The Core

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Lecture Outline

- Coalitional Games and the Core
  - The non-transferable utility ("NTU") formulation
  - The transferable utility ("TU") formulation
- TU Core
  - Interpretations
  - Various Applications
- NTU Core Applications
  - Matching problems
  - Exchange economies

Coalitional Games

- A coalitional game is a pair (N,w) where
  - N is the set of players
  - For each $S \subset N$, $w(S) \subset R^S$, interpreted as the set of payoff vectors or "value allocations" for coalition S.
- Interpretations
  1. Points in w(S) are payoffs the members of S can achieve on their own, as in an exchange economy. This is our standard interpretation.
  2. Points in w(S) are payoffs the members of S can guarantee for themselves by some feasible action.
- Alternative: Games in partition function form...?!

Feasibility, Blocking & Core

- A value allocation $x \in R^N$ is...
  - ..."feasible" if $x \in w(N)$
  - ..."blocked" by coalition S if there is a point in w(S) that is strictly improving for every member of S

$$\exists \hat{x} \in w(S), \forall i \in S, \hat{x}_i > x_i$$

  - ...in the "core" if it is feasible and unblocked.
- More precisely,

$$Core(N,w) = \{x \in w(N) : \neg(\exists S \subset N, \exists \hat{x} \in w(S), \forall i \in S, \hat{x}_i > x_i)\}$$

Cohesiveness

- To say that feasibility means $x \in w(N)$ implicitly assumes a condition we call "cohesiveness."
- An NTU game is "cohesive" if for every partition $\{S_1,\ldots,S_k\}$ of N

$$\left[x_{S_j} \in w(S_j), j = 1,\ldots,k\right] \Rightarrow x \in w(N)$$

- We limit attention to cohesive games.

Coalitional Games with Transfers

- A coalitional game with transferable utility (TU) is a pair (N,v) where
  - N is the set of players
  - v maps "coalitions" (subsets $S \subset N$) to real numbers subject to $v(\varnothing)=0$.
  - The TU game (N,v) is an alternative description of the NTU game (N,w) in which

$$w(S) = \left\{x \in \Re^S : \sum_{i \in S} x_i \le v(S)\right\}$$

Cohesiveness

- The TU game (N,v) is cohesive if for every partition $(S_1,\ldots,S_k)$ of N

$$\sum_{j=1}^{k} v(S_j) \le v(N)$$

- This means roughly that "the coalition of the whole can do anything that its subcoalitions can do separately."

TU Core

- TU games are special cases of NTU games, and the core is defined in the corresponding way. If (N,v) is a TU game and (N,w) is the corresponding NTU representation, then:

$$Core(N,v) = Core(N,w)$$
$$= \left\{x \in \Re^N : x \in w(N), \neg\left(\exists S \subset N, \exists \hat{x} \in w(S), \forall i \in S, \hat{x}_i > x_i\right)\right\}$$
$$= \left\{x \in \Re^N : \sum_{i \in N} x_i = v(N), \neg\left((\exists S \subset N)\sum_{i \in S} x_i < v(S)\right)\right\}$$
$$= \left\{x \in \Re^N : \sum_{i \in N} x_i = v(N), (\forall S \subset N)\sum_{i \in S} x_i \ge v(S)\right\}$$

Core Payoffs as CE Prices

- Market interpretation of the coalitional game
  - Each player brings an indivisible bundle of resources
  - Brokers bid to buy players' resources
- Competitive equilibrium prices x satisfy
  - Zero total profits for the brokers

$$v(N) - \sum_{i \in N} x_i = 0$$

  - No missed profit opportunities

$$v(S) - \sum_{i \in S} x_i \le 0 \text{ for all } S \subset N$$

- Note well
  - x is a CE price vector if and only if $x \in \mathrm{Core}(N,v)$
  - Works even though market is non-anonymous: resources cannot be separated from their owners.

Some TU Examples

Simple Games

Convex Games

Vickrey-Clarke-Groves

"Simple" Games

- Definitions:
  - "Winning" coalitions S can enforce their desired outcome: v(S)=1.
  - "Losing" coalitions S cannot: v(S)=0.
  - "Veto" players are players that are part of every winning coalition.
- Observe: If $x \in \mathrm{Core}(N,v)$, then $x_i \le v(N)-v(N\backslash i)$. (Why?)
- Therefore,
  - Only veto players earn positive payoffs in the core.
  - If there are no veto players, the core is empty.

"Convex" Games, 1

- A game (N,v) is "convex" if for all S,T $v(S)+v(T) \le v(S \cap T)+v(S \cup T)$.
- Intuition: players are <u>complements</u>: if $i \notin S' \supset S''$, then $v(S' \cup \{i\})-v(S') \ge v(S'' \cup \{i\})-v(S'')$.
  - Proof: Take $S=S'$ and $T=S'' \cup \{i\}$ and apply the definition.
- "Greedy algorithm":
  - List the players in some order, say 1,…,|N|.
  - Set $x_i=v(\{1,\ldots,i\})-v(\{1,\ldots,i-1\})$.

“Convex” Games, 2

- Theorem. If (N,v) is a convex game, then each of the |N|! greedy value allocations is in Core(N,v).
- Proof. Fix an ordering of the players. Let coalition $T=\{j_1,\ldots j_n\}$ be listed in ascending order.

$$\begin{aligned}\sum_{k=1}^{n} x_{j_k} &= \sum_{k=1}^{n}\left[v(1,\ldots,j_k)-v(1,\ldots,j_k-1)\right]\\ &\geq \sum_{k=1}^{n}\left[v(j_1,\ldots,j_k)-v(j_1,\ldots,j_{k-1})\right]\\ &= v(j_1,\ldots,j_n)=v(T)\end{aligned}$$

- True or false?: “The core of a convex game is the convex hull of the |N|! ‘greedy’ value allocations.”

An Aside

- The defining condition is a version of “supermodularity.”
  - Given a a partially ordered set (Z,≤) on which the meet and join, $x\wedge y=\sup\{z: z\leq x, z\leq y\}$ and $x\vee y=\inf\{z: z\geq x, z\geq y\}$ are well-defined, a function $f{:}Z\rightarrow R$ is supermodular if $f(x)+f(y)\leq f(x\wedge y)+f(x\vee y)$.
  - In this example, ≤, ∧ and ∨ are ⊂, ∩, and ∪.
  - Supermodular objective functions characterize choice variables that are complements.
  - Supermodular dual (cost or expenditure) functions characterize choice variables that are substitutes.

Exercise

- Suppose each player is endowed with a vector of resources z_i in R^k. Coalition S can produce $v(S)=f(\Sigma_{i\in S} z_i)$.
  - If k=1, under what conditions on f is the game (N,v) “convex” for very possible endowment vector z?
  - If k=|N| and player i has 1 unit of resource i and no units of other resources, under what conditions on f is (N,v) convex?
  - If k is general, under what conditions on f is the game (N,v) convex for every possible endowment vector z?

Vickrey-Clarke-Groves

- Formulation
  - There is a finite set of “decisions” D controlled by player 0, who can exclude participation.
  - Each player i>0 has a valuation $v_i(d)$ for each $d\in D$.
  - $v(S)=0$ if $0\notin S$; otherwise

$$v(S)=\max\left\{\sum_{i\in S} v_i(d) : d\in D\right\}$$

  - “Pivot” mechanism payoff:

$$x_i = v(N)-v(N\setminus i)$$

Vickrey-Clarke-Groves & the Core

- Players are "substitutes" if

$$i \notin T \supset S \Rightarrow v(S \cup \{i\}) - v(S) \geq v(T \cup \{i\}) - v(T)$$

- The Vickrey payoff is given by:

$$x_i(S) = \begin{cases} v(S) - v(S \setminus i) \text{ for } i > 0 \\ v(S) - \sum_{i \in S, i \neq 0} x_i(S) \end{cases}$$

- Theorem. Players are substitutes if and only if for all $S \subset N$, $x(S) \in \text{Core}(S,v)$.

Exercise

- Prove the preceding theorem.
- Suppose each player is endowed with a vector of resources z_i in R^k. Coalition S can produce $v(S)=f(\Sigma_{i \in S} z_i)$.
  - If k=1, under what conditions on f are the players "substitutes" in the game (N,v) for very possible endowment vector z?
  - If k is general, under what conditions on the technology f are the players "substitutes" for every possible endowment vector z?

NTU Applications

Matching problems

Competitive equilibrium in core

Matching Examples

- These are models in which the players have preferences over the other(s) with whom they are matched, but no money changes hands
  - The Gale-Shapley "marriage problem."
  - The "roommate problem."
  - The "college admissions" problem.
  - …

The "Marriage Problem"

- Players i=1,...,m are "men" and players m+1,...,N are "women."
  - A "match" is a mapping from f:N→N such that $f=f^{-1}$ and such that each woman is matched to a man or to herself and symmetrically each man...
  - Each player's utility function depends only on the player's own match.
  - For coalition S, the set of feasible utility profiles are those corresponding to the feasible matches among the members of S.
- Is the core non-empty?
  - Assume all preferences are strict.

Stable Matches & the Core

- A match f is "stable" if two conditions hold:
  1. it is individually rational: for all m and w:
     $$U_m(f(m)) > U_m(m) \,\&\, U_w(f(w)) > U_w(w)$$
  2. there is no man m and woman w for whom:
     $$U_m(w) > U_m(f(m)) \,\&\, U_w(m) > U_w(f(w))$$
- Theorem. A value allocation is in the core of the "marriage problem" game if and only if the corresponding match is stable.
- Proof. If the match f is unstable as above, then it is "blocked" by coalition {m,w}. If the value allocation is blocked by coalition S, then for each i∈S, (i,f(i)) is unstable.

Gale-Shapley Theorem

- Theorem. There exists a stable match in the marriage problem.
- Proof. Apply a "deferred acceptance algorithm," as follows:
  - Each player reports his/her preferences to the "yenta"—a computer routine. There are two versions, depending on which side "makes offers." We study the "women offer" version...

Yenta's Procedure

1. Each woman makes an offer to the first choice on her list.
2. Each man reviews his current offers and holds onto the best one (but "defers acceptance").
3. The rejected women cross the rejecting man off their lists and make offers to the best remaining choice.
4. If any new offers are made at this round, return to step 2. Otherwise, stop and the tentative match is finalized.

Analysis

- After each round, each man's utility for the tentative match rises.
- Therefore, at the final match,
  - No man ever prefers to be matched with a woman he has previously rejected.
  - No woman prefers to be matched to a man to whom she has not yet offered.
  - The match is stable.

Competitive Equilibrium and the NTU Core

- A cluster of ideas
  - Trading areas: can countries in a free trade regime benefit from special agreements?
  - Firm: can producers in competitive markets benefit by "integrating" and trading on special terms?
  - Can a coalition of players facing given prices all be strictly better off by trading among themselves instead?
    - Hint: Think about Arrow's proof of the first theorem of welfare economics.

CE is in the Core

- Consider an exchange economy (N,L,u,ω)
  - N is the set of agents/players
  - L is the number of goods
  - $U_n:R_+^L \rightarrow R$ is agent n's utility function, which is increasing and continuous
  - ω_n is agent n's endowment vector
- Assume that a competitive equilibrium exists. Mimic Arrow's argument for the first welfare theorem
  - Any allocation z_S that is strictly preferred by every member i of coalition S has $p \cdot z_i > 0$ at the competitive equilibrium prices and is therefore infeasible for the coalition.