more likely. It turns out that, $\dot{\succeq}$ is a *qualitative probability*, defined as follows:

**Definition 6** *A relation $\dot{\succeq}$ between the events is said to be a* qualitative probability *iff*

1. *$\dot{\succeq}$ is complete and transitive;*

2. *given any $B, C, D \subset S$ with $B \cap D = C \cap D = \emptyset$, we have*

$$B \dot{\succeq} C \iff B \cup D \dot{\succeq} C \cup D;$$

3. *$B \dot{\succeq} \emptyset$ for each $B \subset S$, and $S \dot{\succ} \emptyset$.*

**Exercise 2** *Show that, under the postulates P1-P5, the relation $\dot{\succeq}$ defined in Definition 5 is a qualitative probability.*

**Quantifying the qualitative probability assassments** Savage uses *finitely-additive probability measures on the discrete sigma-algebra*:

**Definition 7** *By a* probability measure*, we mean a function $p : 2^S \to [0, 1]$ with*

1. *if $B \cap C = \emptyset$, then $p(B \cup C) = p(B) + p(C)$, and*

2. *$p(S) = 1$.*

We would like to represent our qualitative probability $\dot{\succeq}$ with a (quantitative) probability measure $p$ in the sense that

$$B\dot{\succeq}C \iff p(B) \geq p(C) \qquad \forall B, C \subseteq S. \tag{QPR}$$

**Exercise 3** *Show that, if a relation $\dot{\succeq}$ can be represented by a probability measure, then $\dot{\succeq}$ must be a qualitative probability.*

When $S$ is finite, since $\dot{\succeq}$ is complete and transitive, by Theorem 2, it can be represented by some function $p$, but there might be no such function satisfying the condition 1 in the definition of probability measure. Moreover, $S$ is typically infinite. (Incidentally, the theory that follows requires $S$ to be infinite.)

We are intersted in the preferences that can be considered coming from an agent who evaluates the acts with respect to their expected utility, using a utility function on $Z$ and a probability measure on $S$ that he has in his mind. Our task at this point is to find what probability $p(B)$ he assigns to some arbitrary event $B$. Imagine that we ask this person whether $p(B) \geq 1/2$. Depending on his sincere answer, we determine whether $p(B) \in [1/2)$ or $p(B) \in [0, 1/2, 1]$. Given the interval, we ask whether $p(B)$ is in the upper half or the lower half of this interval, and depending on his answer, we obtain a smaller interval that contains $p(B)$. We do this ad infinitum. Since the length of the interval at the $n$th iteration is $1/2^n$, we learn $p(B)$ at the end. For example, let's say that $p(B) = 0.77$. We first ask if $p(B) \geq 1/2$. He says Yes. We ask now if $p(B) \geq 3/4$. He says Yes. We then ask if $p(B) \geq 7/8$. He says No. Now, we ask if $p(B) \geq 13/16 = (3/4 + 7/8)/2$. He says No again. We now ask if $p(B) \geq 25/32 = (3/4 + 7/8)/2$. He says No. Now we ask if $p(B) \geq 49/64$. He says Yes now. At this point we know that $49/64 \cong 0.765 \leq p(B) < 25/32 \cong 0.781$. As we ask further we get a better answer.

This is what we will do, albeit in a very abstract setup. Assume that $S$ is *infinitely divisible under* $\dot{\succeq}$. That is, $S$ has

- a partition $\{D_1^1, D_1^2\}$ with $D_1^1 \cup D_1^2 = S$ and $D_1^1 \dot{\sim} D_1^2$,
- a partition $\{D_2^1, D_2^2, D_2^3, D_2^4\}$ with $D_2^1 \cup D_2^2 = D_1^1$, $D_2^3 \cup D_2^4 = D_1^2$, and $D_2^1 \dot{\sim} D_2^2 \dot{\sim} D_2^3 \dot{\sim} D_2^4$,
- $\vdots$

- a partition $\left\{D_n^1, \cdots, D_n^{2^n}\right\}$ with $D_n^1 \cup D_n^2 = D_{n-1}^1, \ldots, D_n^{2k-1} \cup D_n^{2k} = D_{n-1}^k, \ldots,$ and $D_n^1 \dot{\sim} \cdots \dot{\sim} D_n^{2^n}$,
- $\vdots$

ad infinitum.

| $S$ | | | |
|---|---|---|---|
| $D_1^1$ | | $D_1^2$ | |
| $D_2^1$ | $D_2^2$ | $D_2^3$ | $D_2^4$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

**Exercise 4** *Check that, if $\dot{\succeq}$ is represented by some $p$, then we must have $p(D_n^r) = 1/2^n$.*

Given any event $B$, for each $n$, define

$$k(n, B) = \max\left\{r | B \dot{\succeq} \bigcup_{i=1}^{r} D_n^i\right\},$$

where we use the convention that $\cup_{i=1}^r D_n^i = \emptyset$ whenever $r < 1$. Define

$$p(B) := \lim_{n\to\infty} \frac{k(n, B)}{2^n}. \tag{4}$$

Check that $k(n, B)/2^n \in [0, 1]$ is non-decreasing in $n$. Therefore, $\lim_{n\to\infty} k(n, B)/2^n$ is well-defined.

Since $\dot{\succeq}$ is transitive, if $B \dot{\succeq} C$, then $k(n, B) \geq k(n, C)$ for each $n$, yielding $p(B) \geq p(C)$. This proves the $\Longrightarrow$ part of (QPR) under the assumption that $S$ is infinitely-divisibile. The other part ($\Leftarrow$) is implied by the following assumption:

**P 6'** *If $B \dot{\succ} C$, then there exists a finite partition $\{D^1, \ldots, D^n\}$ of $S$ such that $B \dot{\succ} C \cup D^r$ for each $r$.*

Under P1-P5, P6' also implies that $S$ is infinitely-divisibile. (See the definition of "tight" and Theorems 3 and 4 in Savage.) Therefore, P1-P6' imply (QPR), where $p$ is defined by (4).

**Exercise 5** *Check that, if $\dot{\succeq}$ is represented by some $p'$, then*

$$\frac{k(n, B)}{2^n} \leq p'(B) < \frac{k(n, B) + 1}{2^n}$$

*at each $B$. Hence, if both $p$ and $p'$ represent $\dot{\succeq}$, then $p = p'$.*

Postulate 6 will be somewhat stronger than P6'. (It is also used to obtain the continuity axiom of Von Neumann and Morgenstern.)

**P 6** *Given any $x \in Z$, and any $g, h \in F$ with $g \succ h$, there exists a partition $\{D^1, \ldots, D^n\}$ of $S$ such that*

$$g \succ h_i^x \text{ and } g_i^x \succ h$$

*for each $i \leq n$ where*

$$h_i^x(s) = \begin{cases} x & \text{if } s \in D^i \\ h(s) & \text{otherwise} \end{cases} \text{ and } g_i^x(s) = \begin{cases} x & \text{if } s \in D^i \\ g(s) & \text{otherwise} \end{cases}.$$

Take $g = f_B$ and $h = f_C$ (defined in (3)) to obtain P6'.

**Theorem 5** *Under P1-P6, there exists a unique probability measure $p$ such that*

$$B \dot{\succeq} C \iff p(B) \geq p(C) \qquad \forall B, C \subseteq S. \tag{QPR}$$

In Chapter 5, Savage shows that, when $Z$ is finite, Postulates P1-P6 imply Axioms 2-4 of Von Neumann and Morgenstern —as well as their modeling assumptions such as only the probability distributions on the set of prizes matter. In this way, he obtains the following Theorem:[3]

**Theorem 6** *Assume that $Z$ is finite. Under P1-P6, there exist a utility function $u : Z \to R$ and a probability measure $p : 2^S \to [0, 1]$ such that*

$$f \succeq g \iff \sum_{z \in Z} p(\{s | f(s) = z\}) u(z) \geq \sum_{z \in Z} p(\{s | g(s) = z\}) u(z)$$

*for each $f, g \in F$.*

[3] For the inifinte prize-set $Z$, we need the infinite version of the sure-thing principle:

**P 7** *If we have $f \succeq g(s)$ given $B$ for each $s \in B$, then $f \succeq g$ given $B$. Likewise, if $f(s) \succeq g$ given $B$ for each $s \in B$, then $f \succeq g$ given $B$.*

Under P1-P7, we get the expected utility representation for general case.