Auctions 4:

Interdependent values

Multi-unit auctions

(partial lectures)

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1 Interdependent (common) values.

- Each bidder receives private signal $X_i \in [0, w_i]$. ($w_i = \infty$ is possible)

- $(X_1, X_2, \ldots, X_n)$ are jointly distributed according to commonly known F ($f > 0$).

- $$V_i = v_i(X_1, X_2, \ldots, X_n).$$

  $$v_i(x_1, x_2, \ldots, x_n) \equiv E\left[V_i \mid X_j = x_j \text{ for all } j\right].$$

  Typically assumed that functional forms $\{v_i\}_{i=1}^N$ are commonly known.

- $v_i(0, 0, \ldots, 0) = 0$ and $E[V_i] < \infty$.

- Symmetric case:

  $$v_i(x_i, \mathbf{x}_{-i}) = v(x_i, \mathbf{x}_{-i}) = v(x_i, \pi(\mathbf{x}_{-i})).$$

2 Brief analysis

- Common values / Private values / Affiliated values / Interdependent values.
- Winner's curse.
- Second-price auction: Pivotal bidding—I bid what I get if i just marginally win.
- First-price auction: "Usual" analysis—differential equation, ...
- English auction: See below.
- Revenue ranking: English $>$ SPA $>$ FPA.

  (!) Interdependency and affiliation are important for the first part.

3 Second-price auction

Define

$$v(x,y) = E\left[V_1 \mid X_1 = x,\ Y_1 = y\right].$$

Equilibrium strategy

$$\beta^{\mathrm{II}}(x) = v(x,x).$$

Indeed,

$$\begin{aligned}\Pi(b,x) &= \int_0^{\beta^{-1}(b)} \left(v(x,y) - \beta(y)\right) g(y|x)dy \\ &= \int_0^{\beta^{-1}(b)} \left(v(x,y) - v(y,y)\right) g(y|x)dy.\end{aligned}$$

Π is maximized by choosing $\beta^{-1}(b) = x$, that is, $b = \beta(x)$.

4 Example

1. Suppose S_1, S_2, and T are uniformly and independently distributed on $[0,1]$. There are two bidders, $X_i = S_i + T$. The object has a common value
$$V = \frac{1}{2}(X_1 + X_2).$$

2. In this example, in the first price auction:
$$\beta^{\mathsf{I}}(x) = \frac{2}{3}x, \qquad E[R^I] = \frac{7}{9}.$$

3. In the second-price auction $v(x,y) = \frac{1}{2}(x+y)$ and so
$$\beta^{\mathsf{II}}(x) = x, \qquad E[R^I] = \frac{5}{6}.$$

5 Linkage principle

Define
$$W^A(z,x) = E\left[P(z) \mid X_1 = x,\ Y_1 < z\right]$$
expected price paid by the winning bidder when she receive signal x but bids z.

Proposition: (Linkage principle):

Let A and B be two auction forms in which the highest bidder wins and (she only) pays positive amount. Suppose that symmetric and increasing equilibrium exists in both forms. Suppose also that

1. for all x, $W_2^A(x,x) \geq W_2^B(x,x)$.
2. $W^A(0,0) = W^B(0,0) = 0$.

Then, the expected revenue in A is at least as large as the expected revenue in B.

So,the greater the linkage between a bidder's own information and how he perceives the others will bid the greater is the expected price paid upon winning.