Learning 5:

Adjustment with persistent noise

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1 Adjustment Process

1.1 Game

- N – population size.
- 2×2 symmetric game. (A, B) – actions.
- Suppose there are 3 NE:

  (A, A); (B, B); $(\alpha^* A + (1-\alpha^*)B, \alpha^* A + (1-\alpha^*)B)$.
- Suppose $\alpha^* < \frac{1}{2} \Rightarrow (A, A)$ – risk-dominant NE.

	A	B
A	2,2	0,0
B	0,0	1,1

Here $\alpha^* = \frac{1}{3}$.

1.2 State space

- $\theta_t \in \Theta = [0, \ldots, N]$ – # of players using A.

- Denote

$$u_A(\theta_t) = \frac{\theta_t}{N} u(A, A) + \frac{N - \theta_t}{N} u(A, B); \; u_B(\theta_t) = \ldots.$$

1.3 Deterministic process

- "Darwinian" dynamics: $\theta_{t+1} = P(\theta_t)$, where

$$sgn(P(\theta_t) - \theta_t) = sgn(u_A(\theta_t) - u_B(\theta_t)).$$

- Ex^0: Best-response dynamics:

$$\theta_{t+1} = BR(\theta_t) = \begin{cases} N, & \text{for } u_A(\theta_t) > u_B(\theta_t), \\ \theta_t, & \text{for } u_A(\theta_t) = u_B(\theta_t), \\ 0, & \text{for } u_A(\theta_t) < u_B(\theta_t). \end{cases}$$

1.4 Noise

- 2ε – probability that a player "mutates" (is replaced) (*after her intended choice), independent across players.

- Note: even if only 1 players "consciously" adjusts at a time, there is a positive probability that the whole population mutates at once.

- Clearly P^ε is ergodic.

1.5 Limiting distribution (in Ex^0)

- N^* is $\arg\min_m (m > N\alpha^*)$;
- $BR(\theta_t \geq N^*) = A$;
- $D_A = \{\theta \geq N^*\}$, $D_B = \{\theta < N^*\}$.
- Only basins of attraction matter: Intentional play depends on which of the two states θ_t is and not on θ_t itself.

2 Result

Proposition: If N is large enough so that $N^* < \frac{N}{2}$, then limit φ^* of invariant distributions puts a point mass on $\theta_t = N$, corresponding to all players playing A.

Proof:

1. For any $\theta_t \in D_A$ ($\in D_B$) probability distribution $P^\varepsilon(\theta_t)$ is the same — the problem can be reduced to two states.

2. Define

$$\begin{aligned} q_{BA} &= \Pr(\theta_{t+1} \in D_B | \theta_t \in D_A); \\ q_{AB} &= \Pr(\theta_{t+1} \in D_A | \theta_t \in D_B). \end{aligned}$$

3. Solve

$$\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} 1-q_{AB} & q_{AB} \\ q_{BA} & 1-q_{BA} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$$

and find

$$\frac{\varphi_2}{\varphi_1} = \frac{q_{BA}}{q_{AB}}.$$

4. Take $\lim_{\varepsilon \to 0}$ of $\frac{\varphi_2}{\varphi_1}$.

To change $A \to B$, at least $N - N^*$ mutations into B are needed; for $B \to A$ at least N^* mutations must happen:

$$\begin{aligned} q_{BA} &\approx \binom{N}{N^*} \varepsilon^{N-N^*} (1-\varepsilon)^{N^*}; \\ q_{AB} &\approx \binom{N}{N^*} \varepsilon^{N^*} (1-\varepsilon)^{N-N^*}. \end{aligned}$$

Thus $\frac{\varphi_2}{\varphi_1} \to 0$ as $\varepsilon \to 0$.

3 Summary

- Selection of risk-dominant equilibrium as the unique long-run steady-state in 2×2 games (almost all models).

- “Learning” procedures tend to select equilibria that are relatively robust to mutations — different from Pareto efficiency.

	A	B
A	2,2	$-a$,0
B	0,$-a$	1,1

(B, B) is risk-dominant if $1 + a > 2$.

- Probabilities (ratios of them) of escaping basins of attraction matter.

4 Local interaction (Ellison)

- If the system starts near "wrong" equilibrium the expected time of adjustment may be quite large.

  In KMR model: the probability of escaping is $\approx \varepsilon^{N^*}$.

- Goal: to explain why stochastic adjustment processes might select the risk-dominant equilibrium in an economically relevant time frame.

- Players located on the circle and interact only with neighbors.

- Player selects an action and is matched randomly with one of the two neighbors.

- Observation: Pair of adjacent As wins the population.

4.1 Adjustment process

1. 2×2 symmetric game. (A, B) – actions.

2. $\Theta = \{A, B\}^N$.

3. Deterministic process: player with A switches its neighbors to A.

   Steady states: "All A", "All B", "$ABAB \ldots - BABA \ldots$" cycle.

4. Noise: Probability 2ε of mutating.

5. Limiting distribution: "All A",

   Convergence: Minimal cost of transition from "all B" is 2 if N is even and is 1 if N is odd. (number of mutations it takes to switch to "all A".)