Learning 3:

Replicator dynamics & Adjustment with persistent noise

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Road Map

1. Definition
2. Asymptotic behavior of Replicator dynamics
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   (b) RD vs. ESS
   (c) RD vs. Perfect equilibria
3. Generalization of RD
4. Learning foundations of RD
   (a) Social learning
   (b) Stimulus-response
5. Adjustment models with persistent randomness

1 Notation

- $G = (S, A)$ a symmetric, 2-player game where
- S is the strategy space;
- $A_{i,j} = u_1(s_i, s_j) = u_2(s_j, s_i)$;
- $x, y \in \Delta$ are mixed strategies; $u(x, y) = x^T Ay$;
- $u(ax + (1 - a)y, z) = au(x, z) + (1 - a)u(y, z)$.

2 ESS

Definition: A (mixed) strategy x is said to be *evolutionarily stable* iff, given any $y \neq x$, there exists $\epsilon_y > 0$ s.t.

$$u(x, (1 - \varepsilon)x + \varepsilon y) > u(y, (1 - \varepsilon)x + \varepsilon y),$$

for each ε in $(0, \epsilon_y]$.

Fact: x is evolutionarily stable iff, $\forall y \neq x$,

1. $u(x, x) \geq u(y, x)$, and
2. $u(x, x) = u(y, x) \Longrightarrow u(x, y) > u(y, y)$.

3 Replicator dynamics

- $p_i(t) = \#$people who plays s_i at t.

- $p(t) =$ total population at t.

- $x_i(t) = \frac{p_i(t)}{p(t)}$; $x(t) = (x_1(t), \dots x_k(t))$.

- $$\dot{x}_i = [u(s_i, x) - u(x, x)]\, x_i = u(s_i - x, x) x_i.$$

4 RD in Rock-Scissors-Paper game

	R	S	P
R	1,1	2+a,0	0,2+a
S	0,2+a	1,1	2+a,0
P	2+a,0	0,2+a	1,1

- Unique Nash Equilibrium (s^*, s^*), where $x^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

- Define $h(x) = \log(x_1 x_2 x_3)$.

- $$\dot{h}(x) = \frac{a}{2}\left(3\,\|x\|^2 - 1\right).$$

- $\min_x \|x\| = 1$, $\arg\min = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

4.1 Dynamics

Three scenarios:

1. $a = 0$ — original RSP; all trajectories are cycles.

2. $a < 0$ — x^* is unstable.

3. $a > 0$ — x^* is stable.

5 Rationalizability

- $\boldsymbol{\xi}(t, x_0)$ is the solution to replicator dynamics starting at x_0.

Theorem: If a pure strategy i is strictly dominated (by y), then $\lim_t \xi_i(t, x_0) = 0$ for any interior x_0.

Proof: Define $v_i(x) = \log(x_i) - \sum_j y_j \log(x_j)$. Then,

$$\begin{aligned} \frac{dv_i(x(t))}{dt} &= \frac{\dot{x}_i}{x_i} - \sum_j y_j \frac{\dot{x}_j}{x_j} \\ &= u(s_i - x, x) - \sum_j y_j u(s_j - x, x) \\ &= u(s_i - y, x) \leq -\epsilon < 0. \end{aligned}$$

Hence, $v_i(\boldsymbol{\xi}(t, x_0)) \to -\infty$, so $\xi_i(t, x_0) \to 0$.

Theorem: If i is not rationalizable, then $\lim_t \xi_i(t, x_0) = 0$ for any interior x_0.

6 Theorems

Theorem: Every ESS x is an asymptotically stable steady state of replicator dynamics.

(If the individuals can inherit the mixed strategies, the converse is also true.)

Proof: Define $C = supp(x)$, $Q = \{y | C \subset supp(y)\}$, $H(y) = \sum_{i \in C} x_i \log(y_i)$.

1. x is a local maximum of H, and
2. $\exists$ a neighborhood $n(x)$ s.t. H is increasing along any trajectory in $Q \cap n(x)$.

   $$\dot{H} = \sum_{i \in C} x_i \frac{\dot{y}_i}{y_i} = \sum_{i \in C} x_i u(s_i - y, y) = u(x - y, y) > 0.$$

NE $\rightarrow$ Steady state in RD;

Stable SS in RD $\rightarrow$ NE.

Theorem: If x is an asymptotically stable steady state of replicator dynamics, then (x, x) is a perfect Nash equilibrium.

Proof:

1. (x, x) is a Nash equilibrium.
   (a) x is stable => $\dot{x}_i = u(s_i - x, x)x_i = 0$.
   (b) Suppose $(x, x) \notin NE$.
   (c) $\exists i \notin supp(x) : u(s_i - x, x) > 0$. [by 1 and 2]
   (d) $\exists \delta > 0, n(x) : u(s_i - y, y) > \delta \; \forall y \in n(x)$.
   (e) $\xi_i(t, y^0) > y_i^0 e^{\delta t}$ if $\xi_i(\cdot, y^0)$ remained in $n(x)$.
2. x is not weakly dominated (since ASS).

7 Non-ESS asymptotic stability

	L	M	R
L	0,0	1,-2	1,1
M	-2,1	0,0	4,1
R	1,1	1,4	0,0

- $NE = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$; mutant $= \left(0, \frac{1}{2}, \frac{1}{2}\right)$.

- RD is asymptotically stable.

- Note: If mixed strategies can be inherited, $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ becomes instable.

8 General dynamics

Definition: A process is *payoff monotone* iff, at each interior x,

$$u(s_i, x) > u(s_j, x) \Leftrightarrow \frac{\dot{x}_i}{x_i} > \frac{\dot{x}_j}{x_j}.$$

Theorem: Under any any "regular" payoff monotone dynamics, if strategy i is eliminated by the process of iterated pure strategy strict dominance, then $\lim_t x_i(t) = 0$.

9 Social learning

- Ask around; if the other person does better, adopt his strategy.

Emulation dynamics ("medium-enhancing"):

Player 2 is a dummy, $p(L) = \frac{1}{3}$.

	L	R
U	9,0	0,0
D	2,0	2,0

- Ask around; if the other makes u' and you make u, then switch with probability max$\{0, b(u\prime - u)\}$.

- Aspiration levels.

10 Stimulus-response

- $u(x, y) \in [0, 1]$

- $x_i^k(t+1) = (1-\gamma u(s^k(t), \cdot))x_i^k(t) + F(s^k(t), i)\gamma u(s^k(t), \cdot)$, where

$F(s^k(t), i) = 1$ if $s^k(t) = i$,

$F(s^k(t), i) = 0$ otherwise.

- Result: As γ goes to 0, trajectories converge to the RD trajectories.